Zeitschrift: IABSE reports of the working commissions = Rapports des commissions de travail AIPC = IVBH Berichte der Arbeitskommissionen

Band: 29 (1979)

Artikel: Finite element aspects of concrete cracking

Autor: Argyris, J.H. / Faust, G. / Willam, K.J.

DOI: https://doi.org/10.5169/seals-23564

IV

Finite Element Aspects of Concrete Cracking

Eléments finis et fissuration du béton

Gesichtspunkte zur Rissberechnung im Beton

J.H. ARGYRIS G. FAUST K.J. WILLAM

Institut für Statik und Dynamik der Luft- und Raumfahrtkonstruktionen
Universität Stuttgart
Stuttgart, Fed. Rep. of Germany

SUMMARY
Two techniques are examined for the finite element analysis of cracking: The smeared approach and the discrete crack formulation are compared and illustrated with the example of a thick-walled concrete ring for which extensive experimental results have been made available by the IBIW III at the Technical University Munich.

RESUME
Deux procédés de calcul de la fissuration par la méthode des éléments finis sont examinés: Le modèle „barbouillé" et la formation discrète de fissures sont comparés et illustrés par l'exemple d'un anneau de béton à paroi épaisse, pour lequel de nombreux résultats de mesures existent.

ZUSAMMENFASSUNG
Zwei Verfahren werden untersucht zur Rissberechnung mit finiten Elementen: Das verschmierte Modell und die diskrete Rissformulierung werden am Beispiel eines dickwandigen Betonringes illustriert, an dem umfangreiche Messdaten des IBIW III, Technische Universität München, vorliegen.

INTRODUCTION

The knowledge of the inelastic behaviour and collapse of reinforced shell structures has been improved and increased in the past by experimental research. Early theoretical estimations of the collapse load were mainly based on the investigation into collapse modes and the use of a kinematical method similar to that known as the theory of yield lines for plates. This method permits to find an upper bound solution of the collapse load problem, but it is hardly suitable for reinforced concrete shells of more complex geometry and loading conditions.

Requirements on more generally applicable techniques led to the further development of numerical methods. One of the main ways of despribing mathematically the inelastic behaviour of structures and calculating the collapse loads is based on the two fundamental theorems on lower and upper bounds and on the use of optimization procedures for linear and non-linear programming. A detailed description of the limit analysis of shells of revolution under axi-symmetric loads as an optimization problem by means of the Ritz method, the Bubnov-Galerkin method, the collocation method, the difference method and the method of finite elements is given in /1/.

METHOD OF SOLUTION

For rotationally symmetric shells under axi-symmetric one-parameter loading the collapse load intensity is found by solving the problem

$$\max_{p,N_\Psi,N_\varphi,M_\Psi,M_\varphi}\ \min_{\dot{v}_\Psi,\dot{v}_\zeta,\lambda,\lambda_p}\ \Lambda(p,N_\Psi,N_\varphi,M_\Psi,M_\varphi,\dot{v}_\Psi,\dot{v}_\zeta,\lambda,\lambda_p) \tag{1}$$

$$p \geqq 0 \qquad \text{on } R_p$$

or

$$\min_{\dot{v}_\Psi,\dot{v}_\zeta,\lambda,\lambda_p}\ \max_{N_\Psi,N_\varphi,M_\Psi,M_\varphi,p}\ \Lambda(p,N_\Psi,N_\varphi,M_\Psi,M_\varphi,\dot{v}_\Psi,\dot{v}_\zeta,\lambda,\lambda_p) \tag{2}$$

$$\lambda \geqq 0 \qquad \text{in } V$$

$$\lambda_p \geqq 0 \qquad \text{on } R_p$$

with the Lagrange functional

$$\Lambda = p + 2\pi\Bigg[\int_{s_o}^{s_n} r_o \Big\{\big[\dot{v}_\Psi' - r_\Psi^{-1}\dot{v}_\zeta\big]N_\Psi + r_o^{-1}\big[\cos\Psi\dot{v}_\Psi - \sin\Psi\dot{v}_\zeta\big]N_\varphi$$

$$+\big[-(r_\Psi^{-1}\dot{v}_\Psi)' - \dot{v}_\zeta''\big]M_\Psi + r_o^{-1}\big[-r_\Psi^{-1}\cos\Psi\dot{v}_\Psi - \cos\Psi\dot{v}_\zeta\big]M_\varphi$$

$$+ \lambda\left[c-f(N_\Psi,N_\varphi,M_\Psi,M_\varphi)\right] - \left[\bar{p}_\Psi\dot{v}_\Psi+\bar{p}_\zeta\dot{v}_\zeta\right]p + \lambda_p p \Big\} ds \tag{3}$$

$$- p\Big\{\left[r_o P_\Psi \dot{v}_\Psi\right]_{R_{p\Psi}} + \left[r_o P_\zeta \dot{v}_\zeta\right]_{R_{p\zeta}} + \left[r_o M_s \dot{\chi}_\Psi\right]_{R_{p\chi}}$$
$$- \left[r_o n_s N_\Psi \dot{v}_\Psi\right]_{R_{v\Psi}} - \left[r_o n_s Q_\Psi \dot{v}_\zeta\right]_{R_{v\zeta}} - \left[r_o(-n_s)M_\Psi \dot{\chi}_\Psi\right]_{R_{v\chi}}\Big]$$

In these expressions N_Ψ, N_φ are the normal forces, M_Ψ, M_φ the bending moments in the meridional and circumferential direction, respectivly, $\dot{v}_\Psi$, $\dot{v}_\zeta$ the velocities of displacements tangential and perpendicular to the middle surface of the shell, $\dot{\chi}_\Psi$ the slope velocity, λ, λ_p flow parameters, and n_s the cosinus of the normal direction at the edges of the shell.

Along the meridian $s_o \leq s \leq s_n$ the shell is subdivided into n intervals of the length Δs. The internal forces, displacement velocities, and flow parameters are approximated by functions of discrete argument (fig.).

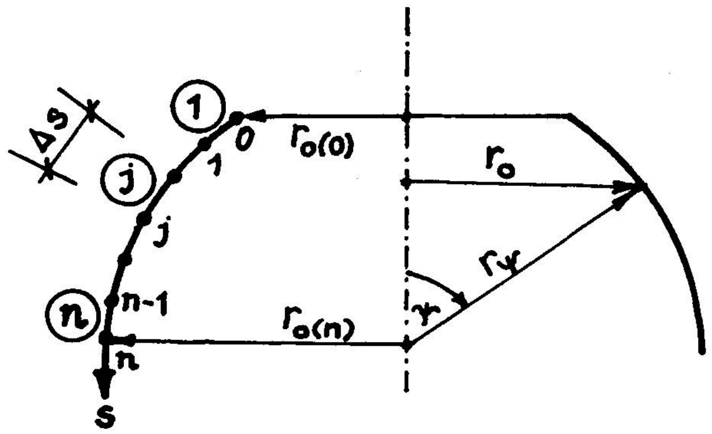

FIG. 1

Assuming that N_Ψ, N_φ, M_φ are constant within each interval j (j=1,...,n), they can be written as vectors in the form

$$N_\Psi(s) \longrightarrow \vec{N}_\Psi = \left[\tilde{N}_{\Psi(1)},\ldots,\tilde{N}_{\Psi(j)},\ldots,\tilde{N}_{\Psi(n)}\right]^T$$
$$N_\varphi(s) \longrightarrow \vec{N}_\varphi = \left[\tilde{N}_{\varphi(1)},\ldots,\tilde{N}_{\varphi(j)},\ldots,\tilde{N}_{\varphi(n)}\right]^T$$
$$M_\varphi(s) \longrightarrow \vec{M}_\varphi = \left[\tilde{M}_{\varphi(1)},\ldots,\tilde{M}_{\varphi(j)},\ldots,\tilde{M}_{\varphi(n)}\right]^T$$

Both displacement velocities and meridional bending moment were approximated by their values at the discrete points j (j=0,...,n)

$$\dot{v}_\Psi(s) \longrightarrow \vec{\dot{v}}_\Psi = \left[\dot{v}_{\Psi(0)},\dots,\dot{v}_{\Psi(j)},\dots,\dot{v}_{\Psi(n)}\right]^T$$
$$\dot{v}_\zeta(s) \longrightarrow \vec{\dot{v}}_\zeta = \left[\dot{v}_{\zeta(0)},\dots,\dot{v}_{\zeta(j)},\dots,\dot{v}_{\zeta(n)}\right]^T$$
$$M_\Psi(s) \longrightarrow M_\Psi = \left[M_{\Psi(0)},\dots,M_{\Psi(j)},\dots,M_{\Psi(n)}\right]^T$$

The actual loading is replaced by concentrated forces $pP_{\Psi(0)},\dots,$ $p\bar{P}_{\Psi(n)}$ and $p\bar{P}_{\zeta(0)},\dots,p\bar{P}_{\zeta(n)}$ at the points j.

As to the yield condition

$$f(N_\Psi,N_\varphi,M_\Psi,M_\varphi) \leqq c$$

it will in general be more convenient to express it by a set of linear inequalities

$$a_{Fi}^{N_\Psi} N_\Psi + a_{Fi}^{N_\varphi} N_\varphi + a_{Fi}^{M_\Psi} M_\Psi + a_{Fi}^{M_\varphi} M_\varphi \leqq b_{Fi} \quad (i=1,\dots,m) \qquad (4)$$

For shells under consideration it is justified to neglect the interaction between circumferential and meridional response of the shell, and for practical calculations it will be sufficient to approximate the yield locus in each direction by 6 or 7 straight lines (fig.2)

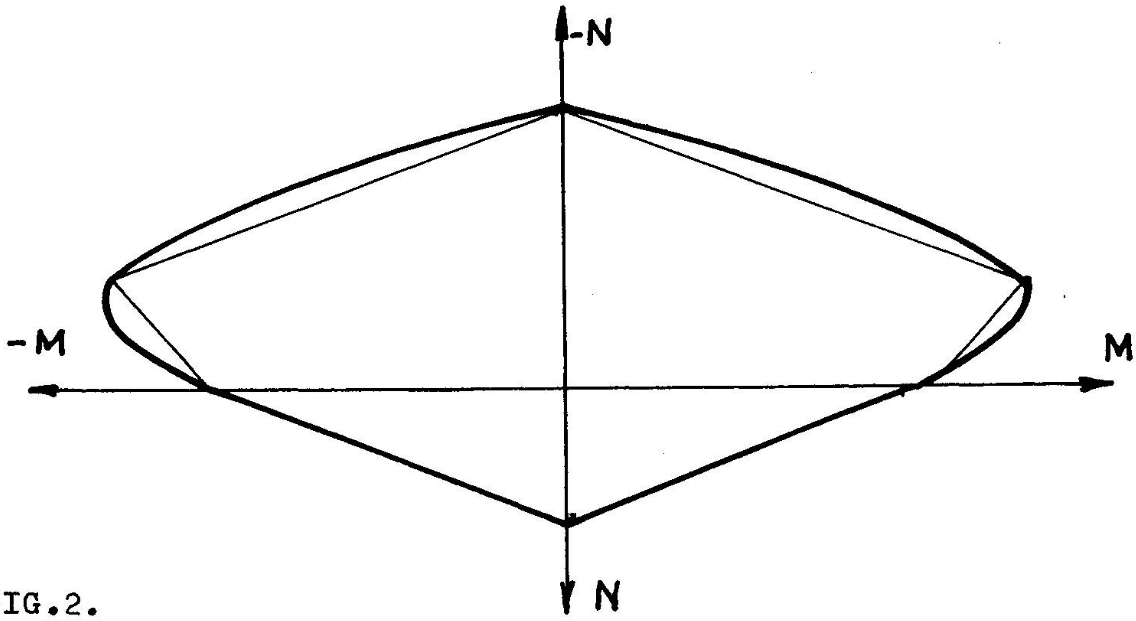

FIG.2.

Thus, instead of one flow parameter λ a set of parameters λ_i (i=1,...,m) is obtained. Assuming all λ_i to be constant within each interval j (j=1,...,n) they can be written in the form

$$\lambda(s) \longrightarrow \vec{\lambda} = [\vec{\tilde{\lambda}}_1,\dots,\vec{\tilde{\lambda}}_i,\dots,\vec{\tilde{\lambda}}_m]^T$$

where

$$\lambda_i = [\tilde{\lambda}_{i(1)},\dots,\tilde{\lambda}_{i(j)},\dots,\tilde{\lambda}_{i(n)}]^T$$

By replacing the integration by a summation and the differential operators by finite differences the Lagrange functional (3) is transformed into

$$\Lambda(p,N_\Psi,N_\varphi,M_\Psi,M_\varphi,v_\Psi,v_\zeta,\lambda,\lambda_p) \longrightarrow \Lambda(p,\vec{\tilde{N}}_\Psi,\vec{\tilde{N}}_\varphi,M_\Psi,\vec{\tilde{M}}_\varphi,\dot{v}_\Psi,\dot{v}_\zeta,\lambda,\lambda_p) \qquad (5)$$

Thus the original problem (1),(2) is reduced to the minimization or maximization of a function.

With the conditions

$$\frac{\Lambda}{\dot{v}_{\Psi(j)}} = 0\,, \quad \frac{\Lambda}{\dot{v}_{\zeta(j)}} = 0\,, \quad \frac{\Lambda}{\lambda_{i(j)}} = 0 \qquad (6)$$

the static formulation of the collapse load problem is finally obtained

$$p \longrightarrow \text{maximum} \qquad (7.1)$$

with

$$(\Delta s)^{-1}E_{1,\Psi}D_3^{(1)}\tilde{R}_o\vec{\tilde{N}}_\Psi - E_{7,\Psi}\tilde{C}_\Psi\vec{\tilde{N}}_\varphi - (\Delta s)^{-1}E_{1,\Psi}R_\Psi^{-1}D_1^{(1)}R_oM_\Psi$$
$$+ E_{7,\Psi}\tilde{R}_\Psi^{-1}\tilde{C}_\Psi\vec{\tilde{M}}_\varphi + (\Delta s)^{-1}E_{1,\Psi}R_o\vec{\tilde{P}}_\Psi\, p = 0 \qquad (7.2)$$

$$E_{7,\zeta}R_\Psi^{-1}R_o\vec{\tilde{N}}_\Psi + E_{7,\zeta}\tilde{S}_\Psi\vec{\tilde{N}}_\varphi + (\Delta s)^{-2}E_{1,\zeta}D_1^{(2)}R_o\vec{M}_\Psi$$
$$- (\Delta s)^{-1}E_{1,\zeta}D_3^{(1)}\tilde{C}_\Psi\vec{\tilde{M}}_\varphi + (\Delta s)^{-1}E_{1,\zeta}R_o\vec{\tilde{P}}_\zeta\, p = 0 \qquad (7.3)$$

$$\tilde{A}_{Fi}^{N_\Psi}\vec{\tilde{N}}_\Psi + \tilde{A}_{Fi}^{N_\varphi}\vec{\tilde{N}}_\varphi + \tilde{A}_{Fi}^{M_\Psi}E_7\vec{M}_\Psi + \tilde{A}_{Fi}^{M_\varphi}\vec{\tilde{M}}_\varphi \leqq \vec{\tilde{b}}_{Fi} \quad (i=1,\dots,m) \qquad (7.4)$$

$$-(\Delta s)^{-1}r_{o(0)}E_5\vec{M}_\Psi + (\Delta s)^{-1}r_{o(0)}\bar{M}_{s(0)}\, p = 0 \quad \text{on } R_p \qquad (7.5)$$

$$(\Delta s)^{-1}r_{o(n)}E_6\vec{M}_\Psi + (\Delta s)^{-1}r_{o(n)}\bar{M}_{s(n)}\, p = 0 \quad \text{on } R_p \qquad (7.6)$$

$$p \leqq 0 \qquad (7.7)$$

Considering the conditions

$$\frac{\Lambda}{N_{\Psi(j)}} = 0\ ,\quad \frac{\Lambda}{N_{\varphi(j)}} = 0\ ,\quad \frac{\Lambda}{M_{\Psi(j)}} = 0\ ,\quad \frac{\Lambda}{M_{\varphi(j)}} = 0\ ,\quad \frac{\Lambda}{p} = 0 \tag{8}$$

the kinematic formulation is found to be

$$\sum_{i=1}^{m} \vec{\tilde{b}}_{Fi}\, \vec{\tilde{\lambda}}_i^{*} \longrightarrow \text{minimum} \tag{9.1}$$

with

$$-(\Delta s)^{-1}\tilde{R}_o D_4^{(1)} E_{1,\Psi}^{T}\vec{v}_\Psi + \tilde{R}_o\tilde{R}_\Psi^{-1} E_{7,\zeta}^{T}\vec{\dot{v}}_\zeta + \sum_{i=1}^{m} \tilde{A}_{Fi}^{N_\varphi}\, \vec{\tilde{\lambda}}_i^{*} = 0 \tag{9.2}$$

$$-\tilde{C}_\Psi E_{7,\Psi}^{T}\vec{v}_\Psi + \tilde{S}_\Psi E_{7,\zeta}^{T}\vec{\dot{v}}_\zeta + \sum_{i=1}^{m} \tilde{A}_{Fi}^{N_\varphi}\vec{\tilde{\lambda}}_i^{*} = 0 \tag{9.3}$$

$$(\Delta s)^{-1} R_o D_2^{(1)} R_\Psi^{-1} E_{1,\Psi}^{T}\vec{v}_\Psi + (\Delta s)^{-2} R_o D_1^{(2)} E_{1,\zeta}^{T}\vec{\dot{v}}_\zeta$$

$$-\left[(\Delta s)^{-1} r_{o(0)} E_5^T \dot{\lambda}_{\Psi(0)} + (\Delta s)^{-1} r_{o(n)} E_6^T \dot{\lambda}_{\Psi(n)}\right] R_{p\chi} + \sum_{i=1}^{m} E_7 \tilde{A}_{Fi}^{M_\Psi}\vec{\tilde{\lambda}}_i^{*} = 0 \tag{9.4}$$

$$\tilde{R}_\Psi^{-1}\tilde{C}_\Psi E_{7,\Psi}^{T}\vec{v}_\Psi + (\Delta s)^{-1}\tilde{C}_\Psi D_4^{(1)} E_{1,\zeta}^{T}\vec{\dot{v}}_\zeta + \sum_{i=1}^{m} \tilde{A}_{Fi}^{M_\varphi}\, \vec{\tilde{\lambda}}_i^{*} = 0 \tag{9.5}$$

$$(\Delta s)^{-1}\vec{\bar{P}}_\Psi^T R_o E_{1,\Psi}^{T}\vec{v}_\Psi + (\Delta s)^{-1}\vec{\bar{P}}_\zeta^T R_o E_{1,\zeta}^{T}\,\vec{v}_\zeta$$

$$+\left[(\Delta s)^{-1} r_{o(0)}\bar{M}_{s(0)}\dot{\lambda}_{\Psi(0)} + (\Delta s)^{-1} r_{o(n)}\bar{M}_{s(n)}\dot{\lambda}_{\Psi(n)}\right] R_{p\chi} \geqq 1 \tag{9.6}$$

$$\begin{aligned} \tilde{\lambda}_{i(0)}^{*} &= 0{,}5\ r_{o(0)}\tilde{\lambda}_{i(0)} = 0 \\ \tilde{\lambda}_{i(j)}^{*} &= r_{o(j)}\tilde{\lambda}_{i(j)} = 0 \quad (i=1,\dots,m),(j=1,\dots,n-1) \\ \tilde{\lambda}_{i(n)}^{*} &= 0{,}5\ r_{o(n)}\tilde{\lambda}_{i(n)} = 0\ . \end{aligned} \tag{9.7}$$

In this equations are

$$D_1^{(1)}=\begin{bmatrix} -0,5 & +0,5 & & & \\ -0,5 & 0 & +0,5 & & \\ & \cdots & \cdots & \cdots & \\ & & -0,5 & 0 & +0,5 \\ & & & -0,5 & +0,5 \end{bmatrix}, \quad D_3^{(1)} = -D_4^{(1)^T} = \begin{bmatrix} +1 & & & \\ -1 & +1 & & \\ & \cdots & \cdots & \\ & & -1 & +1 \\ & & & -1 \end{bmatrix}$$

$$D_1^{(2)}=\begin{bmatrix} -1 & +1 & & & \\ +1 & -2 & +1 & & \\ & \cdots & \cdots & \cdots & \\ & & +1 & -2 & +1 \\ & & & +1 & -1 \end{bmatrix} \qquad \begin{aligned} E_5 &= [1\ 0\ \ldots\ 0\ 0] \\ E_6 &= [0\ 0\ \ldots\ 0\ 1] \end{aligned}$$

$\tilde{R}_o$, $\tilde{R}_\Psi$, $\tilde{C}_\Psi$, $\tilde{S}_\Psi$, $\tilde{A}_{Fi}^{N_\Psi}$, $\tilde{A}_{Fi}^{N_\varphi}$, $\tilde{A}_{Fi}^{M_\Psi}$, $\tilde{A}_{Fi}^{M_\varphi}$, R_Ψ are diagonal matrices with the diagonal elements $\tilde{r}_{o(j)}$, $\tilde{r}_{\Psi(j)}$, $\cos\tilde{\Psi}_{(j)}$, $\sin\tilde{\Psi}_{(j)}$, $\tilde{a}_{Fi(j)}^{N_\Psi}$, $\tilde{a}_{Fi(j)}^{N_\varphi}$, $\tilde{a}_{Fi(j)}^{M_\Psi}$, $\tilde{a}_{Fi(j)}^{M_\varphi}$ $(j=1,\ldots,n)$, and $r_{\Psi(j)}$ $(j=0,\ldots,n)$.

According to the boundary conditions, the shell edges will either belong to R_p if they are loaded by external forces $p\bar{P}_{\Psi(k)}$, $p\bar{P}_{\zeta(k)}$ and bending moments $p\bar{M}_{s(k)}$ $(k=0,n)$, or to R_v if slopes and displacements are restricted. Depending upon the boundary condtions the matrices $E_{1,\Psi}$, $E_{1,\zeta}$, $E_{7,\Psi}$, $E_{7,\zeta}$ are obtained by modification of the matrices

$$E_1 = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \quad \text{and} \quad E_7 = \begin{bmatrix} 0,5 & & & & \\ 0,5 & 0,5 & & & \\ & \cdots & \cdots & \cdots & \\ & & & 0,5 & 0,5 \\ & & & & 0,5 \end{bmatrix}$$

For the solution of the collapse load problem in the form (7) or (9) various optimization techniques for linear programming are available.

The coefficient matrix will be formed very easily, because only a few of the matrix elements differs from zero.

Besides the collapse load intensity, the field of displacements and internal forces in the plastified zones are found without any additional computations. This is a certain advantage in comparison to methods, where the unknown functions are represented in form of series.
As the static and kinematic formulations are dual to each other the same collapse load intensity will be obtained. Therefore, it is not possible to characterize the results as upper or lower bound solutions without additional considerations.

The method described in this paper was used for the investigation into reinforced concrete cylindrical shells under various loading conditions. The calculations were carried out by an electronic computer of the type ROBOTRON 300.

REFERENCES

/1/ Raue, E.: Beitrag zur Berechnung rotationssymmetrischer Flächentragwerke nach der Theorie des Grenzgleichgewichts, Hochschule für Architektur und Bauwesen Weimar, Dissertation (Promotion B), 1975.