Zeitschrift: IABSE reports of the working commissions = Rapports des commissions de travail AIPC = IVBH Berichte der Arbeitskommissionen

Band: 29 (1979)

Artikel: Optimization of reinforcement in slabs by means of linear programming

Autor: Pedersen, H.

DOI: https://doi.org/10.5169/seals-23562

IV

Optimization of Reinforcement in Slabs by Means of Linear Programming

Optimalisation de l'armature des plaques par la programmation linéaire

Optimierung der Bewehrung von Platten mit Hilfe der linearen Programmierung

H. PEDERSEN
Structural Engineer
Cowiconsult
Virum, Denmark

SUMMARY
This paper is concerned with the development of a numerical method for designing reinforced concrete slabs by linear programming. The total amount of reinforcement necessary in a slab with given concrete dimensions, subjected to given loads and with a given arrangement of the reinforcing bars, is minimized. The linear programming problem is formulated using the principles of the lower bound method. The continuum problem is discretized by means of equilibrium finite element types. The linearized yield conditions are established in a number of discrete points to ensure a safe stress field.

RESUME
Le développement d'une méthode numérique est présenté pour le dimensionnement de dalles en béton armé à l'aide de la programmation linéaire. Le volume total de l'armature nécessaire pour une dalle dont la géométrie, le chargement et le système d'armature sont donnés, est minimalisé. Le programme linéaire est formulé en appliquant les principes de la méthode statique de la théorie des charges ultimes. Le problème continu est discrétisé par des éléments finis du type modèle équilibre. Les conditions d'écoulement linéarisées sont établies pour un nombre de points afin d'assurer la stabilité du champ de contraintes.

ZUSAMMENFASSUNG
Die Entwicklung eines numerischen Verfahrens zur Bemessung von Stahlbetonplatten mit Hilfe der linearen Programmierung wird dargestellt. Der insgesamt erforderliche Aufwand an Bewehrung für eine Platte mit gegebenen Abmessungen, gegebenen Lasten und einer vorgewählten Bewehrungsanordnung wird minimiert. Gestützt auf die statische Methode wird das lineare Programm formuliert. Das kontinuierliche Problem wird mit Finiten Elementen vom Gleichgewichtstyp diskretisiert. Die Stabilität des Spannungszustandes wird gesichert, indem die linearisierten Fliessbedingungen für die Anzahl diskreter Punkte aufgestellt werden.

1. INTRODUCTION

Bound methods of limit analysis have proved to be powerful tools for the determination of the ultimate limit load for many structures. Also in the calculations of reinforced concrete slabs plastic theory has been successfully used for years. Johansen [1] formulated the yield-line theory leading to upper bound solutions which theoretically are unsafe solutions.

Since then, many attempts have been made to create safe methods. The establishment of a general plastic theory for reinforced concrete slabs has made it possible to use lower bound methods giving results on the safe side. The major part of this work was done by Nielsen [2] in the early sixties. Based on the theory of perfectly plastic materials, Nielsen [2] formulated the general yield conditions for orthotropic slabs.

The great advantages of using lower bound methods are really achieved when computerizing the construction of the equilibrium solutions. The statically admissible stress fields can be created for instance by means of finite element methods. Overall equilibrium requirements result in a set of linear equations in the stress parameters. A safe stress field is ensured by establishing the yield conditions in a set of discrete points in each element.

Many authors have adopted this approach in the evaluation of the bearing capacity for a given slab subjected to proportional loading. As the yield conditions are not linear, the optimization problem is non-linear. If the yield conditions are linearized, one gets a case of linear programming, LP, which can be solved by general available standard routines. Calculations of this kind for concrete slabs have been performed by Anderheggen and Knöpfell [3] and Knöpfell [4]. For materials governed by the Tresca yield criteria, Faccioli and Vitiello [5] have carried out a similar calculation.

Most of the numerical methods leading to LP-problems include both upper- and lower bound techniques expressing the anology between the duality of the limit analysis theorems and the duality theorem of LP.

It has been concluded by the author [6] that with the present knowledge of the existing optimization techniques - especially concerning reliability and efficiency - an automatic limit design method for practical use should be based on LP. Moreover, it is stated that using LP no results can be achieved by upper bound methods that cannot be produced by lower bound methods alone. The most economic design will often be obtained by varying the reinforcement over the slab area. This case can be treated directly by the lower bound method but can hardly be done by upper bound methods. Further, it has been demonstrated by the author [7] that based on lower bound techniques the limit analysis problem as well as the limit design problem can easily be handled by means of the same computer programme. Thus the LP-problem should be formulated using the principles of the lower bound method.

The general approach adopted here in the development of a rational and safe design method was at first presented by Wolfensberger [8], who used Hillerborgs [9] strip method to generate a parametric moment field.

In this paper, only thin slabs are dealt with. The Kirchoff plate theory is adopted and the material is assumed to be rigid plastic. The optimization criteria is the minimum of the total amount of tensile reinforcement. The arrangement of the reinforcing bars can be chosen a priori to ensure a design for direct practical use. Arbitrarily reinforced slabs with given directions of the reinforcing bars can be handled. Slabs with various geometry and different types of boundary conditions, together with column supports are dealt with. The design is carried out for a set of given loading cases.

As examples to illustrate the method designs of an isotropic square built-in slab and a flat plate construction are shown.

2. MATHEMATICAL FORMULATION OF THE DESIGN PROBLEM

Equilibrium element types are used to descretize the continuum problem. The moment field for the slab is represented by the NM global parameters contained in the vector $\underset{\sim}{M}$. The moment field for an element, e, is given by the nm parameters $\underset{\sim}{m}^e$. The relation between these and the global parameters is for each element given by:

$$\underset{\sim}{m}^e = \underset{\sim}{G}^e \times \underset{\sim}{M} \tag{2.1}$$

By means of interpolation functions the moments referring to a global x-y-system, $\underset{\sim}{m} = (m_x, m_{xy}, m_y)^T$, can be expressed in the form:

$$\underset{\sim}{m} = \underset{\sim}{a}^e \times \underset{\sim}{m}^e \tag{2.2}$$

For each element the boundary forces, $\underset{\sim}{r}^e$, necessary to express the required statical continuity conditions are calculated from:

$$\underset{\sim}{r}^e = \underset{\sim}{k}^e \times \underset{\sim}{m}^e \tag{2.3}$$

By means of (2.1-3) all equilibrium requirements such as internal requirements, continuity requirements along the element boundaries and the statical boundary conditions lead to a set of linear equations in the global parameters:

$$\underset{\sim}{K} \times \underset{\sim}{M} = \underset{\sim}{P} \tag{2.4}$$

where the vector $\underset{\sim}{P}$ represents the effects of the given loads.

The design variables, which are the steel areas, for the slab are established in the ND-dimensional design vector $\underset{\sim}{D}$. For each element the plastic moments, $\underset{\sim}{m}_p^e = (m_{Fx}, m_{Fy}, m_{Fxy}, m'_{Fx}, m'_{Fy}, m'_{Fxy})$, see section 4, are expressed by the relevant steel areas by:

$$\underset{\sim}{m}_p^e = \underset{\sim}{E}^e \times \underset{\sim}{D} + \underset{\sim}{m}_{p,o}^e \tag{2.5}$$

where $\underset{\sim}{m}_{p,o}^e$ are plastic moments due to given reinforcement.

By means of (2.1-2) and (2.5) the linearized yield conditions set up in some a priori selected points for each element, lead to a set of linear inequalities in the global moment parameters and the design variables:

$$\underset{\sim}{R}^1 \times \underset{\sim}{M} + \underset{\sim}{R}^2 \times \underset{\sim}{D} \leq \underset{\sim}{R}_o \tag{2.6}$$

where $\underset{\sim}{R}_o$ expresses the contribution of given reinforcement in (2.5).

Several explicit linear constraints in the design variables such as given intervals for certain steel areas, desired linear relationships between different steel areas etc. can be handled. For clarity, such relationships will be assumed to be included in (2.6).

The total amount of reinforcement is expressed as a linear function in the design variables:

$$Z = \underset{\sim}{c} \times \underset{\sim}{D} \tag{2.7}$$

In this way, the design problem is formulated as a case of LP:

$$\text{minimize:} \quad Z = \underset{\sim}{c} \times \underset{\sim}{D}$$

$$\text{subject to} \quad \underset{\sim}{K} \times \underset{\sim}{M} = \underset{\sim}{P} \quad \text{and:} \quad \underset{\sim}{R}^1 \times \underset{\sim}{M} + \underset{\sim}{R}^2 \times \underset{\sim}{D} \leq \underset{\sim}{R}_o \tag{2.8}$$

3. ELEMENT TYPES, STATICAL EQUATIONS

For many equilibrium slab elements used in elastic calculations the corresponding stress field can be used here directly to generate parametric statically admissible fields. Two such element types are given below, i.e. the triangular element with a constant moment field, TRIC, and the triangular element with a linear moment field TRIL. A direct way can also be used to derive applicable element types considering only the statical properties. An example of this is given by the rectangular element RECT. For the element matrices the reader is referred to reference [6].

Triangular element, TRIC

The geometrical and statical properties of the element are shown in Fig. 3.1. The element parameters can be chosen as:

$$\underset{\sim}{m}^{e} = (m_{b1}, m_{b2}, m_{b3})^{T} \qquad (3.1)$$

The only vertical forces acting at an element are the concentrated corner forces due to discontinuities in the torsional moments. These are established in the vector:

$$\underset{\sim}{r}^{e} = (P_1, P_2, P_3)^{T} \qquad (3.2)$$

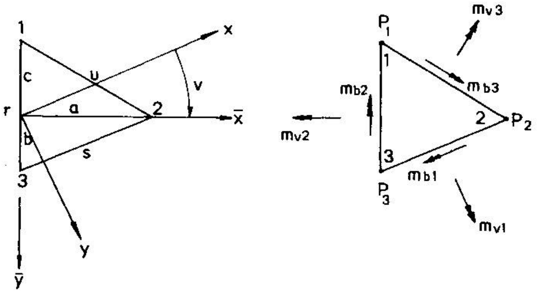


Fig. 3.1: Triangular element, TRIC

It is thus seen that only concentrated forces acting at the nodes of the element mesh can be handled.

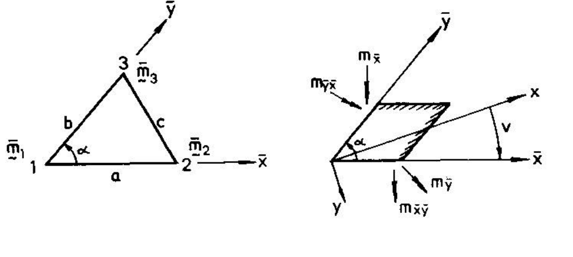
Triangular element, TRIL

This element is due to Veubeke [10], who used it in elastic slab analysis.

The moment field is linear and can thus be represented by nine parameters for each element. The analysis is most easily carried out in a local ablique reference system as shown in Fig. 3.2 where sign conventions and the nine parameters are also shown. The parameters are:

$$\underset{\sim}{m}^{e} = (\bar{\underset{\sim}{m}}_1, \bar{\underset{\sim}{m}}_2, \bar{\underset{\sim}{m}}_3)^{T}$$

where $\bar{\underset{\sim}{m}} = (m_{\bar{x}}, m_{\bar{x}\bar{y}}, m_{\bar{y}})$ (3.3)



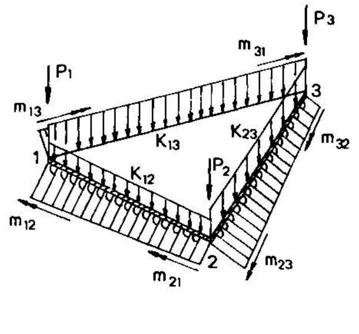


Fig. 3.2 Triangular element, TRIL. Fig. 3.3 Boundary forces.

The corresponding boundary and nodal forces, shown in Fig. 3.3, are contained in the following vector:

$$\underset{\sim}{r}^e = (m_{12}, m_{21}, m_{13}, m_{31}, m_{23}, m_{32}, K_{12}, K_{13}, K_{23}, P_1, P_2, P_3)^T \quad (3.4)$$

Line loads with constant intensity along element boundaries and concentrated loads acting at the nodes of the element mesh can be handled.

Rectangular element, RECT

The slab area is subdivided into rectangular elements by lines parallel to the axes of the global reference system.

The moment field for an element is given by the ten parameters:

$$\underset{\sim}{m}^e = (m_x^1, m_x^2, m_{xy}^1, m_{xy}^2, m_{xy}^3, m_{xy}^4, m_y^1, m_y^2, p_x, p_y)^T \quad (3.5)$$

These parameters are shown in Fig. 3.4. The variation of the m_x-moment is parabolic in the x-direction and constant in the y-direction. Analogous for the m_y-moment. The torsional moment m_{xy} is represented by a hyperbolic paraboloid and is thus linear at the element boundaries.

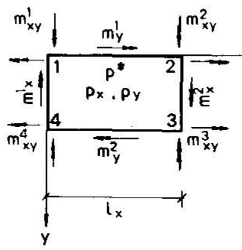


Fig. 3.4 Rectangular element, RECT

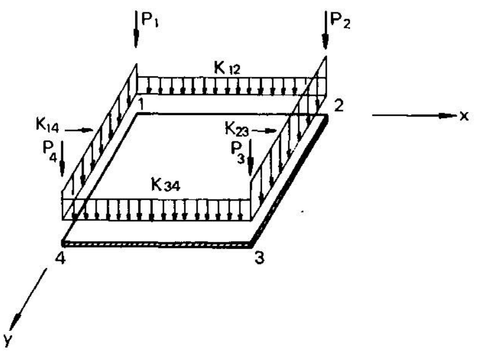


Fig. 3.5 Boundary forces.

The Kirchoff' shear forces and the concentrated forces acting at the corners of the element, see Fig. 3.5, are expressed by the following vector:

$$\underset{\sim}{r}^e = (K_{12}, K_{23}, K_{34}, K_{14}, P_1, P_2, P_3, P_4)^T \quad (3.6)$$

One equation needs to be satisfied to ensure internal equilibrium. From the $\underset{\sim}{a}^e$-matrix one gets directly:

$$m_{xy}^1 + m_{xy}^3 - m_{xy}^2 - m_{xy}^4 = \tfrac{1}{2}\, l_x l_y\,(p^* - p_x - p_y) \quad (3.7)$$

Distributed loads with constant intensity for each element, line loads acting at element boundaries and concentrated loads acting at the nodes can be handled.

4. YIELD CONDITIONS

Yield conditions for arbitrarily reinforced concrete slabs have been derived by many authors on the basis of Johansen's suggestions for the moments in a yield-line. For instance, this has been done by Bræstrup [11], who used polar diagrammes to formulate the yield conditions.

$$\left.\begin{aligned} \varphi_1 &= -(m_{Fx} - m_x)(m_{Fy} - m_y) + (m_{xy} - m_{Fxy})^2 \le 0 \\ \varphi_2 &= -(m'_{Fx} + m_x)(m'_{Fy} + m_y) + (m_{xy} - m'_{Fxy})^2 \le 0 \end{aligned}\right\} \quad (4.1)$$

Here, m_{Fx} is the numerical value of the positive yield moment in pure bending in an x-section and m'_{Fx} is the numerical value of the negative yield moment in that section. Analogous for m_{Fy} and m'_{Fy} . m_{Fxy} and m'_{Fxy} are the plastic torsional moments due to reinforcement at the bottom and the top respectively. The yield conditions (4.1) include the conditions for orthotropic slabs ($m_{Fxy} = m'_{Fxy} = 0$) as a special case, see reference [2].

A safe linearization of the yield conditions has been suggested by Wolfensberger [8], leading to the following eight inequalities:

$$\left.\begin{aligned} - m_{Fx} + m_x \pm (m_{xy} - m_{Fxy}) &\leq 0 \\ - m_{Fy} + m_y \pm (m_{xy} - m_{Fxy}) &\leq 0 \\ - m'_{Fx} - m_x \pm (m_{xy} - m'_{Fxy}) &< 0 \\ - m'_{Fy} - m_y \pm (m_{xy} - m'_{Fxy}) &< 0 \end{aligned}\right\} \qquad (4.2)$$

This linearization has also been used in reference [3] and [4].

As the relationships between the plastic moments and the corresponding steel areas are assumed to be linear, which is a good approximation for such small degrees of reinforcement for which the yield conditions (4.1) are valid, the linearized conditions (4.2) are also linear in the design variables. The plastic moments are assumed to be constant within each element and are given by the vector $\underset{\sim}{m}^e$ as shown in section 2. For example, the linear expression for m_{Fx} is:

$$m_{Fx} = F_1 \cos^2 u_1 + \dots + F_{ND} \cos^2 u_{ND} \qquad (4.3)$$

where u_i is the angle relative to the x-axis for the reinforcement with steel area D_i. If the reinforcement with steel area D_i is not to be represented at the bottom of the element the F_i-factor for m_{Fx}, m_{Fy} and m_{Fxy} is set to zero.

From the given arrangement of reinforcing bars formulated in (4.3), the linear objective function is derived automatically. The c_i-value simply represents the slab area in which the reinforcement with corresponding steel area D_i is extended.

For the element type with constant moment field, TRIC, the linearized yield cindi-tions only have to be established at one point per element. For the element with linear moment field, TRIL, establishment in the three corners of each element will ensure overall fulfillment. For the rectangular element, RECT, the yield conditions (4.2) are set up in the corners and in the centre of the element. For no loading cases this alone can ensure a true lower bound solution. For the solution obtained the yield conditions (4.1) are checked in a finer mesh and the solution is proportioned if needed to fulfil (4.1) in all check-points.

5. NUMERICAL TREATMENT OF THE LP-PROBLEM

The LP-problem stated in (2.8) can be solved directly by means of many LP-codes. However, it can be considerably reduced, and an easy way of treating different loading cases simultaneously can be obtained by solving the linear equations (2.4) first. By means of a rank-method, some moment parameters, $\underset{\sim}{M}_a$, can be expressed in terms of the redundancies, $\underset{\sim}{M}_u$, by:

$$\underset{\sim}{M}_a = \underset{\sim}{K}^r \times \underset{\sim}{M}_u + \underset{\sim}{M}_o \qquad (5.1)$$

where $\underset{\sim}{M}_o$ represents a particular solution for the actual loading case.

The LP-problem is then reduced to:

$$\left.\begin{aligned} &\text{minimize:} && Z = \underset{\sim}{c} \times \underset{\sim}{D} \\ &\text{subject to:} && \underset{\sim}{R}_u \times \underset{\sim}{M}_u + \underset{\sim}{R}^2 \times \underset{\sim}{D} \le \underset{\sim}{R}_o \end{aligned}\right\} \qquad (5.2)$$

where the particular solution is now introduced into the $\underset{\sim}{R}_o$-vector.

If the slab is to be designed according to different loading cases, one only has to establish (5.1) once. For each loading case, i, the particular solution, $\underset{\sim}{M}_o^i$, is calculated and the right-hand sides, $\underset{\sim}{R}_o^i$, in the linear constraints are obtained by substitution in (2.6). By this procedure the following LP-problem is formulated:

$$\left.\begin{aligned} &\text{minimize:} && Z = \underset{\sim}{c} \times \underset{\sim}{D} \\ &\text{subject to:} && \underset{\sim}{R}_u \times \underset{\sim}{M}_u^1 + \underset{\sim}{R}^2 \times \underset{\sim}{D} \le \underset{\sim}{R}_o^1 \\ & && \vdots \\ &\text{and:} && \underset{\sim}{R}_u \times \underset{\sim}{M}_u^n + \underset{\sim}{R}^2 \times \underset{\sim}{D} \le \underset{\sim}{R}_o^n \end{aligned}\right\} \qquad (5.3)$$

Solving this problem, the global optimum (according to the linear model) will be obtained.

An approximate optimum can be achieved by solving the LP-problem:

$$\left.\begin{aligned} &\text{minimize:} && Z = \underset{\sim}{c} \times \underset{\sim}{D} \\ &\text{subject to:} && \underset{\sim}{R}_u \times \underset{\sim}{M}_u + \underset{\sim}{R}^2 \times \underset{\sim}{D} \le \underset{\sim}{R}_o^{min} \\ &\text{and:} && R_{o,j}^{min} = \min\,(R_{o,j}^1, \dots, R_{o,j}^n) \end{aligned}\right\} \qquad (5.4)$$

The design, considering more than one loading case, can also be carried out by successive calculations. For the loading case, i, all reinforcement quantities as obtained earlier are treated as given through $\underset{\sim}{m}_p^o$, and only necessary additional reinforcement, if any, is determined.

The computer time needed to solve the LP-problem, either (5.3) or (5.4), can be reduced by solving the corresponding dual LP-problem. Concerning this, the reader is referred to reference [7].

Numerical calculations using the described method have been performed on the IBM/360-system at the Technical University of Denmark, Copenhagen, using the MPS/360 linear programming code.

6. RESULTS

The design of two different types of concrete slabs with orthogonal reinforcement (orthotropic slabs) is shown.

Square built-in slab

The isotropic square slab with clamped edges is designed using the RECT-element ($m_{Fx} = m_{Fy} = m'_{Fx} = m'_{Fy} = m_F$). The load is uniformly distributed and denoted by p. The result is represented by the quantity $\rho = pl^2/m_F$. In Fig. 6.1 the result is shown as a function of the mesh size.

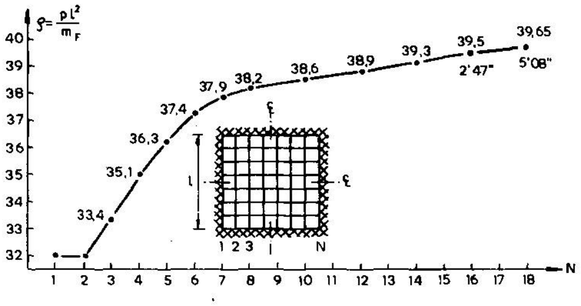


Fig. 6.1 Rectangular element.

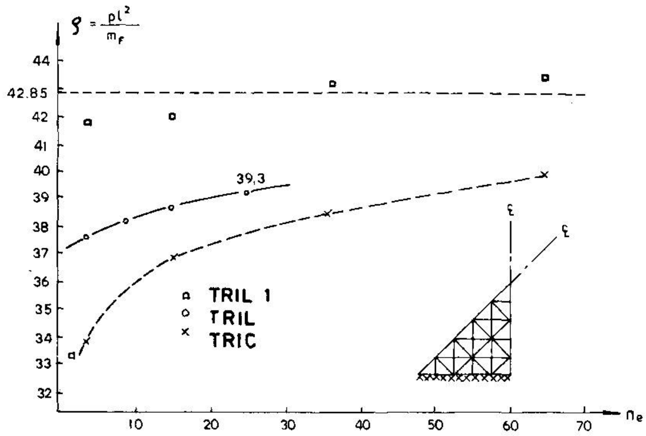


Fig. 6.2 Triangular elements.

This slab has also been calculated by means of triangular elements. The results are given in Fig. 6.2 where n_e is the number of elements in the considered eighth part of the slab.

The results obtained by the TRIC-element have been determined by Anderheggen and Knöpfell [3], who have also given the results according to the element here called TRIL 1. This element has a linear moment field with continuous torsional moments along element boundaries. Considering linear displacement fields for each element, statical equilibrium requirements are established using virtual work methods. In this way, overall equilibrium can never be ensured. Thus the results obtained do not represent true bound solutions.

In comparison, it should be mentioned that Chan [12], using non-linear optimization has determined the value 41.78 for a computer time considerably larger than those met here using linear programming. The exact solution of this problem is ρ=42.851 and was found by Fox [13].

Flat Plate Structure

For the uniformly loaded flat plate structure shown in Fig. 6.3, the total amount of steel is minimized for different arrangement of the reinforcement. The RECT element is used. The three cases: a) $m_{Fx} = m_{Fy} = m'_{Fx} = m'_{Fy} = m_F \sim D_1$, b) $m_{Fx} = m_{Fy} = m_F \sim D_1$, $m'_{Fx} = m'_{Fy} = m'_F \sim D_2$ and c) $m_{Fx} = m_{Fy} = m_F \sim D_1$, $m'_{Fx} = m'_{Fy} = 0$ all give the same total amount of reinforcement denoted by A. In case a) is found $m_F = 0.0685\ pl^2$.

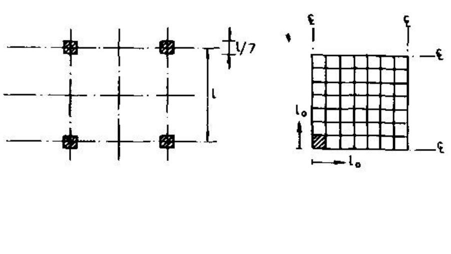


Fig. 6.3 Flat plate structure.

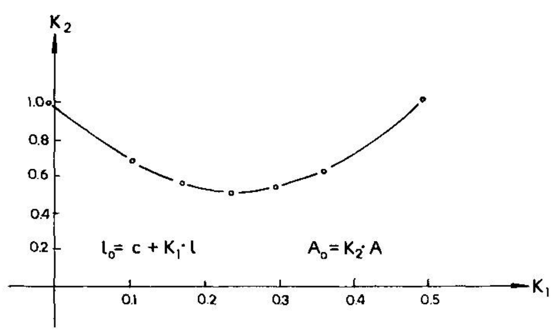


Fig. 6.4 Restricted top reinforcement.

Johansen [14] suggests from yield-line calculations that if the bottom reinforcement is homogenous throughout the slab, and the top reinforcement is homogenous only in a square at the column (zero elsewhere), the latter must have an extension from the centre of the column given by $l_o = c + 0.3\ l$, where c is the radius of the circle with the same area as the cross section of the column. This problem has been calculated for different values of l_o (l_o is shown in Fig. 6.3). The corresponding amounts of reinforcement, A_o, are given in Fig. 6.4 as a function of l_o.

The results show that a minimum is obtained for $l_o \sim c + 0.23\ l$, for which the amount of reinforcement is reduced to $A_o \sim 0.5\ A$.

7. FINAL REMARKS

Results show that linear programming methods require longer computer times compared to those required for calculations on linear elasticity. However, continued developments in the field of electronic computers can be expected to result in reduced prices so that practical design will be able to profit by these methods in the near future.

Moreover, the advantages of these automatic methods in the area of practical design should be emphasized. In this case, the alternative methods are normally not the very sophisticated methods available in the field of structural analysis.

Concerning the finite element discretization, it should be mentioned that procedures like the one adopted by Anderheggen and Knöpfell [3], leading to approximate bound solutions, will probably be successful in practical design methods. This is due to the fact that concerning calculations in practice, one will often accept a design which is safe for a loading case a little different from the prescribed one.

ACKNOWLEDGMENT

The present paper is based on a part of a Ph.D.-thesis (lic. techn.) written under the general supervision of Prof., dr.techn. M.P. Nielsen, Structural Research Laboratory, Technical University of Denmark.

REFERENCES

1 Johansen, K.W.: Brudlinieteorier, København, 1943.

2 Nielsen, M.P.: Limit Analysis of Reinforced Concrete Slabs, Acta Polytechnica Scandinavia, Copenhagen, 1964.

3 Anderheggen, E. and Knöpfell, H.: Finite Element Limit Analysis using Linear Programming, Int. I. Solids Structures, Vol. 8, pp. 1413-1431, 1972.

4 Knöpfell, H.: Berechnung Starr-plastichen Platten Mittels Finiten Elementen, Diss. No. 5034, ETH, Zürich, 1973.

5 Faccioli, E. and Vitiello, E.: A Finite Element Linear Programming Method for the Limit Analysis of Thin Plates, Int. J. Numm. Eng., Vol. 5, pp. 311-325, 1973

6 Pedersen, H.: Optimering af Jernbetonplader, Ph.D.-thesis, Structural Research Laboratory, Technical University of Denmark, to be published.

7 Pedersen, H.: Optimum Design of Thin Concrete Plates, Proc. Int. Symp. Discr. Meth. Eng., CISE-SEGRATE, Milan, pp. 374-389, 1974.

8 Wolfensberger, R.: Traglast und optimale Bemessung von Platten, Diss., ETH, Zürich, 1964.

9 Hillerborg, A.: Strimlemetoden, Stockholm, 1959.

10 Veubeke, B.F. and Sander, G.: An Equilibrium Model for Plate Bending, Int. J. Solid Structures, Vol. 4, pp. 447-468, 1968

11 Bræstrup, M.W.: Yield-line Theory and Limit Analysis of Plates and Slabs, Magazine of Concrete Research, Vol. 27, No. 71, pp. 99-106, 1970.

12 Chan, H.S.Y.: The Collapse Load of Reinforced Concrete Plates, Int. J. Numm. Meth. Eng., Vol. 5, pp. 57-64, 1972.

13 Fox, E.N.: Limit Analysis for Plates: The Exact Solution for a Clamped Square Plate of Isotropic Homogenous Material Obeying the Square Yield Criterion and Loaded by Uniform Pressure. Phil. Trans. Roy. Soc., Vol. 277A, pp. 121-155, 1974.

14 Johansen, K.W.: Yield-line Formulae for Slabs, Cement and Concrete Association, London 1972.