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Application of the Yield-Line Theory for Reinforced Concrete Slabs allowing for Membrane Effects

Application de la théorie des lignes de rupture aux dalles en béton armé en considérant les effets de membrane

Anwendung der Fliessgelenklinientheorie bei Stahlbetonplatten mit Berücksichtigung der Wirkung von Membrankräften

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SUMMARY
The paper presents two methods for calculating the influence of in-plane forces on the load bearing capacity of reinforced concrete slabs. The results are examined by a finite element analysis of an example.

RESUME
On présente deux méthodes pour calculer l'influence des effets de membrane sur la charge ultime des dalles en béton armé. Les résultats sont comparés avec ceux d'une analyse utilisant la méthode des éléments finis, dans un cas concret.

ZUSAMMENFASSUNG
Es werden zwei Methoden zur Untersuchung des Einflusses von Normalkräften auf die Traglast von Stahlbetonplatten vorgestellt. Die Ergebnisse werden an einem Beispiel durch eine Vergleichsrechnung nach der Finite Elemente Methode überprüft.

1. INTRODUCTION

The load bearing capacity of reinforced concrete slabs with restrained edges is higher than predicted by the conventional yield-line theory. The prevention of the outward expansion causes an in-plane compressive force within the slab which provides a higher moment capacity than is assumed by the yield-line theory. In the presented paper the solution of several research workers (for example Morley [1]), who have extended the yield-line theory by including the in-plane forces as generalized stresses for the assumption of a rigid perfectly - plastic material, is compared to a more realistic solution in which the condition of the inextensibility of the slab parts is deleted. It is assumed that the slab will behave elastically within its plane. An approach is developed to estimate the in-plane forces due to constraints which are induced by the reduction of the strains and cracking during slab deflection.

The results of the extended yield-line theory including in-plane forces are examined by a finite element analysis of reinforced concrete slabs. With a program system that was developed in Darmstadt [2], [3], the realistic behaviour of reinforced concrete slabs can be calculated by considering material and geometrical nonlinearities. The computational methods are demonstrated by a rectangular slab under uniform load, supported along three edges. In Fig. 1 the system and the idealization of the slab into elements is shown.

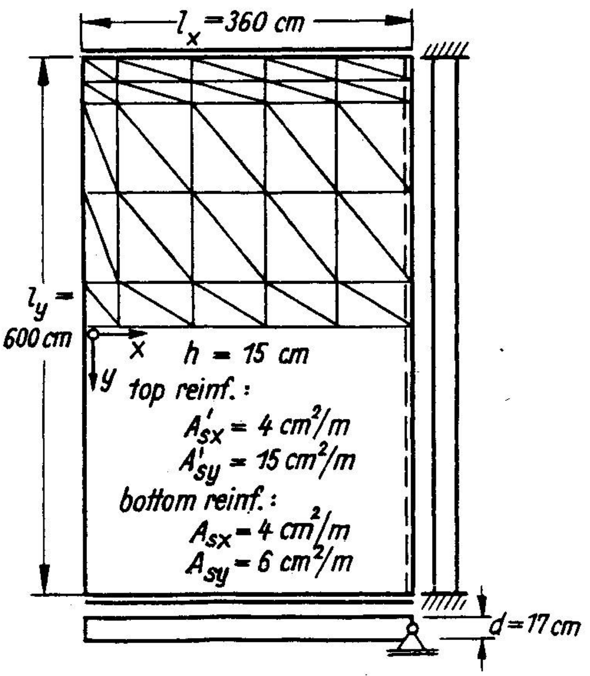

Fig. 1 System and Finite Element Idealization

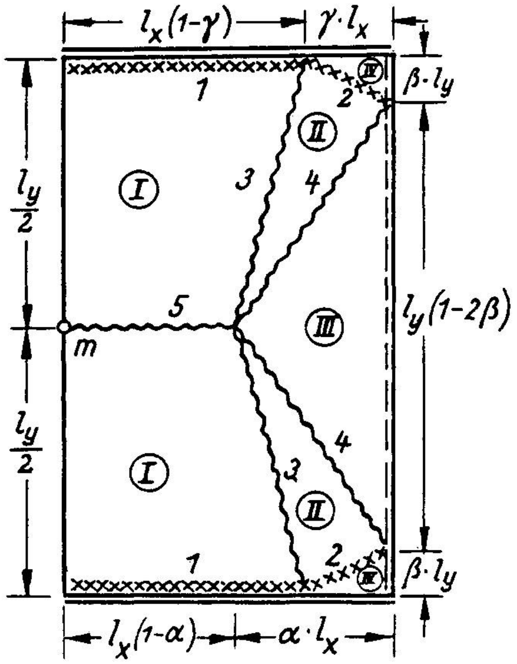

Fig. 2 Yield-Line Pattern with Three Parameters

2. LOAD BEARING CAPACITY INCL. INPLANE FORCES ASSUMING IDEAL-PLASTIC BEHAVIOUR

For the general case of a transversely loaded thin slab the failure will start in a flexural mode. The in-plane forces induced as a secondary effect have only little influence on the failure mode, so that the yield line pattern of the conventional yield line theory for pure bending can be used as a failure mechanism. In Fig. 2 the chosen three-parameter yield-line pattern is shown. With the dimensions indicated in Fig. 1, the yield strength of reinforcement, β_S = 420 N/mm^2 and the characteristic strength of the concrete, β_R = 25 N/mm^2, the lowest upper bound for the ultimate load is by application of the conventional yield-line theory p_0 = 40,6 kN/m^2 with the inherent parameters α = 0,556, β = 0,094 and γ = 0,188.

In extension of the conventional yield-line theory, not only the bending moments but also the resulting in-plane forces normal to the yield lines are used as generalized stresses. By applying the principale of virtual work, the energy dissipated by the in-plane forces along the yield lines must be added to the dissipation density per unit length of the yield line

$$d_i = m \cdot \dot{\Theta} + n \cdot \dot{\Delta} \tag{1}$$

The moments m and forces n are connected by a moment-force-interaction as yield-condition. With a stress-distribution as shown in Fig. 3 the resulting moments and forces normal to the yield line are

$$m_{nn} = A_{sx} \beta_S e_x \cos^2\varphi + A_{sy} \beta_S e_y \sin^2\varphi + \beta_R a \left(b_o - \tfrac{1}{2} a\right) \tag{2}$$

$$n_{nn} = A_{sx} \beta_S \cos^2\varphi + A_{sy} \beta_S \sin^2\varphi - \beta_R \cdot a \tag{3}$$

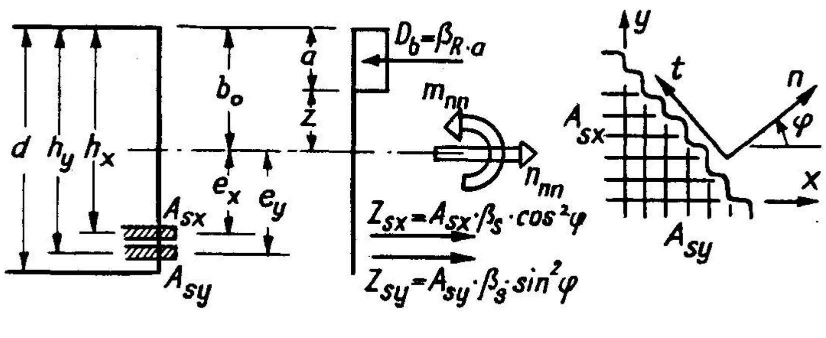

Fig. 3 Stress-Distribution Normal to Yield-Line

By elimination the depth a of the compression zone, the yield criterion becomes

$$F(m_{nn}, n_{nn}) \equiv m_{nn} + \left(b_o - A_{sx}\frac{\beta_S}{\beta_R}\cos^2\varphi - A_{sy}\frac{\beta_S}{\beta_R}\sin^2\varphi\right)n_{nn} + \frac{n_{nn}^2}{2\beta_R} = m_{nnF} \tag{4}$$

where m_{nnF} stands for the yield-moment of the conventional yield-line theory. By application of the flow rule (normality law), a relation between the rate of rotation and the rate of deformation within the reference plane as an internal compatibility condition in the yield line is developed

$$\frac{\dot{\Delta}}{\dot{\Theta}} = b_o - a = z \tag{5}$$

This condition states that the relative rotation axis between two slab parts is identical with the neutral axis. A rotation of the rigid slab parts is only possible, if the rotation axes are horizontal, so that the height of each of the relative rotation axis during the actual deflection under developed failure mechanism can be defined by one parameter, for example the depth of the compression zone a at any point of the yield-line. In Fig. 4 the failure mechanism is shown in the elevation and the rates of displacements of the relative rotation axes by a virtual rate of deflection $\dot{w}_m$ are indicated.

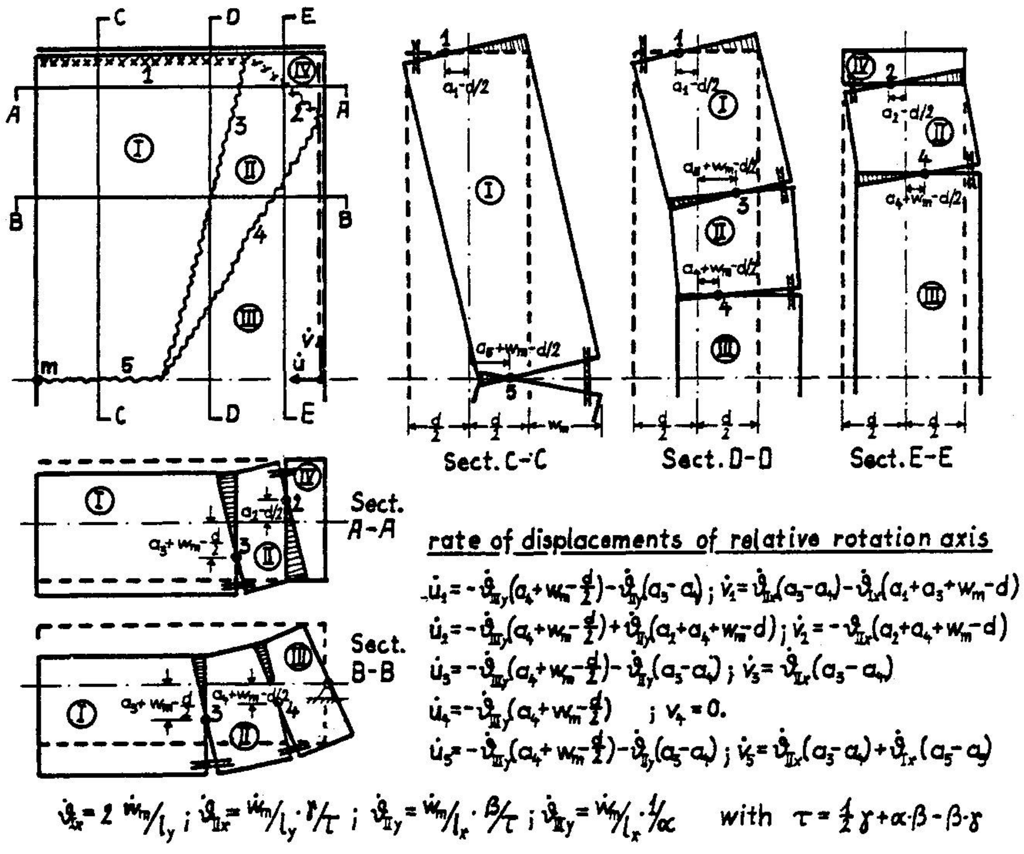

Fig. 4 Failure Mechanism in the Elevation

Eqs. (1), (2), (3) and (5) define the internal energy that is dissipated per unit length of a yield-line during a virtual rate of plastic rotation

$$d_i = \left[A_{sx}\beta_s(h_x - a)\cos^2\varphi + A_{sy}\beta_s(h_y - a)\sin^2\varphi + \frac{1}{2}a^2\beta_R\right]\cdot\dot{\Theta} \tag{6}$$

By equating the total internal and external virtual work, the depth of the compression zones of the five yield-lines, a_1 to a_5, remain unknown. If the clamped edges are fixed in their plane, the rate of displacements of yield-lines nr. 1 and 2 and because of symmetry also of nr. 5 must be zero. By the three conditions $\dot{v}_1 = 0$, $\dot{v}_2 = 0$ and $\dot{v}_5 = 0$ ($\dot{v}$ see Fig. 4) three parameters can be eliminated. The remaining two parameters must be computed by minimizing the ultimate load. This results in a failure load dependent on the actual deflection

$$p_1\left[\frac{kN}{m^2}\right] = 84{,}7 - 3{,}8\ w_m + 0{,}16\ w_m^2 \tag{7}$$

3. ESTIMATION OF THE SELF-INDUCED IN-PLANE FORCES FOR ELASTIC PLATE STRETCHING

The derivation in the preceeding section assumes that the stress-strain-relation of the concrete as well as the reinforcement is ideally-plastic. In reality, however, the stresses of the concrete in the compression zone remain in the elastic range. The flexural response of the slab is well represented by plastic behaviour as soon as the yield point of the reinforcement is exceeded. The in-plane forces, however, are transferred across the yield line mainly by the concrete, so that even in the plastic range of the reinforcement, the plate is deformed elastically in its plane.

To compute the in-plane forces which are induced prior to failure the cracked slab is considered to be an elastic orthotropic panel. It is assumed that the concrete only transfers compression stresses and that shear stresses are possible only in the uncracked zone. Poisson's effect is neglected. If the cross-section remains plane after deflection, the relations between the resultant stresses and the strains within the reference plane are:

$$n_{xx} = K_{xx}\left[\varepsilon_{xx} - (b_o - b_{sx})\left|\frac{\partial^2 w}{\partial x^2}\right|\right]\ ;\ n_{yy} = K_{yy}\left[\varepsilon_{yy} - (b_o - b_{sy})\left|\frac{\partial^2 w}{\partial y^2}\right|\right]$$
$$n_{xy} = \frac{1}{2}a_{xy}E_b\left[\varepsilon_{xy} - \left(b_o - \frac{1}{2}a_{xy}\right)\left|\frac{\partial^2 w}{\partial x\partial y}\right|\right] \tag{8}$$

with $K_{xx} = (A_{sx} + A'_{sx})E_s + a_{xx}E_b\ ;\ K_{yy} = (A_{sy} + A'_{sy})E_s + a_{yy}E_b$ (9)

and the distance of the elastic centroid from the compression side of the slab

$$b_{sx}=\frac{1}{K_{xx}}\left[E_s\left(A_{sx}h_x+A'_{sx}h'_x\right)+\frac{1}{2}E_b a_{xx}^2\right];\ b_{sy}=\frac{1}{K_{yy}}\left[E_s\left(A_{sy}h_y+A'_{sy}h'_y\right)+\frac{1}{2}E_b a_{yy}^2\right] \quad (10)$$

It is assumed that the deflection surface w(x,y) is known and that the in-plane forces do not influence the deflection. Then eqs. (8) to (10), together with the equilibrium conditions and the strain deformation relations within the reference plane

$$\varepsilon_{xx}=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2;\ \varepsilon_{yy}=\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2;\ \varepsilon_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{1}{2}\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y} \quad (11)$$

form a complete system of equations to resolve the planar problem. An exact solution of the problem, however, is impossible as the depth of the compression zone varies over the slab region and is a function of the induced in-plane forces. As an approximation for the active part of the section of the concrete, the compression zone of the cross-section under uniaxial, elastic bending and normal forces is chosen

$$a_{xx}=-\left(A_{sx}+A'_{sx}\right)\frac{E_s}{E_b}+\sqrt{\left(A_{sx}+A'_{sx}\right)^2\frac{E_s^2}{E_b^2}+2\left(A_{sx}h_x+A'_{sx}h'_x\right)\frac{E_s}{E_b}-\frac{2n_{xx}}{E_b\left|\frac{\partial^2 w}{\partial x^2}\right|}} \quad (12)$$

To estimate the membrane forces along the yield-lines, further assumptions are necessary. If a horizontal movement of only the clamped edges is prevented and the slab is able to deform freely in the x-direction, the in-plane forces in this direction may be neglected, n_{xx} = 0, and n_{yy} may be constant in the y-direction.

As boundary condition, the elongation of the slab in the y-direction must be zero, so that the integration yields

$$\int_{l_y}\frac{\partial v}{\partial y}dy=0=n_{yy}\cdot\int_{l_y}\frac{dy}{K_{yy}}-\frac{1}{2}\int_{l_y}\left(\frac{\partial w}{\partial y}\right)^2dy+\int_{l_y}\left(b_o-b_{sy}\right)\left|\frac{\partial^2 w}{\partial y^2}\right|dy \quad (13)$$

With a deflection surface $w=w_m\cdot\left(1-2\frac{x^3}{l_x^3}+\frac{x^4}{l_x^4}\right)\cdot\left(1-8\frac{y^2}{l_y^2}+16\frac{y^4}{l_y^4}\right)$, (14)

the in-plane forces are determined by eq. (13). The solution of eq. (13) is possible only by an iterative process resulting in

$$n_{yy}\left[kN/cm\right]=0{,}21\,w_m^2\left(1-2\frac{x^3}{l_x^3}+\frac{x^4}{l_x^4}\right)^2-3{,}6\,w_m\left(1-2\frac{x^3}{l_x^3}+\frac{x^4}{l_x^4}\right) \quad (15)$$

Now the load bearing capacity of the slab can be calculated by conventional yield-line theory with a yield moment m_{nn} that is related to the in-plane force $n_{nn} = n_{yy} \sin^2\varphi$ by eq. (4). The action of the in-plane forces at the deformed system is taken into account, if the variation of the position of the cross-section to the reference plane during the deflection (see Fig. 4) is considered. Numerical calculations result in

$$p_2 \left[kN/m^2 \right] = 40{,}6 + 9{,}57\, w_m - 1{,}83\, w_m^2 + 0{,}112\, w_m^3 - 0{,}0022\, w_m^4 \qquad (16)$$

4. COMPARISON OF YIELD-LINE SOLUTION WITH THE RESULTS OF FINITE ELEMENT COMP.

To demonstrate the influence of edgerestraint on the load-bearing capacity, in Fig. 5 the strain distributions of the elements along the free edge are shown and compared with the freely movable system at the same load stage. In Fig. 6 the variation of the in-plane forces with increasing deflection are shown. The agreement of the approximation with the results of the FEM is sufficient. Fig. 7 shows the non-dimensionalized load-deflection curves. Although the assumption of elastic plate stretching yields better results than by rigid ideal-plastic behaviour, the ultimate load is overestimated. In the finite element approach failure is reached when the compression stresses of the concrete violate a failure criterion in the biaxial stress state. This happens for the slab with fixed edges before a flexural failure in the plastic range is evident. In [4] the assumptions of the extended yield-line theory are described in detail together with further examples.

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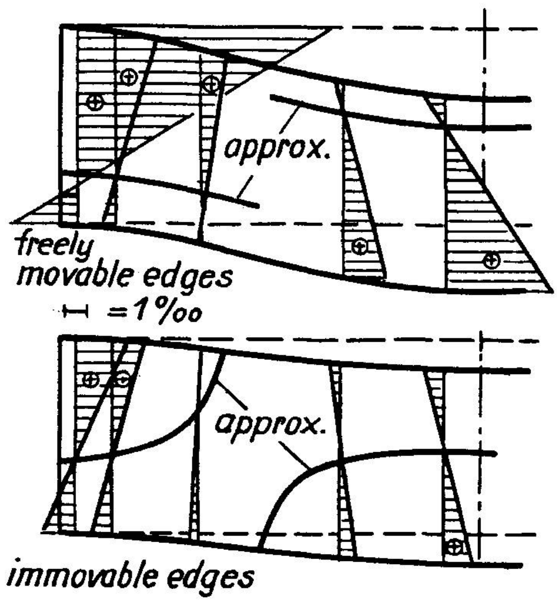

Fig. 5 Strain-Distribution and Assumed Compression Zones

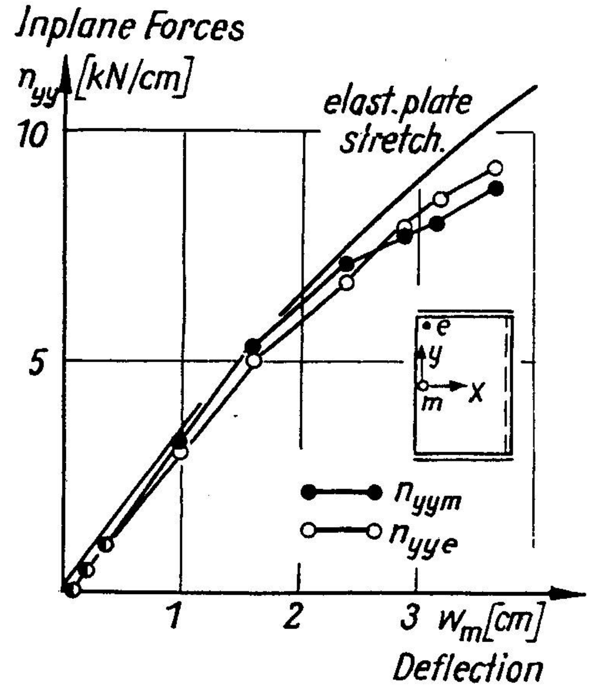

Fig. 6 Development of In-Plane Forces

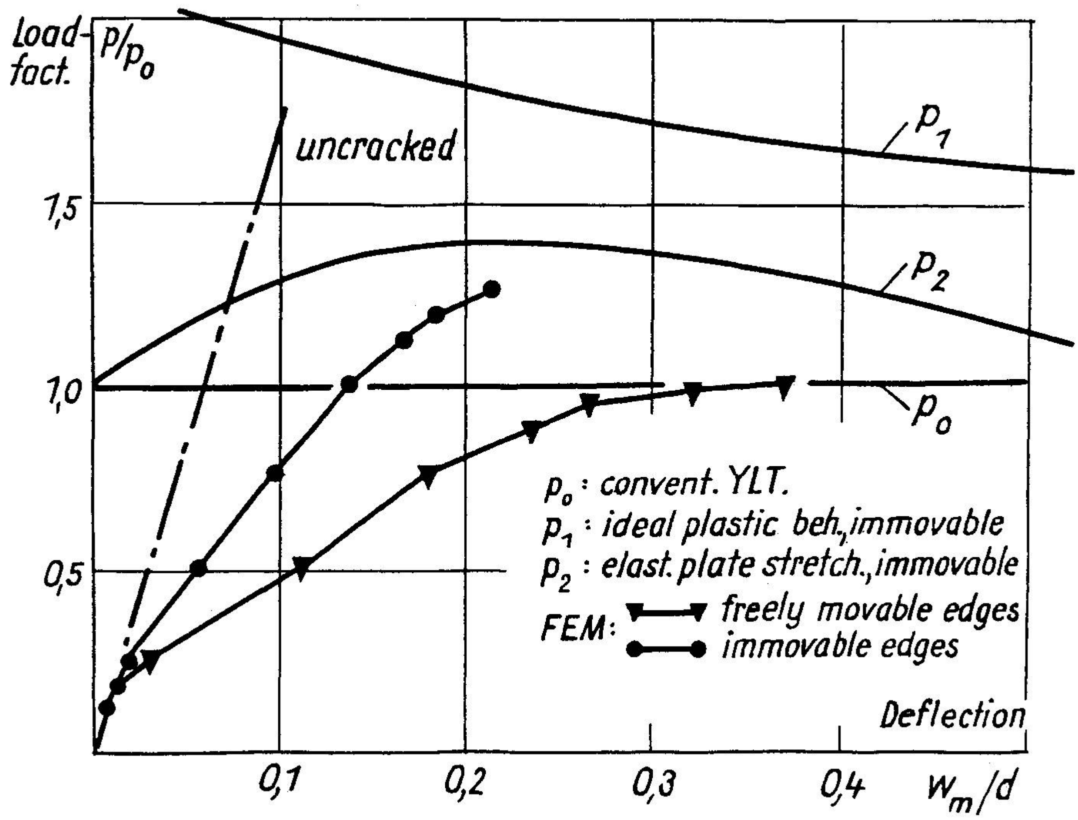

Fig. 7 Load-Deflection Relations