Zeitschrift: IABSE reports of the working commissions = Rapports des commissions de travail AIPC = IVBH Berichte der Arbeitskommissionen

Band: 29 (1979)

Artikel: Nodal forces as real forces

Autor: Clyde, D.H.

DOI: https://doi.org/10.5169/seals-23548

III

Nodal Forces as Real Forces

Les forces nodales en tant que forces réelles

Knotenkräfte als wirkliche Kräfte

D.H. CLYDE
Professor of Civil Engineering
University of Western Australia
Nedlands, W. Australia

SUMMARY
The conventional approach to nodal forces in yield line theory is re-examined because it leads to breakdown cases and other anomalies. It is shown that the true nodal forces are vertical shears at and parallel to strength discontinuities. The existence of these forces was demonstrated for elastic plates by Thomson and Tait but is now shown to be a general statical requirement of shear flow closure. The resulting insight enables the fundamental errors associated with invalid and breakdown cases to be demonstrated.

RESUME
La méthode traditionnelle des forces nodales de la théorie des lignes de rupture est réexaminée parce qu'on obtient dans certains cas des contradictions et d'autres anomalies. On montre que les forces nodales vraies sont des forces de cisaillement existant le long des lignes de discontinuité de la résistance. L'existence de ces forces a été démontrée par Thomson et Tait pour les plaques élastiques. Dans le présent article, on montre que ces forces correspondent à une condition statique générale d'après laquelle les forces de cisaillement doivent être continues. Les conclusions obtenues permettent de montrer les erreurs fondamentales associées aux cas contradictoires de l'application de la méthode des forces nodales.

ZUSAMMENFASSUNG
Die herkömmliche Methode der Knotenkräfte innerhalb der Fliessgelenklinientheorie wird neu betrachtet, da sie in gewissen Fällen zu Widersprüchen und anderen Unregelmässigkeiten führen kann. Es wird gezeigt, dass die wahren Knotenkräfte Querkräften entsprechen, die entlang von Widerstands-Diskontinuitätslinien auftreten. Das Vorhandensein solcher Kräfte wurde für elastische Platten durch Thomson und Tait nachgewiesen. Hier wird gezeigt, dass sie der allgemeinen statischen Forderung nach einem geschlossenen Schubfluss entsprechen. Die grundlegenden Fehler, welche mit Fällen widersprüchlicher Ergebnisse bei der Anwendung der Methode der Knotenkräfte verbunden sind, können mit Hilfe der neu gewonnenen Erkenntnisse aufgezeigt werden.

1. INTRODUCTION

Nodal forces arise in the equilibrium method of yield line theory. A nodal force is a concentrated internal transverse force which, under special circumstances, must be inserted at an end of a straight internal section. Johannsen [1] established the existence of such forces and formulated rules for their determination. Later workers have attempted to improve the rigour of the rules but have, nevertheless, found that breakdown cases exist for which solutions using these rules do not agree with solutions using the alternative work method.

Conflict in the results of different solution methods for properly posed problems in structural mechanics indicates a lack of rigour in setting up one or both of the methods. Fox [2,3] demonstrated that the assumptions of yield line theory can provide the basis of a rigorous rigid-plastic analysis. He constructed coincident upper and lower bound solutions for certain problems whose intractability had earlier suggested an inconsistency between the failure criterion of yield line theory and rigorous plasticity theory [4]. It should be noted that Fox's solutions include zones of finite curvature within which the slabs deform into general developable surfaces whereas in yield line theory only one such surface, the cone, is used. Given the conflict mentioned above and the evidence that a properly formulated solution method has demonstrated the essentially well-posed nature of the problems one must examine the basis of the methods which produce the conflicting solutions. The equilibrium method is so named because a separate equilibrium equation is written for each rigid slab element. The following requirements are satisfied as well as equilibrium :

(i) The forces on the internal boundaries satisfy the failure criterion which in force space [5] is given by

$$M_{xy}^{2} - (M^{x}-M_{x})(M^{y}-M_{y}) \leqslant 0 \qquad M_{xy}^{2} - (M_{x}-\overline{M}^{x})(M_{y}-\overline{M}^{y}) \leqslant 0 \tag{1}$$

(ii) Application of the flow rule to (1) defines associated curvature rates [5]. It may readily be shown that the relative rotations about the yield line of the two rigid segments abutting the yield line correspond to concentrations of one class of curvature rates which satisfy the flow rule.

(iii) The layout of yield lines is such that these relative rotations taken in conjunction with the boundary conditions form a compatible mechanism.

2. NODAL FORCES

Johannsen established that simultaneous solution of the equilibrium equations is theoretically equivalent to the extremised solution of a global equilibrium equation most conveniently written using virtual work.

The most readily demonstrated justification for nodal forces is that there are cases where the two methods give different results and that these can be reconciled by inserting nodal forces of such a magnitude as to cause the equilibrium method solution to co-incide with the work method solution. This method is inferential and has been used by some [6] as a de facto basis for establishment for their magnitude. Johannsen's analysis which established rules for their determination is also inferential and is based on small perturbations of the layout from which it can be shown that nodal forces are required for stationariness.

Despite the indirect determination of nodal forces Johannsen also gave a physical explanation, i.e. "In addition (to the normal moment) a torsional moment and shear stress act on the yield line. These can be resolved into two single forces, one at each end of the section". A counter-example to refute Johannsen's physical explanation is readily found.

Criterion (1) is in the form containing only force variables. Implicit in its derivation is the requirement that the applied normal moment M_n does not exceed the normal moment capacity M^n i.e.

$$M_x \cos^2\alpha + M_y \sin^2\alpha + 2M_{xy} \sin\alpha\cos\alpha \leqslant M^x\cos^2\alpha + M^y \sin^2\alpha \qquad (2)$$

The limit of the inequality may be shown to lead to :

$$M_{nt} = (M^x - M^y) \sin\alpha\cos\alpha \qquad (2a)$$

Since for isotropic reinforcement $M^x = M^y$, M_{nt} must be zero everywhere on an isotropic yield line and $\frac{\partial M_{nt}}{\partial t}$ will thus also be zero. Furthermore translation of the yield line should not lead to violation of (2) so that $\frac{\partial M_n}{\partial n} = 0$ on the yield line. Hence neither twisting moment nor shear force may exist on an isotropic yield line. Nevertheless a classic example of a nodal force occurs where an isotropic yield line meets a free edge and has magnitude M cotα [5]. Thus Johannsen's explanation breaks down. Direct application along the yield line of Thomson-Tait [7] statical equivalence [6] must also fail where there are no forces to which the nodal force can be statically equivalent. Somewhat surprisingly, however, the author has found that the physical basis of the nodal force can be found in other portions of Thomson and Tait's work by examination, not of the yield line which being internal must satisfy conditions of continuity, but of the boundary itself.

3. BOUNDARY CONDITIONS AT AN EDGE

Thomson and Tait utilised equivalence as a mathematical device to reduce three boundary conditions at a free edge to two. Considering a free edge x = a (Fig.1) the apparent consequences of the absence of surface tractions $M_x = 0$, $M_{xy} = 0$, $Q_x = 0$ may be reduced to two namely $M_x = 0$ and $Q_x - \frac{\partial M_{xy}}{\partial y} = 0$. Although Thomson and Tait used statics to derive this result subsequently they showed that there is a local disturbance due to twisting moment at the free edge of an elastic plate which dies out rapidly as one moves away from the edge. They invoked this solution to confirm that St.Venant's principle applies to the statically equivalent forces.

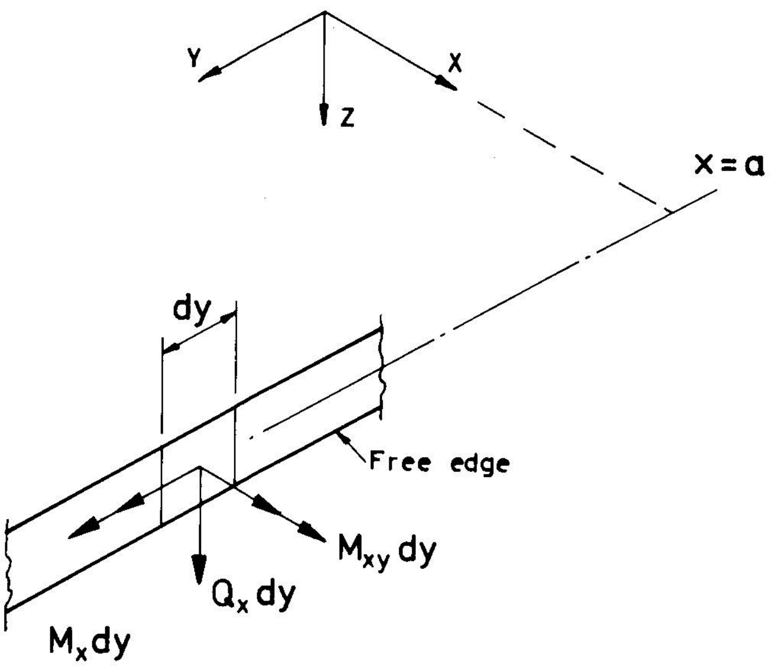

Figure 1. Apparent vanishing forces at free edge.

The analysis which demonstrates the rapid decay provides further insight. It is the treatment of anticlastic curvature produced by alternating upward and downward corner loads on a square plate as a limiting case of St. Venant torsion. They observed that only one half of the total torsional moment arises from the M_{xy} stresses, the other half being due to "two tangential tractions distributed over areas of the edge infinitely near the ends acting perpendicularly to the plate towards opposite parts". These transverse forces exist because the opposing horizontal shear flows which constitute the M_{xy} couple require closure at a stress free edge (Fig.2). Popov [8] has added clarification to this requirement.

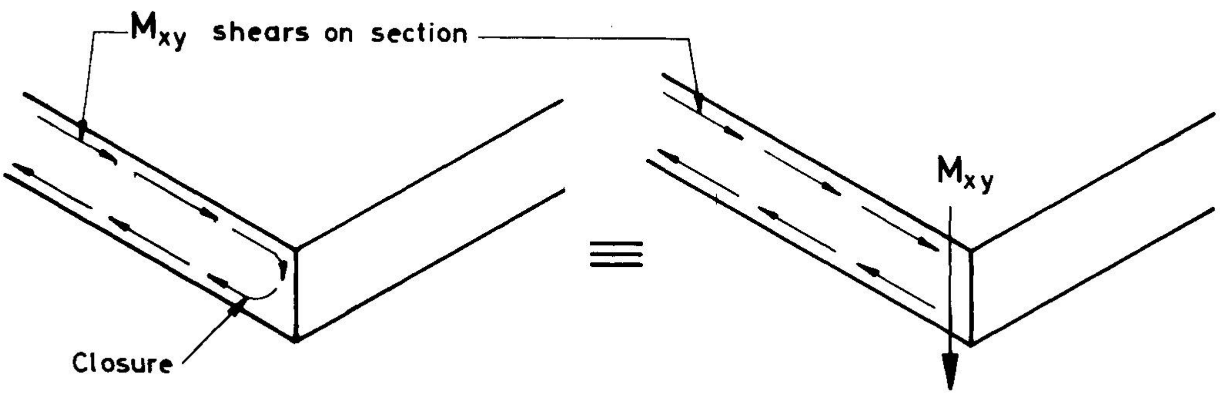

Figure 2. Equivalence of shear flow closure.

While Thomson and Tait's conclusions were based on an elastic solution, the requirement is one of statics and is independent of the nature of the stress distribution which goes to make up M_{xy}. This will now be demonstrated. A twisting moment M_{xy} on a section normal to the free edge is associated with a conjugate M_{xy} on a section parallel to the edge. Assume that the limits of this ideal flexural behaviour occur on a face Δx inside the free edge and parallel to it (Figure 3).

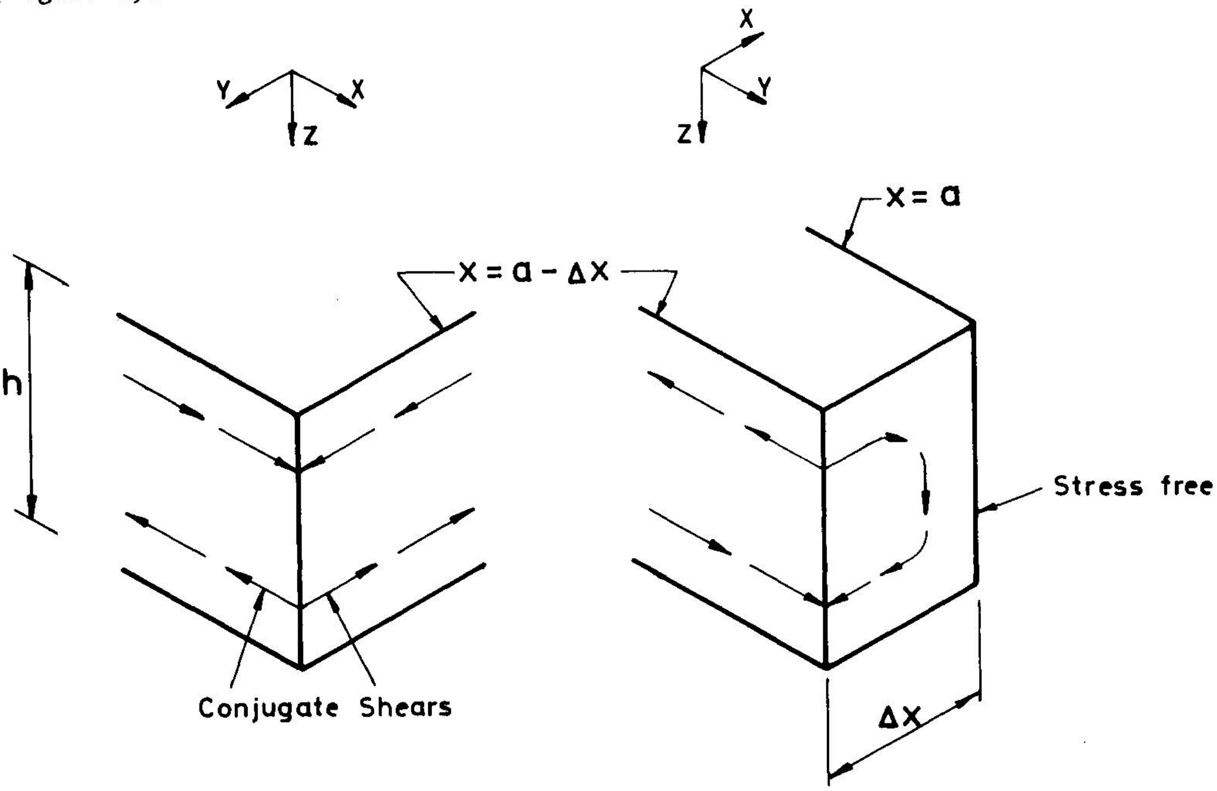

Figure 3. Isolation of Edge Strip.

In the strip Δx wide by h deep the transition from M_{xy} stresses to the stress-free edge state must occur, i.e. over this interval :

$$\int_0^{\Delta x} \frac{\partial \tau_{xy}}{\partial x} dx = \tau_{xy}\Big|_{\Delta x} \qquad (4) \qquad \text{where } M_{xy}\Big|_{\Delta x} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{xy}\Big|_{\Delta x} z\, dz \qquad (5)$$

hence

$$M_{xy} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \int_0^{\Delta x} \frac{\partial \tau_{xy}}{\partial x} z\, dx\, dz \qquad (6)$$

In order to maintain equilibrium parallel to the edge, the differential equation of equilibrium :-

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} = 0 \qquad (7)$$

must be satisfied everywhere. If $\sigma_y = 0$ (i.e. ignoring bending which may be superposed later) this reduces to

$$\frac{\partial \tau_{xy}}{\partial x} = - \frac{\partial \tau_{yz}}{\partial z} \qquad (8)$$

Thus the rate of change of horizontal shear τ_{xy} over the width Δx generates vertical shears τ_{yz} which integrate to yield a vertical force Z_y as follows:-

$$Z_y = \int_0^{\Delta x} \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{yz}\, dz\, dx = \int_0^{\Delta x} \Big[\tau_{yz}\, z \Big]_{-\frac{h}{2}}^{+\frac{h}{2}} dx - \int_0^{\Delta x} \int_{-\frac{h}{2}}^{+\frac{h}{2}} z \frac{\partial \tau_{yz}}{\partial z} dz\, dx \qquad (9)$$

Since $\tau_{yz} = 0$ at $z = \pm \frac{h}{2}$ the first term is zero and substituting (8) in the second yields :-

$$Z_y = \int_0^{\Delta x} \int_{-\frac{h}{2}}^{+\frac{h}{2}} z \frac{\partial \tau_{xy}}{\partial x} dz\, dx = M_{xy}\Big|_{\Delta x} . \qquad (10)$$

This vertical shear force in the edge strip is a physical reality which transcends mere statical equivalence. Because it is a force and not a stress it is invariant under change of angle of the cutting section relative to the edge. Thus for any but the normal section it is not related to the twisting moment on the internal face and it is quite possible to have zero twisting moment in a skew face but to have an edge shear force. This somewhat surprising result clarifies the vertical equilibrium of a 45° corner triangle of the square plate case placed in a state of pure torsion by upward and downward corner forces [9].

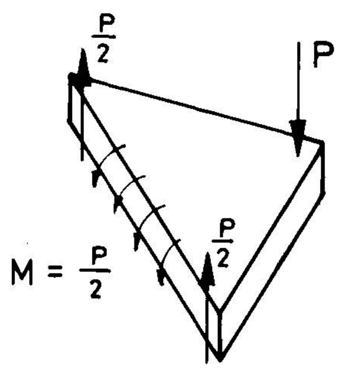

Figure 4. Vertical equilibrium in pure torsion.

4. NODAL FORCES AS REAL FORCES

The nodal force where a yield line meets a free edge is the shear force in the edge strip. It is determined by the twisting moment on faces parallel to and normal to the edge. The twisting moment is readily calculated from the conditions that $M_x = 0$ on the edge and that the normal and twisting moments on the yield line are defined by the failure criterion to be :

$$M_{xy} = - M^x \cot \alpha \tag{11}$$

The problem of a yield line intersecting an internal step change in mesh strength was posed and explored by Jones [6]. If the yield line makes an angle α with the internal boundary on which M^x steel normal to the boundary is reduced from $M^x{}_1$ to $M^x{}_2$ the conditions for satisfaction of the failure criterion on both sides of the change may be obtained in the fashion of equation (11) and are :-

$$(M_{xy_1} - M_{xy_2}) = - (M_1^x - M_2^x) \cot \alpha \tag{12}$$

The band of vertical shear is this time just inside the stronger zone and runs parallel to the boundary between the zones. The magnitude of the shear force is equal to the change in twisting moments and at a yield line becomes the nodal force determined by (12). A practical implication of the above conclusions are that such bands of shear should be included in the reinforcement design considerations.

5. INVALID AND BREAKDOWN CASES

The author has not been able to find further rigorous examples of nodal forces and has separated other pseudo nodal forces into invalid cases, which superficially appear to obey the Johannsen rules, and breakdown cases, which are neither rigorous nor obey the Johannsen rules for their determination.

5.1. Invalid nodal force. This is the intersection of sagging and hogging yield lines. It is possible to devise failure mechanisms which are kinematically admissible and appear to satisfy the failure criterion on all yield lines but fail to satisfy equilibrium at their intersection. The angle of intersection must be such that the failure moments also satisfy the transformation of axes (equilibrium) equation. For example, for isotropic reinforcement this requires orthogonality. For isotropic cases where non-orthogonal intersections are used the interpretation of Johannsen's rules which is widely accepted suggests a nodal force of

$$k = - (M - \overline{M}) \cot \alpha \tag{13}$$

Similarities between this and (12) are deceptive because in this case neither yield line represents a strength discontinuity at which a step change in twisting moment can validly be invoked. Efforts by the author to produce solutions with artificial shear along one yield line led to the conclusion that this merely removes the violation of the yield condition by an amount Δx from the yield line since M_n would not change significantly over a small distance. The assumed nodal force is thus seen as a device which superficially localises the fundamental violation of statics at the intersection but also removes it an infinitesimal amount to one side of one yield line.

The conclusion is that the lack of rigour is implicit in both the equilibrium and the work method. In part this results from the exclusion of Fox's zones of finite curvature from the repertoire of mechanisms available in yield line theory.

5.2. Breakdown cases. These occur where the mechanism is overconstrained [6], and may or may not include invalid intersection angles of yield lines. The overconstraint leads to the necessity for twisting moments and shears on yield lines independently of the angle of intersection problem and thus to apparent forces which do not even superficially appear to satisfy Johannsen's rules.

Of the particular classes of problem which have been explored the following are the three most important :

5.2.1 The re-entrant free edge. Because a free edge is involved at one end of the yield line this is the problem about which most meaningful observations can be made. While real nodal forces are the closure shear flows associated with twisting moments immediately inside a free edge it may readily be shown that an angular discontinuity in a free boundary no twisting moment may exist if all internal moments are required to be continuous. Hence no nodal force may exist at the re-entrant corner of a free boundary. Symmetrical cases show zero apparent nodal force and are the rigorous case of a yield line passing through a re-entrant corner of a free edge.

5.2.2. The re-entrant support. This generates a kinematic requirement for an intersection of a hogging with a sagging yield line which, in general, does not even superficially obey Johannsen's rules and hence goes beyond the case proved above to be invalid. It becomes an overconstrained case for which force transfer between segments is required in order to give the appearance of reconciliation.

5.2.3. The Maltese-Cross failure pattern for a square slab. Wood [6] attributes the posing of the problem to Nylander. This time the penalty of overconstraint is a twisting moment along the yield line associated with extremised solution. The comments under 5.1. above apply with respect to shifting the violation of the failure criterion an infinitesimal distance from the yield line if nodal forces at the centre are postulated as the mode of rectifying the unbalance. The extremised work solution thus cannot be considered as a valid one.

6. CONCLUSIONS

In order to understand the nature of nodal forces it has been necessary to introduce a more comprehensive analysis of boundary conditions than the standard Thomson-Tait one of statical equivalence. As a result it is concluded :

6.1. An internal shear force exists in a narrow strip parallel to a free or simply supported edge which is numerically equal to the twisting moment immediately inside the edge on faces parallel to and normal to the edge.

6.2. A shear force exists in a strip parallel to an internal strength discontinuity which is numerically equal to the step change across the discontinuity of the twisting moment on faces parallel to and normal to the discontinuity.

6.3. At the intersection of a yield line with an edge or internal strength discontinuity the shear force is known as a nodal force.

6.4. Various other cases including the intersection of sagging and hogging yield lines, re-entrant free edges, re-entrant supported edges and the Maltese-Cross failure mechanism for square slabs give rise to false nodal forces due to deficiencies in the posing of the problems.

6.5. Reinforcement requirements at free and simply supported edges and strength discontinuities will need to be re-examined in the light of the existence of the shear face demonstrated in this paper.

7. NOTATION

a	Constant, value of x
h	Slab thickness
n,t,z	Local co-ordinates normal and tangential to section
x,y,z	Cartesian co-ordinates

M Basic symbol for moments per unit length with the following variants :-

M	Isotropic yield moment (sagging)
$\overline{M}$	Isotropic yield moment (hogging)
M_x, M_y, M_{xy}	Applied moments in Cartesian co-ordinates stress resultants of σ_x, σ_y, τ_{xy} respectively.
M_n, M_t, M_{nt}	Applied moments in local co-ordinates.
M^x, M^y	Yield moments in Cartesian co-ordinates (sagging)
$\overline{M}^x, \overline{M}^y$	Yield moments in Cartesian co-ordinates (hogging)
M^n	Yield moment referred to normal co-ordinates

P A concentrated force

Z_y The vertical stress resultant of τ_{yz}

α Yield line orientation relative to M^x yield line

θ Slab parameter

8. REFERENCES

1. JOHANNSEN, K.W. Yield-line Theory. Doctoral Thesis Danmarks Tekniske Hojskole. 1943. English translation, Cement and Concrete Assoc.London 1962.

2. FOX, E.N. Limit analysis for plates: a simple loading problem involving a complex exact solution. Phil.Trans.Roy.Soc.,Vol.272 A.,1972, pp.1057-1073.

3. FOX, E.N. Limit analysis for plates: the exact solution for a clamped square plate of isotropic homogeneous material obeying the square yield criterion and loaded by uniform pressure. Phil.Trans.Roy.Soc.,Vol.277 A., 1974, pp.121-155.

4. WOOD, R.H. Some controversial and curious developments in the plastic theory of structures. Engineering Plasticity, ed.Heyman,J. and Leckie,F.A., Cambridge, 1978.

5. NIELSEN, M.P. The theory of plasticity for reinforced concrete slabs. Introductory Report,IABSE Colloquium on Plasticity in Reinforced Concrete, Copenhagen, 1979 . pp.93-114.

6. Recent Advances in Yield Line Theory. Special Publication of Magazine of Concrete Research, Cement and Concrete Assoc., London, 1965.

7. THOMSON, W. and TAIT, P.G. Natural Philosophy . 1883. Vol.1. Part 2.

8. POPOV, E.P. Kelvin's solution of the torsion problem. Journal of the Engineering Mechanics Div. Proc. of the Am.Soc. of Civ. Engineers. Vol.96, No.EM6 Dec. 1970. pp.1005-1012.

9. MOFFLIN, D. Nodal Forces in Yield Line Theory. Honours Thesis, University of Western Australia, 1978.