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Circular Slabs with Limited Plastic Flow Capacity

Dalles circulaires à capacité d'écoulement plastique limitée

Kreisplatten mit begrenzter plastischer Verformbarkeit

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SUMMARY
A method of elasto-plastic analysis for reinforced concrete circular slabs with a limited plastic rotation capacity is presented. The procedure used is numerical, but analytical elastic solutions already known are utilized as much as possible here. Examples show the relationships between ultimate bending strengths and plastic flow abilities for such slabs.

RESUME
Une méthode d'analyse élasto-plastique est présentée pour des dalles circulaires en béton armé et à capacité de rotation plastique limitée. Il s'agit d'une méthode numérique, mais des solutions analytiques connues pour des plaques élastiques sont utilisées autant que possible. Les relations entre la résistance ultime à la flexion et la capacité de rotation plastique sont illustrées par quelques exemples.

ZUSAMMENFASSUNG
Eine Methode zur elastisch-plastischen Berechnung von Kreisplatten aus Stahlbeton mit einer begrenzten plastischen Rotationsfähigkeit wird dargestellt. Das verwendete Verfahren ist numerisch, doch wird soweit wie möglich auf bekannte analytische Lösungen für elastische Platten zurückgegriffen. Mit Beispielen werden die Beziehungen zwischen den Biegebruchlasten und der plastischen Verformbarkeit solcher Platten erläutert.

1. INTRODUCTION

The classical plastic design method has been developed on the assumption that a structural material or element consists of a perfectly plastic substance which can flow plastically under a constant yield stress until a whole or a part of structure becomes unstable by forming a collapse mechanism. The plastic flow capacity of a concrete being a significant structural material, however, is not unlimited. It is well known that the stress-strain curve of a concrete undergoing a uniaxial compression rises to a strain of about 0.25% and afterward falls gradually to a strain of about 0.35% when a crushing failure occurs. Then, the maximum plastic strain can not be anticipated to exceed 0.3% in uniaxial compression.

On the other hand, the plastic rotation capacity of a reinforced concrete beam is predominated by the plastic extension of the reinforcing steel bars when its reinforcement is very small but by the plastic contraction of the concrete when the reinforcement is rather large. Hence, the plastic rotation capacity becomes smaller as a reinforcing steel ratio becomes larger.

A similar circumstance is naturally supposed to exist in a reinforced concrete slab. This paper is intended to investigate on the effect of the limited plastic flow capacity on the ultimate load carrying capacity for a reinforced concrete circular slab. The rectangular yield curve in the bending moment plane is used here as the initial plastic flow condition of the slab-section, and the subsequent yield curves in an unstable plastic region after a considerable plastic flow are determined according to a piecewise linear strain softening theory. The stress-rate *versus* strain-rate relations in both stable and unstable plastic regions are derived by using the associated flow rule of the plasticity, and consequently the fundamental differential equation concerning load-rate and deflection-rate is obtained.

On the other hand, this paper also presents a new method for the numerical solution of the fundamental differential equation, which is different from the well-known finite element method and finite difference one. The method is developed under the idea that the effect of plastic flow can be replaced with an addition of the self-equilibrating virtual loads resulting from the deviatoric part from the moment distribution given by the linear elastic solution, and the elasto-plastic solution, therefore, can be given by the superposition of the elastic solutions for both the actual and the virtual load distributions.

2. DEFLECTION-RATE EQUATION

A circular slab subjected to axially symmetric loads is dealt with here. Idealizing the moment-curvature relation of the slab section, the curve including linear elastic, perfectly plastic and strain softening parts will be obtained as Fig. 1. Assuming a rectangular yield curve which is based upon the shear fracture of concrete by the Mohr-Coulomb's theory and the tensile plastic flow of steel bars and, using the normality law on plastic strain-rates postulated by A.C.Palmer *et al.* [1], the loading surfaces after a plastic flow are supposed as Fig. 2. Namely, when yielding at the line A B, the yield surface diminishes to A'B', A"B", ••••••, and when yielding at the line A C, to E'C', E"C", ••••••, and when yielding at the corner A, to D', D", ••••••. Hence, the expression of the loading function can be written as

$$\max [\mid M_r \mid - M_{or}(e_r) , \mid M_\theta \mid - M_{o\theta}(e_\theta)] = 0 \qquad (1)$$

where M_r, M_θ, $M_{or}(e_r)$, and $M_{o\theta}(e_\theta)$ are radial and circumferential moments and those bending strengths in the polar coordinates (r, θ), respectively and e_r and e_θ are parameters representing plastic curvatures. The Hooke's law and the associated flow rule of the plasticity give the following relations of moment-rates and curvature-rates :

$$\begin{Bmatrix} \dot{M}_r \\ \dot{M}_\theta \end{Bmatrix} = D \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \dot{\kappa}_r - d\mu_r \cdot \partial F_r/\partial M_r \\ \dot{\kappa}_\theta - d\mu_\theta \cdot \partial F_\theta/\partial M_\theta \end{Bmatrix} \tag{2}$$

where ν = Poisson's ratio, D = flexural rigidity, $\dot{\kappa}_r$, $\dot{\kappa}_\theta$ = curvature-rates, and $d\mu_r$, $d\mu_\theta$ = plastic flow coefficients ; and

$$F_r = |\, M_r \,| - M_{or}(e_r) \ , \ F_\theta = |\, M_\theta \,| - M_{o\theta}(e_\theta) \ ,$$

$$M_{or}(e_r) = \bar{M}_{or} - \beta_r D e_r \ , \ M_{o\theta}(e_\theta) = \bar{M}_{o\theta} - \beta_\theta D e_\theta \ ,$$

$$e_r = \int_0^t d\mu_r \ , \ e_\theta = \int_0^t d\mu_\theta \ ,$$

where β_r, β_θ = strain softening rates which may be related to the coefficient $\beta/(1-\beta)$ indicated in Fig. 1. The plastic flow coefficients are expressed as follows :

for the corners A, D', D", ······· ;

$$\begin{Bmatrix} d\mu_r \\ d\mu_\theta \end{Bmatrix} = \frac{1}{(1-\beta_r)(1-\beta_\theta) - \nu^2} \begin{bmatrix} 1 - \beta_\theta - \nu^2 & -\nu\beta_\theta \\ -\nu\beta_r & 1 - \beta_r - \nu^2 \end{bmatrix} \begin{Bmatrix} \dot{\kappa}_r \\ \dot{\kappa}_\theta \end{Bmatrix} ; \tag{3}$$

for the lines A B, A'B', A"B",······· ;

$$F_\theta = 0, \quad d\mu_\theta = (\dot{\kappa}_\theta + \nu\dot{\kappa}_r)/(1 - \beta_\theta), \quad d\mu_r = 0 \quad ; \tag{4}$$

for the lines A C, E'C', E"C", ······· ;

$$F_r = 0, \quad d\mu_r = (\dot{\kappa}_r + \nu\dot{\kappa}_\theta)/(1 - \beta_r), \quad d\mu_\theta = 0 \quad ; \tag{5}$$

and the expressions for the other lines and corners will easily be obtained by exchanging appropriately the signs of the coefficients in the above expressions. Eqs. (3), (4), (5) and others will be represented for simplification as

$$\begin{aligned} d\mu_r &= \phi_{11}\dot{\kappa}_r + \phi_{12}\dot{\kappa}_\theta \\ d\mu_\theta &= \phi_{21}\dot{\kappa}_r + \phi_{22}\dot{\kappa}_\theta \end{aligned} \tag{6}$$

Using the relation between curvature-rates and deflection-rate in an axially symmetric bending, namely

$$\begin{aligned} \dot{\kappa}_r &= - \partial^2\dot{w}/\partial r^2 \\ \dot{\kappa}_\theta &= - \partial\dot{w}/r\partial r \end{aligned} \tag{7}$$

and substituting Eq. (2) into the following equilibrium equation :

$$\frac{\partial^2 \dot{M}_r}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}(2\dot{M}_r - \dot{M}_\theta) = -\dot{p}(r) \tag{8}$$

where $\dot{p}(r)$ = a load-rate distribution, the fundamental differential equation about deflection-rate can be derived as follows :

$$D\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 \dot{w} = \dot{p}(r) + \frac{\partial^2}{\partial r^2}\dot{X}_1 + \frac{1}{r}\frac{\partial}{\partial r}\dot{X}_2 \tag{9}$$

where

$$\left.\begin{aligned} \frac{\dot{X}_1}{D} &= (\phi_{11}\phi_a + \nu\phi_{21}\phi_b)\frac{\partial^2 \dot{w}}{\partial r^2} + (\phi_{12}\phi_a + \nu\phi_{22}\phi_b)\frac{1}{r}\frac{\partial \dot{w}}{\partial r} \\ \frac{\dot{X}_2}{D} &= \left[(2-\nu)\phi_{11}\phi_a + (2\nu-1)\phi_{21}\phi_b\right]\frac{\partial^2 \dot{w}}{\partial r^2} + \\ &+ \left[(2-\nu)\phi_{12}\phi_a + (2\nu-1)\phi_{22}\phi_b\right]\frac{1}{r}\frac{\partial \dot{w}}{\partial r} \\ \text{in which} \quad \phi_a &= \partial F_r/\partial M_r \quad \text{and} \quad \phi_b = \partial F_\theta/\partial M_\theta \ . \end{aligned}\right\} \tag{10}$$

The second and the third terms in the right side of Eq. (9) mean the addition of the self-equilibrating virtual load-rate distributions resulting from a deviatoric part from a linear elastic moment distribution, namely the moment redistributions due to plastic flow.

3. METHOD OF SOLUTION

The differential operator of the left side in Eq. (9) is the same as an elastic problem, and the unknown moment-rate redistributions ($\dot{X}_1$, $\dot{X}_2$) in a plastic region, therefore, are determined by the solution of the following simultaneous integro-differential equation :

$$\left[(\phi_{11}\phi_a + \nu\phi_{21}\phi_b)\frac{\partial^2}{\partial r^2} + (\phi_{12}\phi_a + \nu\phi_{22}\phi_b)\frac{1}{r}\frac{\partial}{\partial r}\right] \times \\ \times \left[\dot{p}w_o + \int_{R_p}\frac{\partial^2}{\partial \xi^2}\dot{X}_1\overline{w}(r,\xi)d\xi + \int_{R_p}\frac{1}{\xi}\frac{\partial}{\partial \xi}\dot{X}_2\overline{w}(r,\xi)d\xi\right] = \frac{\dot{X}_1}{D} \tag{11}$$

$$\left\{\left[(2-\nu)\phi_{11}\phi_a + (2\nu-1)\phi_{21}\phi_b\right]\frac{\partial^2}{\partial r^2} + \right. \\ \left. + \left[(2-\nu)\phi_{12}\phi_a + (2\nu-1)\phi_{22}\phi_b\right]\frac{1}{r}\frac{\partial}{\partial r}\right\} \times \\ \times \left[\dot{p}w_o + \int_{R_p}\frac{\partial^2}{\partial \xi^2}\dot{X}_1\overline{w}(r,\xi)d\xi + \int_{R_p}\frac{1}{\xi}\frac{\partial}{\partial \xi}\dot{X}_2\overline{w}(r,\xi)d\xi\right] = \frac{\dot{X}_2}{D} \tag{12}$$

where $\dot{p}$ = an actual load-rate intensity, w_o = the elastic solution for the actual load-rate distribution with the unit intensity, $\bar{w}(r,\xi)$ = the elastic solution for a circular line load with the unit intensity at the position r = ξ within a plastic region, and R_p = the plastic region.

Now consider a method of numerical solution for Eqs. (11) and (12). Dividing the radial region of a circular slab by a net of sufficiently fine meshes and considering the case when only the part of one mesh becomes of plasticity, the distributions of the $\dot{X}_1$ and $\dot{X}_2$ within the mesh may be assumed to consist of the continuous three parabolic curves as shown in Fig. 3, because equilibrium conditions require that both the moment redistributions and their first derivatives related to shearing forces must be continuous within a whole plastic region and at an elastic-plastic boundary. Thus the virtual load-rate distributions being equivalent to these moment-rate redistributions are given from Eq. (9) as Fig. 3, which are naturally self-equilibrating. When a region including a number of meshes is plasticized, therefore, the deflection-rate of the slab is obtained by the superposition of the elastic solutions for both the actual load-rate distributions and the virtual load-rate ones mentioned above. Namely,

$$\left.\begin{aligned} \dot{w} &= \dot{p}w_o + \sum_i \dot{z}_{1i}\bar{w}_{1i} + \sum_i \dot{z}_{2i}\bar{w}_{2i} \\ \dot{\kappa}_r &= -\frac{\partial^2 \dot{w}}{\partial r^2} = -\dot{p}\frac{\partial^2 w_o}{\partial r^2} - \sum_i \dot{z}_{1i}\frac{\partial^2 \bar{w}_{1i}}{\partial r^2} - \sum_i \dot{z}_{2i}\frac{\partial^2 \bar{w}_{2i}}{\partial r^2} \\ \dot{\kappa}_\theta &= -\frac{1}{r}\frac{\partial \dot{w}}{\partial r} = -\frac{\dot{p}}{r}\frac{\partial w_o}{\partial r} - \sum_i \dot{z}_{1i}\frac{1}{r_i}\frac{\partial \bar{w}_{1i}}{\partial r} - \sum_i \dot{z}_{2i}\frac{1}{r_i}\frac{\partial \bar{w}_{2i}}{\partial r} \end{aligned}\right\} \quad (13)$$

where $\bar{W}_{1i}$ and $\bar{W}_{2i}$ represent the elastic deflections due to the virtual load-rate distributions with the unit intensity, $\dot{Z}_{1i}$ = 1 and $\dot{Z}_{2i}$ = 1, at the mesh point, i, which will easily be obtained by the integration of the solution for a circular line load given by the well-known literatures, e.g., Timoshenko's book [2], and the summation is executed over all meshes in plastic regions. Substituting the rate-equation (13) into Eq. (10) and using the relations, $\dot{Z}_{1i} = 4\dot{X}_1/\Delta r^2$ and $\dot{Z}_{2i} = 4\dot{X}_2/[\Delta r(2r_i-\Delta r)]$, shown in Fig. 3, a simultaneous equation about the unknowns, $\dot{Z}_{1i}$, $\dot{Z}_{2i}$, i = 1,2,·······, are obtained,and the substitution of its solution into Eq. (13) consequently determines the deflection-rate and the curvature-rates resulting from the actual load-rate, $\dot{p}$. Finally, the total deflection, the total curvatures, and the total moments are obtained from the integration of Eqs. (13) and (2) by making use of the forward difference method of sufficiently short intervals about load increment or central deflection-increment, which may be familiar in the elasto-plastic numerical analysis for a solid [3].

4. RELATIONSHIP BETWEEN PLASTIC FLOW CAPACITIES AND ULTIMATE BENDING STRENGTHS

Numerical calculations for the load *versus* central deflection curves for circular slabs subjected to partially or entirely uniform loads are carried out by setting the following material constants :

yield stress of steel, f_s = 275 N/mm^2; compressive strength of concrete, f_c = 27.5 N/mm^2; secant modulus of elasticity of concrete, $E_c = 1.4 \times 10^4$ N/mm^2; reinforcing steel ratio, p = 0.008; ν = 1/6; d/a = 0.1, d = effective depth, a = radius; strain softening rates, $\beta_r = \beta_\theta = \beta/(1-\beta)$; $D = E_c d^3/12(1-\nu^2)$; $\bar{M}_{or}/E_c d^2 = \bar{M}_{o\theta}/E_c d^2 = \bar{M}_o/E_c d^2 = 1.5 \times 10^{-4}$(isotropic reinforcements); and $\overline{M}_o = f_s d^2 p \cdot (1 - 3p \cdot f_s / 4f_c)$;

and by taking the 11 dividing net points along the radius.

Fig. 4 shows the relationships between the plastic flow ability and the ultimate load in which K_e = limit elastic curvature, $\theta_e(=dK_e)$ = a standardized rotation at the built-in edge, K_p = limit plastic curvature, θ_p = limit plastic rotation at the built-in edge, and P/P_o = the ratio of the ultimate load of the slab to that of the perfectly plastic slab. On the curves for the clamped slabs in Fig. 4, the dotted lines indicate that the limitation of plastic rotation at the built-in edge is more dominant than that of plastic curvature at the center for reducing their ultimate loads. From the figure it can be observed that the influence of plastic flow ability upon the ultimate load carrying capacities becomes larger as the loaded area becomes smaller. Figs. 5 and 6 indicate the relationship between the magnitude of plastic curvature or rotation and the load *versus* deflection curves obtained for the uniformly loaded slabs, and Fig. 7 shows the variations of moment distributions along the radius for the simply supported and uniformly loaded slab with $\bar{\alpha}$ = 0.25 and $\bar{\beta}$ = 0.1, in which the numerals within circles indicate the correspondence to those in Fig. 5. It can be seen that the load *versus* deflection curves for the clamped slab are sensitively influenced by a limit rotation capacity in the built-in edge, and after the built-in edge plastically fails, they follow those for the simply supported slabs.

5. CONCLUSIONS

An elasto-plastic analysis for reinforced concrete circular slabs with limited plastic curvature or rotation abilities due to the restriction of compressive plastic flow of concrete has been carried out by using the associated plastic flow rule of the plasticity which is related to the loading functions considering a strain softening effect. Here, a fundamental differential equation concerning load-rate and deflection-rate has been derived, and a new solution method being different from the finite difference method for this equation has been presented, in which elastic analytical solutions previously known can be utilized as possible. Numerical calculations have revealed the relationships between the load carrying capacities and the plastic flow abilities for reinforced concrete circular slabs with typical material constants.

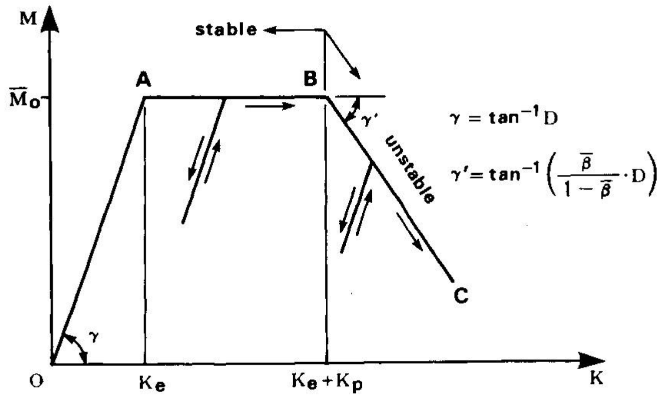

Fig. 1 Idealized Moment-curvature Curve.

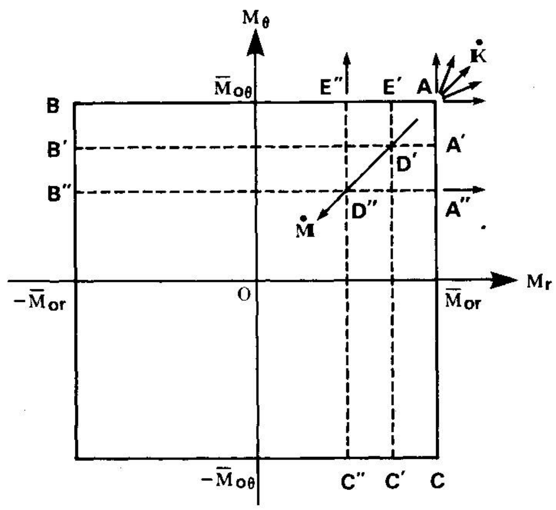

Fig. 2 Loading Surfaces.

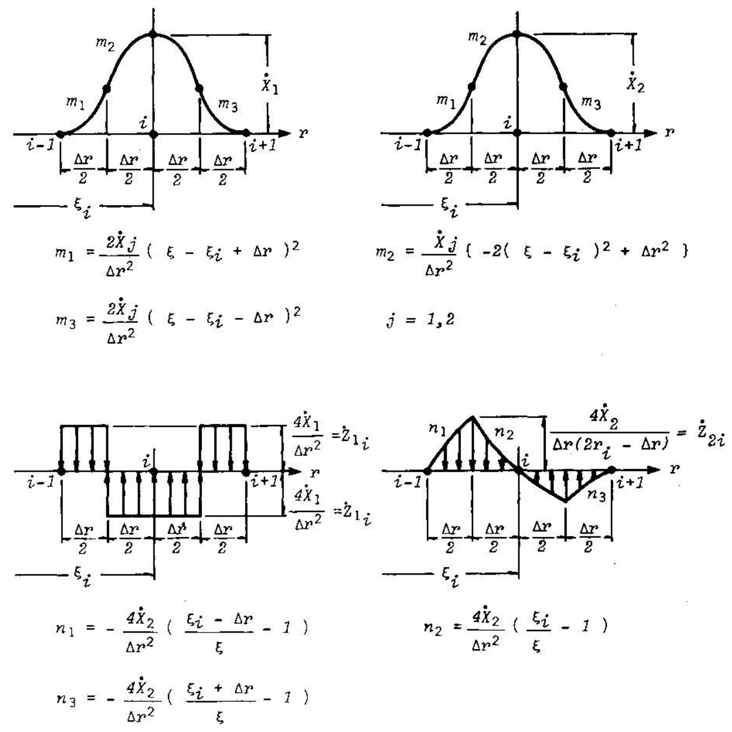

Fig. 3 Moment Redistributions and Virtual Load Distributions Corresponding to Them.

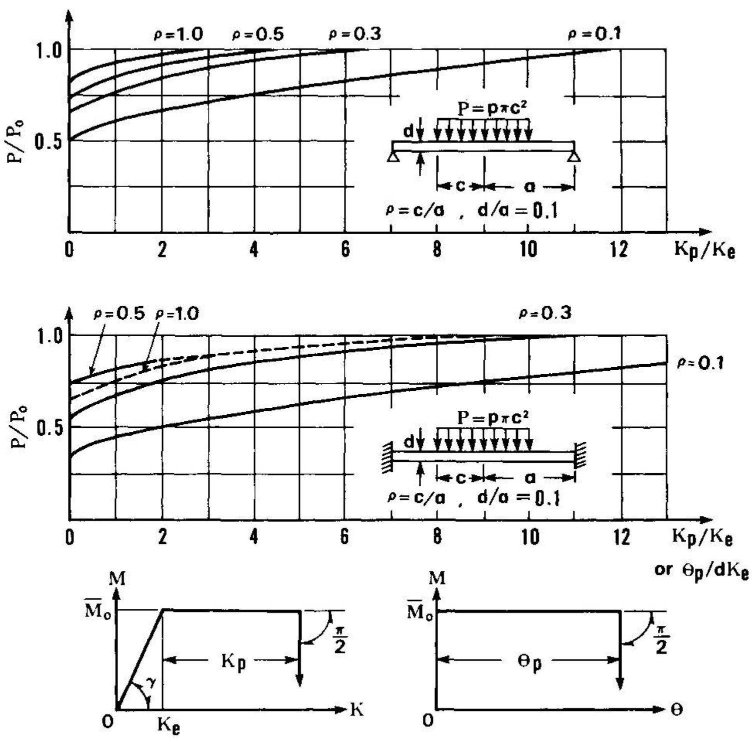

Fig. 4 Relationships Between Ultimate Loads and Limit Plastic Curvature or Limit Plastic Rotation.

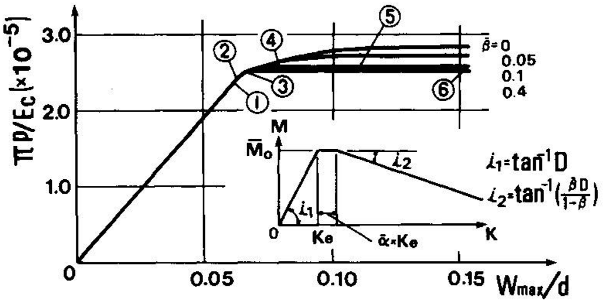

Fig. 5 Load-deflection Curves for the Simply Supported Slab under a Uniform Load, Where $\bar{\alpha}$ = 0.25 is set.

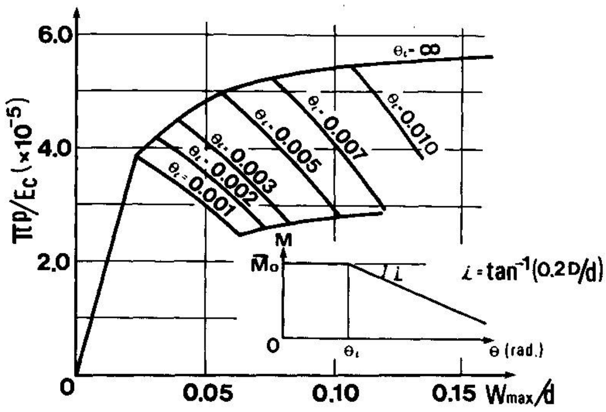

Fig. 6 Load-deflection Curves for the Clamped Slab under a Uniform Load.

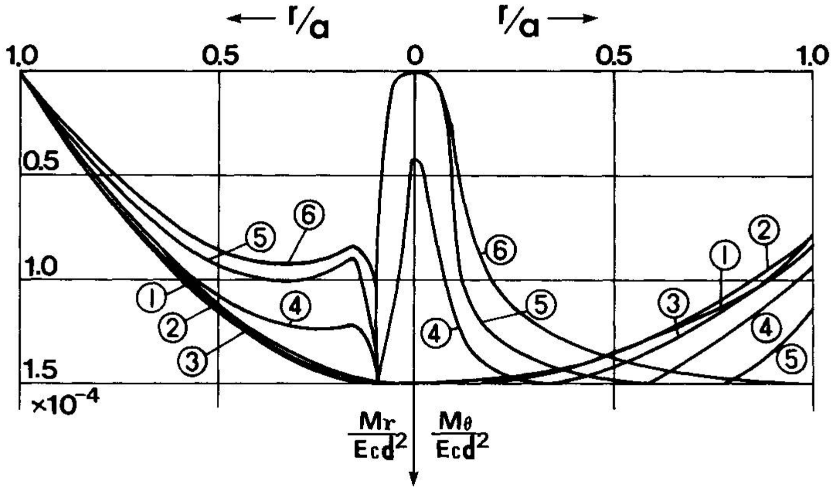

Fig. 7 Variations of Bending Moments along the Radius for the Simply Supported and Uniformly Loaded Slab with $\bar{\alpha}$ = 0.25 and $\bar{\beta}$ = 0.1 in Fig. 5.

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