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III

Complete Limit Analysis Solutions and Yield Line Theory

La solution exacte de l'analyse limite et la théorie des lignes de rupture

Vollständige Lösungen nach Traglastverfahren und Fliessgelenklinientheorie

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SUMMARY
Yield line theory solutions for a circular plate subjected to point loads are compared with the exact solutions obtained by integration of the plastic plate equations for parabolic stress regimes. Differences between the approaches are discussed.

RESUME
Des solutions obtenues selon la théorie des lignes de rupture pour une plaque circulaire soumise aux forces concentrées sont comparées avec la solution exacte provenant de l'intégration des équations des plaques plastiques en régime parabolique. Les différences entre ces deux méthodes sont présentées.

ZUSAMMENFASSUNG
Fliessgelenklinienlösungen für eine Kreisplatte unter Einzellasten werden mit den vollständigen Lösungen verglichen, welche durch Integration der für parabolische Spannungsfelder plastischer Platten geltender Gleichungen erhalten werden. Unterschiede zwischen den beiden Betrachtungsweisen werden erörtert.

1. INTRODUCTION

For plates obeying the Johansen yield criterion the complete limit analysis solutions are available in several cases, [1]. This criterion is usually employed in the yield line theory to obtain an upper bound to the collapse load since the respective calculations concern solely the mechanism of motion under the limit load, [2].

We intend to show on a simple example similarities and differences between the yield line theory solutions and those which give for the Johansen criterion the full information about the collapse load, the collapse mode, and the stress field at collapse, [4], [5].

For the maximum principal moment yield criterion the complete solutions can be obtained for various cases of loading of simply supported plates. A perfectly plastic plate whenever it goes plastic, is totally or partially in the parabolic, isotropic or hyperbolic stress regime or its stress field remains below the yield point, whenever the respective part of the plate remains rigid [4]. The type of stress regime depends on the equation of the yield surface [5], [6].

Stress discontinuities may occur across the line separating different stress regimes, [4], [5], [7]. Discontinuities of the tangent to the deflected surface are admitted by the maximum principal moment yield criterion. This property is used in the yield line theory to generate collapse mechanisms with hinge lines, [2], [8], [9].

The complete solution of a limit analysis problem for a plate consists in finding the collapse load intensity and the associated field of moments and shear forces satisfying
- the internal equilibrium requirements and the prescribed stress boundary conditions
- not violating the yield condition

The complete solution also contains
- the displacement velocity field specifying a kinematically admissible collapse mechanism associated with positive energy dissipation at the plate collapse, [4].

Complete solutions may differ from the results obtained employing the yield line theory both in the collapse load multiplier and in the yield pattern because of the difference in the set of equations used. A yield line solution does not specify the stress field in the plate at collapse as it disregards the differential equations of equilibrium, which are taken into account in any complete solution. It is known that the yield line theory gives upper bounds to the collapse load. In the yield line theory any collapse mode consists of developable surfaces which, in fact, correspond to parabolic stress regimes of the method giving complete solutions, [3], [4].

A circular simply supported plate subjected to two point loads furnishes the case when the yield line theory results differ in comparison with the complete solution.

2. GENERALITIES

We consider a perfectly plastic plate, isotropic both in "positive" and "negative" bending, with the yield moments M_o and M'_o respectively. We shall refer the field variables either to the cartesian or to the polar coordinate systems.

The comparison of the exact and yield line solutions will be made for a simply supported plate, Fig. 1a. The point loads Q_1 and Q_2 are applied at the equal distances A from the plate center. The following dimensionless quantities will be used.

$$\rho = \frac{r}{R}, \quad a = \frac{A}{R}, \tag{2.1}$$

$$\Lambda = \frac{M_o}{M_o}, \quad w = \frac{W}{R},$$

$$m_i = \frac{M_i}{M_o}, \quad t_i = \frac{T_i R}{M_o},$$

$$i = x,\ y,\ xy,\ r,\ \theta,\ r\theta \tag{2.2}$$

$$q_1 = \frac{Q_1}{M_o}, \quad q_2 = \frac{Q_2}{M_o},$$

$$\alpha = \frac{q_2}{q_1}, \quad 0 \leqslant \alpha \leqslant 1 \tag{2.3}$$

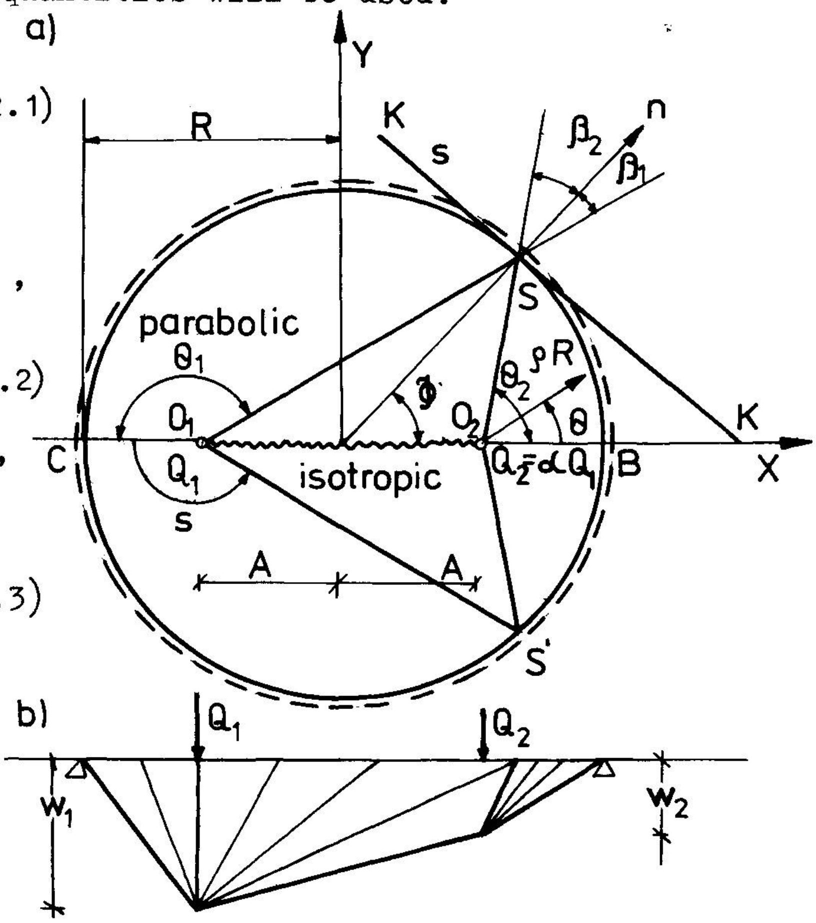

Fig. 1

where W, M_i, T_i denote the deflection velocity, the bending moments and the shear forces respectively whereas R stands for the reference length which equals to the plate radius.

Moreover,

$$q = (1 + \alpha)\, q_1 \tag{2.4}$$

denotes the dimensionless total collapse load of the plate. The q_+, q_{1+}, q_{2+} will stand for dimensionless limit loads corresponding to yield line theory solutions. The ratio of the deflection velocities of the points of load application, Fig. 1b, is $\beta = w_2/w_1$.

The maximum principal moment yield condition of the Johansen criterion represents a square yield locus in the plane of principal bending moments

$$\frac{m_2-m_1}{2} \pm \frac{m_1+m_2}{2} \mp 1 = 0, \qquad \frac{m_2-m_1}{2} \pm \frac{m_1+m_2}{2} \pm \Lambda = 0 \tag{2.5}$$

or a pair of intersecting cones in the moment space m_x, m_y, m_{xy}.

3. COMPLETE SOLUTION

The complete solution of the considered plate under the criterion (2.5) consists of the parabolic zones $SCS'O_1$ and $S'BSO_2$ joined by the isotropic regime $SO_1S'O_2$. In the isotropic zone the dimensionless stress field is

$$m_x = m_y = m_r = m_\theta = 1, \quad t_x = t_y = t_r = t_\theta = 0 \tag{3.1}$$

Thus any direction is principal and the zone caries no transverse loading. The velocity field is arbitrary and is subjected to the condition that the Gaussian curvature of the deflected surface is non-negative there [3], [4].

In the parabolic zones $SCS'O_1$ and $S'BSO_2$ the bending moments and the shear forces expressed in the polar coordinates with origin each time at the point of loading, Fig. 1a, are respectively

$$m_\theta = 1, \quad m_r = -\frac{a^2\sin^2\theta}{1-a^2\sin^2\theta}, \quad t = 0, \quad t_r = -\frac{1}{\varrho}\,\frac{1}{1-a^2\sin^2\theta} \tag{3.2}$$

The velocity of deflection in a parabolic zone is bounded by a developable surface, [4]. For the point loading considered two conical surfaces $SCS'O_1$ and $S'BSO_2$ are obtained. The vertices of the cones are at the points of the load application, Fig. 1b.

Along the lines O_1S, O_2S, and symmetrically, discontinuities in the radial moment appear, as it can be seen when comparing the results (3.1) and (3.2). Between the parabolic and the isotropic regimes on the lines SO_1 and SO_2 in Fig. 1a, there is a continuous transition of the circumferential derivative of the deflection velocity. The triangular part AO_1O_2 of the isotropic range rotates with respect to the axis K-K, Fig. 1a, which is tangent to the plate boundary at S. Geometrical considerations lead to the conclusion that the flat element O_1SO_2 is tangent to the sectors of cones SO_1S' and SO_2S'. The points of load application have the vertical velocities related as follows

$$\beta = \frac{w_2}{w_1} = \frac{1 - a\cos\phi}{1 + a\cos\phi} \tag{3.3}$$

The collapse loads are calculated considering the shear force along a circumferential trajectory [4], [7].

$$q_1 = 2\int_0^S (-t)\,ds = 2\int_0^1 (-t)\,\varrho\,d\theta \tag{3.4}$$

and the results are respectively

$$q_1 = \frac{2}{\sqrt{1-a^2}}\left[\pi - \arctan\sqrt{1-a^2}\,\frac{\sin\phi}{a+\cos\phi}\right] \tag{3.5}$$

$\Gamma = 0$ if $\phi \leqslant \arccos a$ $\quad \Gamma = \pi$ if $\phi > \arccos a$

$$q_2 = \frac{2}{\sqrt{1-a^2}}\left[\gamma - \arctan\sqrt{1-a^2}\,\frac{\sin\phi}{a+\cos\phi}\right] \tag{3.6}$$

The obtained parabolic solutions are valid for the following positions of the load application points

$$0 \leqslant a \leqslant k\sqrt{\frac{\Lambda}{1+\Lambda}} \tag{3.7}$$

where $k = \sec\theta_i$, $\theta_i < \pi/2$, $k = 1$, $\theta_i > \pi/2$, $i = 1,2$. The angle specifying the meeting point S of the parabolic regimes is given by the equation

$$\alpha = \frac{\gamma - \arctan\left(\sqrt{1-a^2}\cdot\dfrac{\sin\phi}{a-\cos\phi}\right)}{\pi - \arctan\left(\sqrt{1-a^2}\cdot\dfrac{\sin\phi}{a+\cos\phi}\right)} \tag{3.8}$$

In Fig. 2 this angle is specified in terms of the ratio of the loads applied. This allows to derive the shape of the central isotropic zone for the given α and the load application point a.

The deflection rates at the points of loading are not equal if the loads are not equal, $\alpha \neq 1$. The shape of the deflected surface is indicated in Fig. 1b. It is seen that for the load ratio $\alpha = 1$ is for any position of the loading $\beta = 1$.

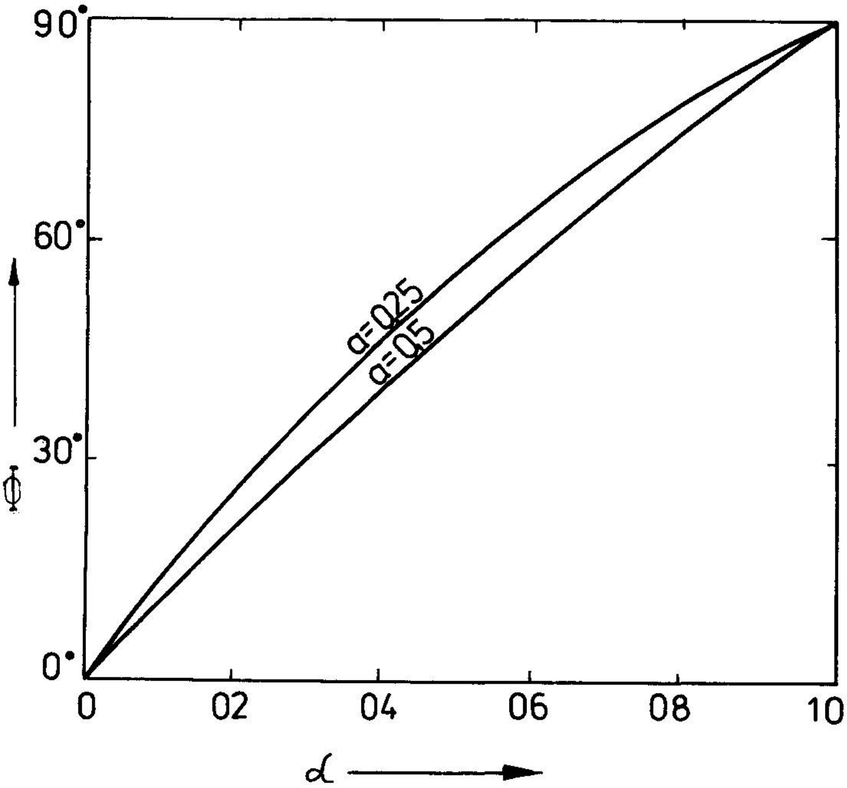

Fig. 2

At the point S a concentrated reaction appears, namely

$$V = \tan\beta_1 + \tan\beta_2 = \frac{2a\sin\phi}{1 - a^2\cos^2\phi} \tag{3.9}$$

and the reaction on the boundary is $t = t_n + \partial m_{ns}/\partial s$ where m_{ns} denotes the twisting moment appearing along the simply supported edge.

4. YIELD LINE SOLUTION

To this end a kinematically admissible deformation mode is assumed first. For the considered plate and the loading the deflected surface consists of conical elements Fig. 3. The zones SO_1S and $S'O_2S'$ are sectors of cones with the vertices at the points of load application. The zone SKS corresponds to a cone with the vertex on the plate boundary. There is a ridge O_1O_2 on the deflected surface, consisting of intersection of the cone SKS with the symmetric one.

In the yield line theory, when a continuous field of yield lines is considered, the dissipations due to the internal forces is expressed as follows

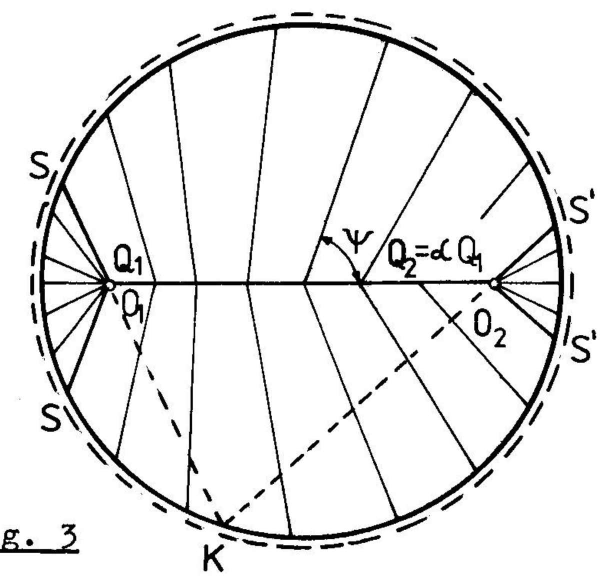

Fig. 3

$$D = M_o w_o \int_A \varkappa_\theta \, dA = M_o w_o \left\{ \int_{\theta_1}^{\theta_2} \left[1 + \left(\frac{\varrho'}{\varrho}\right)^2\right] d\theta - \left[\frac{\varrho'}{\varrho}\right]_{\theta_1}^{\theta_2} \right\} \tag{4.1}$$

or

$$D = M_o w_o \left\{ \int_{\theta_1}^{\theta_2} \left[1 + \left(\frac{z'}{z}\right)^2\right] d\theta - \left[\left(1 - \frac{z_1}{z}\right) \frac{z}{z}\right]_{\theta_1}^{\theta_2} \right. \tag{4.2}$$

respectively to the situations shown in Figs 4a and 4 b. The results concern the collapse mechanism of Fig. 3. The solution (4.1) concerns the cones with vertices at O_1 and O_2, whereas (4.2), [10], gives the dissipations on the collapse mode in the form of a conical surface with its vertex on the plate boundary as shown in Fig.3.

For a given ratio of the loads a bound to the load carrying capacity is

$$q_+ = \min \frac{2(1+\alpha)}{\sqrt{1-a^2}\,(C+\alpha D)} \left\{ C\left(\Omega + \arctan \frac{\sqrt{1-a^2}\,\sin\psi}{a-\cos\psi}\right) + \right.$$

$$\left. + D\left(\arctan \frac{\sqrt{1-a^2}\,\sin\psi}{a+\cos\psi}\right) + \frac{4a\sin\psi}{\sqrt{1-a^2}} \right\}, \qquad 0 \leqslant \alpha \leqslant 1 \tag{4.3}$$

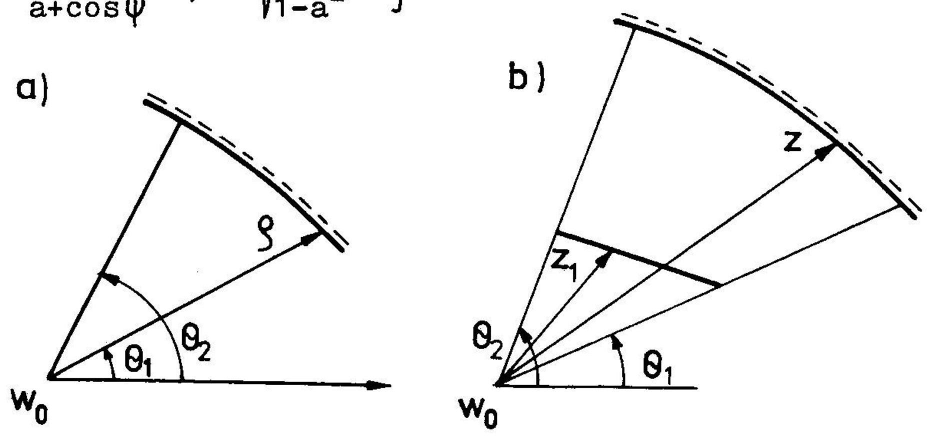

Fig. 4

where $C = 1 + a\cos\psi$, $D = 1 - a\cos\psi$
and $\Omega = 0$ if $\arccos a \leq \psi \leq \pi/2$, $\Omega = \pi$ if $\psi \leq \arccos a$

Another collapse mode consisting of conical parts SO_1S and $S'O_2S'$ joined by the cylindrical surfaces can be conceived. The vertices of the cones SKS' are at the infinity, [10]. The respective results are in [11].

In Fig. 5 the collapse load intensities are shown for the considered collapse mode as well as the exact result.

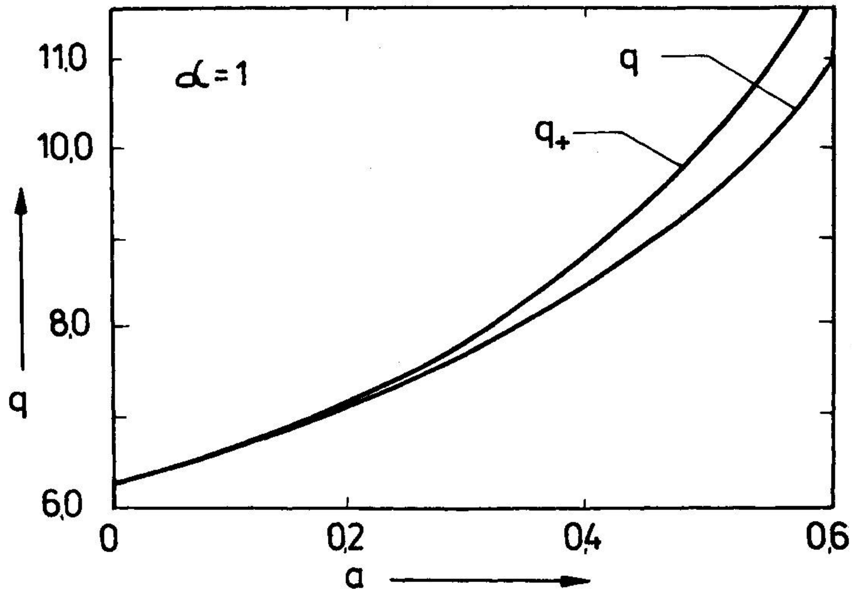

Fig. 5

5. COMPARISONS

Comparing the complete solution with the considered kinematical solutions one can conclude that for the load ratio $0 < \alpha < 0.5$ the differences between the solutions is of order of few per cent only. The largest admissible excentricity for the studied complete solution involving parabolic and isotropic regimes is for $\alpha = 1$, Fig. 6. The results are given in the table. The considered kinematically admissible collapse modes of the yield line theory and the velocity field corresponding to the exact limit analysis solution of a rigid-perfectly plastic plate are compared in Fig. 7 at $\alpha = 1$ for the load excentricity $a = 0.5$.

The analysis of complete and kinematically admissible solutions of the considered plate problem suggests that experiments should be made regarding the existence of an isotropic zone as well as to its extent, and regarding the data concerning the limit load and the largest differences should appear for the load excentricity $a = 0.7$, close to the limiting case of applicability of the parabolic-isotropic solution. For $\alpha > \sqrt{\Lambda/(1+\Lambda)}$ a hyperbolic zone must appear and the exact solution is not known. In the considered case of "layered isotropy", $\Lambda = 1$, the hyperbolic zone appears for $a > 0.707$.

The exact solution allows to assess the reaction distribution along the support. A concentrated force V appears at the point S, where two parabolic zones meet the isotropic region.

Load carrying capacity q at $\alpha = 1$

a	exact	yield line theory
0	6.283	6.283
0.5	9.674	10.170
0.7	13.141	14.760

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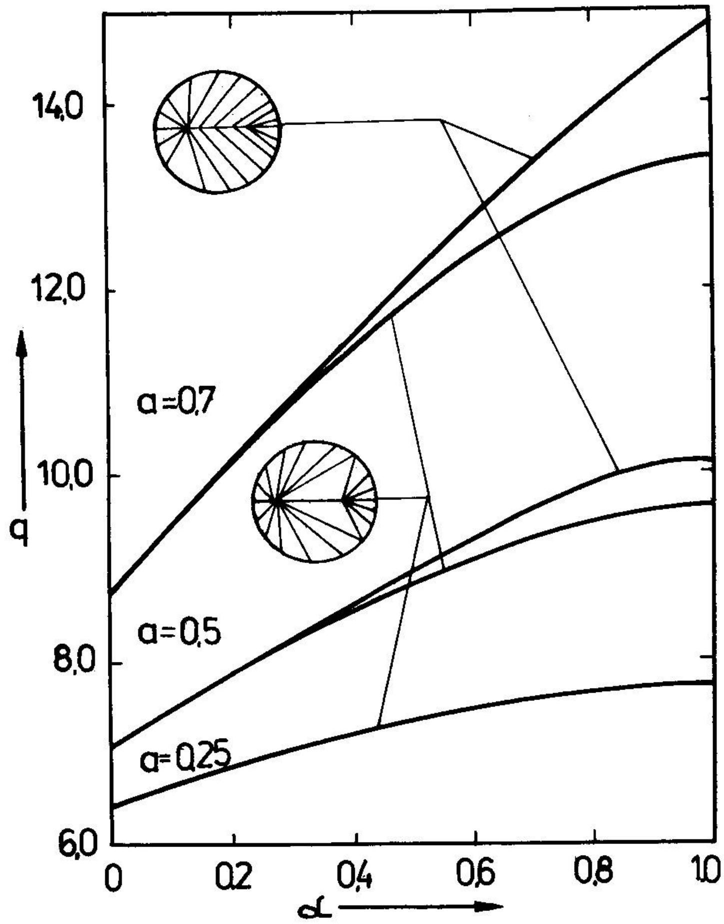

Fig. 6

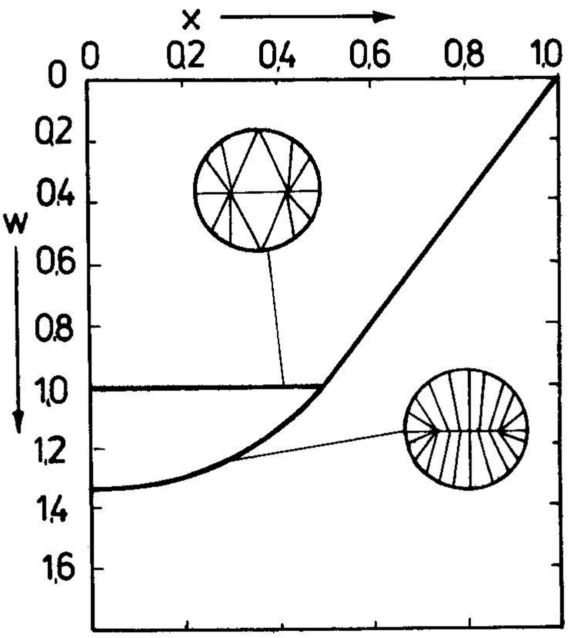

Fig. 7

practice and operating between the Faculte Polytechnique de Mons and the Institute of Fundamental Technological Research, Warsaw.

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