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Torsion-Bending-Shear in Concrete Beams: A Kinematic Model

Poutres en béton armé soumises à la torsion, à la flexion et au cisaillement. Une solution cinématique

Torsion-Biegung-Schub in Stahlbetonbalken. Eine kinematische Lösung

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SUMMARY
A kinematic solution according to the theory of plasticity is presented for reinforced concrete beams loaded in combined torsion, bending and shear.

RESUME
La méthode cinématique de la théorie de la plasticité est appliquée pour déterminer la résistance des poutres soumises à la torsion, à la flexion et à l'effort tranchant.

ZUSAMMENFASSUNG
Für Stahlbetonbalken unter Torsion, Biegung und Querkraft wird eine Lösung nach der kinematischen Methode der Plastizitätstheorie dargestellt.

1. INTRODUCTION

A kinematic model for beams loaded in combined torsion and bending has recently been presented by Peter Müller and Bruno Thürlimann [1] - [3]. The model clarifies some contradictions in the theory for torsion which have earlier been discussed by the writer [4], [5].

In this paper the kinematic model of Müller-Thürlimann is extended to include the effect of vertical shear as well. The extension is based on the same principles as the writer has earlier used in a kinematic model for torsion-bending shear based on skew bending [4] - [9].

The presentation below follows the same outline as the one in Bruno Thürlimanns paper [1]. The same general assumptions are made i.e. the concrete and the reinforcement are rigid perfectly plastic materials. The concrete is governed by a square yield criterion. The reinforcement bars have a yield stress of $\pm f_y$ and carry forces in axial directions only. Local and bond failures are excluded.

2. KINEMATIC MODEL

A kinematic model for combined torsion, bending and shear is presented in Figs. 1 and 2. In Fig. 1 general notations are given and in Fig. 2 the kinematic model is presented. In the model, there are two cracks, ABC and DEF, see Fig. 2a. The right half of the beam rotates around the axis AD through the crack ends in the top of the beam. In the bottom of the beam a parallelogram, BCEF, is cut out. The rotation around the axis AD is notated $\dot{\omega}$. The rotation is possible if the axis AD is parallel to the diagonal CF in the paralleogram. This implies the condition that $\ell_{AD}=\ell_{CF}$, see Fig. 2b. Further, the angle β of the rotation axis AD follows from the following geometric conditions.

$$\ell_{CF} = b \cot \alpha_4 + h \cot \alpha_2 - \ell_{AD} + h \cot \alpha_6 + b \cot \alpha_4$$

With $\ell_{CF} = \ell_{AD}$ and $\cot \beta = \ell_{AD}/b$ we obtain

$$\cot \beta = \cot \alpha_4 + \frac{h}{2b} (\cot \alpha_2 + \cot \alpha_6) \tag{2.1}$$

In order to express the energy dissipation, the velocity components of point B (equal to point E) are needed, see Fig.2c.

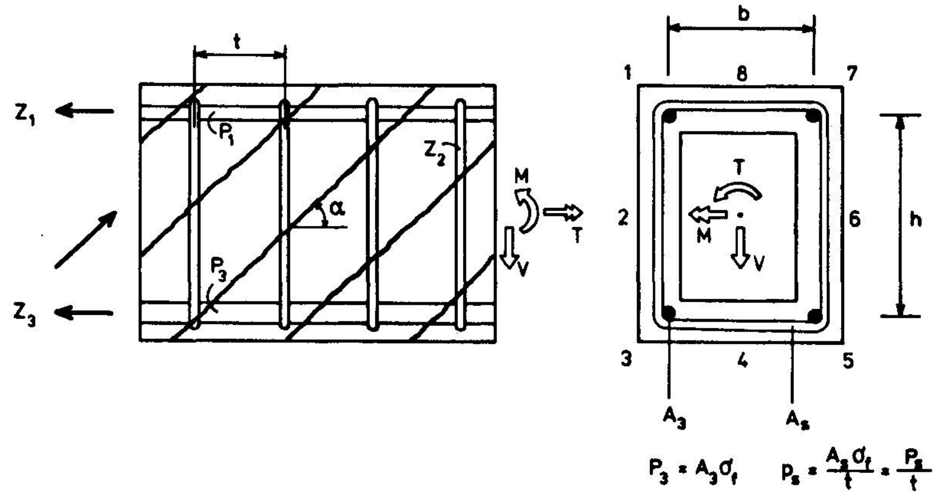

Fig. 1 Rectangular beam with box-section. Notations

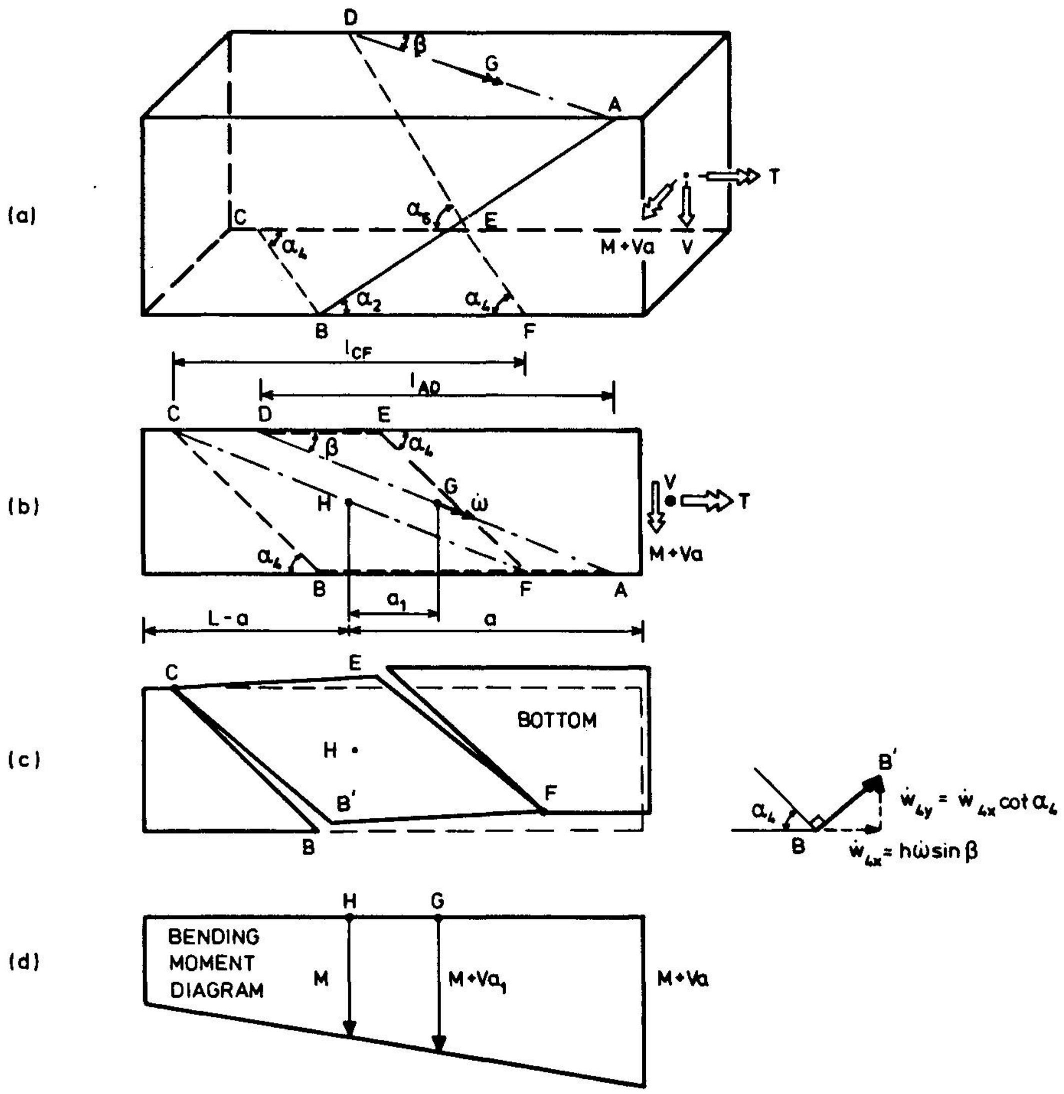

Fig. 2 Kinematic failure model: (a) General view; (b) Model seen from above; (c) Deformations in bottom; (d) Bending moment diagram.

$$\text{Wall 4:} \quad \dot{w}_{4x} = h\dot{\omega} \sin\beta \qquad \dot{w}_{4y} = \dot{w}_{4x} \cot\alpha_4$$

$$\text{Wall 2:} \quad \dot{w}_{2x} = h\dot{\omega} \sin\beta \qquad \dot{w}_{2y} = \dot{w}_{2x} \cot\alpha_2 \tag{2.2}$$

3. WORK EXPRESSIONS

Using the reinforcements shown in Fig. 1. the *internal work* in the cracks can be written as

$$L_{in} = \dot{w}_{4x}(P_3+P_5) + \tfrac{1}{2}p_s h\dot{w}_{2x} \cot^2\alpha_2 + p_s b\dot{w}_{4x}\cot^2\alpha_4 + \tfrac{1}{2}p_s h\dot{w}_{6x} \cot^2\alpha_6 \tag{3.1}$$

The *external work* carried out when the beam rotates around the axis AD can be written in the following way. As a vertical shear force, V, is present, the bending moment is varying. It has the value M at the reference point H in the middle of the bottom parallelogram, BCEF, see Fig. 2b, where the longitudinal reinforcement bars Nos. 3 and 5 are crossed by cracks. The distance in the longitudinal direction of the beam between the point H and the midpoint G on the rotation axis is a_1, and consequently the applied bending moment at point G will be $M + Va_1$. The external work equation then takes the form

$$L_{ex} = (M + Va_1)\, \dot{\omega} \sin\beta + T\, \dot{\omega} \cos\beta \tag{3.2}$$

where a_1 from Fig. 2b with cot β from Eq. (2.1) can be written as

$$a_1 = b \cot\alpha_4 + h \cot\alpha_2 - b \cot\beta = \frac{h}{2}(\cot\alpha_2 - \cot\alpha_6) \tag{3.3}$$

The failure mechanism is goverened by the three inclinations α_2, α_4 and α_6. In the general case these three angles are independent of each other. This general case is lengthy to handle. In order to simplify the deductions, the following assumption will be made regarding the relationship between the angles

$$\left.\begin{aligned} \cot\alpha_2 &= \cot\alpha_T + \cot\alpha_V \\ \cot\alpha_4 &= \cot\alpha_T \\ \cot\alpha_6 &= \cot\alpha_T - \cot\alpha_V \end{aligned}\right\} \tag{3.4}$$

The angles α_T and α_V are here two independent variables.

The assumption is motivated by the fact that the failure mechnism will in this way correspond with a probable stress distribution in the beam.

The expression for the internal work L_{in} in Eq. (3.1) can now be rewritten as

$$L_{in} = \dot{\omega} \sin \beta \ [h(P_3 + P_5) + \frac{1}{2} p_s h^2 (2 \cot^2 \alpha_T + 2 \cot^2 \alpha_V) + p_s bh \cot^2 \alpha_T] =$$

$$= \dot{\omega} \sin \beta [h(P_3 + P_5) + p_s h(b+h) \cot^2 \alpha_T + p_s h^2 \cot^2 \alpha_V] \qquad (3.5)$$

In order to simplify the expression for the external work L_{ex} in Eq. (3.2), we first rewrite the expression for cot β in Eq. (2.1) and the expression for the distance a_1 in Eq. (3.3)

$$\cot \beta = \cot \alpha_T + \frac{h}{2b} 2 \cot \alpha_T = \frac{b+h}{b} \cot \alpha_T \qquad (3.6)$$

$$a_1 = h \cot \alpha_V \qquad (3.7)$$

The external work can now be written

$$L_{ex} = \dot{\omega} \sin \beta [M + T \frac{b+h}{b} \cot \alpha_T + Vh \cot \alpha_V] \qquad (3.8)$$

The internal work in Eq. (3.5) shall be equal to the external work in Eq. (3.8)

$$M + T\frac{b+h}{b}\cot \alpha_T + Vh \cot \alpha_V = h(P_3+P_5) + p_s h(b+h)\cot^2\alpha_T + p_s h^2 \cot^2 \alpha_V \qquad (3.9)$$

4. MINIMIZATION

If T and V are fixed, the minimum value of M with respect to the angles α_T and α_V follows from differentiations of Eq. (3.9) with respect to cot α_T and cot α_V

$$\frac{\partial M}{\partial \cot \alpha_T} + T \frac{b+h}{b} = 2p_s h(b+h) \cot \alpha_T$$

$$\frac{\partial M}{\partial \cot \alpha_V} = 0 \text{ gives } \cot \alpha_T = \frac{T}{2bh} \cdot \frac{1}{p_s} \qquad (4.1)$$

$$\frac{\partial M}{\partial \cot \alpha_T} + Vh = 2p_s h^2 \cot \alpha_V$$

$$\frac{\partial M}{\partial \cot \alpha_V} = 0 \text{ gives } \cot \alpha_V = \frac{V}{2h} \cdot \frac{1}{p_s} \qquad (4.2)$$

For the case of pure bending (T = V = 0), pure torsion (M = V = 0) and pure shear (M = T = 0), Eq. (3.9) with Eqs. (4.1) and (4.2) gives

$$M_0 = h(P_3 + P_5) \;; \quad T_0 = 2bh\sqrt{\frac{P_3+P_5}{b+h}\,p_s} \;; \quad V_0 = 2h\sqrt{\frac{P_3+P_5}{h}\,p_s} \tag{4.3}$$

Using Eqs. (4.1) to (4.3), Eq. (3.9) can now be rewritten as

$$M/M_0 + (T/T_0)^2 + (V/V_0)^2 = 1 \tag{4.4}$$

This is the same solution as has earlier been obtained with a static approach [4], [6], [10]. Hence ther is an identity between the kinematic model presented here and earlier presented static methods.

If the three inclinations α_2, α_4 and α_6 are retained as independent variables in the work expressions, Eq. (3.9) can be written in the following way.

$$M + T\left(\frac{h}{2b}\cot\alpha_2 + \cot\alpha_4 + \frac{h}{2b}\cot\alpha_6\right) + V\frac{h}{2}(\cot\alpha_2 - \cot\alpha_6) =$$

$$= h(P_3+P_5) + \frac{1}{2}p_s h^2(\cot^2\alpha_2 + \cot^2\alpha_6) + p_s bh\cot\alpha_4 \tag{4.5}$$

A minimization of M with respect to the angles α_2, α_4 and α_6 will then give

$$\left.\begin{aligned} \cot\alpha_2 &= \left(\frac{T}{2bh} + \frac{V}{2h}\right)\frac{1}{p_s} = \cot\alpha_T + \cot\alpha_V \\ \cot\alpha_4 &= \frac{T}{2bh}\cdot\frac{1}{p_s} = \cot\alpha_T \\ \cot\alpha_6 &= \left(\frac{T}{2bh} - \frac{V}{2h}\right)\frac{1}{p_s} = \cot\alpha_T - \cot\alpha_V \end{aligned}\right\} \tag{4.6}$$

Hence, the shear flow from torsion and shear are acting in the same direction in side 2, and in opposite directions in side 4. This is in agreement with the assumption in Eq. (3.4).

5. DISCUSSION

The inter-action equation presented, Eq. (4.4), is deduced for point H in Fig. 2c. As can be seen from the moment diagram in Fig. 2d, the bending moment is higher in point G and in every point to the right of point H in the figure. The failure mechanism presented for point H is for this reason not stable [3].

A failure mechanism will start to develop in the area with the highest loads, that is, in the right end of the beam element in Fig.2. However, in the right end of the beam a support or a concentrated load may be situated. This will influence and change the failure mechanism. High concrete stresses will occur and they may cause the failure. For this reason the failure model will be more complicated in the vicinity of a support or a concentrated load.

To be correct according to the theory of plasticity, the effect of warping should be considered [3]. However, Paul Lüchinger has shown that for a rectangular beam, as is studied here, the effect of warping may be neglected [10]. If the warping is considered, it will at worst give a slightly higher load-carrying capacity.

Although the presented kinematic model is not stable for mispan cross-sections, it does give a rather good prediction of the type of cracks and deformations that has been observed in tests see Fig. 3 [7], [8]. The model also gives an identical load-carrying capacity as earlier presented static methods [4], [6], [10]. For these reasons, the writer considers the presented kinematic model to be a step in the direction of a better understanding of the interaction between torsion, bending and shear. To be able to give a complete solution to the problem, the effects of supports and of loading conditions must be studied. Here the concrete compression strength must be entered as an essential parameter.

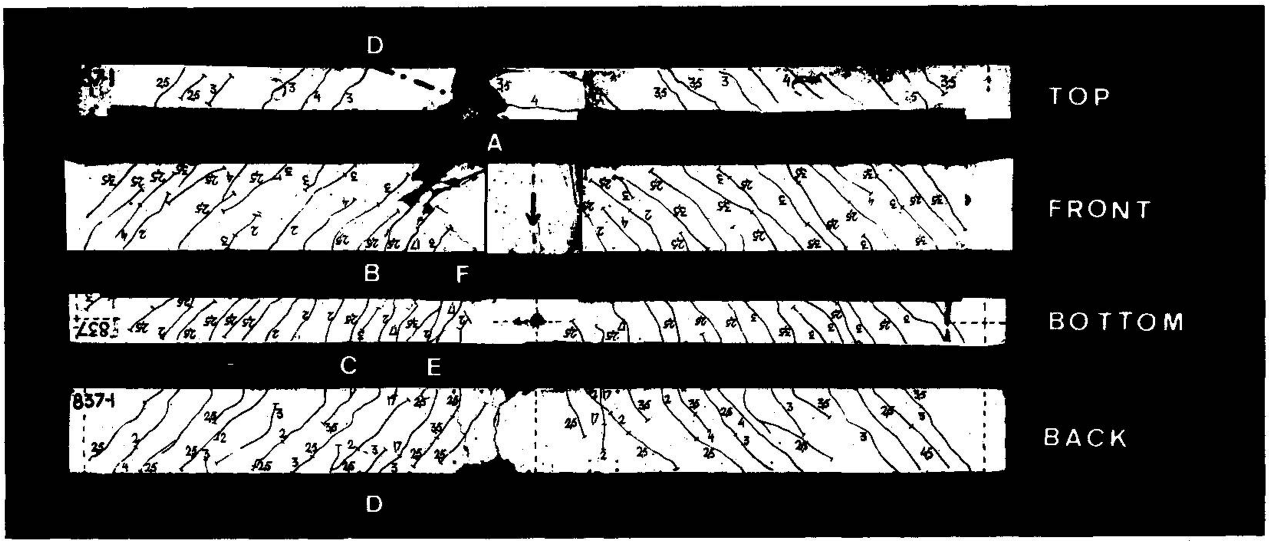

Fig. 3 Crack pattern and failure mechanism for a beam loaded in combined torsion, bending and shear. (Beam 1-1A in [4] and [7]). The beam is loaded in mid-span with an eccentric point-load acting downwards. The numerals along the cracks refer to the applied load when this part of the crack became visible (in Mp = MN/100). In the left part of the beam two failure cracks ABC and FED are indicated as well as a rotation hinge AD, compare with Fig. 2. (The beam is rectangular with $b \times h \times l = 100 \times 200 \times 3300$ mm. The stirrup capacity is $p_s = 0.236$ MN/m. The relation between the bending moment M, the torsional moment T and the vertical shear force V in the failure section is M:T:Vh = 1:0.5:0.2).

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