**Zeitschrift:** IABSE reports of the working commissions = Rapports des commissions de travail AIPC = IVBH Berichte der Arbeitskommissionen

**Band:** 29 (1979)

**Artikel:** Plastic analysis of torsion and shear in reinforced concrete

**Autor:** Mueller, P.

**DOI:** https://doi.org/10.5169/seals-23540

II

# Plastic Analysis of Torsion and Shear in Reinforced Concrete

Analyse plastique du béton armé soumis à la torsion et au cisaillement

Plastische Berechnung für Torsion und Schub im Stahlbeton

**P. MUELLER**
Dr.sc.techn., dipl. Ing. ETH
Massachusetts Institute of Technology
Cambridge, Massachusetts, U.S.A.

**SUMMARY**
Collapse mechanisms for reinforced concrete beams in torsion, bending and shear are presented. The mechanisms complete the truss model solution to an exact plastic solution. The shear strength of deep beams and of beam regions near supports or point loads is explored.

**RESUME**
Des mécanismes de ruine pour des poutres en béton armé sous torsion, flexion et effort tranchant sont présentés. Ces mécanismes complètent le modèle de treillis pour constituer une solution plastique exacte. On étudie la résistance ultime à l'effort tranchant des régions d'appuis ou d'application des charges et des poutres courtes.

**ZUSAMMENFASSUNG**
Kollapsmechanismen für Stahlbetonbalken unter Torsion, Biegung und Querkraft werden dargestellt. Die Mechanismen vervollständigen die Fachwerkmodellösung zu einer plastizitätstheoretisch exakten Lösung. Die Schubtragfähigkeit von Balken in Lasteinleitungsbereichen und von wandartigen Trägern wird untersucht.

## 1. INTRODUCTION AND ASSUMPTIONS

In recent years the theory of plasticity has considerably contributed to both the understanding of shear transfer and the determination of the shear strength in reinforced (prestressed or mild) concrete [1]. This paper reports on some results of two studies [2,3] contributing to a rational and consistent plastic theory for reinforced concrete beams, deep beams and shear walls. First collapse mechanisms for beams in torsion, bending and shear are presented. Second some plane stress problems related with the shear strength of beams near supports and of deep beams are treated.

It is assumed that webs and flanges of beams and walls can be modeled as elastic-perfectly-plastic membranes governed by the yield criterion derived from the no tensile strength, square yield criterion for concrete (Figs. 1 and 2). The states of stress and motion are described in terms of membrane forces per unit length and in-plane velocities. In the next section it is moreover assumed that the collapse is initiated by yielding of the reinforcement, i.e., the membranes are plastified in yield regime I. For discontinuity lines of the velocity field that are compatible with yield regime I, the term collapse crack is used. Collapse cracks are straight lines. According to the flow rule for yield regime I, they must open normally to their direction.

## 2. COLLAPSE MECHANISMS FOR BEAMS IN TORSION, BENDING AND SHEAR

### 2.1 One-Dimensional Beam Theory

The derivation of yield criteria and flow rules for the differential beam element in terms of generalized stresses and strains (stress-resultants, curvature, twist, etc.) marks the transition from solid mechanics to one-dimensional beam theory. This transition is usually treated as a two-dimensional problem in the plane of a cross-section. Interaction relations are established through an investigation of possible states of stress and strain rates in a fully plastic cross-section. To this end simplifying assumptions with respect to the states of stress and deformation must always be made. The first usual assumption is that the transverse normal stresses, which do not enter the equilibrium conditions for a cross-section, are zero. The remaining assumptions are contained in the constitutive equations relating the stresses and strain rates in a cross-section to the generalized stresses and strain rates.

In the (static) truss model approach [1] it is assumed that the shear flow is constant along the perimeter of a thin-walled closed cross-section subjected to torsion or constant over the depth of a web subjected to shear. It is shown in Ref. [3] that these assumptions are consistent with the following kinematic assumptions for the deformed fully plastic cross-section.

- the cross-sectional shape is not distorted
- plane distribution of the longitudinal strain rates
- warping of the cross-section out of its plane is unrestrained in the case of beams with closed, thin-walled cross-section subjected to uniform torsion and bending; warping of the web is unrestrained in the case of beams with single-symmetric thin-walled cross-section subjected to bending and shear.

Using these assumptions the corresponding interaction relations can also be found readily via the power of dissipation of the cross-section [3]. The resulting expressions for the distribution of the shear strain rates and for the warping function are completely analagous in form to those found in elastic beam theory for uniform torsion. While this sectional approach shows that the resulting interaction relations fully comply with usual assumptions of beam theory, it gives no indication under what conditions unrestrained warping can be assumed.

## 2.2 Spatial, Discontinuous Collapse Mechanisms

Due to the simplifying assumptions of the sectional approach, it is often not possible to find stress and velocity fields that satisfy the boundary, equilibrium and compatibility conditions of plane stress theory and that exhibit the stress and strain rate distribution found in the sectional approach. Indeed, for beams in bending and shear, it has been demonstrated in Ref. [3] that only the discontinuous collapse mechanisms shown in Fig. 3 can be associated with the generalized strains of the sectional approach. For beams in uniform torsion and bending such velocity fields exist, but they do not represent the shortest possible mechanisms, which are again discontinuous.

The mechanisms shown in Fig. 3 represent also the basic elements of the discontinuous collapse mechanisms for beams in torsion and bending with thin-walled closed cross-section of polygonal shape. Fig. 4a shows the basic torsional collapse mechanism for a beam with quadrilateral convex cross-section. The relation between the inclination of the skew axis of rotation AE and the inclinations of the collapse cracks ABC and EFG follows from the flow rule of perpendicular collapse crack opening and from the assumption that the beam ends undergo rigid body motion only. From kinematics it is easily found that the four collapse crack tips and the axis of rotation must lie in one plane. Assuming an unrestrained longitudinal extension of the mechanism, the same interaction relations result as from the (static) space truss model approach [2,3]. Each mechanism with axis of rotation in one of the four walls yields one interaction relation. If all longitudinal reinforcement is yielding at collapse, any linear combination of the four mechanisms is also feasible, in particular also a mechanism with pure twist and elongation as shown in Fig. 4b.

It is worthwhile to relate the kinematic assumptions for the spatial mechanisms to those of the more usual sectional approach. The assumption of a rigid body motion of the beam ends corresponds evidently to the assumptions of a plane distribution of the longitudinal strain rates and of no distortion of the sectional shape. In both approaches these assumptions ensure that the power of the external loads can be uniquely expressed in terms of the bending and torsional moments and the rates of curvature and twist or skew rotation, respectively. The assumption, finally, that the mechanisms of Fig. 4 can longitudinally spread in an unrestrained manner, corresponds to the assumption of unrestrained warping for a fully plastic cross-section. To illustrate the last statement, Fig. 5 shows a beam in uniform torsion with uniform stirrup reinforcement. The yield strength of the four longitudinal corner bars is $Z_y$ within the test region L and much higher outside. The longitudinal distance between the points B and E, where the open collapse cracks cross the corner bars, is denoted by $\ell$ and the length that would develop in a uniformly reinforced beam by $\ell_u$. For $\ell_u \leq L$, the mechanism is unrestrained, the shear flow is constant along the perimeter of the cross-section and the beam acts in pure St. Venant or uniform torsion. For $\ell_u > L$ the mechanism is restrained and a higher ultimate torque results. The exact collapse mechanism is characterized by $\ell = L$. The shear flow jumps at the corners, the forces in the corner bars vary (Fig. 5b) and, hence, the beam acts in nonuniform torsion.

It is common practice in reinforced concrete theory to treat solid cross-sections as box-sections, because test results have shown that solid and box-shaped beams with identical dimensions and reinforcement have practically the same ultimate torque. It is worthwhile therefore to investigate the collapse crack propagation into the core of the beam. Because the tips A, C, D, F of the two collapse cracks and the skew axis of rotation lie in one plane (Fig. 5a), collapse cracks propagate along the three planes ABC, DEF and ADCF into the core. While the collapse cracks ABC and DEF are externally visible, the collapse crack along plane ADCF opens only internally thus creating a cavity. Assume that a skew bending crack opens along plane ADCF, while the front end of the beam rotates about axis AD (Fig. 5a). As shown in Fig. 6, the two crack edges are then in position ADCF' (Fig. 6a) and ADC'F (Fig. 6b). If the tetraeder ABCF rotates about axis AC from position ACF' into position ACF and the tetraeder CDEF about axis DF from position C'DF into position CDF, the skew bending crack is completely closed again along its perimeter. There remain the externally visible collapse cracks ABC, DEF and a cavity as shown in Fig. 6c. All the interior collapse crack surfaces are planes, and opening occurs everywhere normal to these planes. Consequently the core concrete does not contribute to the power of dissipation. Clearly, there is no difference in torsional resistance between box beams and solid beams with convex quadrilateral cross-sections, when failure is initiated by yielding of the reinforcement. In both cases only an outermost concrete layer is effective. This effective concrete layer, along with the reinforcement, is modelled in the present approach as a two-dimensional membrane. The membrane forces of the truss model and the in-plane velocity components of the mechanisms Figs. 4 to 6 are compatible and represent therefore an exact solution.

So far, only quadrilateral cross-sections have been treated. However, generalization to arbitrary, convex, polygonal thin-walled cross-sections is straightforward. Again, the basic mechanisms are characterized by a skew axis of rotation in one of the walls as shown in Fig. 7a. Any other mechanism compatible with yield regime I is a linear combination of such mechanisms. Starting from the skew axis of rotation, a sequence of overlapping collapse cracks forms that span over two walls (Fig. 7a and b). The relation between the inclination $\beta$ of the axis of rotation and the inclinations $\alpha_i$ of the collapse cracks follows again from the flow rule of perpendicular collapse crack opening and from the assumption that the beam ends undergo rigid body motion only.

Consider the tip of the collapse crack starting at corner i. Due to the flow rule, the in-plane velocities $\dot{u}_i$, $\dot{v}_i$ must be related by (Fig. 7d)

$$a_i \dot{v}_i = L_i \dot{u}_i \tag{1}$$

Because the front end of the beam undergoes a rigid body motion only, the velocities in Eq. (1) are given by

$$\dot{u}_i = \dot{\omega} \sin\beta \, z_i \quad , \tag{2a}$$

$$\dot{v}_i = \dot{\omega} \cos\beta \, r_i - \dot{\omega} \sin\beta \, x_i \cos\delta_i \quad , \tag{2b}$$

where all terms are defined in Fig. 7. Finally, according to Fig. 7b, the length $L_i$ of the mechanism in wall i is given by

$$L_i = x_i - x_{i-1} + a_i \cot\alpha_i + a_{i-1} \cot\alpha_{i-1} \tag{3}$$

Introducing Eqs. (2) and (3) in Eq. (1) and noting that $z_{i+1} = z_i + a_i \cos\delta_i$ , there results the following recurrence formula for $x_i$.

$$z_{i+1} x_i = a_i r_i \cot\beta - z_i (a_i \cot\alpha_i + a_{i-1} \cot\alpha_{i-1}) + z_i x_{i-1} \tag{4}$$

Eq. (4) merely states that the skew axis of rotation and the crack tips in corner i and i+1 of wall i lie in one plane. Using Eq. (4) and starting with $x_1 = 0$, the coordinates $x_i$ of the front tips of each crack can be consecutively calculated. The condition that the last collapse crack must meet again the skew axis of rotation, finally yields

$$\cot\beta = \frac{1}{A_o} \sum_i \bar{z}_i a_i \cot\alpha_i \quad , \tag{5}$$

where $A_o$ denotes the area enclosed by the cross-section and $a_i$, $\bar{z}_i$ are the depth and the z-coordinate of the middle line of wall i (Fig. 7d). For a beam with a regular polygonal cross-section and equal stirrup reinforcement in each wall the following simple relations hold: $\cot\beta = 2 \cot\alpha$, $L_i = 2a \cot\alpha$. Using Eq. (5), again the interaction relations of the space truss model can be derived [3].

The derivation has only involved the in-plane velocities. A study including the velocity components normal to the walls reveals the following. The collapse cracks, the corner lines and the straight lines joining the crack tips (Fig. 7b) form the boundaries of the wall regions that move as rigid bodies. Contrary to quadrilateral sections, vectors of relative rotation are not only present in the lines joining the crack tips but moreover in the corner lines. The vectors of relative rotation between the crack edges point from the starting to the end point of the crack and have the same magnitude in both walls. Thus one collapse crack opens indeed along one plane.

## 3. PLANE STRESS PROBLEMS

The critical cross-sections are often located at points of applications of concentrated loads or at supports, where the simplifying assumptions of beam theory are hardly met. In deep beams the effect of shear transfer through strut action becomes significant. In these cases realistic theoretical values for the ultimate strength in bending and shear can only be expected from a plane stress approach that takes into account the variable concrete stress fields in the web. For underreinforced beams with variable depth the exact collapse mechanisms are basically the same as in Fig. 3. The corresponding concrete stress may vary, however, significantly over the depth of a cross-section. The mechanisms presented in the preceding section demonstrate that adjacent cross-sections can theoretically not be designed independently. In engineering practice, however, the needed reinforcement is always determined on a sectional basis. Thus the conditions have to be established, on which a sectional design ensures stable, statically admissible stress fields in the web.

These problems are treated in detail in Ref. [3] along with a discussion of the general stress and velocity fields in fully plastic reinforced concrete walls.

Fig. 8 summarizes some results of an investigation on possible concrete compression fields in a thin web near the base of a cantilever beam (Fig. 8a) and near a concentrated load (Fig. 8b). The web is assumed to be in yield regime I. The trajectories of the compression fields in the concrete form non-centered fans and are determined from the boundary condition of zero transverse normal stress along boundary AB and from the assumption that the concrete compressive strength is reached along boundary BC (Fig. 8a) or CD (Fig. 8b). Results are presented for several values for the ratio of the allowable nominal shear stress and the effective concrete strength, $\tau/\kappa\beta_p$, and for the selected inclination $\alpha_{dim}$ used to determine the stirrup reinforcement. The table in Fig. 8a shows that the additional longitudinal reinforcement needed in section BC due to the presence of shear may be of comparable order of magnitude as in the free span, although codes do usually not require such additional reinforcement in this case. The table in Fig. 8b presents values for the load transfer length e that is needed to ensure stable, statically admissible stress fields in the web below a concentrated load. These values may be of considerable magnitude. Clearly, these results show that further study is needed in this area.

Fig. 9 shows a deep beam after attainment of the ultimate load. A lower bound for the collapse load has been derived on the basis of the stress fields indicated in Fig. 9b [3]. The stirrups and the distributed longitudinal reinforcement are assumed to yield. The main longitudinal reinforcement is in tension everywhere and reaches the yield strength in the points A and A'. The concrete compression field consists of a central strut and two non-centered fans. The concrete strength is reached in the strut and along the lines AC' and A'C. The experimental ultimate load was 174.5 kips or 85% of the theoretical ultimate load found from bending theory. The lower bounds derived from the stress fields as outlined above are 168 kips and 182 kips depending on whether the concrete cover of the reinforcement is assumed to be ineffective or effective, respectively. Neglecting the cover is equivalent to a 30% reduction in concrete strength. This reduction results only in a 8% decrease in the ultimate load. Theoretically approximately 30% of the shear force is carried by strut action. While this result indicates that the investigated stress fields may be realistic, realistic collapse mechanisms have still to be found.

## 4. CONCLUSIONS

The collapse mechanisms for beams in bending and torsion or bending and shear presented in this paper complete the (static) space truss model solution to an exact solution within the framework of a membrane theory as outlined above. In particular, they clarify the kinematic assumptions the resulting interaction relations are based on. This is of importance insofar as, first, conclusions with respect to the applicability of plastic theory can only be drawn from the comparison of test results with an exact or sufficiently bounded value for the theoretical collapse load. Second, kinematic assumptions can be verified more easily experimentally than static assumption.

The spatial mechanisms show that the warping term in the sectional approach reflects primarily the fact that the boundaries between rigid and yielding regions cannot lie in a cross-section normal to the beam axis. The finite length of the shortest possible mechanism implies that the interaction relations actually describe the strength of a whole beam region. This has to be kept in mind when detailing the reinforcement. The knowledge of the collapse mechanisms helps to detect weak reinforcing details. The finite length of the collapse mechanisms must also be appropriately considered in the analysis of test results and the design of test specimens. While the length of the mechanism is important for the reinforcement details, it is normally short compared to the beam length. This means that warping restraints have only a very localized effect. Such restraints have no effect at all on the strength of regions more distant than the minimal mechanism length.

Results from an investigation on the general stress fields in the web of beams and deep beam in bending and shear have been briefly summarized. They indicate that further work is needed on regions of application of concentrated loads and reactions and on deep beams. The lower bounds calculated for a deep beam exhibiting strut action indicate that the effort might be worthwhile.

## ACKNOWLEDGEMENT

The reported results were obtained in the course of a research project under the supervision of Dr. B. Thürlimann, ETH Zürich, Switzerland.

## REFERENCES

1. Thürlimann, B.: "Plastic Analysis of Reinforced Concrete Beams," IABSE Colloquium Kopenhagen 1979, Denmark, Introductory Report.
2. Mueller, P.: "Failure Mechanisms for Reinforced Concrete Beams in Torsion and Bending," Publications, International Association for Bridge and Structural Engineering (IABSE), Vol. 36-II, p. 147, 1976.

3. Mueller, P.: "Plastische Berechnung von Stahlbetonscheiben und -balken" (Plastic Analysis of Reinforced Concrete Shear Walls and Beams), Institut fur Baustatik und Konstruktion, ETH Zurich, Bericht Nr. 83, Birkhauser Verlag Basel und Stuttgart, 1978.

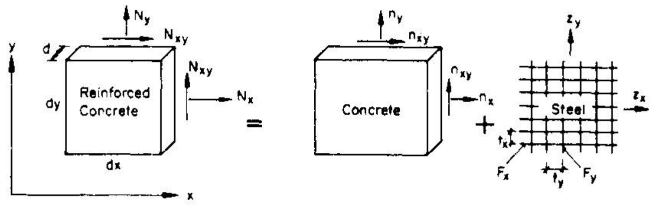



Fig. 1 Notation

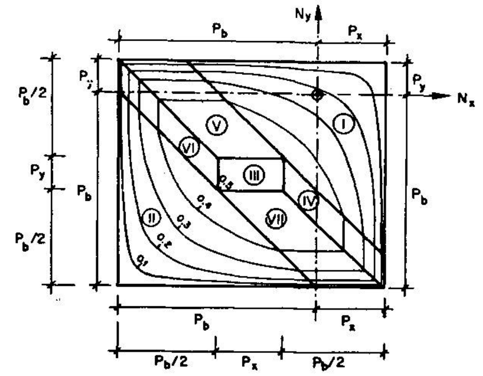


Fig. 2 Yield Criterion

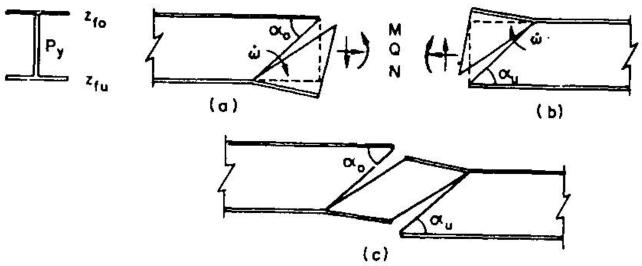



Fig. 3 Collapse Mechanisms for Underreinforced Beams in Bending and Shear

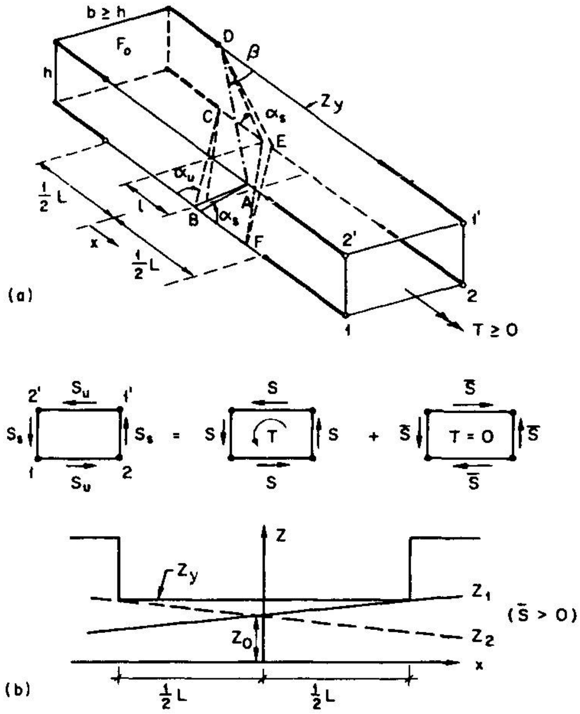



Fig. 5 Effect of Warping Restraints

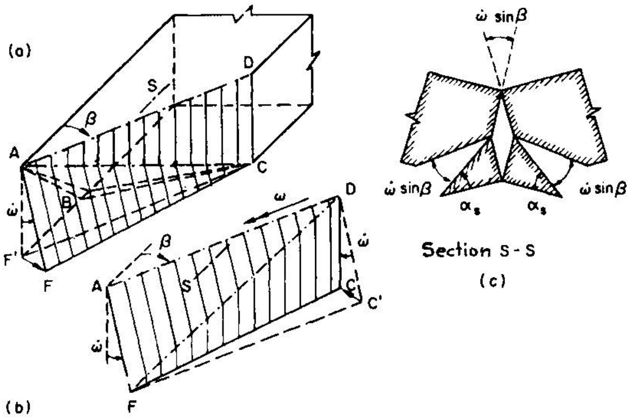



Fig. 6 Basic Torsional Mechanism for Solid Beams

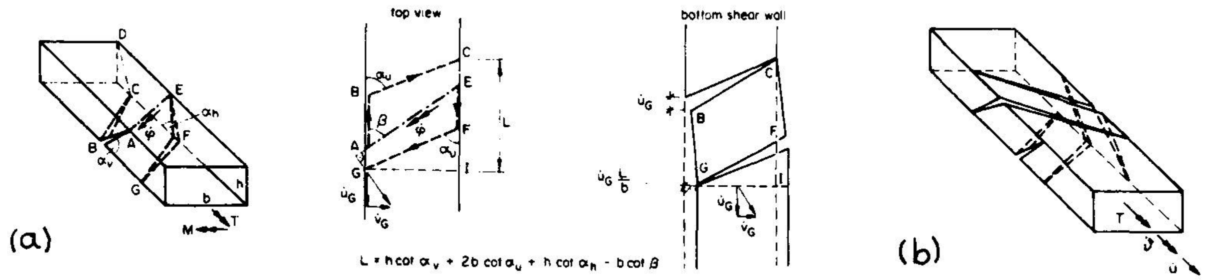



Fig. 4 Basic Torsional Collapse Mechanism (a) and Combined Mechanism (b)

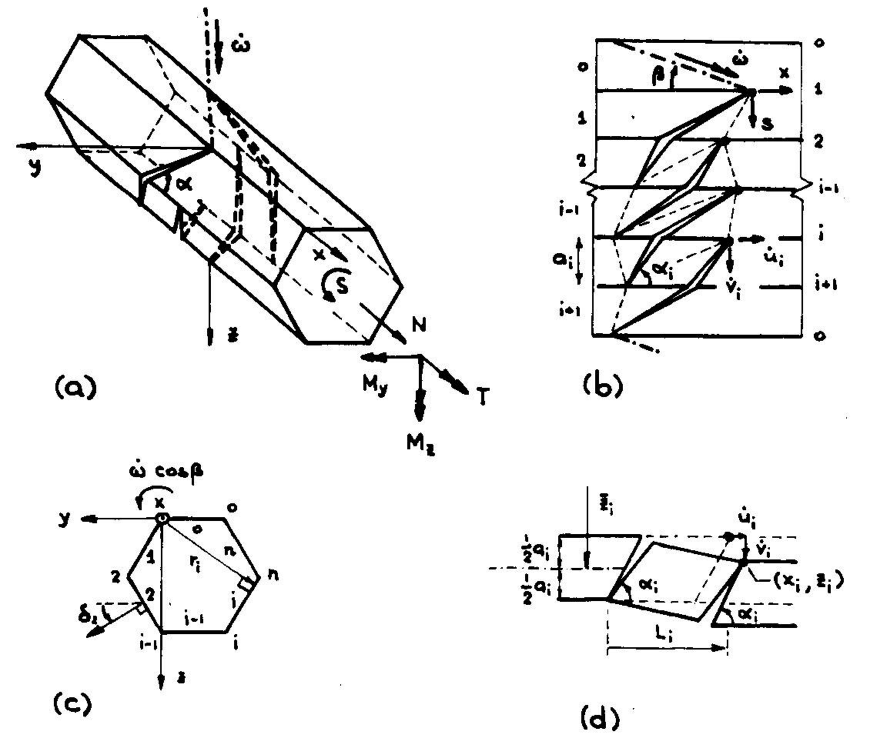
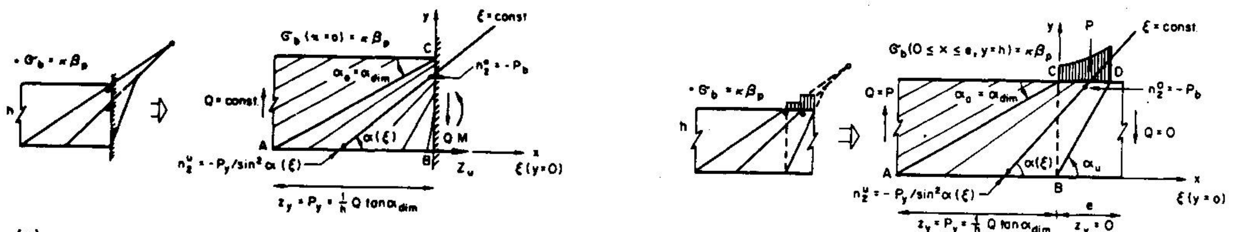



Fig. 7
Basic Torsional
Mechanism for Beams
with Polygonal
Thin-walled Sections



(a)

$\cot\alpha_u^*$ ; $(Z_u = \frac{M}{h} + \frac{1}{2} Q \cot\alpha_u^*)$

| $\tan\alpha_{dim}$ \ $\tau/\kappa\beta_p$ | $\frac{60}{225}$ | $\frac{60}{0.7 \cdot 225}$ | $\frac{50}{0.7 \cdot 225}$ |
|---|---|---|---|
| 3/5 | .47 | .94 | .63 |
| 1 | .28 | .50 | .36 |

(b)

$\cot\alpha_u$ ; $(e = h \cot\alpha_u)$

| $\tan\alpha_{dim}$ \ $\tau/\kappa\beta_p$ | $\frac{60}{225}$ | $\frac{60}{0.7 \cdot 225}$ | $\frac{50}{0.7 \cdot 225}$ |
|---|---|---|---|
| 3/5 | .69 | 1.23 | .90 |
| 1 | .42 | .66 | .52 |

Fig. 8 Stress Fields Near Support (a) and Concentrated Load (b)

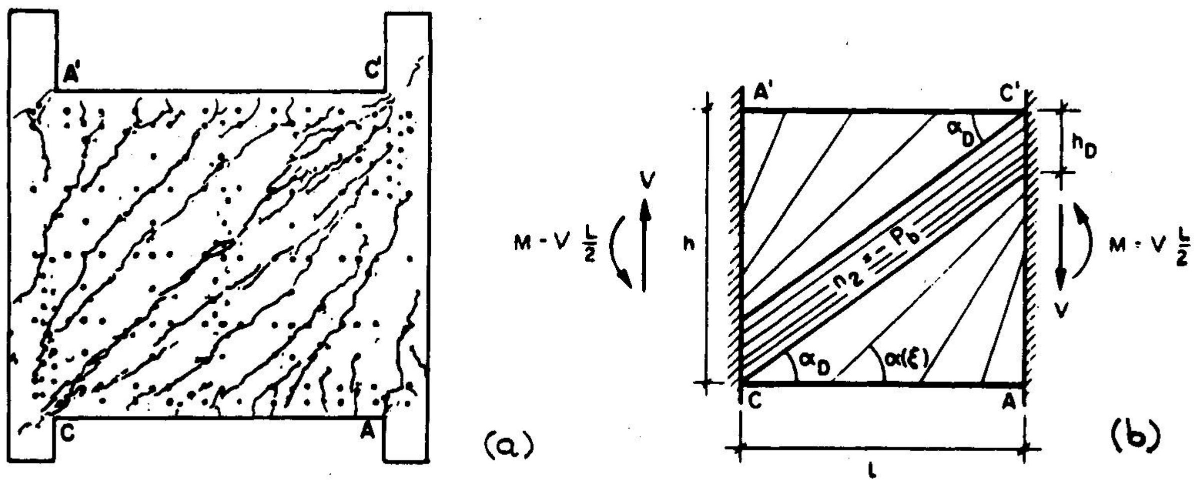



Fig. 9 Test Specimen of Deep Beam (a) and Investigated Stress Fields (b)