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A General Yield Criterion for Orthogonally Reinforced Concrete Slab Elements

Critère général d'écoulement pour des éléments de plaques en béton armé avec armature orthogonale

Eine allgemeine Fliessbedingung für orthogonal bewehrte Betonplattenelemente

P.J. COOKSON
Project Engineer
Burks Green & Partners
Newark, England

SUMMARY
A general criterion for orthogonally reinforced concrete slab elements yielding under combined bending, twisting and membrane forces is derived. An upper yield surface is obtained in parametric form, and an approximate closed form criterion is proposed. The approximate criterion reduces to the commonly accepted criteria for membrane forces, and bending and twisting moments, when acting alone as special cases.

RESUME
Un critère général d'écoulement est développé pour des éléments de plaques en béton armé avec armature orthogonale sollicités par des actions combinées de flexion et de membrane. Une surface d'écoulement circonscrite est obtenue sous forme paramétrique et un critère approché est proposé. Pour les actions isolées de flexion ou de membrane, le critère approché est semblable aux critères généralement acceptés dans ces cas.

ZUSAMMENFASSUNG
Eine allgemeine Fliessbedingung für orthogonal bewehrte, durch kombinierte Biege- und Membrankräfte beanspruchte Betonplattenelemente wird entwickelt. Eine umschriebene Fliessfläche wird in parametrischer Form dargestellt, und eine Näherung dafür in analytisch geschlossener Form wird vorgeschlagen. Die Näherung führt für reine Beanspruchung durch Membrankräfte beziehungsweise durch Biege- und Drillungsmomente auf die allgemein anerkannten entsprechenden Bedingungen.

1. INTRODUCTION

Reinforced concrete structures composed of thin-walled elements often carry external loads which produce membrane forces in the plane of the elements in combination with bending and twisting moments about axes in the plane. To be able to apply plasticity theory to such structures at the ultimate limit state it is necessary to know the yield criterion of a slab element under combined stress resultants.

This paper is concerned with obtaining such a yield criterion for a concrete slab element which has been reinforced orthogonally with layers of steel in the (x,y) directions parallel to the slab mid-plane. The sign convention for the stress resultants per unit width is shown in figure(1), for rectangular coordinates (n,t) rotated θ clockwise from the (x,y) directions.

It will be assumed that the transverse shear forces (Vt,Vn) and stresses normal to the plane of the element have negligible effect on the strength. The general yield criterion is then defined by six independent stress resultants, and may be represented by a closed convex yield surface in six-dimensional space. Previous work on such yield criteria is described by Morley[1].

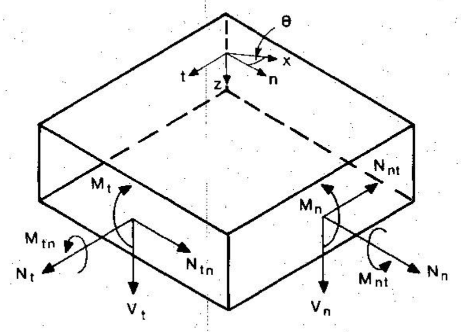

Fig.1 Sign Convention for Stress Resultants

The commonly accepted criterion for yielding to occur under bending and twisting alone is that the bending moment on a given cross-section should reach a value which is a function of the yield moments in the x and y directions alone. This "normal moment criterion", which was adopted by Johansen for his yield line theory, has been adequately justified by a method which considers the respective contributions of the steel and the concrete to the stress resultants for a specified set of strain rates. (see e.g. ref[2]).

A suitable criterion for combined bending, twisting and membrane forces which is analagous to the normal moment criterion is not obvious. In order to obtain such a criterion recourse is made to the method of postulating a set of strain rates, and from an evaluation of the contribution of each layer of material in the slab, the stress resultants are obtained by integrating across the slab thickness.

A yield criterion using this approach has been obtained [3] for a set of strain and curvature rates $\dot{\varepsilon}_t, \dot{\kappa}_{nt}, \dot{\kappa}_t = 0$ and $\dot{\varepsilon}_{nt}, \dot{\varepsilon}_n, \dot{\kappa}_n \neq 0$, where $\dot{\varepsilon}_n, \dot{\varepsilon}_t, \dot{\varepsilon}_{nt}$ are the mid-depth strain-rates, and $\dot{\kappa}_{nt}, \dot{\kappa}_n, \dot{\kappa}_t$ are the curvature rates: positive directions corresponding to the positive direction of the appropriate stress resultant. These strain rates describe classical yield lines, but with the additional possibility of in-plane shear deformation in the yield line. A relationship Φ(Mn,Nn,Nnt) = 0 is obtained between the bending moment, normal force and in-plane shear force acting in combination on a yielding cross-section of the slab. If Mn,Nn,Nnt are then expressed in terms of Mx,My,Mxy,Nx,Ny,Nxy and θ ,the angle between the x and n axes(fig.1) a one parameter set of hyperplanes is defined in (Mx......Nxy) space.The hypersurface enveloped by this one parameter family represents an "upper yield surface",which is a convex surface circumscribing the actual yield surface of the body,and is defined by the set of hyperplanes which touch the true yield surface.Mathematically the envelope is obtained by eliminating θ between the expressions for $\Phi = 0$ and $\partial\Phi/\partial\theta = 0$.Braestrup[4] and Save[5] have obtained upper yield surfaces in this manner for the yielding of plates under bending and twisting alone.

The six-dimensional yield surface obtained represents a total yield surface in (Mx....Nxy) space,even though it is based on a strain rate field having one principal curvature rate and one strain rate in the direction of zero curvature rate equal to zero.The generalisation is valid because other combinations of strain rate and curvature rate can be described when the flow rule is applied to the intersections of the hypersurfaces which constitute the upper yield surface.

In the following a yield criterion in parametric form is derived,a safe approximation to this criterion is proposed,and finally the approximate surface is transposed into (Mx....Nxy) space.No rigorous proof of the continuity or the convexity of the yield surface is given.

2.PARAMETRIC YIELD CRITERION

The yield criterion derived below in parametric form is described in more detail elsewhere[3,6],but a brief outline is included here for completeness.

If it is assumed that transverse shear and stresses normal to the slab plane are unimportant,then the yield criteria for the steel and the concrete are required in plane stress.The steel is taken to carry stresses only in the original bar directions,and it yields at a stress $\pm\sigma_y$ .The concrete is taken to be an isotropic rigid plastic material with negligible tensile strength,and with a square principal stress yield criterion as shown in figure(2).

Under an applied set of stress resultants the element is taken to be yielding with a strain rate set defined by $\dot{\varepsilon}_{nn}, 2\dot{\varepsilon}_{nt}, \dot{\kappa}_{nn}$,as the only non-zero strain rates.Under such conditions,assuming plane sedtions remain plane,the strain rates at a distance z from the median plane(fig.1)are given by:

$$\dot{e}_t = \dot{\varepsilon}_{tt} = 0$$

$$\dot{e}_{nt} = \dot{\varepsilon}_{nt} = \text{constant}$$

$$\dot{e}_n = \dot{\varepsilon}_{nn} + \dot{\kappa}_{nn} z$$

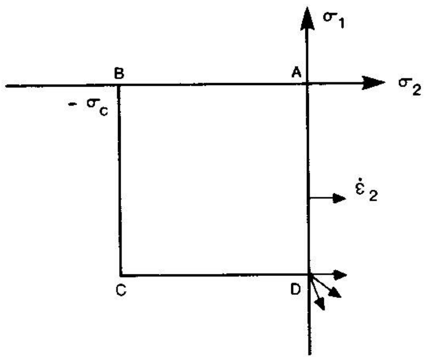

Fig.2 Yield Criterion for Concrete

From an examination of Mohr's circle it can be seen that for these conditions the principal strain rates are non-zero and of opposite sign.Therefore from the normality law the corresponding stress states in the concrete must be defined by corners B and D of the yield criterion. Assuming that the concrete is isotropic the principal directions of stress and strain rate coincide, and the stress values at level z can be defined as follows:

$$\sigma_n = -\frac{\sigma_c}{2}\left[1 - \frac{\dot{e}_n}{\sqrt{(\dot{e}_n^2 + 4\dot{e}_{nt}^2)}}\right] \qquad \tau_{nt} = \sigma_c \cdot \frac{\dot{e}_{nt}}{\sqrt{(\dot{e}_n^2 + 4\dot{e}_{nt}^2)}}$$

The contribution of the concrete to the stress resultants can now be found from the integrals:

$$M_n^c = \int_{-h/2}^{h/2} \sigma_n z . dz \qquad N_n^c = \int_{-h/2}^{h/2} \sigma_n dz \qquad N_{nt}^c = \int_{-h/2}^{h/2} \tau_{nt} dz$$

Similar uniquely defined expressions exist for M_t^c, N_t^c, M_{nt}^c, as σ_t is uniquely determined when $\dot{e}_{nt} \neq 0$.
It can then be shown that

$$M_n^c = \sigma_c \cdot \left(\frac{h}{2}\right)^2 \cdot \left[\frac{\varphi - \psi - \chi(\lambda+\mu)}{(\lambda-\mu)^2}\right]$$

$$N_n^c = \sigma_c \cdot \frac{h}{2} \cdot \left[\frac{\chi}{\lambda-\mu} - 1\right]$$

$$N_{nt}^c = \sigma_c \cdot \frac{h}{2} \cdot \frac{\psi}{\lambda-\mu}$$

where $\lambda = (\dot{e}_n/2\dot{e}_{nt})$ at $z = +h/2$ and $\mu = (\dot{e}_n/2\dot{e}_{nt})$ at $z = -h/2$
and

$$\chi = [1+\lambda^2]^{1/2} - [1+\mu^2]^{1/2} \qquad \varphi = \lambda[1+\lambda^2]^{1/2} - \mu[1+\mu^2]^{1/2}$$

$$\psi = \log_e\left[\frac{\lambda + (1+\lambda^2)^{1/2}}{\mu + (1+\mu^2)^{1/2}}\right]$$

Again similar parametric equations can be defined for the reactions M_t^c, N_t^c, M_{nt}^c. In order to obtain a closed form expression for the yield criterion in generalized stress space it would be necessary to eliminate λ, μ from the above expressions, and then add the steel contribution to obtain the total stress resultants. It would be difficult to eliminate λ and μ directly so a simple conservative approximation to the parametric surface is sought.

Approximations which are within 10% of the values given by the parametric surface are discussed elsewhere[3]. However in the following an alternative approximation is proposed which is more conservative, but which allows a simple transformation of the yield surface into (Mx....Nxy) space.

3 APPROXIMATE YIELD CRITERION

The following conditions are used as an approximate relationship between the concrete stress resultants:

$$N_{nt}^{c\,2} + N_n^c . [\sigma_c h + N_n^c] + 2M_n^c . \sigma_c = 0 \qquad \text{for} \quad \theta_n \geq 0 \quad \ldots . (1)$$

$$N_{nt}^{c\,2} + N_n^c . [\sigma_c h + N_n^c] - 2M_n^c . \sigma_c = 0 \qquad \text{for} \quad \theta_n \leq 0$$

The two paraboloids defined by equations (1) correspond to the approximation suggested by Janas and Sawczuk[7] for a material with a maximum normal stress condition at failure. These relationships are plotted for given values of N_n^c in fig.(3) together with the values obtained from the parametric equations for the same values of N_n^c. It can be seen that under certain combinations of N_{nt}^c, M_n^c, the approximate surface is very conservative. For $N_{nt}^c=0$ or $M_n^c=0$, however, the parametric values and the approximate relationships coincide: but the approximate surface has a corner when $M_n^c=0$ whilst the parametric surface is smooth.

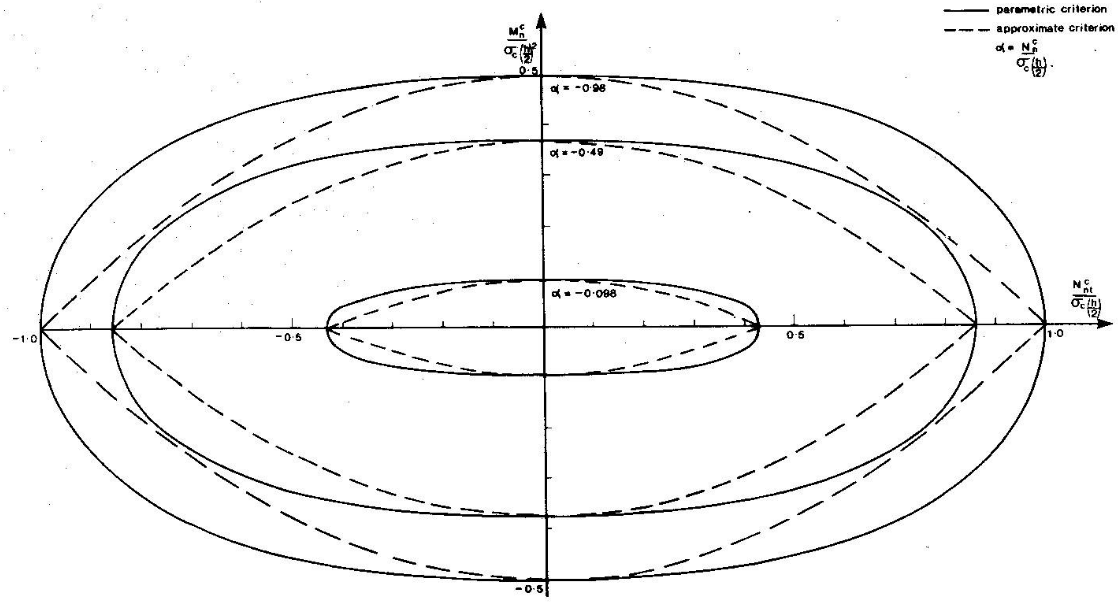

Fig.3 Yield Surfaces for Concrete Stress Resultants

The total stress resultants are obtained by adding the steel contributions M_n^s, N_n^s, N_{nt}^s, and equations (1) become:

$$(N_{nt}-N_{nt}^s)^2 + (N_n-N_n^s)\cdot[\sigma_c h+(N_n-N_n^s)] + 2(M_n-M_n^s)\cdot\sigma_c = 0 \quad \text{...(2a)}$$

$$(N_{nt}-N_{nt}^s)^2 + (N_n-N_n^s)\cdot[\sigma_c h+(N_n-N_n^s)] - 2(M_n-M_n^s)\cdot\sigma_c = 0 \quad \text{...(2b)}$$

It is noted that equations (2a,2b) are also valid for $\dot{\varepsilon}_{nt}=0$ i.e. a convential yield line. Under these circumstances σ_n at a given depth z is $-\sigma_c$ or 0, but σ_t is indeterminate as the stress states are no longer confined to the corners in fig.(2) and M_t^c is therefore unknown. M_{nt}^c, on the other hand, is known to be zero.

4. TRANSFORMATION OF THE YIELD CRITERION TO (Mx,My...Nxy) SPACE

To define the yield locus in (Mx...Nxy) space Mn etc. are first expressed in terms of (Mx,My,.....Nxy,θ), giving a function $\Phi=0$. A yield surface envelope is then obtained by eliminating θ between the equations $\Phi=0$ and $\partial\Phi/\partial\theta=0$.

The steel reinforcement is treated as plates of constant thickness at the level of the bar centres which are only able to carry stress in the original bar directions, and which yield at a stress $\pm\sigma_y$. The slab has k layers of reinforcement in the x direction, each layer of area a_i per unit length and at depth $z=d_i$ from the median plane. A_x and H_x are defined by:

$$A_x = \sum_k a_i r_i \qquad H_x = \sum_k d_i a_i r_i$$

where r_i is the ratio of the stress in layer (i) to the yield stress σ_y .The ratio r_i when determinate takes the value of ±1.Similar expressions also define A_y and H_y .The steel contributions can now be defined:

$$N_n^s = \sigma_y(A_x\cos^2\theta + A_y\sin^2\theta) \qquad N_{nt}^s = \sigma_y(A_x - A_y).\sin\theta\cos\theta$$

$$M_n^s = \sigma_y(H_x\cos^2\theta + H_y\sin^2\theta) \qquad M_{nt}^s = \sigma_y(H_x - H_y).\sin\theta\cos\theta$$

Substitution of these values into equation(2a),together with the known expressions for Nn etc.in terms of Nx,Ny,Nxy,θ gives,with some rearrangement:

$$\begin{aligned}\Phi = &[N_{xy}^2 + \sigma_c h(N_x - A_x\sigma_y) + (N_x - A_x\sigma_y)^2 + 2(M_x - H_x\sigma_y)\sigma_c]\cos^2\theta \\ &+[N_{xy}^2 + \sigma_c h(N_y - A_y\sigma_y) + (N_y - A_y\sigma_y)^2 + 2(M_y - H_y\sigma_y)\sigma_c]\sin^2\theta \\ &-[N_{xy}.\{(N_x - A_x\sigma_y) + (N_y - A_y\sigma_y) + \sigma_c h\} + 2M_{xy}\sigma_c]\sin 2\theta = 0 \quad ..(3)\end{aligned}$$

Elimination of θ between the expression for Φ and $\partial\Phi/\partial\theta = 0$ gives:

$$\begin{aligned}&[N_{xy}^2 + \sigma_c h(N_x - A_x\sigma_y) + (N_x - A_x\sigma_y)^2 + 2(M_x - H_x\sigma_y)\sigma_c].[N_{xy}^2 + \sigma_c h(N_y - A_y\sigma_y) \\ &+ (N_y - A_y\sigma_y)^2 + 2(M_y - H_y\sigma_y)\sigma_c] = [N_{xy}.\{(N_x - A_x\sigma_y) + (N_y - A_y\sigma_y) + \sigma_c h\} \\ &\qquad + 2M_{xy}\sigma_c]^2 \quad ....(4a)\end{aligned}$$

and for hogging principal curvature the equivalent expression is:

$$\begin{aligned}&[N_{xy}^2 + \sigma_c h(N_x - A_x\sigma_y) + (N_x - A_x\sigma_y)^2 - 2(M_x - H_x\sigma_y)\sigma_c].[N_{xy}^2 + \sigma_c h(N_y - A_y\sigma_y) \\ &+ (N_y - A_y\sigma_y)^2 - 2(M_y - H_y\sigma_y)\sigma_c] = [N_{xy}.\{(N_x - A_x\sigma_y) + (N_y - A_y\sigma_y) + \sigma_c h\} \\ &\qquad - 2M_{xy}\sigma_c]^2 \quad ....(4b)\end{aligned}$$

The general yield criterion defined by equations (4a,4b)can be used to determine which combinations of the six stress resultants (Mx,My,Mxy,Nx,Ny,Nxy) will cause yielding of a given orthogonally reinforced concrete slab element.

5.SPECIAL CASES FROM THE GENERAL YIELD CRITERION

5.1.Nx = Ny = Nxy = 0

Under these conditions equation (4a) reduces to:

$$[M_x - \sigma_y\{H_x + A_x(\frac{h}{2} - \frac{A_x\sigma_y}{2\sigma_c})\}].[M_y - \sigma_y\{H_y + A_y(\frac{h}{2} - \frac{A_y\sigma_y}{2\sigma_c})\}] = M_{xy}^2$$

which with Ax,Ay,Hx,Hy,constant for all values of θ,becomes:

$$(Mpx - Mx).(Mpy - My) = Mxy^2 \quad ......(5a)$$

where Mpx,Mpy are the ultimate moments for simple bending in the x and y directions respectively.The corresponding expression for negative bending is:

$$(M'px + Mx).(M'py + My) = Mxy^2 \quad ......(5b)$$

where M'px,M'py are the ultimate hogging moments in the x and y directions.

Equations (5a) and (5b) are the commonly accepted expressions for the yield criterion under bending alone.But it should be noted that $\varepsilon_{nt} \neq 0$ on the yield line.

5.2.Mx,My,Mxy=0

If Hx=Hy=0 (which could occur for equal top and bottom reinforcement in a plate)then equations (4a,4b) become:

$$[N_{xy}^2 + \sigma_c h(N_x - A_x\sigma_y) + (N_x - A_x\sigma_y)^2].[N_{xy}^2 + \sigma_c h(N_y - A_y\sigma_y) + (N_y - A_y\sigma_y)^2]$$

$$= N_{xy}^2 [(N_x - A_x\sigma_y) + (N_y - A_y\sigma_y) + \sigma_c h]^2$$

which reduces to:

$$[(N_x - A_x\sigma_y)(N_y - A_y\sigma_y) + \sigma_c h\{(N_x - A_x\sigma_y) + (N_y - A_y\sigma_y) + \sigma_c h\} - N_{xy}^2]$$

$$.[(N_x - A_x\sigma_y)(N_y - A_y\sigma_y) - N_{xy}^2] = 0$$

thus either $N_{xy}^2 = (N_x - A_x\sigma_y).(N_y - A_y\sigma_y)$ ......(6a)

or $N_{xy}^2 = [(N_x - A_x\sigma_y) + \sigma_c h].[(N_y - A_y\sigma_y) + \sigma_c h]$...(6b)

Equations (6a,6b) correspond to the yield criteria for in plane forces derived by Nielsen[8].It should be noted that these equations coincide with the parametric equations and are therefore exact as they satisfy equilibrium,yield and are kinematically admissible.Nielsen derived them on the basis of static admissibility.

6.DISCUSSION OF THE GENERAL YIELD CRITERION

The values to be used for Ax,Ay,Hx,Hy,in equations (4a,4b) are various.When a layer of steel is in tension or compression then the contribution to Ax etc.is clearly defined.However when the strain rate in a given steel layer is zero the steel stress is indeterminate,lying in the range $-\sigma y \leqslant \sigma s \leqslant +\sigma y$.When this occurs the corresponding value of Ax,Ay,Hx,orHy is not uniquely defined for the given set of strain rates.Consequently "flat" zones are produced in the yield surface in 6-D space as each steel layer undergoes a stress change from $-\sigma y$ to $+\sigma y$.

In most practical cases two layers of steel will be provided in the x and y directions;top steel and bottom steel.To consider every combination of tensile,compressive or indeterminate stress would lead to an extremely complex yield surface.It is therfore suggested that a restricted set of steel stresses be considered when a slab has known amounts of top and bottom steel.(The restrictions are unnecessary when using the yield criterion for provision of strength formulae).It is proposed that the top steel is taken to be $+\sigma y$ in both directions,or $-\sigma y$ in both directions,and similarly for the bottom steel.The four permutations which can be introduced into equations (4a,4b) will give eight relationships which define a yield surface in (Mx....Nxy) space.The range of validity of the eight relationships will be defined by the boundaries created from the intersection of each of the eight hypersurfaces.The yield locus thus derived either coincides with,or circumscribes,the locus which would be obtained if proper account were taken of the other combinations of stresses occurring in the steel.Consequently the above assumptions about the steel are unsafe in some circumstances in that they overestimate the resistance of the slab element to certain combinations of applied stress resultants.However for practical reinforcement percentages the error involved will not be great,and will often be offset by the conservatism introduced by the safe approximation of the concrete contributions relative to the parametric values.

The parametric yield criterion was derived by considering a strain rate field which had one principal curvature rate,and one strain rate in the direction of zero curvature rate,equal to zero.Nevertheless the yield criterion obtained for (Mx...Nxy) space is valid for any combination of strain rates and curvature rates because the flow rule applied at the intersection of the surfaces composing the yield criterion will describe the other combinations of strain and curvature rates.The six-dimensional surface thus derived represents an upper yield surface for an orthogonally reinforced concrete plate,when transverse shear and stresses normal to the plane of the plate can be taken to have negligible influence.

The simplified approximate yield criterion could be used for calculations using assumed collapse mechanisms by applying the normality law,but such computation does not strictly represent an upper bound calculation as the yield criterion used is a lower bound approximation.Results for collapse loads thus obtained could therefore be less than or greater than the true collapse load,and subsequent comparison with experiments could give a misplaced confidence in the accuracy of the theory.For upper bound calculations it may be preferable to use approximations closer to the parametric yield surface.Such computation is feasible (see[3]) as the yield surface does not need to be defined in (Mx...Nxy) space in these circumstances.

The range of validity of a lower bound approximation to an upper yield surface is debatable.Nevertheless the parametric equations represent the true yield surface for a useful set of strain rates i.e. generalized yield lines,and the suggested approximations are not too crude for most practical cases.Therefore sufficient grounds exist to accept the resulting yield surface as a useable criterion.The proposed yield criterion is more likely to be of use for lower bound calculations,and in particular for deriving formulae for the provision of reinforcement in slabs.

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