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Collapse Load Analysis of Engineering Structures by Using New Discrete Element Models

Calcul à la ruine de structures, à l'aide de modèles nouveaux d'éléments discrets

Berechnung der Traglast von Baukonstruktionen mit neuen diskreten Elementen

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SUMMARY
Combining advantages of the concept of limit analysis and standard load incremental procedure in existing finite element method, a method of collapse load analysis is introduced in this paper. It should be mentioned that this method is not a method of rigid plastic analysis, but effects of elasticity, finite deformation or instability can be taken into account.
Motivation of development of this method and its theoretical basis will be explained first and justification of the present method is illustrated by several numerical examples including collapse load analysis of concrete slabs.

RESUME
Le rapport présente une méthode de calcul à la ruine combinant les avantages de la théorie des charges limites et du procédé de l'augmentation progressive des charges, tels qu'ils existent dans des méthodes actuelles par éléments finis. Cette méthode n'est pas une méthode de calcul rigide-plastique, car elle tient compte des effets de l'élasticité, de déformations limitées et d'instabilité. Les raisons du développement de cette méthode et ses bases théoriques sont présentées. Plusieurs exemples numériques, parmi lesquels le calcul à la ruine de dalles en béton armé, illustrent et prouvent la valeur de la méthode développée ici.

ZUSAMMENFASSUNG
Eine Methode zur Traglastberechnung wird vorgestellt, welche Vorteile der Traglastverfahren einerseits und der bei der Anwendung der Methode der finiten Elemente üblichen Verfahren der schrittweisen Laststeigerung andererseits kombiniert. Es handelt sich nicht um eine starr-plastische Berechnungsmethode, denn das elastische Verfahren, endliche Verformungen oder Instabilitäten können ebenfalls berücksichtigt werden. Die Entwicklung der Methode wird begründet, und ihre theoretischen Grundlagen werden dargestellt. Anhand mehrerer numerischer Beispiele, unter anderem zur Berechnung der Traglast von Stahlbetonplatten, wird die Anwendung der Methode erläutert.

ABSTRACT

Combining advantages of the concept of limit analysis and standard load incremental procedure in existing finite element method, a method of collapse load analysis is introduced in this paper. It should be mentioned that this method is not a method of rigid plastic analysis, but effects of elasticity, finite deformation or instability can be takn into account.
Motivation of development of this method and its theoretical basis will be explained first and justification of the present method is illustrated by several numerical examples including collapse load analysis of concrete slabs.

1. INTRODUCTION

1.1 Plastic Analysis and its Limitation

Consider any structure or solid subjected to external load (statical or dynamical). As long as the external load is small, it may deform elastically and the induced stresses and strains are so small that their distribution can be determined by well established theory of elasticity.
With the increase of external loads, however, strain distribution may reach the stage where no longer it is small and finite strain distribution will set up locally or partially in the deformed body under consideration, and then the structure may be subjected to large deformation or may buckle in case of ductile materials.
Upon further increase of the load, deformed structures may start to yield and develop so called plastic hinges, hinge lines (or slip lines) or slip surfaces, and plastic zone will grow and spread out and finally cracks may initiate from some overstressed region.
At the final stage of loading or the ultimate load, a certain link mechanism which may usually consist of plastic hinges or plastic hinge lines or slip surfaces will be formed in the structure.
Under such condition the structure may loose stability and it will start to move freely just like a linked rigid bodies and it will be *collapsed.*
According to the theory of plasticity, it can be proved that a collapse load solution can be uniquely determined if the solution satisfies the following three conditions:

(i) equilibrium condition
(ii) plasticity condition
(iii) mechanism condition

In general it is extremely difficult to obtain such a solution analytically and two different procedure of obtaining approximate solutions have been proposed by basing on the well-known upper bound and lower bound theorems in the theory of limit analysis which were originally proposed by Prager, Drucker and many others. These approximate solution procedures have provided a very

powerful tool for collapse load analysis of plane frames, simple plate and shell structures and resulted in development of plastic analysis and design which are now accepted in routine structural design of those engineering structures mentioned.
Practical application of plastic analysis, however, has been limited to collapse load analysis of plane frames and furthermore influences of stability, crack initiation can not be taken into account. Especially dynamic collapse load analysis is still under state of arts and because of this reason, its application to structural dynamics has been quite limited.

1.2 Finite Element Method and Current Status of Development

Advent of the finite element method has changed this situation completely. By using the standard load incremental procedure it is possible to trace step by step the equilibrium state of structures and elasto-plastic stress distribution corresponding to the load condition at the time prescribed and sequence of formation of collapse mechanism.
At the present moment it is not too difficult in principle to obtain the collapse load solution of any complex structure under a certain loading condition by using standard computer programs. Very serious drawback of the finite element method, however, is computing time and cost, especially in nonlinear analysis.
In order to solve such a difficult problems active work has been done all over the world. Actual problems, however, are still far beyond control of any existing method.

1.3 Concept of the Rigid Body-Spring Element

The present author believed that development of a new discrete models might be only a possible way to solve this problem. He tried to find a new physical model rather than mathematical in which essential feature of deformable bodies is retained without introducing highly complex mathematical manipulation. In general when structures or solids reach their ultimate state of loading, they may be yielded, collapsed and crushed into pieces. At the limiting state each part or piece of the structures may move like rigid bodies. Based on such experimental evidences, the following *Rigid Body- Spring* model has been conceived. Consider the bending problem of a beam under lateral loads as shown in Fig. 1.
Within elastic range, deformation is distributed throughout the beam, but once plastic deformation starts either at the point of load application or at the beam ends, strain energy will be absorbed in the narrow portion of a beam where plastic deformation takes place and at the ultimate stage of loading a number of the so-called *plastic hinges* will be formed so that the beam structure will collapse into a link mechanism. This mechanism consists of rigid bars and plastic hinges. In case of bending problems of concrete slabs as shown in Fig. 1, similar experimental evidence will be observed. That is, within the range of elastic bending, deformation is distributed over the whole plate area, however, at the final stage of loading the plate will collapse under a certain mechanism which consists of rigid plate segments, and plastic hinge lines connecting those plate segments.
The so-called *slip line theory* is also well known in also plane stress as well as plane strain problems in the theory of plasticity (See slso Fig. 1). According to this theory, it is assumed that two dimensional solids will move under a certain mechanism which consists of two dimensional rigid segments and slip lines connecting those segments, and along which relative sliding of two neighboring segments will occur. In the following section theoretical basis of these new models will be described.

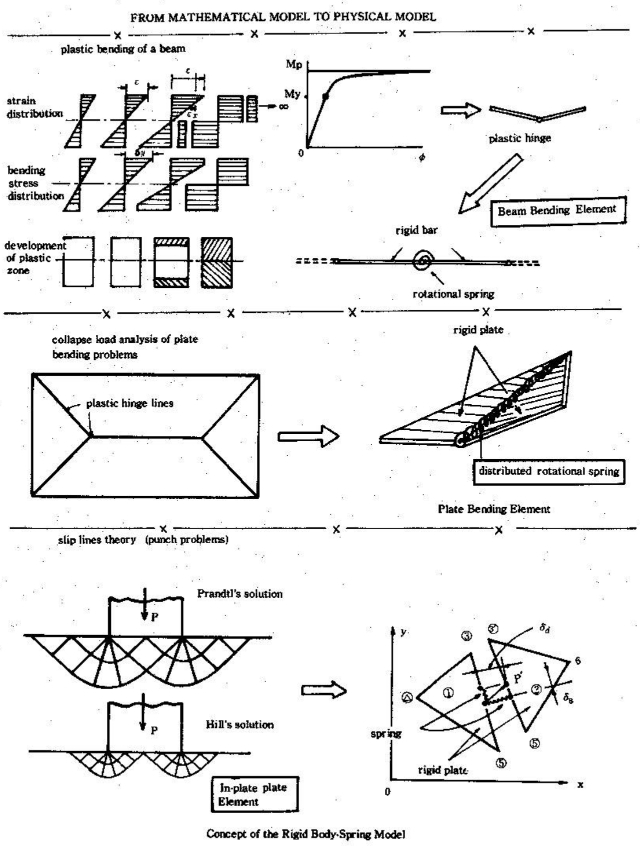


Fig. 1

Concept of the Rigid Body-Spring Model

2. THEORETICAL BASIS OF RIGID BODY-SPRING MODELS

2.1 Physical Basis of Rigid Body-Spring Models

Consider a set of three dimensional rigid bodies of arbitary shape as shown in Fig. 2. They are assumed to be in equilibrium with external loads, and reaction forces are produced by the spring system which is distributed over the contact surface of two adjacent bodies. For further development of new element models, it will be assumed that the contact area is known and fixed*.

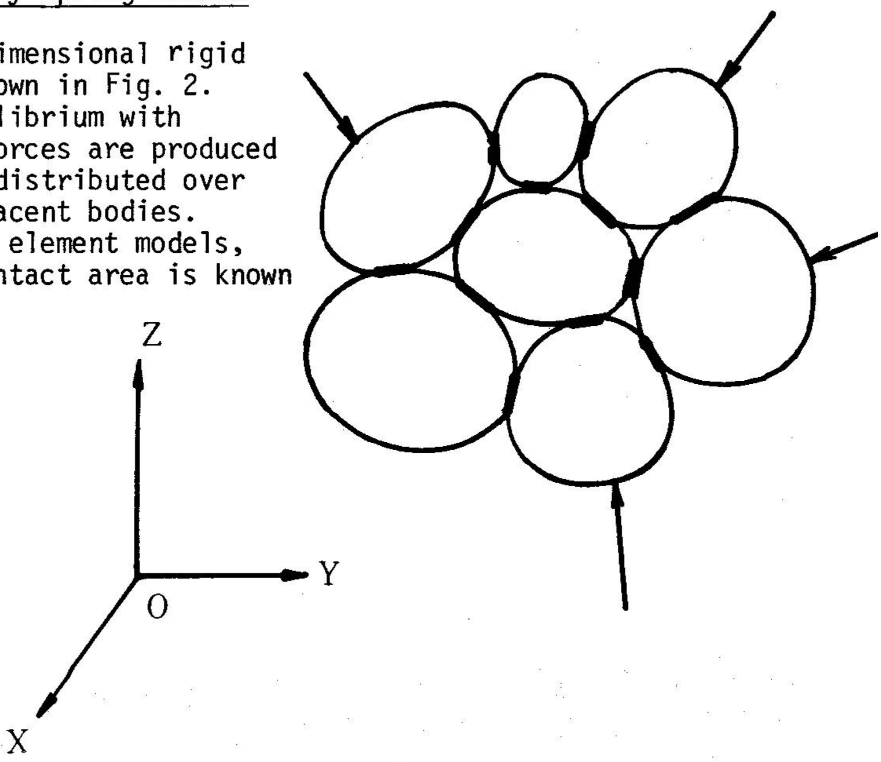


Fig. 2

* It should be mentioned here that in actual contact problem, the contact surface are not known a priori, and therefore it can be determined only in iterative way.

Taking such two rigid bodies under contact, infinitesimal deformation of the spring system is considered. (Fig. 3). Displacement $\boldsymbol{u}$ of an arbitrary point in a rigid body can be given by the following vectorial equation:

$$\boldsymbol{u} = \boldsymbol{u}_G + \boldsymbol{\Theta} \times (\boldsymbol{r} - \boldsymbol{r}_G) \qquad (1)$$

where $\boldsymbol{u}_G$ is displacement vector of the centroid, $\boldsymbol{\Theta}$ is the infinitesimal rotation vector and $(\boldsymbol{r} - \boldsymbol{r}_G)$ is a position vector of arbitrary point with respect to the centroid before deformation.

$$\boldsymbol{u}_G = (u_g, v_g, w_g) \qquad \boldsymbol{\Theta} = (\theta, \phi, \chi) \qquad (2)$$

Denoting the displacement vectors of arbitrary point P(x,y,z) in body (I) and (II) by $\boldsymbol{u}'$, $\boldsymbol{u}''$, respectively, they are given by the following equations:

$$\begin{aligned} \boldsymbol{u}' &= \boldsymbol{u}_1 + \boldsymbol{\Theta}_1 \times (\boldsymbol{r} - \boldsymbol{r}_1) \\ \boldsymbol{u}'' &= \boldsymbol{u}_2 + \boldsymbol{\Theta}_2 \times (\boldsymbol{r} - \boldsymbol{r}_2) \end{aligned} \qquad (3)$$

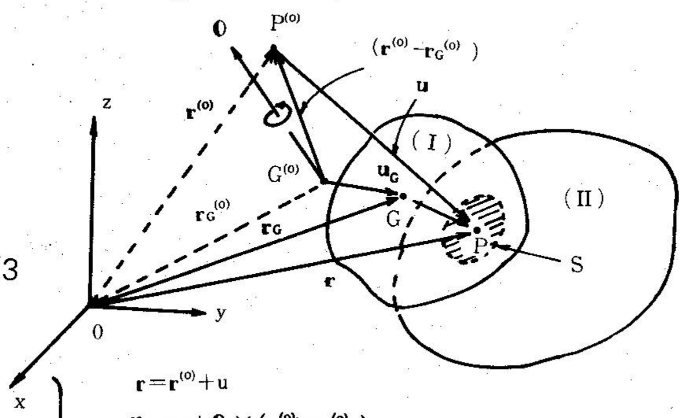


Fig. 3

$\mathbf{r}=\mathbf{r}^{(0)}+\mathbf{u}$

$\mathbf{u}=\mathbf{u}_G+\boldsymbol{\Theta}\times(\mathbf{r}^{(0)}-\mathbf{r}_G^{(0)})$

$\quad=\mathbf{u}_G+\boldsymbol{\Theta}\times(\mathbf{r}-\mathbf{r}_G)$

S: contact surface

superscript (0)' implies the state before deformation

More precisely,

$$\left.\begin{aligned} u' &= u_1 + (z - z_1)\phi_1 - (y - y_1)\chi_1 \\ v' &= v_1 + (x - x_1)\chi_1 - (z - z_1)\theta_1 \\ w' &= w_1 + (y - y_1)\theta_1 - (x - x_1)\phi_1 \end{aligned}\right\} \qquad (4\text{-a})$$

$$\left.\begin{aligned} u'' &= u_2 + (z - z_2)\phi_2 - (y - y_2)\chi_2 \\ v'' &= v_2 + (x - x_2)\chi_2 - (z - z_2)\theta_2 \\ w'' &= w_2 + (y - y_2)\theta_2 - (x - x_2)\phi_2 \end{aligned}\right\} \qquad (4\text{-b})$$

Therefore denoting the point P after displacement in bodies (I) and (II) by P' and P" respectively, the relative displacement vector of the point P can be defined as follows:

$$\overrightarrow{P'P''} = \boldsymbol{u}'' - \boldsymbol{u}' \qquad (5)$$

Denoting the unit normal drawn outward to the contact surface at the point P by $\boldsymbol{n}$, (See Fig. 4) the normal displacement δ_d to the surface S can be given as follows:

$$\delta_d = (\overrightarrow{P'P''}, \boldsymbol{n}) = l(u'' - u') + m(v''-v') + n(w''-w') \qquad (6\text{-a})$$

where

$$\boldsymbol{n} = (l, m, n) \qquad (6\text{-b})$$

Similarly the displacement component δ_s in the tangential plane to the surface can be given by the following equation:

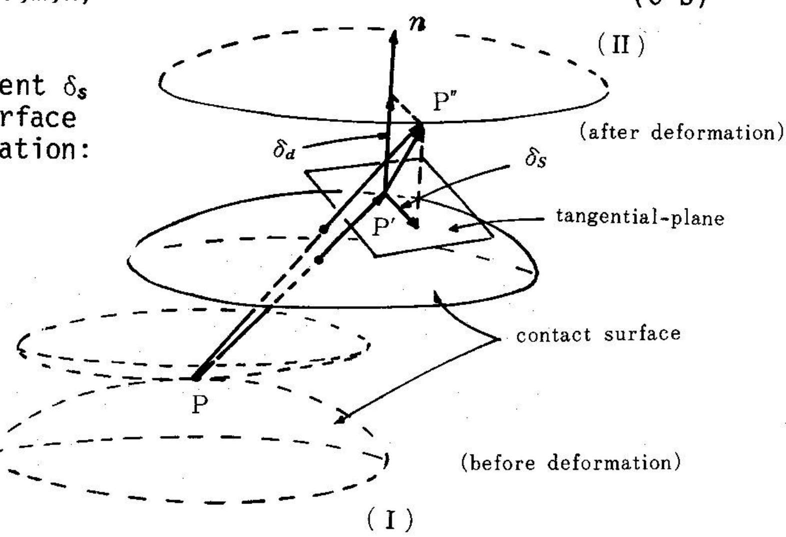


Fig. 4

$$\begin{aligned}\delta_s^2 &= |\boldsymbol{n} \times \overrightarrow{P'P''}|^2 \\ &= \{m(w'' - w') - n(v'' - v')\}^2 + \{n(u'' - u') - l(w'' - w')\}^2 \\ &\quad + \{l(v'' - v') - m(u'' - u')\}^2\end{aligned} \tag{7}$$

Basing on the above preliminaries, strain energy due to the relative displacements (δ_d, δ_s) of the spring system distributed over the contact surface S can be given by the following equation:

$$V = \frac{1}{2} \iint_S (k_d \, \delta_d^2 + k_s \delta_s^2) \, dS \tag{8}$$

Substituting eqs.(6) and (7) into eq.(8) the following equation can be easily obtained:

$$\begin{aligned}V &= \frac{1}{2} \iint_S [k_s(\delta_x^2 + \delta_y^2 + \delta_z^2) + (k_d - k_s)(l^2 \delta_x^2 + m^2 \delta_y^2 + n^2 \delta_z^2 \\ &\quad + 2\, l m \, \delta_x \delta_y + 2\, m n \, \delta_y \delta_z + 2\, n l \, \delta_z \delta_x)] \, dS \\ &= \frac{1}{2} \iint \boldsymbol{\delta}^T \bar{\mathbf{D}} \, \boldsymbol{\delta} \, dS\end{aligned} \tag{9-a}$$

where

$$\begin{aligned}\delta^T &= \lfloor \delta_x , \delta_y , \delta_z \rfloor \\ \delta_x &= u'' - u', \quad \delta_y = v'' - v', \quad \delta_z = w'' - w'\end{aligned} \tag{9-b}$$

$$[\bar{D}] = \begin{bmatrix} k_d l^2 + k_s(1-l^2) & (k_d - k_s) l m & (k_d - k_s) l n \\ (k_d - k_s) l m & k_d m^2 + k_s(1-m^2) & (k_d - k_s) m n \\ (k_d - k_s) l n & (k_d - k_s) m n & k_d n^2 + k_s(1-n^2) \end{bmatrix} \tag{10}$$

Displacement vector $\boldsymbol{\delta}$ can be also expressed by the following matrix equation:

$$\boldsymbol{\delta} = \mathbf{B} \, \mathbf{d} \tag{11}$$

where

$$[\mathbf{B}] = \begin{bmatrix} -1 & 0 & 0 & 0 & -(z-z_1) & (y-y_1) & 1 & 0 & 0 & 0 & (z-z_2) & -(y-y_2) \\ 0 & -1 & 0 & -(z-z_1) & 0 & -(x-x_1) & 0 & 1 & 0 & (z-z_2) & 0 & (x-x_2) \\ 0 & 0 & -1 & -(y-y_1) & (x-x_1) & 0 & 0 & 0 & 1 & (y-y_2) & -(x-x_2) & 0 \end{bmatrix} \tag{12}$$

$$\mathbf{d}^T = \lfloor u_1, v_1, w_1, \theta_1, \phi_1, \chi_1 ; u_2, v_2, w_2, \theta_2, \phi_2, \chi_2 \rfloor \tag{13}$$

Substituting eq.(11) into eq.(9-a), the following expression will be finally obtained:

$$V = \frac{1}{2} \mathbf{d}^T \mathbf{k} \, \mathbf{d} \tag{14-a}$$

where

$$\mathbf{k} = \iint_S \mathbf{B}^T \bar{\mathbf{D}} \, \mathbf{B} \, dS \tag{14-b}$$

where S is area of the contact boundary surface on two adjacent elements. The complete form of the stiffness matrix [K] is already obtained in the author's previous papers.

Applying Castigliano's theorem to eq.(14-a), the following stiffness equation can be derived:

$$\mathbf{R} = \frac{\partial V}{\partial \mathbf{d}} = \mathbf{k}\,\mathbf{d} \qquad (15\text{-a})$$

where $\mathbf{k}$ is a (12 x 12) symmetric matrix given by the following equation:

$$\mathbf{k} = {}_{12}[\, k_{ij} \,]^{12} \qquad (15\text{-b})$$

and $\mathbf{R}$ is nodal reaction vector defined by the following equation:

$$\mathbf{R}^T = \lfloor X_1, Y_1, Z_1, L_1, M_1, N_1;\ X_2, Y_2, Z_2, L_2, M_2, N_2 \rfloor \qquad (15\text{-c})$$

Spring constants k_d and k_s can be determined systematically by using the finite difference expression for strain components as follows:
On the contact surface S shown in Fig. 3 , normal and tangential stresses σ_n τ_{ns} satisfy the following equations:

$$\sigma_n = E'\varepsilon_n^* \ , \qquad \tau_{ns} = G\,\gamma_{ns} \qquad (16)$$

Strain components ε_n and γ_{ns} are approximated by the following finite defference expressions

$$\varepsilon_n = \delta_V/h \quad , \qquad \gamma_{ns} = \delta_H/h \qquad (17)$$

where $h = h_1 + h_2$ is the projection of the vector G_1G_2 on $\mathbf{n}$.
On the other hand, the following relations are obtained from the definition of spring constants

$$\sigma_n = k_d\delta_V \quad , \qquad \tau_{ns} = k_s\,\gamma_{ns} \qquad (18)$$

Therefore comparing eqs. (16) and (18), the following formulae can be derived

$$k_d = E'/h \quad , \qquad k_s = G/h \qquad (19)$$

The stiffness equation defined by eq.(15) must be obtained for each contact surface if a given rigid body (I) has a number contact surfaces with other rigid bodies including the body (II), and for equilibrium of a given total system of rigid bodies, they should be summed up and the final form of the stiffness equation can be given by the following standard form of the finite element method.

$$\mathbf{K}\,\mathbf{U} = \bar{\mathbf{F}} \qquad (20)$$

where

$$\mathbf{K} = \Sigma\,\mathbf{k}, \qquad \mathbf{U} = \Sigma\,\mathbf{d} \ , \qquad \bar{\mathbf{F}} = \Sigma\,\mathbf{f} \qquad (21)$$

$\bar{\mathbf{F}}$ is a given external load vector.
However, care must be exercised in constructon of eq.(20), because in this method the centroid of each rigid body is selected as the node and therefore superposition of stiffness matrices are somewhat different from that of the standard finite element method.

In case where the body (I) is supported by other bodies through its whole boundary surface S, i.e.

$$S = S_1 + S_2 + \ldots\ldots\ldots + S_m$$

this model is idealization of three dimensional elastic continuum as shown in Fig. 5 in which the shape of each element can be chosen arbitrary.

$$* \quad E' = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}$$

The method outlined so far will be called hereafter as the Rigid Bodies-Spring Method (RBSM) or Stiffness Lumping Method (SLM). Using this method stress analysis of deformable bodies under contact will be possible in iterative way, typical application of which is analysis of the rockfill dam.

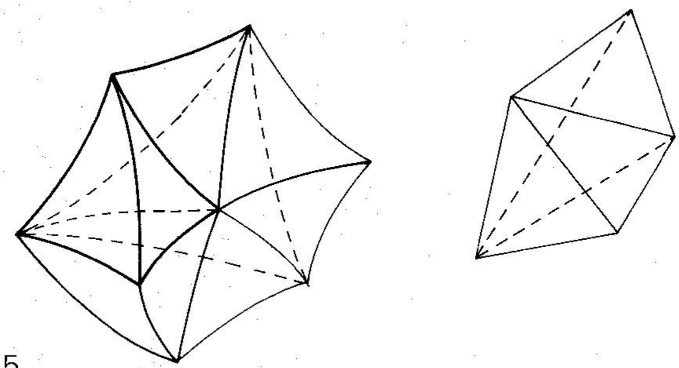


Fig. 5 (a) general form (b) standard form

2.2 Variational Basis of RBSM

RBSM model is originally proposed by basing on physical consideration and therefore it is desperately needed to establish the mathematical basis for those elements although a series of simple bending and vibration analyses were conducted. In the following section the mathematical basis of the present elements will be briefely described. For the sake of simplicity, consider a set of triangular elements in the plane stress problem as shown in Fig. 6.

Each triangular element is assumed that they are conected by a set of boundary elements $\overline{AB}$, $\overline{BC}$, $\overline{CA}$.
Infinitesimal displacement field for each element is also given by the following equation:

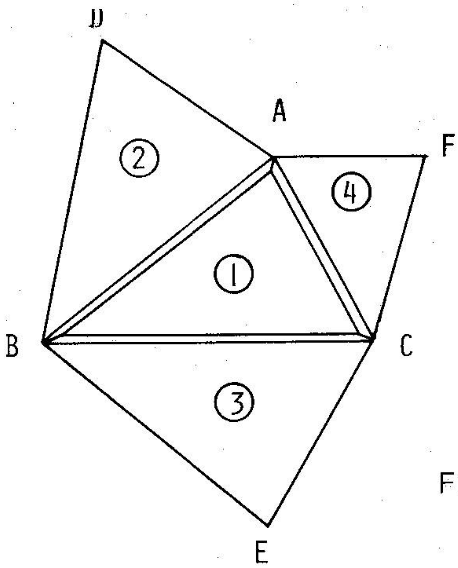


Fig. 6

Fig. 6 A Set of Triangular Plate elements for the pla stress problem

$$\begin{Bmatrix} U \\ V \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -(y-y_G) \\ 0 & 1 & (x-x_G) \end{bmatrix} \begin{Bmatrix} u_G \\ v_G \\ \chi \end{Bmatrix} + \begin{bmatrix} x-x_0 & 0 & \frac{1}{2}(y-y_G) \\ 0 & y-y_0 & \frac{1}{2}(x-x_G) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad (22)$$

Or in compact form:

$$\mathbf{u}(X) = \mathbf{A}(X)\mathbf{d} + \mathbf{B}(X)\boldsymbol{\varepsilon} \qquad (23)$$

where

$$\mathbf{u}(X)^T = \lfloor U(x,y),\ V(x,y) \rfloor$$

$$\mathbf{A}(X) = \begin{bmatrix} 1 & 0 & -(y-y_G) \\ 0 & 1 & (x-x\ ) \end{bmatrix}$$

$$\mathbf{B}(X) = \begin{bmatrix} x-x_G & 0 & \frac{1}{2}(y-y_G) \\ 0 & y-y_G & \frac{1}{2}(x-x_G) \end{bmatrix}$$

and

$$\mathbf{d}^T = \lfloor u_G,\ v_G,\ \chi \rfloor \qquad \boldsymbol{\varepsilon}^T = \lfloor \varepsilon_x,\ \varepsilon_y, \gamma_{xy} \rfloor \qquad (24)$$

Eqs. (22) or (23) implies that the linear displacement field consists of two independent parameters, i.e. the rigid body movement of the centroid $\mathbf{d}$ and uniform strain distribution $\boldsymbol{\varepsilon}$ in the said element.

Since nodal parameters of each element can be assumed independently and continuity condition of displacements along the interface boundary should be imposed by means of Lagrangian multiplier in the variational formulation of total potential energy.
There are several methods of formulation which are called hybrid displacement method and they are discussed clearly in the texts of Professor Washizu, Zienkiewicz, Gallagher and many other's [2],[4],[5].
Here an approach originally proposed by Ping Tong [4] is adopted.
Consider the functional π_{PH} given by the following equation:

$$\pi_{PH} = \sum_{n} \left(\iint_{S_n} \frac{1}{2} \varepsilon^T D \varepsilon \, dS - \iint_{S_n} \bar{b}^T \cdot u \, dS - \int_{C_{\sigma n}} \bar{T} u \, dS - \int_{C_n} (n\sigma)^T (u - u_b) \, dS \right) \tag{25}$$

$\sum_{n}$ implies summation of the total n elements.
The first term of the right hand side of eq.(25) is the strain energy to be stored in the element, the second and the third represent potential energy of external body force $\bar{b}$ and boundary load $\bar{T}$. The last term is the additional potential to be imposed on the displacement field to secure their continuity along the interface boundary lines in which σ is the stress matrix and it can be expressed by

$$\sigma = D \varepsilon \tag{26}$$

where D is the stress-strain matrix, and u_b represents the displacement vector of the boundary elements.
Therefore the functional π_{PH} is a function of d, ε of all elements as well as d_b, ε_b of all boundary elements. As already mentioned before nodal parameters d and ε of each element is assumed independently, variations are taken first with respect to d and ε of a typical element and the following matrix equations can be derived:

(i) with respect to variation δd

$$E_d^T \varepsilon + \bar{Q}_d = 0 \tag{27}$$

where

$$\begin{aligned} E_d^T &= \int_{C_n} A(X)^T \, n \, D \, dS \\ \bar{Q}_d &= \iint_{S_n} A(X)^T \, \bar{b} \, dS + \int_{C_{\sigma n}} A(X)^T \, \bar{T} \, dS \end{aligned} \tag{28}$$

(ii) with respect to variation $\delta\varepsilon$

$$E_d^T d - S D \varepsilon + \bar{Q}_\varepsilon - G^T q = 0 \tag{29}$$

where

$$\left.\begin{aligned} &S_0 = \iint_{S_n} dS \,, \quad S_1 = \int_{C_n} B(X)^T n \, dS \,, \quad S = S_0 - 2S_1 \\ &\bar{Q}_\varepsilon = \iint_{S_n} B(X)^T \, \bar{b} \, dS + \int_{C_n} A(X)^T \, \bar{T} \, dS \\ &G = \lfloor \int_{C_n} D \, n^T A(X) \, dS, \ \int_{C_n} D \, n^T B(X) \, dS \rfloor = \lfloor E_d \,, \ S_1^T D \rfloor \\ &q = \lfloor d_b \,, \ \varepsilon_b \rfloor \end{aligned}\right\} \tag{30}$$

Combining together eqs. (27) and (29), the following matrix equations can be derived:

$$\begin{bmatrix} 0 & E_d \\ E_d^T & -SD \end{bmatrix} \begin{Bmatrix} d \\ \varepsilon \end{Bmatrix} = \begin{Bmatrix} -\bar{Q}_d \\ G^T q - \bar{Q}_\varepsilon \end{Bmatrix} \tag{31}$$

Let's define the inverse matrix of the left hand side of eq.(31) by the following equation:

$$\begin{bmatrix} 0 & \mathbf{E}_d \\ \mathbf{E}_d^T & -\mathrm{S\,D} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{\Phi}_{dd} & \boldsymbol{\Phi}_{d\varepsilon} \\ \boldsymbol{\Phi}_{\varepsilon d} & \boldsymbol{\Phi}_{\varepsilon\varepsilon} \end{bmatrix} \tag{32}$$

Then nodal parameters ($\mathbf{d}$, $\boldsymbol{\varepsilon}$) can be expressed as follows:

$$\left.\begin{aligned} \mathbf{d} &= -\boldsymbol{\Phi}_{dd}\bar{\mathbf{Q}}_d - \boldsymbol{\Phi}_{d\varepsilon}\bar{\mathbf{Q}}_\varepsilon + \boldsymbol{\Phi}_{d\varepsilon}\mathbf{G}^T\mathbf{q} \\ \boldsymbol{\varepsilon} &= -\boldsymbol{\Phi}_{\varepsilon d}\bar{\mathbf{Q}}_d - \boldsymbol{\Phi}_{\varepsilon\varepsilon}\bar{\mathbf{Q}}_\varepsilon - \boldsymbol{\Phi}_{\varepsilon\varepsilon}\mathbf{G}^T\mathbf{q} \end{aligned}\right\} \tag{33}$$

Using eq. (33), element nodal parmenters ($\mathbf{d}, \boldsymbol{\varepsilon}$) can be eliminated from the functional Π_{pH} given by eq.(25).
After some calculations, Π_{pH} can be given in the following form:

$$\Pi_{pH}(\mathbf{d}_b, \boldsymbol{\varepsilon}_b) = -\sum_n \left(\frac{1}{2}\mathbf{q}^T\mathbf{k}\mathbf{q} - \bar{\mathbf{Q}}\,\mathbf{q} + \mathbf{C}_n\right)$$

Needless to say, minimization of $\Pi_{pH}(\mathbf{q})$ with respect to $\mathbf{q}$ will yield the standard equilibrium equation of a given structure in the finite element method.
Summarizing the method proposed, unknown stresses or strains in the elements can be obtained by using the principle of the minimum potential energy under a given boundary displacement $\mathbf{d}_b$ on the element interfaces.
More precisely the element stiffness matrix can be expressed in terms of boundary displacements $\mathbf{d}_b$ and strain components $\boldsymbol{\varepsilon}_b$ as shown in Fig. 7.
It may be the most reasonable approach to derive a new discrete model in which the boundary interface can be regarded as a slip line when the corresponding boundary element is plastically yielded.
There are several variations of this element model, some of which are given by Fig. 8.
As a matter of fact, the following conclusions can be drawn from careful comparative study of the RBSM and Model II:

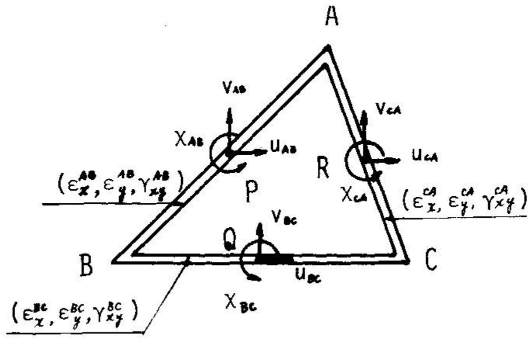


Fig. 7 A New Discrete Element with Boundary Displacements $\mathbf{d}_b$ and Strain Components $\boldsymbol{\varepsilon}_b$

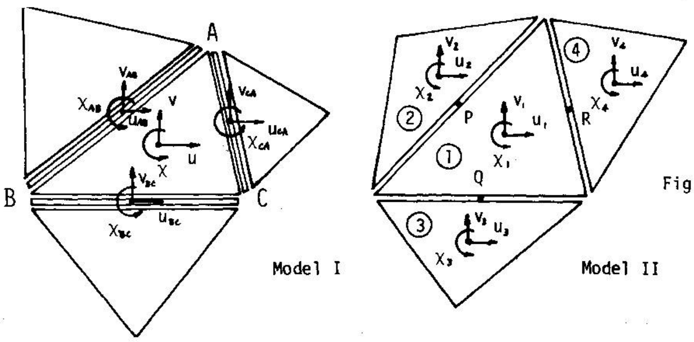


Fig. 8 Two Possible Variations of a New Discrete Element

(i) By two types of spring system whose intensities k_d and k_s , material properties of isotropic solids can be completely represented.

(ii) Convergency of elastic solutions is often considerably influenced by the mesh division. This is a serious disadvantage of the original rigid body-spring element.
Poor convergency of this element may be attributed to lacking of some cross coupling terms among elements 2, 3 and 4 in the stiffness matrix (See Fig. 8).

(iii) It can be expected that the Model II of Fig. 8 might give appropriate base for convergency study of the RBSM.

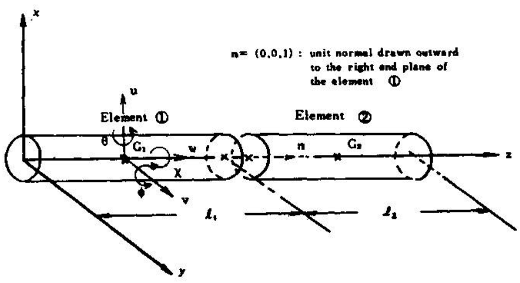
2.3 Stiffness Matrices of Beam and Plate Elements

A series of element matrices are now under development for practical application of the present method. In any element total number of degrees of freedom never exceeds 6 because it is assumed to be rigid.
In case of a beam element, deformation consists of axial, bending (about two principal axes) and torsional deformation, and in bending problem effect of shear deformation can be easily taken into account.
In case of plate and shell problems, membrane stiffness as well as bending stiffness can be defined by this (6 x 6) stiffness matrix.
Consideration of the shear deformation can be also made.
In what follows stiffness matrices of a straight beam element of constant cross section and a flat triangular plate element will be given.

GENERAL STIFFNESS MATRIX $[k_{ij}^{12}]$ OF THREE DIMENSIONAL RIGID BODIES-SPRING ELEMENT

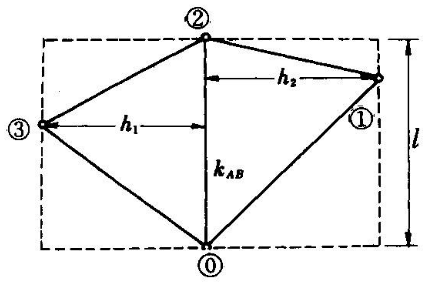
	u_1	v_1	w_1	θ_1	ϕ_1	χ_1	u_2	v_2	w_2	θ_2	ϕ_2	χ_2
X_1	K_1	0	0	0	$\frac{l_1}{2}K_1$	$-K_4$	$-K_1$	0	0	0	$\frac{l_2}{2}K_1$	K_4
Y_1		K_1	0	$-\frac{l_1}{2}K_1$	0	K_3	0	$-K_1$	0	$-\frac{l_2}{2}K_1$	0	$-K_3$
Z_1			K_2	K_4	$-K_6$	0	0	0	$-K_2$	$-K_7$	K_6	0
L_1				$K_{16}+\frac{l_1^2}{4}K_1$	$-K_{18}$	$-\frac{l_1}{2}K_3$	0	$\frac{l_1}{2}K_1$	$-K_7$	$-K_{16}+\frac{l_1l_2}{4}K_1$	K_{18}	$\frac{l_1}{2}K_3$
M_1	S.Y.M				$K_{15}+\frac{l_1^2}{4}K_1$	$-\frac{l_1}{2}K_4$	$-\frac{l_1}{2}K_1$	0	K_6	K_{18}	$-K_{15}+\frac{l_1l_2}{4}K_1$	$\frac{l_1}{2}K_4$
N_1						K_9+K_{10}	K_4	$-K_3$	0	$-\frac{l_2}{2}K_3$	$-\frac{l_2}{2}K_4$	$-K_9-K_{10}$
X_2							K_1	0	0	0	$-\frac{l_2}{2}K_1$	$-K_4$
Y_2								K_1	0	$\frac{l_2}{2}K_1$	0	K_3
Z_2									K_2	K_7	$-K_6$	0
L_2										$K_{16}+\frac{l_2^2}{4}K_1$	$-K_{18}$	$\frac{l_2}{2}K_3$
M_2											$K_{15}+\frac{l_2^2}{4}K_1$	$\frac{l_2}{2}K_4$
N_2												K_9+K_{10}

$K_1=\int k_s ds, K_2=\int k_d ds$
$K_3=\int k_s x ds, K_4=\int k_s y ds$
$K_6=\int k_d x ds,\ K_7=\int k_d y ds$
$K_9=\int k_s x^2 ds, K_{10}=\int k_s y^2 ds$
$K_{15}=\int k_d x^2 ds, K_{16}=\int k_d y^2 ds, K_{18}=\int k_d xy ds$



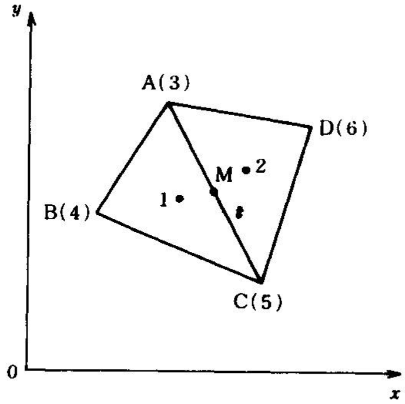
Stiffness matrix of a new plate bending element $\left(\times \frac{k_{AB}}{\Delta_{10}\Delta_{20}}\right)$

	W_0	W_1	W_2	W_3
Z_0	$\frac{\Delta_{20}}{\Delta_{10}}(y_{12}^2+x_{12}^2)+\frac{\Delta_{10}}{\Delta_{20}}(y_{23}^2+x_{23}^2) - 2(y_{12}y_{23}+x_{12}x_{23})$	$\frac{\Delta_{20}}{\Delta_{10}}(y_{12}y_{20}+x_{12}x_{20}) - (y_{23}y_{20}+x_{23}x_{20})$	$\frac{\Delta_{20}}{\Delta_{10}}(y_{12}y_{01}+x_{12}x_{01}) + \frac{\Delta_{10}}{\Delta_{20}}(y_{23}y_{30}+x_{23}x_{30}) - (y_{12}y_{30}+y_{23}y_{01}+x_{12}x_{30}+x_{23}x_{01}$	$\frac{\Delta_{10}}{\Delta_{20}}(y_{23}y_{02}+x_{23}x_{02}) - (y_{12}y_{02}+x_{12}x_{02})$
Z_1		$\frac{\Delta_{20}}{\Delta_{10}}(y_{20}^2+x_{20}^2)$	$\frac{\Delta_{20}}{\Delta_{10}}(y_{20}y_{01}+x_{20}x_{01}) - (y_{20}y_{30}+x_{20}x_{30})$	$y_{20}^2+x_{20}^2$
Z_2			$\frac{\Delta_{20}}{\Delta_{10}}(y_{01}^2+x_{01}^2)+\frac{\Delta_{10}}{\Delta_{20}}(y_{30}^2+x_{30}^2)-2(y_{01}y_{30}+x_{01}x_{30})$	$\frac{\Delta_{10}}{\Delta_{20}}(y_{30}y_{02}+x_{30}x_{02}) - (y_{01}y_{02}+x_{01}x_{02})$
Z_3	SYM.			$\frac{\Delta_{10}}{\Delta_{20}}(y_{30}^2+x_{30}^2)$



	u_1	v_1	θ_1	u_2	v_2	θ_2
X_1	$k_d y_{53}^2+k_s x_{53}^2$					
Y_1	$-(k_d-k_s)x_{53}y_{53}$	$k_d x_{53}^2+k_s y_{53}^2$				
M_1	$k_d y_{53}\Delta_{11}-k_s x_{53}\Delta_{21}$	$-(k_d x_{53}\Delta_{11}+k_s y_{53}\Delta_{21})$	$k_d\Delta_{11}^2+k_s\Delta_{21}^2+k_r l_{53}^2$	SYM.		
X_2	$-(k_d y_{53}^2+k_s x_{53}^2)$	$(k_d-k_s)x_{53}y_{53}$	$-(k_d y_{53}\Delta_{11}-k_s x_{53}\Delta_{21})$	$k_d y_{53}^2+k_s x_{53}^2$		
Y_2	$(k_d-k_s)x_{53}y_{53}$	$-(k_d x_{53}^2+k_s y_{53}^2)$	$k_d x_{53}\Delta_{11}+k_s y_{53}\Delta_{21}$	$-(k_d-k_s)x_{53}y_{53}$	$k_d x_{53}^2+k_s y_{53}^2$	
M_2	$k_d y_{53}\Delta_{22}-k_s x_{53}\Delta_{12}$	$-(k_d x_{53}\Delta_{22}\ k_s y_{53}\Delta_{12})$	$k_d\Delta_{11}\Delta_{22}+k_s\Delta_{21}\Delta_{12}\ k_r l_{53}^2$	$-(k_d y_{53}\Delta_{22}-k_s x_{53}\Delta_{12})$	$k_d x_{53}\Delta_{22}+k_s y_{53}\Delta_{12}$	$k_d\Delta_{22}^2+k_s\Delta_{12}^2+k_r l_{53}^2$

$$\left.\begin{aligned}2\Delta_{11} &= x_{53}(x_{31}+x_{51}) + y_{53}(y_{31}+y_{51})\\ 2\Delta_{12} &= x_{53}(y_{32}+y_{52}) - y_{53}(x_{32}+x_{52})\\ 2\Delta_{21} &= -x_{53}(y_{31}+y_{51}) + y_{53}(x_{31}+x_{51})\\ 2\Delta_{22} &= -x_{53}(x_{32}+x_{52})\quad y_{53}(y_{32}+y_{52})\end{aligned}\right\}$$



3. THEORETICAL BASIS OF NONLINEAR ANALYSIS [7],[8],[9],[13]

In general nonlinear structural problems are coupled problems of large deformation, inelasticity and crack, and they may be solved by using the incremental procedure. In what follows, essentials of solution procedure of nonlinear structural problems will be given.

3.1 Geometrical nonlinear problem

In case of finite displacement, assumption of the infinitesimal angular displacement is no longer valid and eq.(1) should be replaced by the following equation:

$$\mathbf{u}' = \mathbf{u}_G + (\mathbf{T} - \mathbf{I})(\mathbf{r} - \mathbf{r}_G) \tag{35}$$

$\mathbf{T}$ is a coordinate transformation matrix of local coordinates attached to the centroid between before and after deformation as follows:

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{36}$$

or

$$\mathbf{r}' = \mathbf{T}\,\mathbf{r}$$

and $\mathbf{I}$ is an unit matrix.
An unit normal $\mathbf{n}$ drawn outward at P of the element (I) before deformation may be subjected to the similar transformation as follows:

$$\mathbf{n}' = \mathbf{T}\,\mathbf{n} \tag{37}$$

where

$$\mathbf{n} = (l, m, n), \quad \mathbf{n}' = (l', m', n')$$

Consequently components (δ_d, δ_s) of the relative displacement $\overrightarrow{P'P''}$ will be given by the following equation:

$$\left.\begin{aligned} \delta_d &= (\overrightarrow{P'P''}, \mathbf{n}') = l'(u_{II} - u_I) + m'(v_{II} - v_I) + n'(w_{II} - w_I) \\ \delta_s^2 &= (\overrightarrow{P'P''} \times \mathbf{n}')^2 = \{m'(w_{II} - w_I) - n'(v_{II} - v_I)\}^2 \\ &\quad + \{n'(u_{II} - u_I) - l'(w_{II} - w_I)\}^2 + \{l'(v_{II} - v_I) - m'(u_{II} - u_I)\}^2 \end{aligned}\right\} \tag{38}$$

Knowing the strain energy V, and applying the principle of virtual work statical equilibrium equation can be derived where effect of finite rotation of elements is considered. From this equation the following standard incremental form of stiffness equation can be derived after some calculation.

$$(\mathbf{K} + \mathbf{K}_0 + \mathbf{K}_G)\,\mathbf{d}^* = \bar{\mathbf{F}}^* - \mathbf{F}_r \tag{39}$$

where $\mathbf{K}_0$ is the initial strain matrix, $\mathbf{K}_G$ * the geometrical stiffness matrix, $\mathbf{d}^*$, $\bar{\mathbf{F}}^*$ are increments of the displacement and exteral loads respectively, and $\bar{\mathbf{F}}_r$ is an unbalance force due to manipulation error in previous stage of loading. Detail of the derivation is given in the previous papers of the author.

3.2 Material Nonlinearity Problems

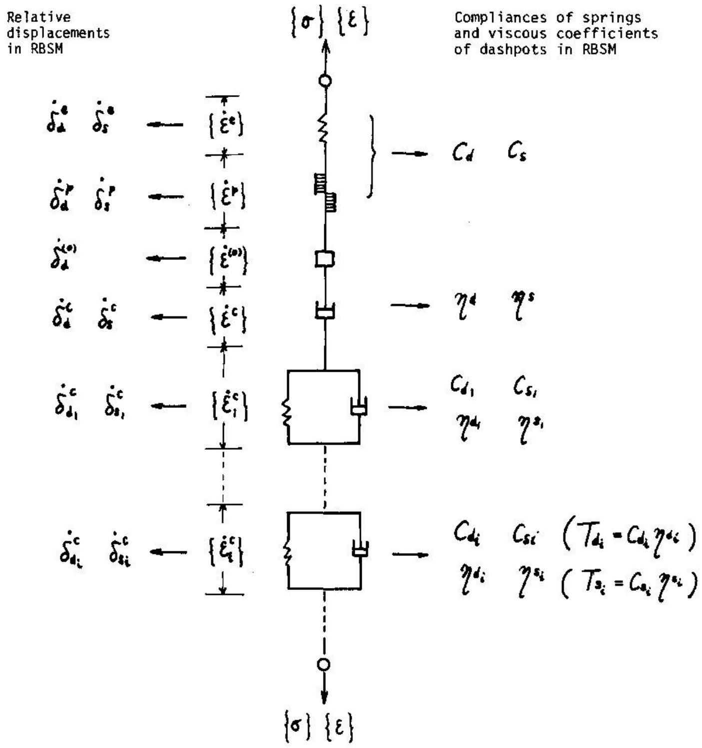
For simplicity, displacement of a given body is assumed to be infinitesimal, and therefore the problem will be reduced to integration of the following stiffness equation based on the well-established incremental procedure.

$$K\,d^* = \bar{F}^* - \bar{F} \tag{40}$$

For integration of eq.(40), yield or failure criterion of a given material should be introduced.
For this purpose the elastic strain energy density of the spring system V_0 is considered, and it is given by the following formula.

$$V_0 = \frac{1}{2}\left(k_n \delta_n^2 + k_s \delta_s^2 \right) = \frac{1}{2}\left(\frac{\sigma_n^2}{k_n} + \frac{\tau_{ns}^2}{k_s} \right) \tag{41}$$

It can be concluded from eq.(41) that if the maximum strain energy criterion is adopted, the material may fail if $V_0 = \sigma_y{}^2/2E$. According to this theory it will be seen that yielding will occur if $\tau_{ns} = \sigma_y/\sqrt{2(1+\nu)}$, while brittle failure will initiate if $\sigma_n = \sqrt{(1+\nu)(1-2\nu)}\sigma_y/(1-\nu)$.
As alternative failure criteria the maximum shearing stress theory may be adopted for ductile materials, while the maximum stress theory can be considered for brittle materials.
To avoid unnecessary confusion in further development, it is assumed that material is ductile and ideal plastic.
Solution of eq.(40) based on this assumption will give generalized solution of limit analysis which is well established in framed structures.
A series of such solutions have been given in previous papers of the authors.
The present method of analysis on the material nonlinear problem can be generalized by replacing the spring system connecting rigid elements by the spring-dashpot system as shown in the Fig. 9
Using such rigid bodies-spring-dashpot system, static and dynamic analysis of viscoelastic-plastic problem under thermal loading may be possible.



$\{\varepsilon^e\}$: elastic strain $\{\varepsilon^p\}$: plastic strain $\{\varepsilon^{(o)}\}$: initial strain (thermal strain)
$\{\varepsilon^c\}$: stationary creep strain $\{\varepsilon_i^c\}$: transient creep strain

Fig. 9 Mechanical Model of Viscoelastic-Plastic Materials

3.3 Consideration of Crack Initiation and Growth in the Present Anaysis

In analysis of the material nonlinear problems described in the last paragraph yield criterion is applied pointwisely on the contact boundary surface. Therefore in component calculation of the stiffness matrix for each contact surface, appropriate scheme of numerical integration should be adopted.
More precisely, for example, k_{11} of the general 3D stiffness matrix is given by

$$k_{11} = \iint \left\{ k_n \ell^2 + k_s (1-\ell^2) \right\} dS$$

And therefore if the boundary surface is curved or k_n , k_s depend on stresses and strain, calculation of k_{11} should be made, for example, by using Gauss' integration scheme, Using such integration scheme it is possible to pursue gradual development of plastic hinge lines, slip lines or slip surfaces on the contact boundary, and the ultimate load can be calculated. In real structures, however, it is usual to consider that initiation and propagation of crack may reduce substantially the ultimate load. At the present moment, the criterion for crack initiation and propagation is not well established and therefore the

following simple criterion is adopted for the time being.
Crack intiation and propagation may take place when the shearing strain γ exceeds γ_b which may be equivalent to the concept of *COD*. It is not difficult to incorporate this criterion with the yield criterion in analysis of material nonlinear problems. As a matter of fact, crack analysis of two dimensional notched plates were conducted by the present authors and reasonable results were obtained. Effect of large scale yielding, however, was not considered in this analysis and therefore more refined analysis will be planned in near future by taking into account of such an effect.

4. SOME EXAMPLES OF COLLAPSE LOAD ANALYSIS [6],[10],[11],[13]

To show validity of the present new elements, a series of numerical anaysis has been conducted and most of the results obtained were reported in the conference proceedings or engineering journals.
Therefore only some new results will be shown here without explanation.

(I)
- (a) collapse analysis of square concrete slabs.
- (b) two dimensional punch problem
- (c) three dimensional elasto-plastic anaysis of a through crack problem.

(II) (d) shake down analysis of a simply supported square plate subjected to variable transverse loads.

(III) (e) collapse analysis of cylindrical shell roofs simply-supported on four edges and subjected to external radial pressure.

(IV) (f) dynamic collapse of automobile front structures.

5. CONCLUSION

Outline is briefly explained on a new discrete method of analysis which has been proposed by the present author. This method may be suitable for analysis of highly nonlinear problems where plasticity, large deformation and crack growth are coupled. Therefore broad application may be expected in future to analysis and design of the reinforced concrete structures where punching shear crack growth, creep etc. are important design prarameters. The followings are conclusion so far obtained from a series of numerical analysis.

(i) Stiffness of a given body is lumped on the contact sufaces of neighbouring rigid elements and yielding or failure is assumed to occur only on these contact surfaces. Consequently the analysis of material nonlinear problems becomes much simpler than that of conventional finite element method.

(ii) Concept of node superposition in the conventional finite element analysis is completely discarded in the present analysis and *slip* due to plastic deformation or frictional force on the contact surface can be easily represented in this method.

(iii) Since the lower order shape function is employed for element stiffness formulation, computing time for stiffness calculation will be considerably reduced to compare with the conventional finite element method.

(iv) Variational formularion of the present method is now under way.
It is expected in near future to give rational basis for this discrete analysis.

(v) Although it can be concluded by a series of test analyses that the present method may be very powerful for the collapse load analysis, accumlation of results of numerical analysis of more realistic structures should be necessary for verification of the method.

ACKNOWLEDGEMENTS

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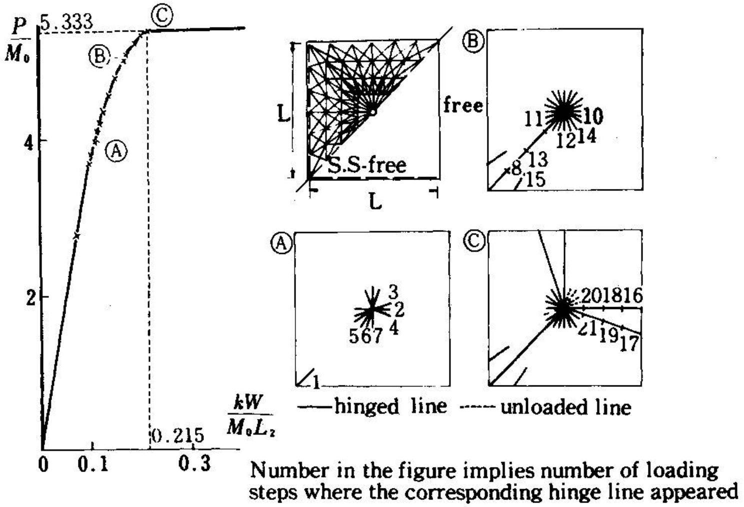


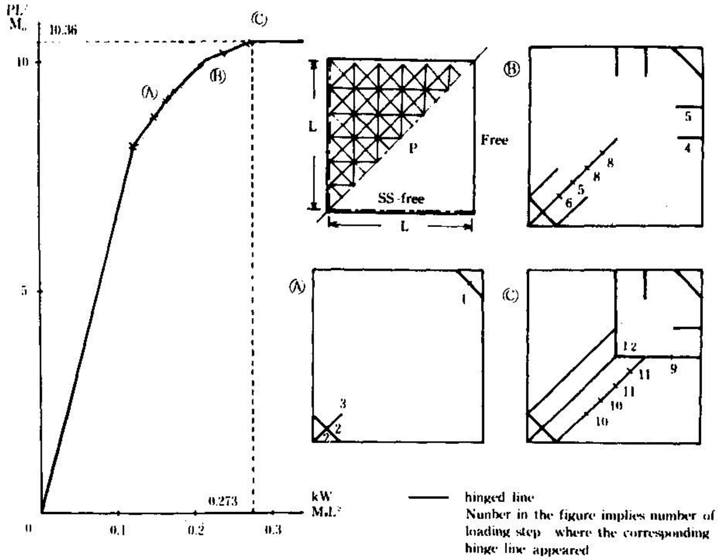
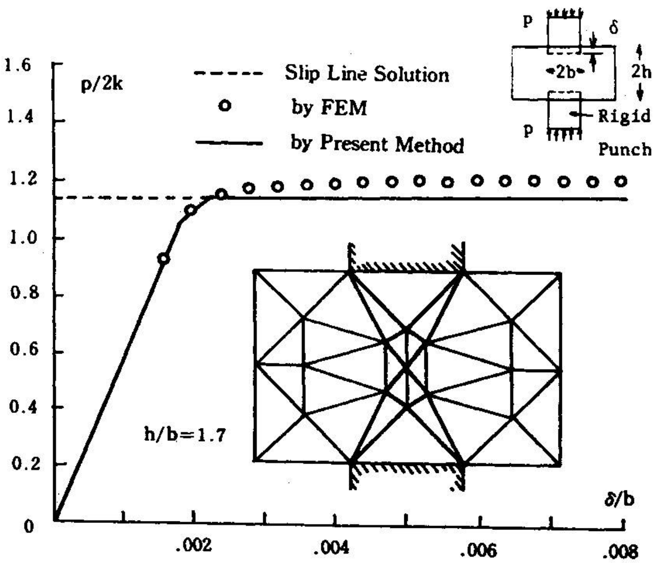
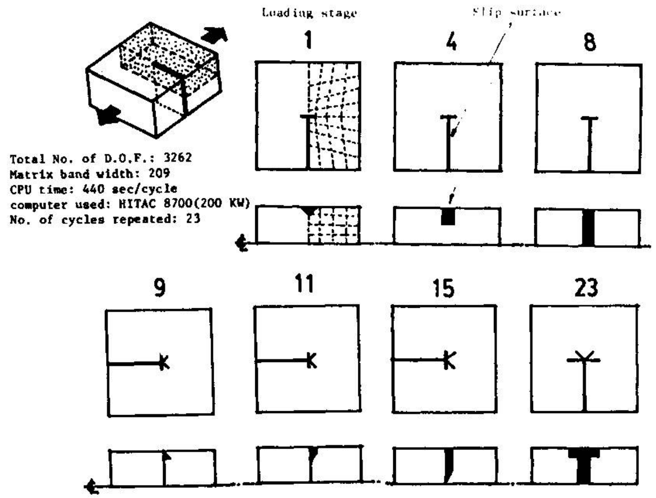


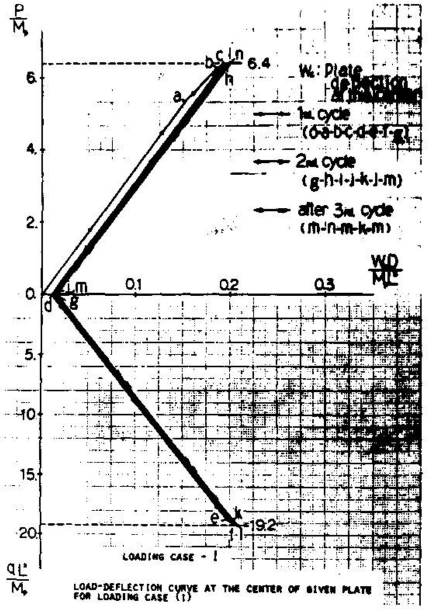
PLATE BENDING COLLAPSE



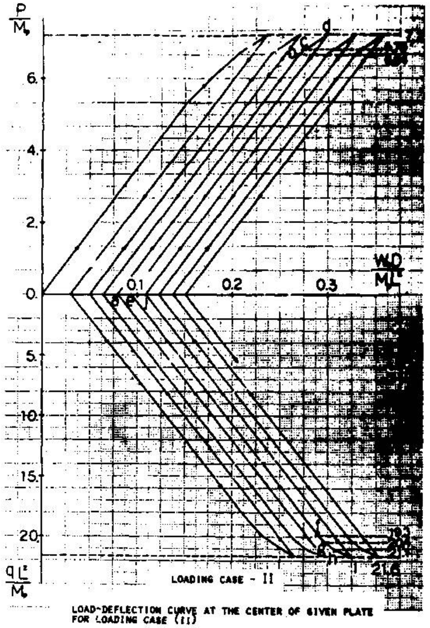
20 PUNCH PROBLEM



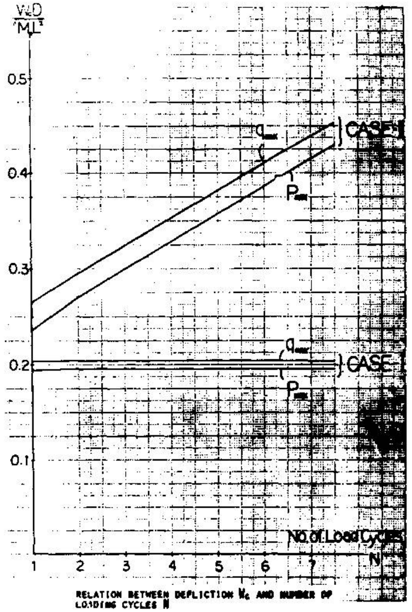
3D CRACK ANALYSIS



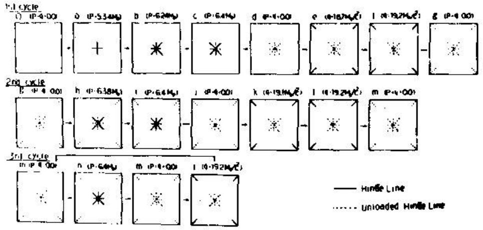
LOAD-DEFLECTION CURVE AT THE CENTER OF GIVEN PLATE FOR LOADING CASE (I)



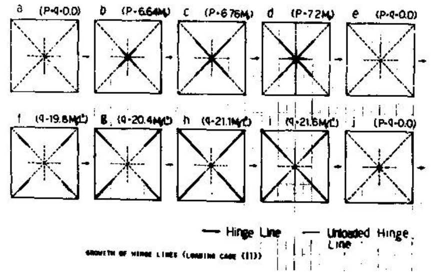
LOAD-DEFLECTION CURVE AT THE CENTER OF GIVEN PLATE FOR LOADING CASE (II)



RELATION BETWEEN DEFLECTION W_c AND NUMBER OF LOADING CYCLES N

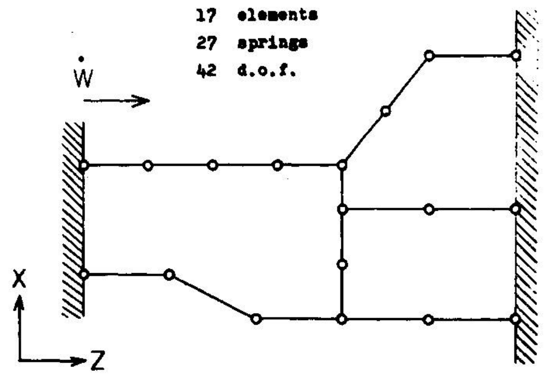


GROWTH OF HINGE LINES (loading case (I))

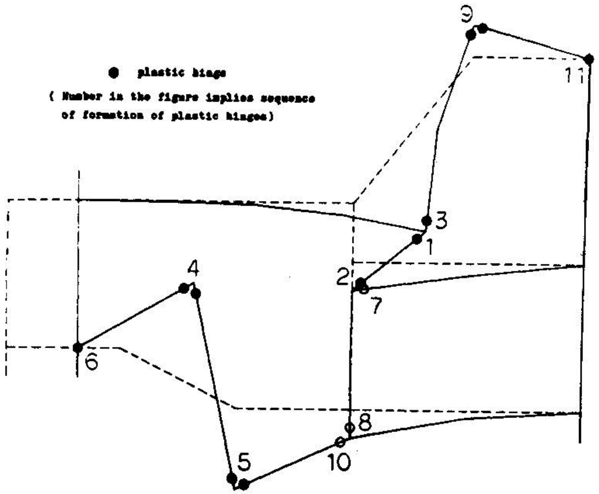


GROWTH OF HINGE LINES (LOADING CASE (II))

SHAKE DOWN ANALYSIS
OF A SIMPLY SUPPORTED
SQUARE PLATE

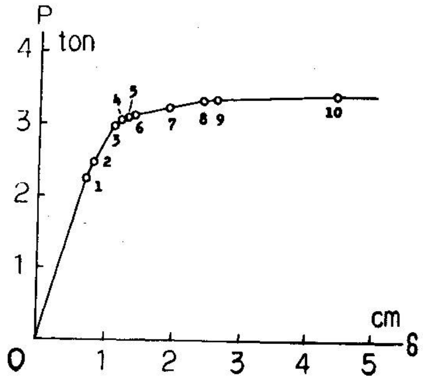


FINITE ELEMENT MODEL OF AUTOMOBILE BODY STRUCTURE



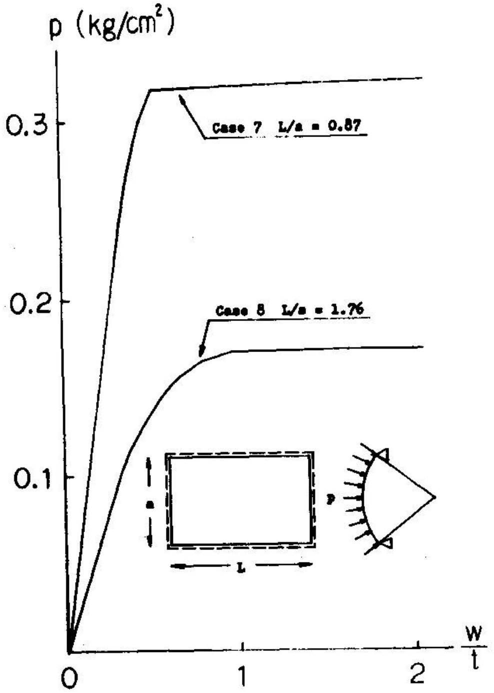
(a) final deflected shape

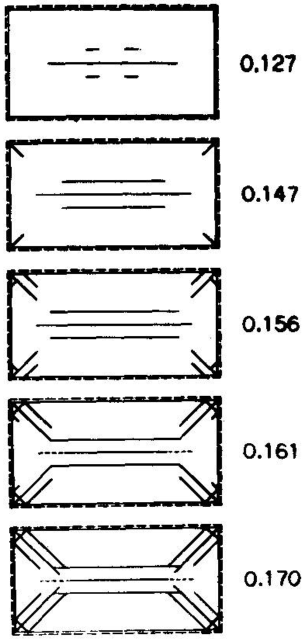
DYNAMIC COLLAPSE OF AUTOMOBILE BODY STRUCTURE



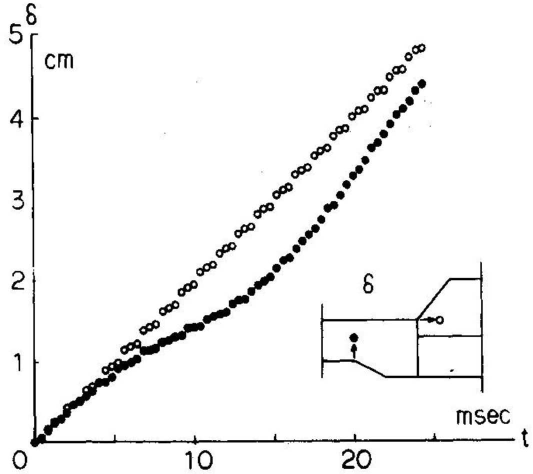
(b) load-deflection curve

STATIC COLLAPSE OF AUTOMOBILE BODY STRUCTURE
(Number in the figure implies number of loading steps where the corresponding hinge appeared.)





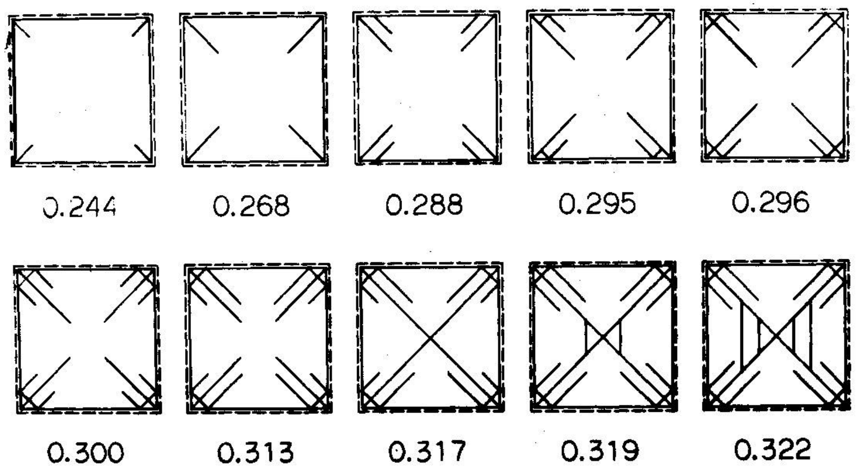
Case 8 L/a = 1.76



(a) time variation of deflection

DYNAMIC COLLAPSE OF AUTOMOBILE BODY STRUCTURE

CRASH ANALYSIS OF AN AUTOMOBILE FRONT BODY



Case 7 L/a = 0.87

COLLAPSE ANALYSIS OF A
SHELL ROOF STRUCTURE