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On the Finite Element Method in the Field of Plasticity

Sur la méthode des éléments finis en plasticité

Zur Methode der finiten Elemente auf dem Gebiet der Plastizität

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SUMMARY
The fundamental aspects of some broadly applicable finite element procedures for the analysis of structures assuming ideal elasto-plastic or rigid-plastic material behaviour are presented and shortly discussed.

RESUME
Les aspects fondamentaux de certains procédés très généraux basés sur la méthode des éléments finis pour l'analyse des structures avec un comportement élasto-plastique ou rigide-plastique du matériau sont présentés et brièvement discutés.

ZUSAMMENFASSUNG
Die Grundprinzipien einiger allgemein anwendbarer, auf die Methode der finiten Elemente sich stützender numerischer Verfahren zur Berechnung elasto-plastischer und starr-plastischer Tragwerke werden geschildert und kurz diskutiert.

1. INTRODUCTION

The main reason for the extraordinary success of the finite element method in structural engineering lies certainly in its very broad applicability to all kind of structure types, loading conditions and material properties. Of course, this is also true in the field of plasticity, where computer based finite element procedures represent powerfull tools for the numerical analysis of complex real life structures.

The aim of the present paper is to present a short state-of-the-art theoretical review of some general finite element procedures assuming ideal elasto-plastic or rigid-plastic material behaviour. In order to confine the discussion to few fundamental questions, no speciic structure type and no specific material will be considered here. It should be clear, however, that much research work in recent years has led to very many different approaches for taking into account plastic deformations, some of them being certainly more straightforward, if possibly less generally applicable, then those discussed here.

It is assumed that the reader is familiar with matrix notation and with the main principles of conventional finite element analysis.

2. FINITE ELEMENT MODELS

Finite element models are used to build parametric fields satisfying prescribed continuity conditions. Parametric fields for the components of the displacement vector $\{u\}$ defining the displacement state and for the stress vector $\{\sigma\}$ defining the stress state within a structure are given in matrix notation by

$$\{u\} = [\Phi]\{U\} , \tag{1}$$
$$\{\sigma\} = [\Psi]\{\Sigma\} , \tag{2}$$

where U- and Σ-components of the global vectors $\{U\}$ and $\{\Sigma\}$ are nodal displacements and stress parameters respectively. The coefficients of the matrices $[\Phi]$ and $[\Psi]$ are shape functions defined piecewise within each element by generally simple analytical functions and satisfying prescribed continuity conditions along the element interfaces.

It is typical of the finite element method to use for the virtual displacements the same assumptions as for the real ones, thus restricting the infinite class of virtual functions considered by conventional virtual work methods to those given by the assumed shape functions of the matrix $[\Phi]$. Denoting virtual quantities with an asterisk the virtual displacement field $\{u^*\}$ is given by

$$\{u^*\} = [\Phi]\{U^*\}, \tag{3}$$

where the U^*'s are virtual displacement parameters.

The strain state within the structure is defined by a strain vector $\{\varepsilon\}$ whose components are obtained from the displacement vector $\{u\}$ applying an operator Δ, i.e. using kinematical strain-displacement relations:

$$\{\varepsilon\} = \{\Delta u\} = [\Delta\Phi]\{U\}. \tag{4}$$

As small displacements shall be assumed, Δ is a linear operator, thus identical relations can be written for the virtual strains $\{\varepsilon^*\}$:

$$\{\varepsilon^*\} = \{\Delta u^*\} = [\Delta\Phi]\{U^*\}. \tag{5}$$

The main problem of finite element structural analysis is that of finding a feasible internal stress distribution $\{\sigma\}$ satisfying equilibrium with the prescribed external loads $\{p\}$. This is often achieved by applying the principle of virtual displacements which says that the internal stresses $\{\sigma\}$ are in equilibrium with the external loads $\{p\}$ when the internal and the external virtual works are equal for all possible values of the virtual displacements $\{u^*\}$:

$$\int_V \{\varepsilon^*\}^T\{\sigma\}\cdot dV = \int_V \{u^*\}^T\{p\}\cdot dV, \tag{6}$$

where $\{\varepsilon^*\}$ and $\{u^*\}$ are kinematically compatible, i.e. $\{\varepsilon^*\}$ is derived from $\{u^*\}$ according to Eq. (5). V is the total volume of the structure consisting of several finite elements. Using the parametric virtual displacement field $\{\varepsilon^*\} = [\Delta\Phi]\{U^*\}$, Eq. (6) leads to

$$\{R\} = \{P\}, \tag{7}$$

where the vector $\{R\}$ of the internal nodal reaction forces due to the stress state $\{\sigma\}$ and the global vector $\{P\}$ of the external nodal loads are defined as follows:

$$\{R\} = \int_V [\Delta\Phi]^T\{\sigma\}\cdot dV, \tag{8}$$

$$\{P\} = \int_V [\Phi]^T\{p\}\cdot dV . \tag{9}$$

Eq. (7) represents a set of generalized equilibrium equations between the internal nodal forces $\{R\}$ and the external nodal loads $\{P\}$ leading, in general, to an only approximate satisfaction of the microscopic equilibrium conditions. To solve the problem of finding $\{\sigma\}$, however, the material behaviour has to be taken into account.

3. IDEAL ELASTO-PLASTIC STRESS-STRAIN RELATIONS

If the stresses are sufficiently small the material is assumed to behave perfectly elastically. The stress-strain relations are then given by Hooke's law

$$\{\sigma\} = [D](\{\varepsilon\} - \{\varepsilon_o\}), \tag{10}$$

where $[D]$ is the material dependent, symmetric and positive-definite "elasticity" matrix. The ε_o's are initial strains (e.g. due to temperature change) which are not directly associated with stresses. For simplicity initial strains shall not be considered here.

Eq. (10) is assumed to be valid only if the following yield conditions are satisfied:

$$f_k(\{\sigma\}) < c_k \qquad (k = 1 \text{ to } K), \tag{11}$$

where the f_k's are generally non-linear functions of the stress components. The c_k's are positive material constants. In the stress space the equations

$$f_k(\{\sigma\}) = c_k \qquad (k = 1 \text{ to } K) \tag{12}$$

piecewise define the yield surface of the material (see Fig. 1). This can, of course, in some cases be defined by a single non-linear function (K = 1). In order to take into account strain hardening or softening effects the c_k's are sometimes assumed to be functions of stress-strain history. For simplicity this shall not be considered here, i.e. ideal elasto-plastic material behaviour with constant c_k's shall be assumed.

If the stresses increase so much as to reach one of the surfaces (Eq. 12) delimiting the yield surface the relations between $\{\sigma\}$ and $\{\varepsilon\}$ change, and in fact it is only possible to give tangential relations between stress increments $d\{\sigma\}$ and strain increments $d\{\varepsilon\}$, the total stresses $\{\sigma\}$ being path dependent functions of the total strains $\{\varepsilon\}$ (non-conservative material behaviour).

It is then convenient to think of the strain increment $d\{\varepsilon\}$ as a sum of an "elastic" increment $d\{\varepsilon_{el}\}$ and a "plastic" increment $d\{\varepsilon_{pl}\}$:

$$d\{\varepsilon\} = d\{\varepsilon_{el}\} + d\{\varepsilon_{pl}\}\ , \tag{13}$$

where $d\{\varepsilon_{el}\}$ produces a stress increment $d\{\sigma\}$ according to Hooke's law, while $d\{\varepsilon_{pl}\}$ acts exactly as the initial strains $\{\varepsilon_o\}$ of Eq. (10), i.e. is not associated with any stress changes:

$$d\{\sigma\} = [D]d\{\varepsilon_{el}\} = [D](d\{\varepsilon\} - d\{\varepsilon_{pl}\}). \tag{14}$$

According to the theory of plasticity the plastic strain increment vector $d\{\varepsilon_{pl}\}$ has to be perpendicular to the yield surface, i.e. parallel to the gradient $\{\text{grad } f_k\}$ of the function $f_k(\{\sigma\})$ for $\{\sigma\}$ given by $f_k(\{\sigma\}) = c_k$, and pointed towards the outside of the allowable stress domain (see Fig. 1):

$$d\{\varepsilon_{pl}\} = \{\text{grad } f_k\}\cdot d\alpha\ , \tag{15}$$

where $d\alpha$ is an arbitrary non-negative constant which can be determined by requiring the stress increment $d\{\sigma\}$ to satisfy the k'th yield condition exactly, i.e. to be parallel to the yield surface:

$$\{\text{grad } f_k\}^T d\{\sigma\} = 0. \tag{16}$$

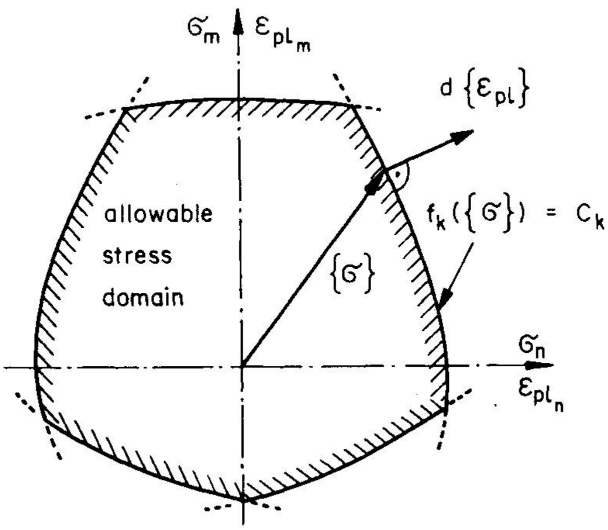

Fig. 1: Yield Surface defined by a set of non-linear yield conditions

From Eqs. (14), (15) and (16) simple algebra leads to a tangential relation between $d\{\sigma\}$ and $d\{\varepsilon\}$ similar to Hooke's law:

$$d\{\sigma\} = [D_T]d\{\varepsilon\}, \tag{17}$$

where $[D_T]$ is a symmetric, positive-semidefinite "tangential" matrix satisfying

$$[D_T]d\{\varepsilon_{pl}\} = [D_T]\{\text{grad } f_k\} = 0. \tag{18}$$

Eq. (17), however, is not really valid for any $d\{\varepsilon\}$. If unloading takes place, i.e. if $d\{\varepsilon\}$ is such that a purely elastic stress increment $d\{\sigma\} = [D]d\{\varepsilon\}$ would point towards the inside of the allowable stress domain:

$$\{\text{grad } f_k\}^T [D]d\{\varepsilon\} < 0, \tag{19}$$

then the material is assumed to behave elastically again ($[D_T] = [D]$).

If, after the stress vector $\{\sigma\}$ has reached the yield surface satisfying the single k'th yield condition exactly, the strains are increased any further, other yield conditions might become satisfied exactly, the stress vector $\{\sigma\}$ reaching "edges" or "corners" of the yield surface. The procedure explained above has then to be generalized for taking into account simultaneously more than one of the conditions (15), (16) and (19).

Of course, all this quite complicates elasto-plastic analysis and it is certainly an advantage if the material behaviour can be described by just a few non-linear yield conditions, possibly by a single one.

4. ELASTO-PLASTIC INCREMENTAL ANALYSIS

In finite element elasto-plastic analysis the primary unknown of the problem is generally chosen to be the displacement state of the structure described by the parametric field of Eq. (1).

As long as the material behaves elastically ($\{\sigma\} = [D]\{\varepsilon\}$) the internal nodal reaction forces $\{R\}$ of the structure can be expressed as linear function of the unknown nodal displacement parameters $\{U\}$:

$$\{R\} = \int_V [\Delta\Phi]^T \{\sigma\} \cdot dV = [K]\{U\} \ , \tag{20}$$

the global linear elastic stiffness matrix $[K]$ being defined by

$$[K] = \int_V [\Delta\Phi]^T [D][\Delta\Phi] \cdot dV. \tag{21}$$

The U's are then found by solving the system of linear equilibrium equations

$$[K]\{U\} = \{P\}. \tag{22}$$

However, when, due to high stress levels in some parts of the structure plastic strains occur, the relations between $\{R\}$ and $\{U\}$ become non-linear. It is then necessary to increase the external loads $\{P\}$ in steps $\Delta\{P\}$ and to find a new stress distribution after each load increase satisfying equilibrium (7) while taking into account the elasto-plastic stress-strain relations discussed above. The most widely accepted iterative algorithm to do so can be described as follows:

a. Initialize $\{U\} := \{P\} := 0$

b. Increase $\{P\} := \{P\} + \Delta\{P\}$ and $\{U\} := \{U\} + \Delta\{U\}$, where $\Delta\{U\}$ is obtained from the solution of the following system of linear equations:

$$[\tilde{K}]\Delta\{U\} = \Delta\{P\}, \tag{23}$$

$[\tilde{K}]$ being a approximation of the stiffness matrix valid for the current load step as explained below.

c. Determine the internal nodal reactions $\{R\}$ according to Eq. (8) from the actual stress state, $\{\sigma\}$ obtained from the incremented strain state $\{\varepsilon\} = [\Delta\Phi]\{U\}$ corresponding to the new $\{U\}$.

d. If, within a prescribed tolerance $\{R\} = \{P\}$ repeat from b.

e. Otherwise apply the nodal loads $\{P\}-\{R\}$ representing unbalanced residual nodal forces obtained from the difference between the external loads $\{P\}$ and the internal reactions $\{R\}$.

A corresponding displacement increase $\Delta\{U\}$ is found by solving the system of linear equations:

$$[\tilde{K}]\Delta\{U\} = \{P\}-\{R\} \tag{24}$$

f. Increase $\{U\} := \{U\}+\Delta\{U\}$ and repeat from c.

As in most cases a limit load is to be found rather than the response of the structure to a prescribed load, the external loads $\{P\}$ have to be increased until an equilibrium stress state can not be found anymore or until the displacements in some parts of the structure grow beyond prescribed "collapse" limits.

Two main questions arise. The first one concerns the stiffness matrix $[\tilde{K}]$ of Eqs. (23) and (24), which, ideally, should describe the relation between $\{R\}$- and $\{U\}$-increments within a load step for the partially plastified structure. Often $[\tilde{K}]$ is approximated by the linear elastic stiffness matrix $[K]$, the method described here being then often called (somehow improperly) the "initial stress method". Fig. 2 shows its basic principle when applied to a single degree of freedom system for a single load step ($\Delta P = P$). Sometimes a better approximation for $[\tilde{K}]$ is used taking into account the changes in stiffness caused by the plastified zones of the structure.

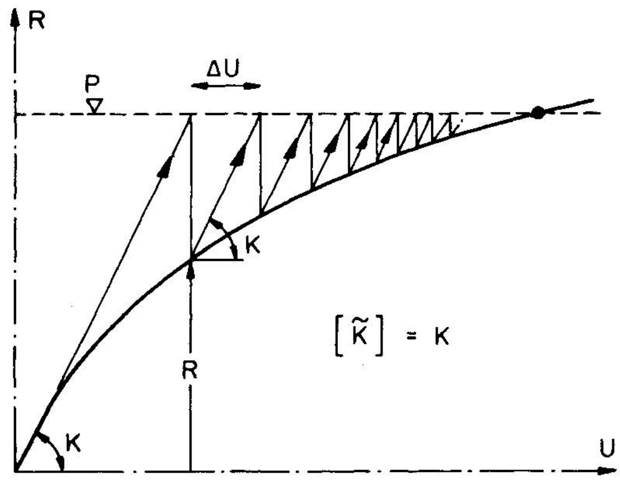

Fig. 2: Initial stress method for a single degree of freedom system

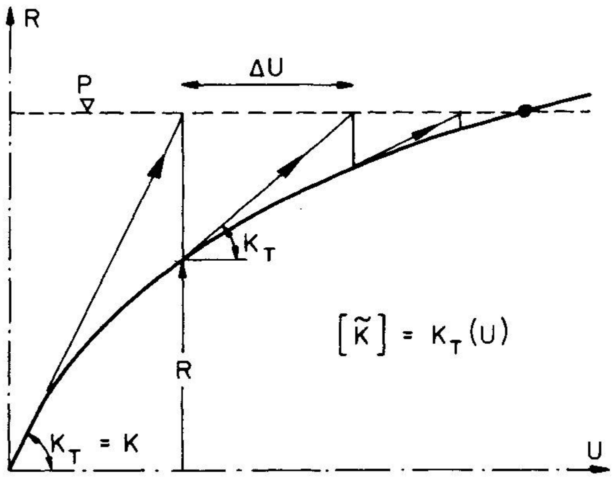

Fig. 3: Newton-Raphson method for a single degree of freedom system

A frequent choice for $[\tilde{K}]$ is the tangent stiffness matrix $[K_T]$ relating infinitesimal $d\{R\}$- and $d\{U\}$-increments at the beginning of a load step or during the iterations within a load step:

$$d\{R\} = [K_T]d\{U\}. \tag{25}$$

The $[K_T]$-matrix can be obtained, like $[K]$ in Eq. (21), from a sum of the contributions of each single element using instead of $[D]$ the elasto-plastic tangential $[D_T]$-matrix defined in Eq. (17):

$$[K_T] = \int_V [\Delta\Phi]^T [D_T][\Delta\Phi]\cdot dV. \tag{26}$$

If $[K_T]$ is used, the solution procedure described above corresponds to the so-called "Newton-Raphson-Method" for the solution of non-linear systems of coupled equations. Fig. 3 shows its basic principle. Obviously a much faster convergence is obtained when using $[K_T]$ instead of $[K]$, however, the computational effort needed at each step will increase very much. In fact not only the numerical evaluation of $[K_T]$ is time consuming, but also a totally new solution of the Eqs. (23) or (24) is needed each time $[K_T]$ is changed, which is not the case when using allways the same elastic stiffness matrix $[K]$. An obvious possibility would be to evaluate $[K_T]$ (or some more or less crude approximation of it) only from time to time, thus using the same $[K_T]$-matrix for several steps. It should be noted, however, that convergence (quite contrary to geometrically non-linear problems) can in many cases be obtained using allways the same linear elastic stiffness $[K]$.

A second question concerns the way the internal reactions $\{R\}$ or their increases $\Delta\{R\}$, which, of course, can also be obtained from a sum of element contributions, are evaluated from the stress increments $\Delta\{\sigma\}$ caused by the strain increments $\Delta\{\varepsilon\}$ associated with $\Delta\{U\}$. Obviously, $\Delta\{\varepsilon\}$ not being infinitesimal, the use of the incremental relations between $d\{\sigma\}$ and $d\{\varepsilon\}$ derived obove (Eq. (17)), will, in general, involve some approximations. Details should not be discussed here, it should be noted, however, that as long as a stress distribution can be found which satisfies equilibrium, i.e. leading to $\{R\} = \{P\}$ violations of the elasto-plastic incremental stress-strain relations are not too disturbing. In fact from the lower-bound theorem of the plasticity theory one knows that the stress distribution obtained can only underestimate the limit load, thus leading to a safe design.

5. RIGID-PLASTIC ANALYSIS

If rigid-plastic material behaviour is assumed the statical (or lower-bound) and the kinematical (or upper-bound) theorems of the theory of plasticity represent powerful tools for the evaluation of a limit load factor λ multiplying given external loads $\{p\}$ and possibly of the shape of the collapse mechanism.

According to the statical theorem a stress state $\{\sigma\}$ has to be found which satisfies equilibrium with the external loads $\lambda\{p\}$ as well as the yield conditions everywhere within the structure. The limit load is then found by maximizing λ.

By introducing a finite element parametric stress field (Eq. (2)) the internal reactions $\{R\}$, which have to equal $\lambda\{P\}$ in order to satisfy equilibrium (7), can be evaluated as linear functions of the unknown nodal stress parameters $\{\Sigma\}$:

$$\{R\} = \int_V [\Delta\Phi]^T\{\sigma\}\cdot dV = [E]\{\Sigma\} = \lambda\{P\}, \tag{26}$$

where $[E]$ is a global "equilibrium"-matrix obtained, as usual, by a sum of element contributions and definded by

$$[E] = \int_V [\Delta\Phi]^T[\Psi]\cdot dV. \tag{27}$$

The stress parameters $\{\Sigma\}$ will also have to satisfy yield conditions. These will have to be checked in Q discrete "checkpoints" throughout the structure, where the stress components assume the values $\{\sigma_q\} = [\Psi_q]\{\Sigma\}$ $(q = 1 \text{ to } Q)$. Allthough the use of the non-linear yield conditions (Eq. (11)) is possible, it is certainly convenient in rigid-plastic analysis to use linear ones, thus introducing polyedrical yield surfaces (see Fig. 4), even if a larger number of inequalities may become necessary. Linear yield conditions are given by:

$$\{f_k\}^T\{\sigma_q\} \leq c_{kq} \qquad (k = 1 \text{ to } K;\ q = 1 \text{ to } Q) \tag{28}$$

or, for all conditions together at a checkpoint q

$$[f]^T\{\sigma_q\} \leq \{c_q\} \qquad (q = 1 \text{ to } Q), \tag{29}$$

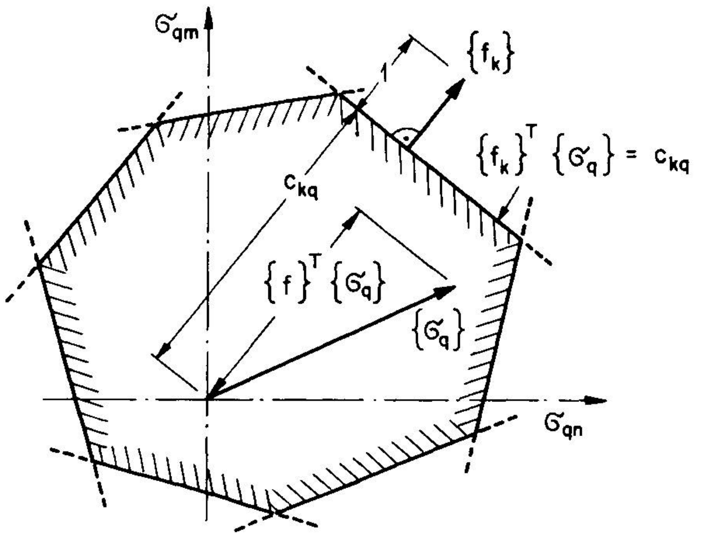

Fig. 4: Yield surface defined by a set of linear yield conditions

where c_{kq} represents the resistance of the structure at a checkpoint q for a stress direction $\{f_k\}$.

From the optimality condition $\lambda \rightarrow$ maximum, from the equilibrium equations (26) and from the linearized yield conditions (29) the following linear program for the unknowns λ and $\{\Sigma\}$ is found (see also Fig. 5):

$$\lambda \rightarrow \text{maximum}$$

$$\{\sigma\} = -\{P\}\lambda + [E]\{\Sigma\} \tag{30}$$

$$0 \leq c_{kq} - [f]^T[\Psi_q]\{\Sigma\} \quad (q = 1 \text{ to } Q).$$

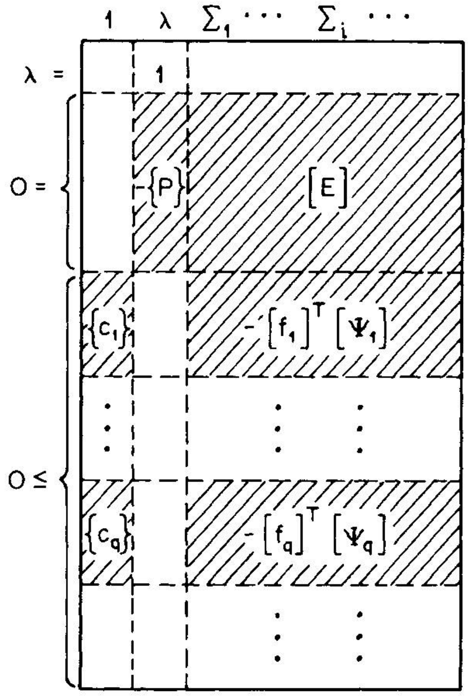

Fig. 5: Tableau of the linear program (30): $\lambda \rightarrow$ maximum

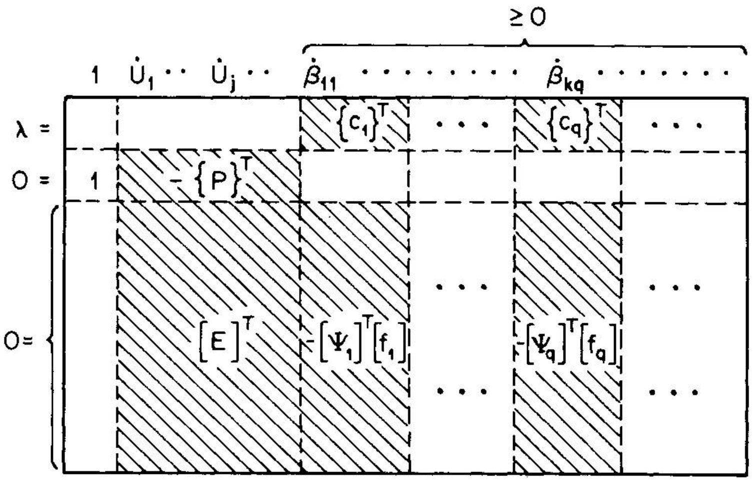

Fig. 6: Tableau of the linear program (31): $\lambda \rightarrow$ minimum

From the kinematical (or upper-bound) theorem the following linear program, whose derivation shall not be given here is found (see also Fig. 6):

$$\begin{aligned} \lambda &= \sum_q \{c_q\}^T\{\dot{\beta}_q\} \rightarrow \text{minimum}, \\ 0 &= 1 - \{P\}^T\{\dot{U}\}, \\ \{\sigma\} &= [E]^T\{\dot{U}\} - \sum_q [\Psi_q]^T[f]\{\dot{\beta}_q\} \\ \{\dot{\beta}_q\} &\geq \{\sigma\} \qquad (q = 1 \text{ to } Q), \end{aligned} \tag{31}$$

where the $\dot{U}$'s are nodal displacement velocity parameters and the $\dot{\beta}_{kq}$'s are generalized strain velocity parameters somehow related to the $d\alpha$'s introduced in Eq. (15) (see Ref. [4]).

The linear programs (30) and (31) are "dual" to each other. The same load factor λ will therefore be obtained. As expected the value of λ only depends on the mathematical model, not on the method of solution used (statical or kinematical approach). A lower bound of the true value of λ will be obtained if the assumed Ψ-functions and the linear inequalities (29) guarantee that microscopic equilibrium conditions and yield conditions are nowhere violated. An upper-bound (at least for the linearized yield condition used) will be obtained if kinematical compatibility conditions are satisfied exactly. In many cases, however, a bound for λ will not be found, but just an approximation of it.

By solving one of the linear programs (30) or (31) the solution of the other one is also known. Numerical values not only for λ but also for the Σ-, $\dot{U}$- and $\dot{\beta}$-parameters are therefore obtained. The displacement velocity parameters $\{\dot{U}\}$ describe the collapse mechanism. The stress parameters $\{\Sigma\}$ define a corresponding state of admissible stresses. However, because this is defined in an unique way, only in the regions and in the directions in which plastic flow occurs, the values of the Σ-parameters will generally not be very meaningful as large portions of the structure may remain rigid during collapse. The $\dot{\beta}$-parameters provide informations on the distribution of plastic flow during collapse.

The procedure described here, while being at least in principle generally applicable, has the disadvantage of being a two-field procedure as independent parametric assumptions both for the stresses and for the displacements have to be introduced. In fact the criteria for chosing these parametric fields are not always clear. Moreover, it would certainly be an advantage not to have any equilibrium equations in the linear program (30), which would be the case if stress assumptions satisfying a priori equilibrium conditions could be found.

This is in some cases possible if parametric finite element fields for stress functions (like Airy's for plate stretching problems) are introduced. The stress components building the vector $\{\sigma\}$ are then derived, generally by differentation, from the stress functions leading to:

$$\{\sigma\} = [\Psi]\{\Sigma\} + \lambda\{\bar{\sigma}\}, \qquad (32)$$

where the columns of the $[\Psi]$-matrix corresponding to the nodal stress function parameters $\{\Sigma\}$ represent homogeneous stress states while $\lambda\{\bar{\sigma}\}$ represents an inhomogeneous stress state satisfying equilibrium with the external loads. From the statical theorem the following linear program is then obtained:

$\lambda \rightarrow$ maximum,

$$\{\sigma\} \leq \{c_q\} - [f]^T\{\bar{\sigma}_q\}\lambda - [f]^T[\Psi_q]\{\Sigma\}, \qquad (q = 1 \text{ to } Q). \qquad (33)$$

Of course from the kinematical theorem a corresponding dual linear program could also be derived which would show that the $\dot{U}$'s, i.e. the collapse mechanism, will not be obtained by this approach (but the $\dot{\beta}$'s will).

An important advantage of the stress function approach is that equilibrium conditions can, in many cases, be satisfied exactly, thus being possible to obtain a true lower-bound of the load factor λ. There are, however, also some drawbacks:

not for all kinds of structure stress functions exist (e.g. not for framed structures); the stress distribution $\{\overline{\sigma}\}$ is easily found if only surface loads along the structure boundaries are present, which is often the case for plate-stretching and rotationally symmetric problems but almost never for plate-bending and shell problems; the assumed parametric fields for the stress functions have often (e.g. in the case of Airy's function) to satisfy stringent continuity conditions at the element interfaces; finally some complications arise for multiply connected domains.

An other, generally applicable approach to obtain stress assumptions satisfying a priori, at least approximately, equilibrium conditions would be to use linear elastic analysis to find both the inhomgeneous stress state $\{\overline{\sigma}\}$ and the homogeneous ones building the columns of the matrix $[\Psi]$. These can be obtained by specifying as load cases any number of different initial strain distributions resulting in an equal number of homogeneous (but not necessarily linearly independent) stress states.

6. ON PLASTIC OPTIMUM DESIGN

If some kinds of relation between the c_{kq}-coefficients representing the resistance of the structure at a checkpoint q for a stress direction $\{f_k\}$ (see Fig. 4) and a "merit"-function M $(\ldots.,c_{kq},\ldots.)$ can be mathematically established, an optimum design problem leading to an optimal distribution of the resistance coefficients c_{kq} for a prescribed design load $\{p\}$ can be formulated. Using, for simplicity, stress assumptions satisfying a priori equilibrium (i.e. Eq. (32) with $\lambda = 1$), the following mathematical program for the unknown c_{kq}- and Σ-coefficients is found:

$$\begin{aligned} & M(\ldots.,c_{kq},\ldots.) \rightarrow \text{optimum}, \\ & \{0\} \leq -[f]^T\{\overline{\sigma}_q\} - [f]^T[\Psi_q]\{\Sigma\} + \{c_q\} \qquad (q = 1 \text{ to } Q). \end{aligned} \tag{34}$$

An obvious difficulty of this approach lies in the choice of the merit function M. An other difficulty arises when several different loading cases govern the design of the structure as different sets of Σ-coefficients defining an "optimal" homogeneous stress state for each of the loading cases considered would have to be determined.

This last difficulty can be avoided when the inhomogeneous stress distributions $\{\sigma_n\}$ for each of the N loading cases considered (n = 1 to N) can be found by linear elastic analysis. This is only possible if the c_{kq}'s , i.e. the plastic resistance distribution within the structure, can be assumed not to have any influence on the elastic stress distribution (e.g. this is possible when looking for an optimal reinforcement distribution in a given concrete structure). From Eq. (28) the yield conditions for k = 1 to K and q = 1 to Q can then be formulated as follows:

$$0 \leq -\max_n(\{f_k\}^T\{\overline{\sigma}_{nq}\}) - \{f_k\}^T\{\Psi_q\}\{\Sigma\} + c_{kq}, \tag{35}$$

where at each checkpoint q and for each stress direction $\{f_k\}$ only the most unfavourable load case n is explicitely checked ($\{\overline{\sigma}_{nq}\}$ represents the elastic stresses due to the n'th load case at a checkpoint q), while for all loading cases together a single "optimal" homogeneous stress distribution defined by the stress parameter vector $\{\Sigma\}$ is introduced. According to the so-called shake-down theorem of the plasticity theory, this procedure will result in the design of a structure capable of stabilising for any conceivable load cycle, i.e. a structure which will behave perfectly elastically after plastic flow has occurred in the first load cycles. But the real advantage of this procedure, when applicable, is that the optimum design problem will be much simplified when several loading cases have to be considered which is, of course, almost always the case.

7. OVERVIEW AND CONCLUSIONS

In the field of plasticity most procedures suggested to date are based on an elasto-plastic approach, the initial stress method, with or without stiffness modification, being certainly the most generally applicable one. In different well-known general purpose finite element computer programs this kind of analysis is implemented. The main advantage of the elasto-plastic approach is that it can provide all needed informations on structural behaviour from working conditions until collapse. Other non-linear effects due to large deformations, crack propagation, creep, contact problems, friction, in fact, at least in principle, to any kinds of material behaviour that can be mathematically described can be taken into account by step-by-step iterative methods. An other important field of application is non-linear dynamic analysis by time-step integration of the dynamic equations.

However, the difficulties involved in an elasto-plastic analysis when applied to real life problems should not be underestimated. The computational effort needed will generally be high as reiterate solutions of large systems of linear equations will be necessary as well as reiterate evaluations of internal forces and stiffness matrices for each element by numerical integration procedures. Modeling problems might also arise as it is often necessary, in order to reduce computing time, to approximate reality by simple models, i.e. by coarse finite element meshes. This requires from the user of the computer program a very clear understanding of the way the program internally works and of the approximations involved. Finally, the interpretation of results and their relation to the actual design of the structure may also present some difficulties.

Rigid-plastic limit load analysis has received, so far, less attention than elasto-plastic analysis. This is probably due to the limited scopes that can be pursued by such an approach, as no information on working stresses or on displacements before collapse can be obtained. For real life problems an additional linear elastic analysis will therefore in most cases be necessary. An other difficulty arising from the rigid-plastic approach is caused by the great computational effort generally needed for solving the large linear programs involved. It is felt that more research work is needed for finding faster solution methods taking advantage of the peculiar nature of the problem. If this succeeds, however, rigid-plastic limit analysis, possibly combined in the same computer program with linear elastic analysis, could well become a widely used tool for everyday's structural engineering, being certainly easier to apply to real life problems than elasto-plastic analysis.

Rigid-plastic optimum design, and actually any kinds of direct optimum design procedures has found very few applications in civil engineering. In fact the prevailing attidude today is that the design of a structure cannot be done in a completely automatic way, but always requires a close interaction between the designer and the computer, which is more a problem of man - machine communication than of the theoretical approach used for the design. In some cases, however, the most important being probably the problem of finding a minimum weight reinforcement distribution for a given concrete structure, plastic optimum design methods can be useful to find the best solution among a narrow choice specified by the designer working in an interactive computer aided design environment.

REFERENCES

[1] O.C. Zienkiewicz: "The Finite Element Method in Engineering Science", McGraw-Hill, 1971.

[2] J.T. Oden: "Finite Elements of Non-linear Continua", McGraw-Hill, 1972.

[3] J. Robinson: "Integrated Theory of Finite Element Methods", John Wiley and Sons, 1973.

[4] E. Anderheggen: "Starr-plastische Traglastberechnungen mittels der Methode der Finiten Elemente", Swiss Federal Institute of Technology Zürich, Institute of Structural Engineering, Report No 32, 1971, Birkhäuser Verlag Basel and Stuttgart.