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Autor: Nielsen, M.P.

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The Theory of Plasticity for Reinforced Concrete Slabs

La théorie de la plasticité pour des dalles en béton armé

Die Plastizitätstheorie von Stahlbetonplatten

M.P. NIELSEN
Professor, Dr. techn.
Structural Research Laboratory, TU
Lyngby, Denmark

SUMMARY
The paper presents a short survey of the plastic theory of reinforced concrete slabs. Only the most fundamental aspects of the theory together with a short introduction to new areas of development have been dealt with.

RESUME
Le rapport présente une revue sommaire de la théorie de la plasticité appliquée aux dalles en béton armé. Seuls les aspects les plus fondamentaux de la théorie ainsi qu'une brève introduction des nouvelles possibilités de développement ont été présentés.

ZUSAMMENFASSUNG
Die Abhandlung bietet eine kurze Übersicht über die heutigen Kenntnisse der Plastizitätstheorie von Stahlbetonplatten. Nur die wesentlichsten Aspekte der Theorie werden behandelt und eine kurzgefasste Einführung zu neueren Entwicklungen wird gegeben.

1. INTRODUCTION

The aim of this paper is to present a survey of what is known in the theory of plasticity for reinforced concrete slabs.

Since the number of papers and books on the slab theory is very great, the references given are some selected papers, which in the authors opinion can be recommended as a starting point for further study of one particular specialized subject.

A number of important aspects of the theory have been left out of discussion because of space limitations. Such problems are rotation capacity problems, the application of the linear elastic solution as a lower bound solution, rules concerning the practical use of yield line theory and several others.

2. HISTORICAL REVIEW

The first contribution to the plastic theory of reinforced concrete slabs was made by the Danish engineer, Aage Ingerslev, [21.1] [23.1]. In 1921, he proposed a method of calculation based upon the assumption of constant bending moments along certain so-called yield lines. Several of Ingerslev's solutions have later proved to be exact, and his very early work has been of fundamental importance to the development of the theory.

Further pioneer work in this field was done by K.W.Johansen, [31.1][32.1][32.2] [43.1][49.1][62.1][72.1]. In his doctoral thesis from 1943 the theory took a very long step towards its final form.

In Johansen's work the yield lines had a geometrical meaning too, i.e. as lines along which a relative rotation of the slab parts meeting at the yield line takes place. Utilizing this he was able to define geometrically admissible yield line patterns and further his introduction of the work equation put him in a position to calculate upper bounds for the load carrying capacity. These contributions were of significance not only in the development of the slab theory but also, in general, in the development of the theory of rigid plastic materials. Mention should also be made of the introduction of the nodal force concept in the so-called equilibrium method, which sometimes considerably facilitates the calculation of upper bound solutions. His nodal force theory has, however, been the subject of some critisism, and several alternative theories have been formulated, see section 4.

Concurrently with Johansens work in Denmark, corresponding work was carried out in Russia, inter alia, by Gvozdev, see [59.1], in which Gvozdev's work is described.

One of the most important theoretical problems left unsolved by Johansen was the establishment of yield conditions. This basic information was not needed by Johansen, since he was able in a more or less intuitive way to find formulas for the work done in a yield line.

Yield conditions in the general case of orthotropic slabs were developed by the author, [63.1][64.1][69.1][71.1], and by Massonnet and Save [63.2], Wolfensberger [64.2], Kemp [65.1] and Morley [66.1].

It turned out that Johansen's formulas for the work in a yield line were in complete agreement with the yield conditions established. Hereby was his upper bound method put into the framework of the general theory of rigid plastic materials.

An early attempt to find a safe method for the calculation of the load carrying capacity was made by Hillerborg, [56.1][59.2]. He proposed to design several types of slabs by assuming the load to be carried only by bending moments in two perpendicular directions. To be economical, this so-called strip method generally requires the reinforcement to be varied through the slab. The strip method has been further developed by Hillerborg himself [74.2] and by others [68.4][68.5].

At the middle of the sixties the slab theory had almost obtained a final form and at that time it appeared as a special and useful case of the general theory of rigid plastic materials.

The developments since then have been concerned with three main subjects.

Firstly the theory as it was developed at the middle of the sixties had only taken account of bending and twisting moments, i.e. the in-plane forces were neglected. This is a more severe restriction in the theory of reinforced concrete slabs than in the classical theory of plates, since strains in the middle plane in a reinforced concrete slab develop, not only because of second order strain effects, but also because of the fact, that as soon as the concrete cracks the neutral axis seldom lies in the middle plane. Therefore the cracking leads to in-plane forces, especially if the slab edges are restrained. The membrane effect was first studied by Ockleston, [55.2].

The membrane effect often leads to a considerably higher load carrying capacity than calculated by taking account of the bending effects only.

Several papers have been published on the subject since, see section 9, but a general, practical design method has not yet been formulated.

Secondly the general development in optimization theory has also touched the reinforced concrete slab theory. The first results were reported by Wood [62.3] and Morley [66.2], who gave an exact solution for the simply supported square slab. Since then considerable progress has taken place and a great number of exact solutions exist, see section 7.

Thirdly the rapid development in automatic data processing has lead to a formulation of automatic design methods also in the reinforced concrete slab theory. One of the first contributions in this field was that of Wolfensberger [64.2]. The subject is now in a rapid development, see section 8, and in the near future one might expect that commercial programs for reinforced concrete slabs based on the theory of plasticity will be available.

3. BASIC EQUATIONS

3.1 Statical conditions

The statical conditions are the same as in the classical thin plate theory, i.e. the generalized stresses are in rectangular coordinates, x, y, the bending moments per unit length m_x and m_y and the twisting moment $m_{xy} = m_{yx}$. Besides we have the shear forces per unit length q_x and q_y. The statical boundary conditions are the so-called Kirchhoff boundary conditions requiring only the statical equivalence of the twisting moment and the shear force on the boundary to correspond to the internal forces.

It is often overlooked that the Kirchhoff boundary conditions in many cases express a physical reality, since the shear stresses arising from the twisting moments really are concentrated along the edges in such a way that it is natural to treat them as concentrated forces.

A stress field satisfying the equilibrium equations and the statical boundary conditions is as usual termed a statically, admissible stress field.

3.2 Geometrical conditions

The generalized strain rates corresponding to the generalized stresses m_x, m_y and m_{xy} are the curvature rates κ_x and κ_y and the rate of twist $2\kappa_{xy}$.

3.3 Yield conditions for orthotropic and isotropic slabs

Yield conditions for slabs can be derived in several ways. The most satisfactory way, in the author's opinion, is to derive the yield conditions on the basis of reasonable assumptions concerning the behaviour of the basic materials, concrete and steel. This was the way used by the author in [63.1] and [64.1], considering the action of bending and twisting moments in a slab. The basic ideas were already partly formulated by Jørgen Nielsen [57.1]. The yield condition can also be derived on the basis of the corresponding yield conditions for plates loaded in their own plane [63.7]. This method was used by the author in [69.1] and [71.1], giving the same result as the first mentioned method.

The yield conditions were derived by Massonet and Save too [63.2], on the basis of Johansen's formulas for the moments in a yield line. Essentially the same method was used by Wolfensberger [64.2] and Kemp [65.1]. The yield conditions have also been studied by Morley, [66.1], along similar lines as the author's.

The concrete is assumed to have a tensile strength equal to zero and a square yield locus.

The reinforcement bars are assumed to be able to carry only tensile or compressive stresses in their own direction.

Considering an orthotropic slab, i.e. a slab reinforced at the top and at the bottom in the same two perpendicular directions x and y , the yield conditions are found to be

$$\left.\begin{aligned} -(m_{Fx}-m_x)(m_{Fy}-m_y)+m_{xy}^2 &\leqq 0 \\ -(m'_{Fx}+m_x)(m'_{Fy}+m_y)+m_{xy}^2 &\leqq 0 \end{aligned}\right\} \qquad (3.3.1)$$

In the equations m_{Fx} is the numerical value of the positive yield moment in pure bending in a section perpendicular to the x-axis and m'_{Fx} is the numerical value of the negative yield moment in pure bending in a section perpendicular to the x-axis. The symbols m_{Fy} and m'_{Fy} have similar meanings. The first equation in (3.3.1) only applies when $m_x \leqq m_{Fx}$ and $m_y \leqq m_{Fy}$. Similarly the second equation only applies when $m_x \geqq -m'_{Fx}$ and $m_y \geqq -m'_{Fy}$.

In a m_x, m_y, m_{xy} - coordinate system, (3.3.1) corresponds to a surface consisting of two intersecting cones as shown in Figure 3.3.1.

The expressions are only valid for relatively small degrees of reinforcement, where the relative extension of the compressive zones in the concrete is small, see [63.1] and [64.1].

As will be seen the above yield conditions only contains bending and twisting moments, i.e. in-plane forces are neglected. This is sometimes a more severe limitation in the theory of reinforced concrete slabs than for metal plates, see section 9. Further the influence of shear forces in the direction of the slab normal is also neglected.

A moment field corresponding to points within or on the yield surface is as usual termed a safe moment field.

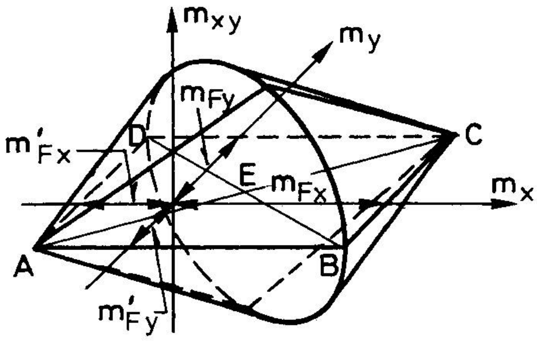

Fig. 3.3.1

The yield condition has been experimentally confirmed by tests on slabs in pure torsion, which gave a very good agreement between theory and tests, [69.1][71.1].

Other tests were also carried out, [63.8][67.2][67.3], but the confidence to the yield conditions derived lies mainly in the agreement between numerous tests on slabs and the load carrying capacity determined on the basis of the yield conditions.

The corresponding yield conditions for plates loaded in their own plane, [63.7], have been tested in several cases, [69.1][71.1].

For a rigid plastic structure with the generalized stresses Q_i, the generalized strain rates q_i and the yield condition $f(Q_i) = 0$, the flow rule is

$$q_i = \lambda \frac{\partial f}{\partial Q_i} \qquad \lambda > 0 \tag{3.3.2}$$

It is assumed that $f < 0$ for stresses, which can be carried by the structure. Geometrically (3.3.2) expresses, that the strain rate vector is an outward normal to the yield surface.

If the yield surface has an edge or a vertex, the strain rate vector is allowed to lie within the angle determined by the limits of the normals of the surface, when the stress vector approaches the edge or the vertex by all ways possible.

For an orthotropic reinforced concrete slab we get for instance in the case where the first expression in (3.3.1) is valid

$$\left.\begin{aligned} \kappa_x &= \lambda(m_{Fy} - m_y) \\ \kappa_y &= \lambda(m_{Fx} - m_x) \\ \kappa_{xy} &= \lambda\, m_{xy} \end{aligned}\right\} \tag{3.3.3}$$

Notice that in this region

$$\kappa_x \kappa_y = \kappa_{xy}^2 \tag{3.3.4}$$

i.e. one principal curvature rate is zero.

A similar conclusion holds if the second expresssion in (3.3.1) is valid.

The expressions along the edge and the vertices of the yield surface shall not be dealt with here.The reader is referred to [64.1] or [63.2].

In the special case $m_{Fx} = m_{Fy} = m_F$, $m'_{Fx} = m'_{Fy} = m'_F$ the slab is isotropic, i.e. the yield condition can be written in terms of principal moments m_1 and m_2 only.

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The yield locus is shown in Figure 3.3.2. The principal curvature rates κ_1 and κ_2 according to the flow rule (3.3.2) are illustrated in the figure too.

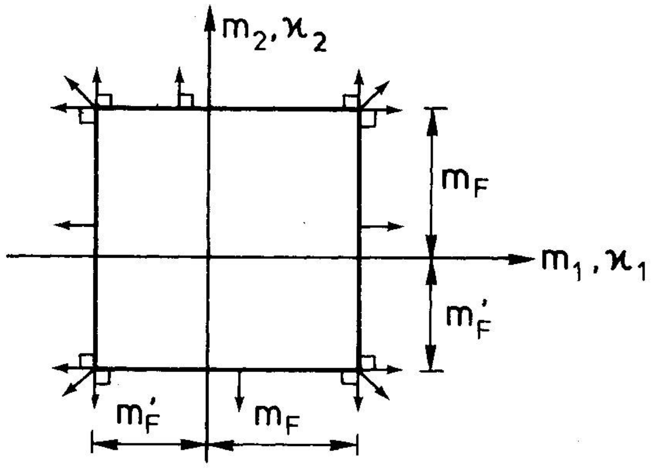


Fig. 3.3.2

This yield locus is often referred to as Johansen's yield locus.

If the problem is to design a slab to carry given bending and twisting moments, one can of course use the expressions for the yield condition to obtain safe values of the yield moments. Alternatively the reinforcement can be determined by means of the formulas:

$$\left.\begin{aligned} m_{Fx} &= m_x + \gamma\,|m_{xy}| \qquad & m_{Fy} &= m_y + \frac{1}{\gamma}\,|m_{xy}| \\ m_{Fx}' &= -\,m_x + \gamma'\,|m_{xy}| \qquad & m_{Fy}' &= -\,m_y + \frac{1}{\gamma'}\,|m_{xy}| \end{aligned}\right\} \tag{3.3.5}$$

where γ and γ' are positive numbers, which can, theoretically, be arbitrarily chosen. The formulas follow immediately from the corresponding reinforcement formulas for plates loaded in their own plane, [63.7][69.1][71.1]. A set of formulas giving optimal reinforcement at the point considered, were developed by the author, [64.1][69.1][71.1].

It should be noted that if there are twisting moments along an edge, not only the top and bottom should be reinforced according to the formulas, but the edge itself should be reinforced, too, for instance by closed stirrups connecting the top and bottom reinforcement.

3.4 Yield conditions for arbitrarily reinforced slabs

For a plate loaded in its own plane and reinforced in several directions forming any angle to each other, it may be shown, [69.1], that the yield condition corresponds to an equivalent orthotropically reinforced plate. For a slab with the same lines of symmetry at the top and at the bottom the yield condition therefore corresponds to an equivalent orthotripic slab.

If the lines of symmetry are not the same at the top and at the bottom, yield conditions can be derived by means of the yield conditions for plates loaded in their own plane transformed to the coordinate system by means of which the yield condition is to be described. Braestrup [70.1] showed that the yield condition may also be formulated in moments referred to axes x,y arbitrarily oriented with respect to any number of reinforcement directions. The yield surface is bi-conical as the one shown on Figure 3.3.1, but the vertices A and C no longer lie in the plane $m_{xy} = 0$. We shall however not pursue this matter further here.

4. UPPER BOUND SOLUTIONS

4.1 Upper bound solutions by the work equation method

The upper bound technique is now well-known and described in several books and papers, see for instance [43.1][53.1][60.3][62.1][62.2][62.3][63.2] and [63.6], therefore we shall here only be concerned with the most fundamental aspects of the theory.

To establish an upper bound solution for the load carrying capacity of a rigid plastic slab, one has to find a geometrically possible deflexion rate field, write down the work equation, which equals the external work and the dissipation, i.e. the internal work carried out by the generalized stresses corresponding to the deflexion rate field. The solution of the work equation gives an upper bound for the load carrying capacity.

Of course the best answer one can get from a geometrically possible deflexion rate field containing more than one geometrical parameter is the one corresponding to the lowest load carrying capacity, therefore the solution found by means of the work equation has to be minimized with respect to the geometrical parameters.

The simplest type of geometrically possible deflexion rate fields is obtained by dealing with deflexion rates corresponding to discontinuities in the angular deflexion along straight lines, i.e. yield lines. These so-called yield line patterns, which were first considered by Johansen [43.1] and Gvozdev, see [59.1], can be easily found for any slab type utilizing the fact that a straight yield line separating two slab parts has to pass through the point of intersection between the axes of rotation for the two slab parts in question.

The dissipation along a yield line can be found by considering a yield line to be a narrow zone with constant curvature rate in one direction only. Let the curvature rate be $\kappa > 0$ in the n-direction forming an angle φ to the x-axis, see Figure 4.1.1.

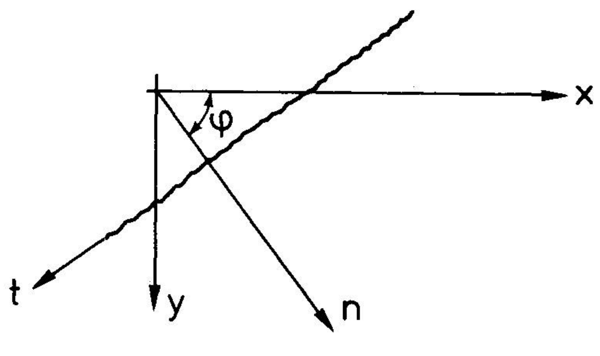


Fig. 4.1.1

Then we have

$$\kappa_x = \kappa\cos^2\varphi \qquad \kappa_y = \kappa\sin^2\varphi \qquad \kappa_{xy} = -\kappa\sin\varphi\cos\varphi \qquad (4.1.1)$$

Inserting these expressions into (3.3.3), and solving the equations with regard to the moments, we get

$$\left.\begin{aligned} m_x &= m_{Fx} - \frac{\kappa\sin^2\varphi}{\lambda} \\ m_y &= m_{Fy} - \frac{\kappa\cos^2\varphi}{\lambda} \\ m_{xy} &= -\frac{\kappa\sin\varphi\cos\varphi}{\lambda} \end{aligned}\right\} \tag{4.1.2}$$

The bending moment m_n is thus

$$m_n = m_x\cos^2\varphi + m_y\sin^2\varphi - 2m_{xy}\sin\varphi\cos\varphi =$$

$$m_{Fx}\cos^2\varphi + m_{Fy}\sin^2\varphi \tag{4.1.3}$$

which is the bending moment in a positive yield line.

In a similar way it is possible to calculate the twisting moment in a yield line. One finds

$$m_{nt} = \frac{1}{2}(m_{Fx} - m_{Fy})\sin 2\varphi \tag{4.1.4}$$

The formulas express the significant result that the bending and twisting moments in a yield line can be calculated as if the principal moments were found in sections coinciding with the directions of the reinforcement, which is naturally not the case at other points of the yield surface, than those corresponding to $m_{xy} = 0$.

These are the formulas intuitively proposed by Johansen, [43.1], which are thus consistent with the yield conditions developed later.

In the special case of an isotropic slab where $m_{Fx} = m_{Fy} = m_F$, we get

$$m_n = m_F \tag{4.1.5}$$

$$m_{nt} = 0 \tag{4.1.6}$$

i.e. the bending moment is independent of the angle φ , and the twisting moment is zero.

Similar expressions are of course valid for a negative yield line.

The dissipation D along the yield lines having the discontinuities θ_n in the angular deflection rates and the arc length ds , is

$$D = \int |m_n||\theta_n| ds \tag{4.1.7}$$

For practical purposes, however, it is simpler to calculate the work done by the external and internal forces on each slab part and thereafter summing over all slab parts. As the work done by an arbitrary system of forces, when it is rotated, is equal to the moment about the axis of rotation times the angle of rotation, the work equation may be written

$$\sum_j M_{ej}\omega_j = \sum_j M_{ij}\omega_j \tag{4.1.8}$$

where M_{ej} is the moment about the axis of rotation of the external load acting on the j'th slab part, M_{ij} is the corresponding moment with opposite sign of the bending and twisting moments along the yield lines, and ω_j is the rotation rate of the j'th slab part.

Even though only the bending moment perform work in a yield line, it is naturally possible to include the work done by the twisting moments, since their contribution vanishes by summation over all the slab parts, see [64.1].

It must be strongly emphasized that although the dissipation in a yield line is always positive, the terms on the left hand side of (4.1.8) may not all be positive.

For a continuous curvature rate field, the dissipation can be found by means of the flow rule and the expression for the internal work.

In the special case of an isotropic slab, the result is

$$D = \frac{1}{2}\iint[\frac{1}{2}(m_F+m_F')(|\kappa_1|+|\kappa_2|) + \frac{1}{2}(m_F-m_F')(\kappa_1+\kappa_2)]dx\,dy \qquad (4.1.9)$$

A simple example is a circular "fan", where a circular area or a part of a circular area is deformed to a cone with vertex in the center. There is a yield line along the limiting circle.

The dissipation in the more general case of a "fan" , where the negative yield line is an arbitrary curve , was derived by Mansfield, [57.2][60.1] and in a more direct way by the author [64.1]. For the case of nonpolar fans, see [67.4].

4.2 Upper bound solutions by equilibrium methods

Instead of using the work equation on yield line patterns Ingerslev [21.1] and Johansen [43.1] formulated an alternative approach based on equilibrium equations for the individual slab parts formed by the yield lines.

The main advantage of the equilibrium method is that the minimizing process in the work equation method is avoided. Using the equilibrium method the necessary algebra is often reduced a great deal compared to the work equation method. Furthermore by the equilibrium method, information is often gained for instance about column reactions and support reactions, information which cannot be delivered by the work equation method. Finally equilibrium equations may also show, how an estimated yield line pattern has to be changed in order to furnish a better result.

Ingerslev simply proposed to establish the necessary number of equilibrium equations by assuming, for an isotrop slab, that the shear forces and the twisting moments in the yield lines were zero, and that the bending moment for a homogeneously reinforced slab was constant. He demonstrated the technique in several examples, for instance the rectangular slab with uniform load. Johansen found that Ingerslev's solutions were in agreement with the minimized value obtained by the work equation. However cases were also found, where the two methods were not in agreement. Such a case is shown in Figure 4.2.1, where a yield line pattern consisting of one positive yield line, originating from a corner, is considered in a rectangular slab with two adjacent edges simply supported and the other two edges free.

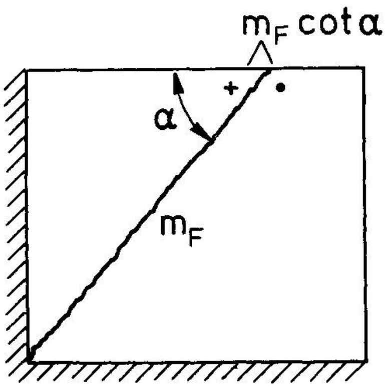


Fig. 4.2.1

The discrepancy was according to Johansen due to the fact that the shear forces and twisting moments are not allways zero in a yield line, and he proceeded to determine the statical equivalence of the shear forces and twisting moments in the form of concentrated forces at the ends of the yield lines, the so-called nodal forces. One important assumption in his calculations was that the bending moment has a stationary value in the yield line.

In the isotropic case considered in Figure 4.2.1, the nodal forces were found to be two numerically equal but opposite directed forces $m_F \cot\alpha$ at the point, where the yield line intersects the free boundary.

The nodal force theory of Johansen was not too convincing, and several attempts were made to improve the theory.

The author, [64.1][65.4], suggested to distinguish between nodal forces, which are simply the usual Kirchhoff boundary forces, and nodal forces, which are the statical equivalence of shear forces in internal yield lines. For isotropic slabs Johansen's theory and the author's gave identical results, while this was not the case for orthotropic slabs.

A sufficient condition for finding identical results by the work equation method and the equilibrium method is, according to the author's theory, that the equilibrium equations for each slab part that has been formed by the yield lines, are satisfied in such a way that a so-called stationary moment field may be found in each slab part. A stationary moment field is a statically admissible, but not necessarily safe, moment field for which, in the isotropic case, the shear forces and twisting moments are zero along all internal yield lines.

There are many cases, for which it is impossible to find a stationary moment field, and in all these cases, it has been found that the nodal forces cannot be determined by means of general formulas.

Some important examples are slabs, for which the number of geometrical parameters are not sufficient to make it possible to satisfy all necessary equilibrium equations, slabs where yield lines end at corners, slabs where yield lines intersect in a statically impossible way (e.g. three positive and one negative yield line) and slabs where a yield line passes point loads.

Alternative theories explaining the limitations of the Johansen nodal force theory have been given by Nylander,[60.2][63.5], Kemp, [65.2], Morley, [65.3], Wood, [65.5], Jones, [65.6] and Møllmann, [65.7].

Nodal forces can also be derived for curved yield lines, [43.1][64.1].

A number of solutions with curved yield lines were obtained numerically by the author, [62.4][63.3].

4.3 Yield line formulas

A collection of solutions for isotropic slabs covering most of the problems met in practice has been worked out by Johansen, [49.1][72.1]. By means of the affinity theorem, see section 6, the solutions can be used for a class of orthotropic slabs too.

To deal in an approximate manner by several loading cases Johansen, [43.1], found some superposition principles, see also [63.2].

5. LOWER BOUND SOLUTIONS

5.1 Introduction

One disadvantage of the upper bound methods for designing a slab is that it is very difficult, if not impossible, to vary the reinforcement in accordance with the stresses. Particularly this disadvantage is felt strongly, when one is concerned with the problem of determining the extent of top reinforcement, which is generally not carried through the whole slab. Also the reinforcement near columns and in supporting beams may constitute a problem when dealing with upper bound methods. Sometimes it is also argued that the upper bound methods are unsafe, since they lead to an overestimation of the load carrying capacity. This is of course, theoretically, correct, but this point is more or less academic, since the membrane effect generally gives a reserve capable of compensating more than necessary for this overestimation.

Nevertheless it is quite natural to study the possibilities of approaching the load carrying capacity from below.

A lower bound solution requires the determination of a statically admissible, safe stress field.

A number of lower bound solutions exists for isotropic and homogeneously reinforced slabs, but it is much more difficult by simple means to obtain a lower bound solution than to obtain an upper bound solution for a slab with given reinforcement.

The problem to find the reinforcement in a given slab is simpler, since then only a statically admissible stress field is required. Knowing this the necessary reinforcement can be determined by means of the formulas (3.3.5), which automatically renders the solution safe.

An extremely simple, statically admissible stress field can sometimes be found using Hillerborg's strip method [56.1][59.2][68.4][68.5][74.2], where only bending in two perpendicular directions is considered.

5.2 The strip method

The idea behind the strip method is that the slab is imagined to carry the load as two sets of beams at right angles to each other. Namely, if m_{xy} is made equal to zero in the equilibrium equation, we get

$$\frac{\partial^2 m_x}{\partial x^2} + \frac{\partial^2 m_y}{\partial y^2} = -p \tag{5.2.1}$$

which is satisfied, if

$$\left.\begin{aligned} \frac{\partial^2 m_x}{\partial x^2} &= -p_x \\ \frac{\partial^2 m_y}{\partial y^2} &= -p_y \\ p_x + p_y &= p \end{aligned}\right\} \tag{5.2.2}$$

The first and second equation in (5.2.2) are simple beam equations. The sub-division of the load per unit area p into p_x and p_y is arbitrary, and need not be the same throughout the slab.

It is rather evident that this simple method will be rather uneconomical, if the slab is homogeneously reinforced. If, however, the reinforcement is varied in accordance with the moment field, the reinforcement volume can easily compete with upper bound solutions, and, as shown by Hillerborg, [74.2], even exact solutions can be obtained.

The strip method can be used for many types of slabs supported on columns, if the moment field library is supplemented by a statically admissible moment field for a rectangular slab, uniformly loaded and supported in the middle on a column. Such a moment field has been developed by Hillerborg. With this moment field for instance, the slab shown in Figure 5.2.1 can be calculated by first assuming the load to be

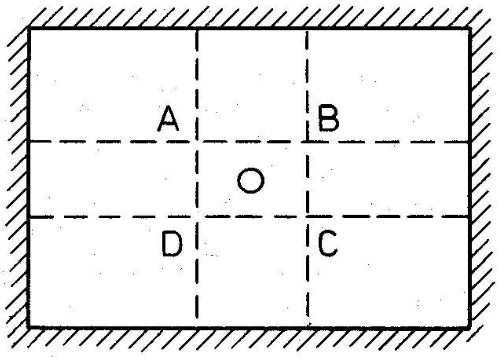


Fig. 5.2.1

transferred to the strips passing over the column. Then these strips are calculated as supported by a uniformly distributed reaction acting on the rectangular part ABCD. Finally these moment fields are superimposed on the moment field for a loading opposite to the reaction, acting on the rectangular part ABCD, which is now imagined as being supported on the column.

The strip method is not as general in its application as the yield line theory, in fact, it has to be altered and adjusted according to the various types of slabs.

5.3 Simple moment fields for rectangular slabs

The equilibrium equation for a slab can in the special case of uniform loading on rectangular slabs be satisfied if the bending moments vary as a parabolic cylindrical surface and if the twisting moments vary as a hyperbolic paraboloid. A moment field of this type was first suggested by Prager, [52.1], in his exact solution for the simply supported square slab, see also [55.1].

It has turned out that many rectangular slabs with different kinds of support conditions can be treated by the use of the above mentioned moment fields.

A number of solutions have been given by Bach and Nielsen, [78.2].

6. EXACT SOLUTIONS

6.1 Exact solutions for isotropic, homogeneously reinforced slabs

To find an exact solution one has to determine a statically admissible,safe moment field. The curvature rate field corresponding to this moment field, according to the flow rule, has to satisfy the compatibility equations and the corresponding deflexion rate has to satisfy the geometrical boundary conditions.

If the yield condition is satisfied in a zone, we might distinguish between 3 types of yield zones.

In type 1 both principal moments are equal to m_F or m_F' . It is easily shown that the equilibrium equations can only be satisfied with $p = 0$. The shear forces are similarly zero. Each section is thus a principal section.

In type 2 the principal moments are $m_1 = m_F$ and $m_2 = -m_F'$. It may be shown that in the case $p = 0$, the principal sections form a Hencky net (slip line net). This static analogy was first pointed out by Johansen, [43.1] and developed further by the author, [62.5][64.1]. A corresponding geometrical analogy was described by Johnson [69.2] and the complete analogy by Collins [71.2].

In a yield zone of type 3 there is only yielding in one principal direction. For instance we might have $m_1 = m_F$, $-m_F' \leq m_2 \leq m_F$, where the equals sign is only valid at certain points. Therefore

$$\left.\begin{aligned} &\kappa_1\kappa_2 = \kappa_x\kappa_y - \kappa_{xy}^2 = 0 \\ &\kappa_1 + \kappa_2 > 0 \quad \text{or} \quad \kappa_1 + \kappa_2 < 0 \quad \text{everywhere} \end{aligned}\right\} \qquad (6.1.2)$$

The general solution to (6.1.2) is developable surfaces. The curves along which the principal curvature is zero are straight lines (generatrices). The possible surfaces are conical, cylindrical and tangential surfaces.

There exists a number of exact solutions for isotropic slabs, some of which are given in Figure 6.1.3.

The solutions a, b and c was given by Johansen, [43.1]. The solution b contains the well-known solution $P = 2\pi(m_F+m_F')$, which is valid for a concentrated force acting on a circular slab with fixed or simply supported edges as special cases. As showed by Haythornthwaite and Shield [58.1], the solution is valid for an arbitrary fixed slab, g.

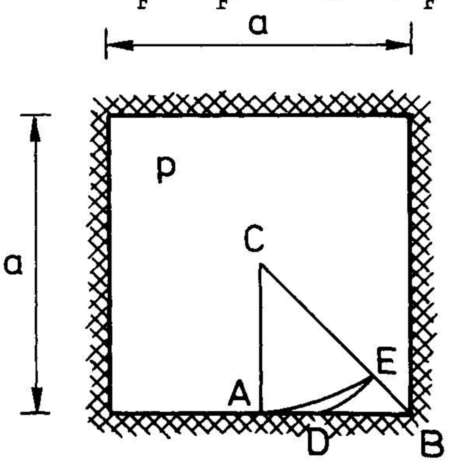
Exact solutions for circular slabs are relatively easy to obtain when the loading is rotationally symmetrical. Mention should be made of an interesting solution obtained by Nylander [59.3], for the case of a slap supported on both an exterior and an interior circular support, where two radial fields, separated by a circular yield line, do not solve the problem as could be expected.

Solution d was given by Prager, [52.1], and solutions e and f by Wood, [62.3]. Johansen gave solution f as an upper bound solution, [43.1]. Solutions h - p are the author's, [62.5][63.4][64.1].

The solution p and some other known solutions have equivalents in the slip line theory.

Ingerslev's yield line solution for the rectangular slab was shown by the author to be exact only if the negative yield moment has a certain value ranging from $m_F' = m_F$ for a square slab to $m_F' = 3m_F$ for a very long slab [64.1].

The clamped square slab for a long time denied its solution. In fact it was being claimed that the problem had no solution according to the present plastic theory, [68.3]. However in 1974 it was shown by Fox, [74.1], that the exact load carrying capacity in the case $m_F = m_F'$ is $p\,a^2/m_F = 42.851$.



$p\,a^2 / m_F = 42.851$

Fig. 6.1.2

The solution turned out to be rather complicated containing one region C A E with a yield zone of type 3 and one region A E D with a yield zone of type 2. Finally there is a rigid portion E B D.

Fox, [72.2], also solved the rectangular simply supported slab with a concentrated force.

Finally a class of solutions was developed by Massonnet [67.1].

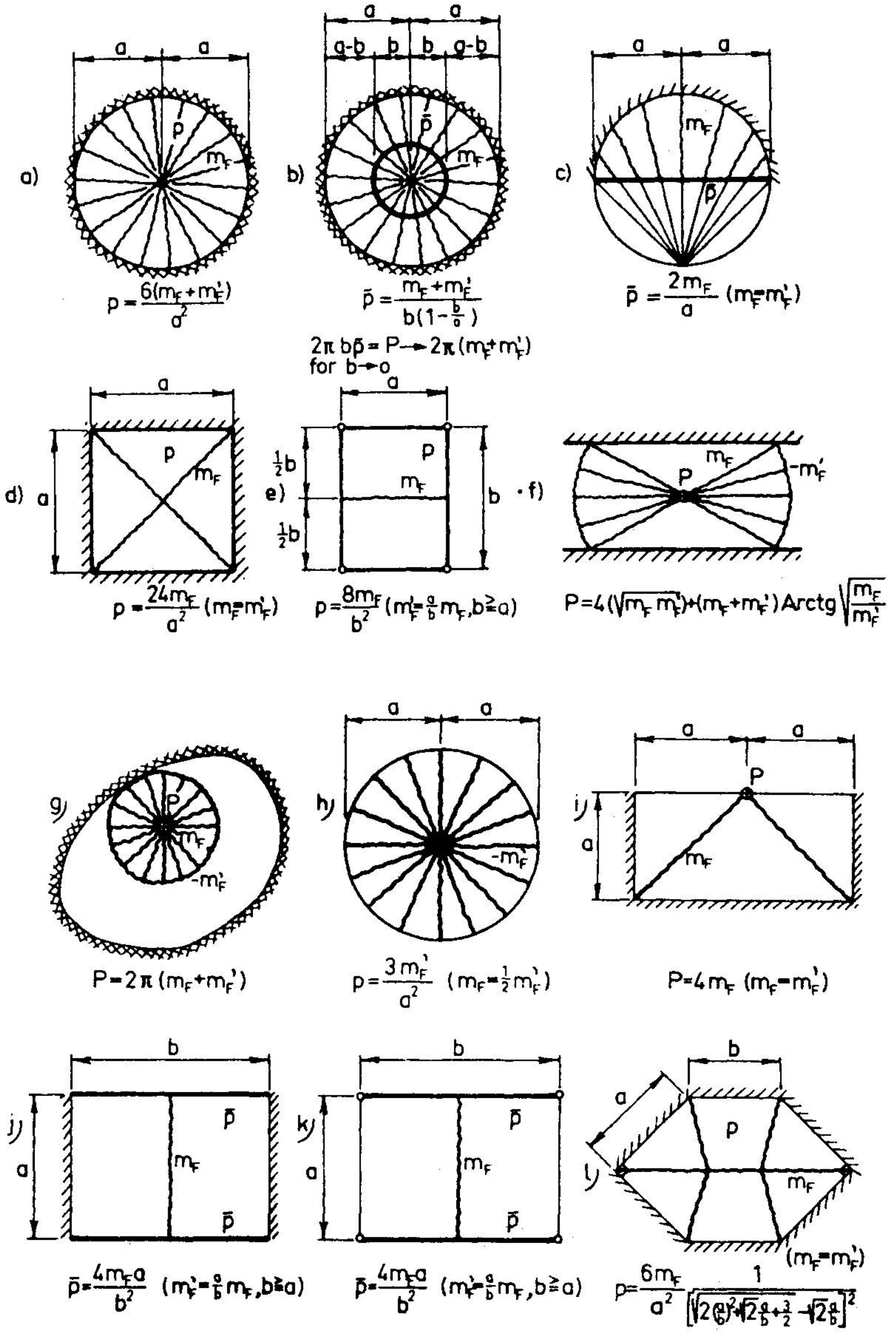


Fig. 6.1.3(continued)

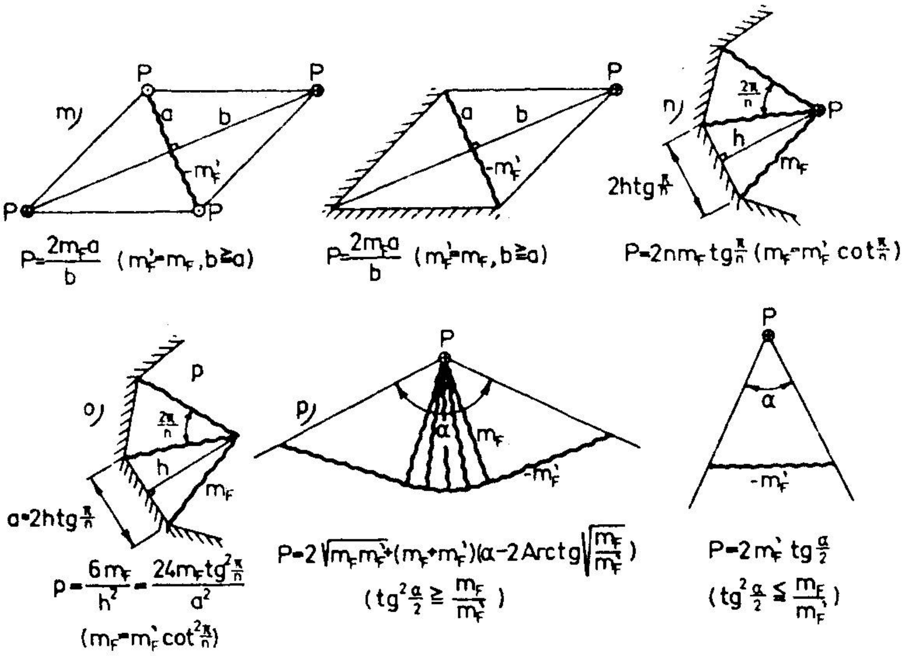


Fig. 6.1.3

6.2 The affinity theorem

For a special class of orthotropic slabs exists an interesting affinity theorem. It was developed by Johansen [43.1] for upper bound solutions and extended to lower bound solutions and exact solutions by the author [64.1].

The special class of orthotropic slabs for which the theorem is valid is characterized by $m_{Fx} = m_F$, $m'_{Fx} = m'_F$, $m_{Fy} = \mu m_F$ and $m'_{Fy} = \mu m'_F$.

The affinity theorem enables one to transform solutions for isotropic slabs to a special but rather general class of orthotropic slabs. This implies that for most practical purposes only calculations for isotropic slabs need to be performed.

7. ANALYTICAL OPTIMUM REINFORCEMENT SOLUTIONS

It is a natural task for a designer to look for one or another kind of optimal solution.

A fundamental question in the plastic theory for reinforced concrete slabs is to find the absolute minimum of the reinforcement volume for a given slab, with a prescribed load.

Considerable progress in answering this question has been gained by the work of Morley, Lowe and Melshers, Rozvany and others. A review paper containing most of the available information has been written by Rozvany and Hill [76.1], to which the reader is referred.

If the slab thickness has been given and if the variation of the compressive zones in the concrete is neglected, the Drucker-Shield criterion for minimum volume of a plastic structure, [56.2], immediately shows that one has to look for a constant principal curvature rate field throughout the slab, to which it is possible to assign a principal moment field corresponding in direction and sign to the curvature field. For many important cases the curvature rate field is the same for a wide class of load configurations on the same slab.

Morley, [66.2], gave a solution for the simply supported square slab, which is illustrated in Figure 7.1.

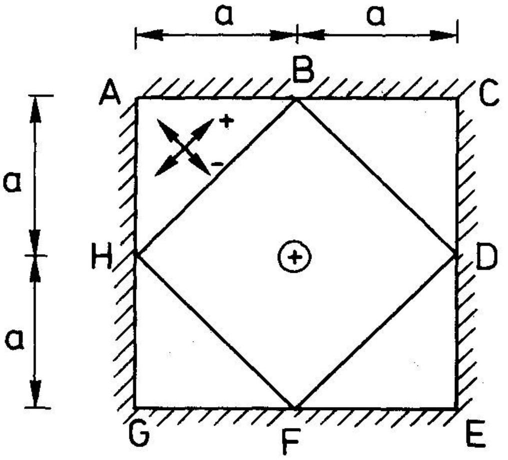


Fig. 7.1

In the region BDFH the two principal curvature rates are positive and equal. In the triangular regions the principal curvature rates are equal and have opposite signs. A load acting in the region BDFH is transferred to the infinitely narrow beams BD, DF, FH and HB by strip action. The strips can be arbitrarily selected. A load acting in the triangular regions can for instance be carried by strips lying under 45° to the edges and spanning from support to support.

A great number of solutions of this kind have been given by Rozvany anf Hill, [76.1].

It will be seen that the reinforcement has to be rather artificially arranged. A special problem is furnished if concentrated reinforcement bands in, theoretically, infinitely narrow beams is required since this might give rise to problems concerning the concrete stresses.

Anyway the optimal solutions are extremely useful as a basis for comparisons with the kind of solutions which for one reason or another are preferred by the designer.

Optimization of reinforcement with such constraints as to render the solutions more practical has also been considered. References may be found in the review paper by Rozvany and Hill, see also section 8.

8. NUMERICAL METHODS

The development of electronical computers has opened up new possibilities for finding approximate solutions to structural problems.

To find lower bound solutions in the plastic theory, one needs to create a sufficiently wide class of statically admissible stress fields and to find the one corresponding to the greatest load factor. Statically admissible stress fields can be created for instance by means of the finite element method, where the stress field within each element is expressed by a number of parameters. Equilibrium requirements within the element, continuity requirements along the element boundaries and the statical boundary conditions lead to a set of linear equations.

If the yield conditions are linearized, one gets a set of linear constraints, which together with the equilibrium equations constitutes a linear programming problem for the determination of the largest load which can be carried by the slab.

A similar method can be used in order to determine optimal reinforcement arrangements both in cases where the reinforcement is allowed to vary from point to point and in cases where the reinforcement arrangement is subject to certain geometrical constraints.

In Figure 8.1 a solution obtained by Pedersen [74.3] for the clamped square slab uniformly loaded ($m_F = m'_F$) is illustrated.

The finite element used was a rectangular element with bending moments varying as a parabolic cylindrical surface and twisting moments varying as a hyperbolic paraboloid, i.e. the load within each element was assumed to be constant.

The linearized yield conditions used were

$$\left.\begin{aligned} m_x \pm m_{xy} &\leqq m_{Fx} \\ -m_x \pm m_{xy} &\leqq m'_{Fx} \\ m_y \pm m_{xy} &\leqq m_{Fy} \\ -m_y \pm m_{xy} &\leqq m'_{Fy} \end{aligned}\right\} \qquad (8.1)$$

These equations were checked at the corners and in the middle of the element. However for the solutions obtained, the correct yield condition (3.3.1) were checked in a finer mesh, and the solution was proportioned if needed to fulfill the correct yield condition in all check points.

The figure shows the load carrying capacity obtained as a function of the mesh size. Also the total computer time is shown for some of the calculations.

As mentioned in section 6.1 the exact solution is $pa^2/m_F = 42.851$, which means that the best numerical solution deviates only a few percent from the exact one.

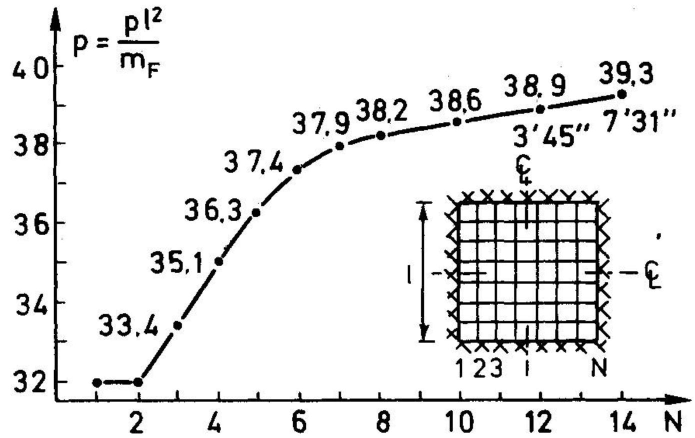


Fig. 8.1

The first calculations of this kind reported in the litterature were these of Wolfensberger, [64.2], whose procedure was very similar to that described above. Also Anderheggen, [72.4], Ceradini, [65.11], Gavarini, [66.4] and Sacchi, [66.5] have adopted such an approach.

Instead of using linear programming for the determination of the load carrying capacity of reinforced concrete slabs, Chan, [72.3], has used quadratic programming, which however led to considerably higher computer times.

Another approach has been used by Bäcklund, [73.1], who determined upper and lower bounds by following the complete behavior of the slab when the load grows from zero to the ultimate value.

Linear programming methods require a large computer and computer times far exceeding those required for linear elastic calculations. Nevertheless it is to be expected that in the near future commercial programs based on the plastic theory of reinforced concrete slabs will be in operation.

9. MEMBRANE ACTION

The theory presented neglects the fact that the strain field, corresponding to bending and twisting moments only, always results in strains in the slab middle surface, and these strains do not generally satisfy the compatibility equations and the geometrical boundary conditions. This leads to in-plane forces in the slab.

Further the rigid plastic theory in its standard formulation (1st order theory) neglects effects of changes in geometry. Since plates and reinforced concrete slabs often are rather flexible structures, the changes in geometry sometimes has a considerable effect on the load carrying capacity. These effects are often called membrane effects, and one speaks about a compressive membrane effect, which often predominates at small deflections and of a tensile membrane effect, which is dominating at larger deflections.

In plane forces arise already in the early stages of cracking.

A uniformly loaded, simply supported square slab often has a load-deflection relationship of a type shown in Figure 9.1. Instead of yielding under constant load, one

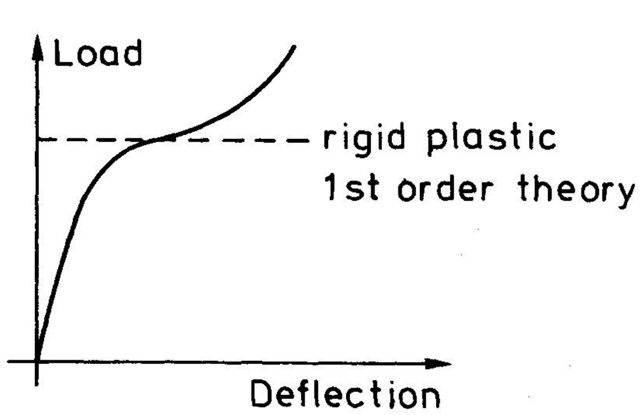


Fig. 9.1

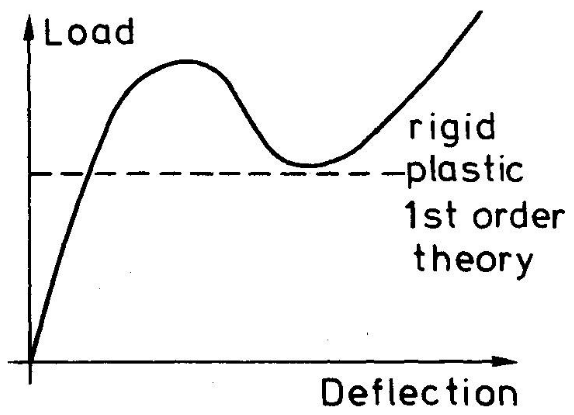


Fig. 9.2

hardly observes anything peculiar at the load corresponding to the rigid plastic 1st order theory. The real collapse load generally is somewhat higher than the rigid plastic 1st order load. Small degrees of reinforcement lead to relatively higher collapse loads compared to the rigid plastic 1st order load than higher degrees of reinforcement.

Quite different behavior is observed for a clamped slab if horizontal displacements are prevented along the edges. A typical load deflection curve is shown in Figure 9.2.

Failure is here by a snap-through action after which the load approximately reaches the rigid plastic 1st order load. Finally the load is again increased through a tensile membrane action. The maximum load may far exceed the rigid plastic 1st order load.

A theoretically correct determination of the full load deflection curve taking account of the elastic deformations, cracking of the concrete, realistic constitutive equations of the concrete until failure and the effect of changes in geometry is extremely complicated and has not yet been obtained.

Estimates of the effect of changes in geometry can however be obtained relatively simple by means of a series of upper bound calculations assuming the form of the deflected slab to be known. For instance a circular slab loaded at the center by a point load can be assumed to deflect as a cone similar to the deflection rate cone found by 1st order rigid plastic theory. Similarly a square slab can be assumed to deflect into a pyramidal form corresponding to the deflection rate form found by 1st order rigid plastic theory, too.

Having fixed the deflected form it is a relatively simple task by means of the usual upper bound technique to calculate the load corresponding to the deflected form assumed. The load carrying capacity of course turns out to be a function of the deflection.

For a simply supported slab, respectively a clamped slab, the load deflection curve obtained in this way will be of the type shown in Figure 9.3 and Figure 9.4.

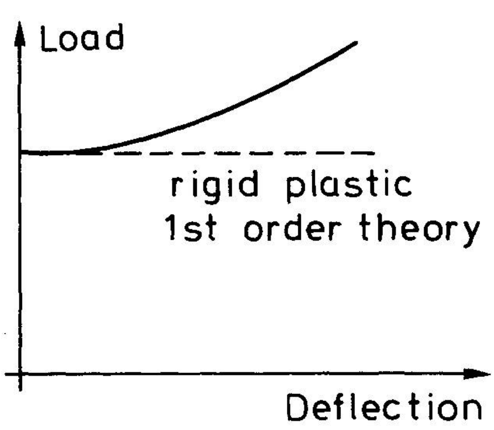


Fig. 9.3

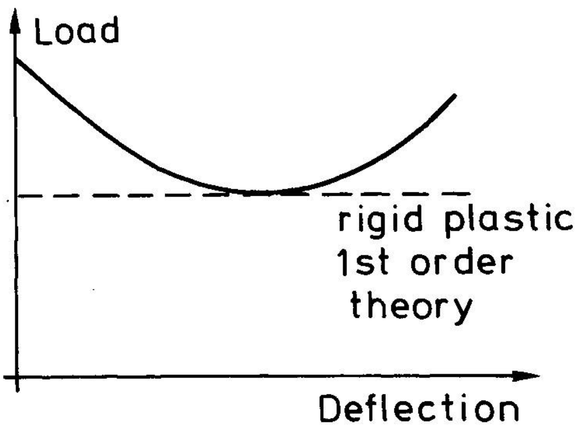


Fig. 9.4

The maximum load found for the clamped slab will not be reached in practice because of the elastic deformations neglected.

As shown by Calladine, [68.2], the calculations in several cases turn out to be very much simpler using the 3-dimensional theory instead of the 2-dimensional theory usually adopted in slab theory.

Because of the great effect of the elastic deformations on the load carrying capacity of clamped slabs, the rigid plastic theory cannot be used with confidence in practice. Since large reserves in load carrying capacity are inherent in the effect of changes in geometry, one of the most urgent needs of slab research is to create a reliable design method capable of utilizing these reserves.

Although already Johansen, [43.1], was aware of the tensile membrane action, the first to demonstrate the great effect of restrained edges was Ockleston, [55.2], who in a test series on a condemned building became aware of a break-down of the rigid plastic 1st order theory for internal slab parts. Several research workers have since that time studied the problem theoretically and experimentally, among them Wood, [62.3] and Park, [64.3]. An upper bound analysis of a type described above were among others performed by Sawczuk, [64.6][65.9], Janas and Sawczuk, [66.3], Morley, [67.6], Janas, [68.1] and, as mentioned already, by Calladine, [68.2]. A litterature survey has been performed by Bäcklund [72.5]. Concerning membrane action, see also [58.2][63.9][64.4][64.5][65.8][65.10][67.5][73.2][75.1][78.1].

NOTATIONS

⊕	Concentrated force
+	Downward-directed concentrated force
●	Upward-directed concentrated force
∿∿∿∿∿	Yield line
⁄⁄⁄⁄⁄⁄⁄	Simply supported edge
xxxxxx	Fixed edge
———	Free edge
━━━	Line load
o	Column without restraint

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