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Session III

Slabs

The Theory of Plasticity for Reinforced Concrete Slabs

La théorie de la plasticité pour des dalles en béton armé

Die Plastizitätstheorie von Stahlbetonplatten

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SUMMARY
The paper presents a short survey of the plastic theory of reinforced concrete slabs. Only the most fundamental aspects of the theory together with a short introduction to new areas of development have been dealt with.

RESUME
Le rapport présente une revue sommaire de la théorie de la plasticité appliquée aux dalles en béton armé. Seuls les aspects les plus fondamentaux de la théorie ainsi qu'une brève introduction des nouvelles possibilités de développement ont été présentés.

ZUSAMMENFASSUNG
Die Abhandlung bietet eine kurze Übersicht über die heutigen Kenntnisse der Plastizitätstheorie von Stahlbetonplatten. Nur die wesentlichsten Aspekte der Theorie werden behandelt und eine kurzgefasste Einführung zu neueren Entwicklungen wird gegeben.

1. INTRODUCTION

The aim of this paper is to present a survey of what is known in the theory of plasticity for reinforced concrete slabs.

Since the number of papers and books on the slab theory is very great, the references given are some selected papers, which in the authors opinion can be recommended as a starting point for further study of one particular specialized subject.

A number of important aspects of the theory have been left out of discussion because of space limitations. Such problems are rotation capacity problems, the application of the linear elastic solution as a lower bound solution, rules concerning the practical use of yield line theory and several others.

2. HISTORICAL REVIEW

The first contribution to the plastic theory of reinforced concrete slabs was made by the Danish engineer, Aage Ingerslev, [21.1] [23.1]. In 1921, he proposed a method of calculation based upon the assumption of constant bending moments along certain so-called yield lines. Several of Ingerslev's solutions have later proved to be exact, and his very early work has been of fundamental importance to the development of the theory.

Further pioneer work in this field was done by K.W.Johansen, [31.1][32.1][32.2] [43.1][49.1][62.1][72.1]. In his doctoral thesis from 1943 the theory took a very long step towards its final form.

In Johansen's work the yield lines had a geometrical meaning too, i.e. as lines along which a relative rotation of the slab parts meeting at the yield line takes place. Utilizing this he was able to define geometrically admissible yield line patterns and further his introduction of the work equation put him in a position to calculate upper bounds for the load carrying capacity. These contributions were of significance not only in the development of the slab theory but also, in general, in the development of the theory of rigid plastic materials. Mention should also be made of the introduction of the nodal force concept in the so-called equilibrium method, which sometimes considerably facilitates the calculation of upper bound solutions. His nodal force theory has, however, been the subject of some critisism, and several alternative theories have been formulated, see section 4.

Concurrently with Johansens work in Denmark, corresponding work was carried out in Russia, inter alia, by Gvozdev, see [59.1], in which Gvozdev's work is described.

One of the most important theoretical problems left unsolved by Johansen was the establishment of yield conditions. This basic information was not needed by Johansen, since he was able in a more or less intuitive way to find formulas for the work done in a yield line.

Yield conditions in the general case of orthotropic slabs were developed by the author, [63.1][64.1][69.1][71.1], and by Massonnet and Save [63.2], Wolfensberger [64.2], Kemp [65.1] and Morley [66.1].

It turned out that Johansen's formulas for the work in a yield line were in complete agreement with the yield conditions established. Hereby was his upper bound method put into the framework of the general theory of rigid plastic materials.

An early attempt to find a safe method for the calculation of the load carrying capacity was made by Hillerborg, [56.1][59.2]. He proposed to design several types of slabs by assuming the load to be carried only by bending moments in two perpendicular directions. To be economical, this so-called strip method generally requires the reinforcement to be varied through the slab. The strip method has been further developed by Hillerborg himself [74.2] and by others [68.4][68.5].

At the middle of the sixties the slab theory had almost obtained a final form and at that time it appeared as a special and useful case of the general theory of rigid plastic materials.

The developments since then have been concerned with three main subjects.

Firstly the theory as it was developed at the middle of the sixties had only taken account of bending and twisting moments, i.e. the in-plane forces were neglected. This is a more severe restriction in the theory of reinforced concrete slabs than in the classical theory of plates, since strains in the middle plane in a reinforced concrete slab develop, not only because of second order strain effects, but also because of the fact, that as soon as the concrete cracks the neutral axis seldom lies in the middle plane. Therefore the cracking leads to in-plane forces, especially if the slab edges are restrained. The membrane effect was first studied by Ockleston, [55.2].

The membrane effect often leads to a considerably higher load carrying capacity than calculated by taking account of the bending effects only.

Several papers have been published on the subject since, see section 9, but a general, practical design method has not yet been formulated.

Secondly the general development in optimization theory has also touched the reinforced concrete slab theory. The first results were reported by Wood [62.3] and Morley [66.2], who gave an exact solution for the simply supported square slab. Since then considerable progress has taken place and a great number of exact solutions exist, see section 7.

Thirdly the rapid development in automatic data processing has lead to a formulation of automatic design methods also in the reinforced concrete slab theory. One of the first contributions in this field was that of Wolfensberger [64.2]. The subject is now in a rapid development, see section 8, and in the near future one might expect that commercial programs for reinforced concrete slabs based on the theory of plasticity will be available.

3. BASIC EQUATIONS

3.1 Statical conditions

The statical conditions are the same as in the classical thin plate theory, i.e. the generalized stresses are in rectangular coordinates, x, y, the bending moments per unit length m_x and m_y and the twisting moment $m_{xy} = m_{yx}$. Besides we have the shear forces per unit length q_x and q_y. The statical boundary conditions are the so-called Kirchhoff boundary conditions requiring only the statical equivalence of the twisting moment and the shear force on the boundary to correspond to the internal forces.

It is often overlooked that the Kirchhoff boundary conditions in many cases express a physical reality, since the shear stresses arising from the twisting moments really are concentrated along the edges in such a way that it is natural to treat them as concentrated forces.

A stress field satisfying the equilibrium equations and the statical boundary conditions is as usual termed a statically, admissible stress field.

3.2 Geometrical conditions

The generalized strain rates corresponding to the generalized stresses m_x, m_y and m_{xy} are the curvature rates κ_x and κ_y and the rate of twist $2\kappa_{xy}$.

3.3 Yield conditions for orthotropic and isotropic slabs

Yield conditions for slabs can be derived in several ways. The most satisfactory way, in the author's opinion, is to derive the yield conditions on the basis of reasonable assumptions concerning the behaviour of the basic materials, concrete and steel. This was the way used by the author in [63.1] and [64.1], considering the action of bending and twisting moments in a slab. The basic ideas were already partly formulated by Jørgen Nielsen [57.1]. The yield condition can also be derived on the basis of the corresponding yield conditions for plates loaded in their own plane [63.7]. This method was used by the author in [69.1] and [71.1], giving the same result as the first mentioned method.

The yield conditions were derived by Massonet and Save too [63.2], on the basis of Johansen's formulas for the moments in a yield line. Essentially the same method was used by Wolfensberger [64.2] and Kemp [65.1]. The yield conditions have also been studied by Morley, [66.1], along similar lines as the author's.

The concrete is assumed to have a tensile strength equal to zero and a square yield locus.

The reinforcement bars are assumed to be able to carry only tensile or compressive stresses in their own direction.

Considering an orthotropic slab, i.e. a slab reinforced at the top and at the bottom in the same two perpendicular directions x and y , the yield conditions are found to be

$$\left.\begin{aligned} -(m_{Fx}-m_x)(m_{Fy}-m_y)+m_{xy}^2 &\leqq 0 \\ -(m'_{Fx}+m_x)(m'_{Fy}+m_y)+m_{xy}^2 &\leqq 0 \end{aligned}\right\} \qquad (3.3.1)$$

In the equations m_{Fx} is the numerical value of the positive yield moment in pure bending in a section perpendicular to the x-axis and m'_{Fx} is the numerical value of the negative yield moment in pure bending in a section perpendicular to the x-axis. The symbols m_{Fy} and m'_{Fy} have similar meanings. The first equation in (3.3.1) only applies when $m_x \leqq m_{Fx}$ and $m_y \leqq m_{Fy}$. Similarly the second equation only applies when $m_x \geqq -m'_{Fx}$ and $m_y \geqq -m'_{Fy}$.

In a m_x, m_y, m_{xy} - coordinate system, (3.3.1) corresponds to a surface consisting of two intersecting cones as shown in Figure 3.3.1.

The expressions are only valid for relatively small degrees of reinforcement, where the relative extension of the compressive zones in the concrete is small, see [63.1] and [64.1].

As will be seen the above yield conditions only contains bending and twisting moments, i.e. in-plane forces are neglected. This is sometimes a more severe limitation in the theory of reinforced concrete slabs than for metal plates, see section 9. Further the influence of shear forces in the direction of the slab normal is also neglected.

A moment field corresponding to points within or on the yield surface is as usual termed a safe moment field.

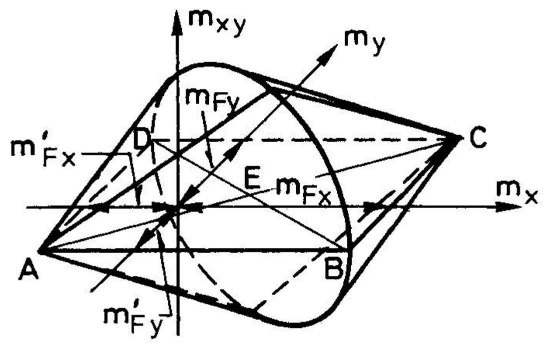


Fig. 3.3.1

The yield condition has been experimentally confirmed by tests on slabs in pure torsion, which gave a very good agreement between theory and tests, [69.1][71.1].

Other tests were also carried out, [63.8][67.2][67.3], but the confidence to the yield conditions derived lies mainly in the agreement between numerous tests on slabs and the load carrying capacity determined on the basis of the yield conditions.

The corresponding yield conditions for plates loaded in their own plane, [63.7], have been tested in several cases, [69.1][71.1].

For a rigid plastic structure with the generalized stresses Q_i , the generalized strain rates q_i and the yield condition $f(Q_i) = 0$, the flow rule is

$$q_i = \lambda \frac{\partial f}{\partial Q_i} \qquad \lambda > 0 \tag{3.3.2}$$

It is assumed that $f < 0$ for stresses, which can be carried by the structure. Geometrically (3.3.2) expresses, that the strain rate vector is an outward normal to the yield surface.

If the yield surface has an edge or a vertex, the strain rate vector is allowed to lie within the angle determined by the limits of the normals of the surface, when the stress vector approaches the edge or the vertex by all ways possible.

For an orthotropic reinforced concrete slab we get for instance in the case where the first expression in (3.3.1) is valid

$$\left.\begin{aligned} \kappa_x &= \lambda(m_{Fy} - m_y) \\ \kappa_y &= \lambda(m_{Fx} - m_x) \\ \kappa_{xy} &= \lambda\, m_{xy} \end{aligned}\right\} \tag{3.3.3}$$

Notice that in this region

$$\kappa_x \kappa_y = \kappa_{xy}^2 \tag{3.3.4}$$

i.e. one principal curvature rate is zero.

A similar conclusion holds if the second expresssion in (3.3.1) is valid.

The expressions along the edge and the vertices of the yield surface shall not be dealt with here.The reader is referred to [64.1] or [63.2].

In the special case $m_{Fx} = m_{Fy} = m_F$, $m'_{Fx} = m'_{Fy} = m'_F$ the slab is isotropic, i.e. the yield condition can be written in terms of principal moments m_1 and m_2 only.

28/7

The yield locus is shown in Figure 3.3.2. The principal curvature rates κ_1 and κ_2 according to the flow rule (3.3.2) are illustrated in the figure too.

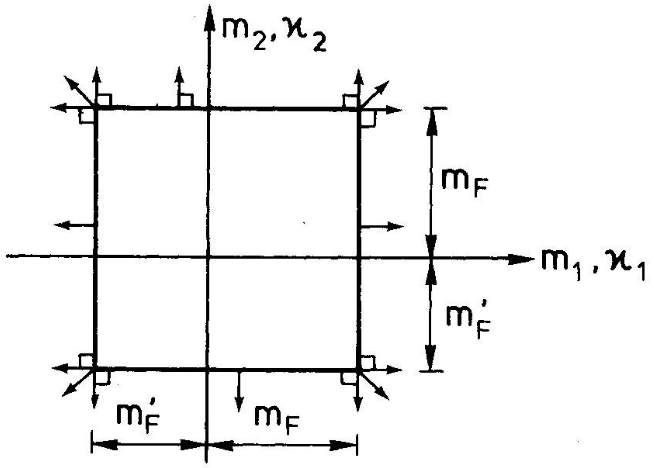


Fig. 3.3.2

This yield locus is often referred to as Johansen's yield locus.

If the problem is to design a slab to carry given bending and twisting moments, one can of course use the expressions for the yield condition to obtain safe values of the yield moments. Alternatively the reinforcement can be determined by means of the formulas:

$$\left.\begin{aligned} m_{Fx} &= m_x + \gamma\,|m_{xy}| & m_{Fy} &= m_y + \frac{1}{\gamma}\,|m_{xy}| \\ m_{Fx}' &= -\,m_x + \gamma'\,|m_{xy}| & m_{Fy}' &= -\,m_y + \frac{1}{\gamma'}\,|m_{xy}| \end{aligned}\right\} \qquad (3.3.5)$$

where γ and γ' are positive numbers, which can, theoretically, be arbitrarily chosen. The formulas follow immediately from the corresponding reinforcement formulas for plates loaded in their own plane, [63.7][69.1][71.1]. A set of formulas giving optimal reinforcement at the point considered, were developed by the author, [64.1][69.1][71.1].

It should be noted that if there are twisting moments along an edge, not only the top and bottom should be reinforced according to the formulas, but the edge itself should be reinforced, too, for instance by closed stirrups connecting the top and bottom reinforcement.

3.4 Yield conditions for arbitrarily reinforced slabs

For a plate loaded in its own plane and reinforced in several directions forming any angle to each other, it may be shown, [69.1], that the yield condition corresponds to an equivalent orthotropically reinforced plate. For a slab with the same lines of symmetry at the top and at the bottom the yield condition therefore corresponds to an equivalent orthotripic slab.

If the lines of symmetry are not the same at the top and at the bottom, yield conditions can be derived by means of the yield conditions for plates loaded in their own plane transformed to the coordinate system by means of which the yield condition is to be described. Braestrup [70.1] showed that the yield condition may also be formulated in moments referred to axes x,y arbitrarily oriented with respect to any number of reinforcement directions. The yield surface is bi-conical as the one shown on Figure 3.3.1, but the vertices A and C no longer lie in the plane $m_{xy} = 0$. We shall however not pursue this matter further here.

4. UPPER BOUND SOLUTIONS

4.1 Upper bound solutions by the work equation method

The upper bound technique is now well-known and described in several books and papers, see for instance [43.1][53.1][60.3][62.1][62.2][62.3][63.2] and [63.6], therefore we shall here only be concerned with the most fundamental aspects of the theory.

To establish an upper bound solution for the load carrying capacity of a rigid plastic slab, one has to find a geometrically possible deflexion rate field, write down the work equation, which equals the external work and the dissipation, i.e. the internal work carried out by the generalized stresses corresponding to the deflexion rate field. The solution of the work equation gives an upper bound for the load carrying capacity.

Of course the best answer one can get from a geometrically possible deflexion rate field containing more than one geometrical parameter is the one corresponding to the lowest load carrying capacity, therefore the solution found by means of the work equation has to be minimized with respect to the geometrical parameters.

The simplest type of geometrically possible deflexion rate fields is obtained by dealing with deflexion rates corresponding to discontinuities in the angular deflexion along straight lines, i.e. yield lines. These so-called yield line patterns, which were first considered by Johansen [43.1] and Gvozdev, see [59.1], can be easily found for any slab type utilizing the fact that a straight yield line separating two slab parts has to pass through the point of intersection between the axes of rotation for the two slab parts in question.

The dissipation along a yield line can be found by considering a yield line to be a narrow zone with constant curvature rate in one direction only. Let the curvature rate be $\kappa > 0$ in the n-direction forming an angle φ to the x-axis, see Figure 4.1.1.

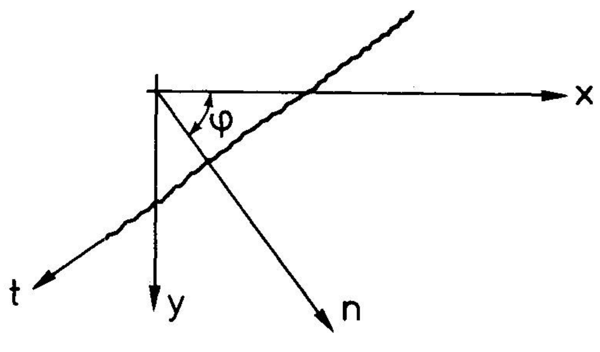


Fig. 4.1.1

Then we have

$$\kappa_x = \kappa\cos^2\varphi \qquad \kappa_y = \kappa\sin^2\varphi \qquad \kappa_{xy} = -\kappa\sin\varphi\cos\varphi \qquad (4.1.1)$$

Inserting these expressions into (3.3.3), and solving the equations with regard to the moments, we get

$$\left.\begin{aligned} m_x &= m_{Fx} - \frac{\kappa \sin^2\varphi}{\lambda} \\ m_y &= m_{Fy} - \frac{\kappa \cos^2\varphi}{\lambda} \\ m_{xy} &= -\frac{\kappa \sin\varphi\cos\varphi}{\lambda} \end{aligned}\right\} \tag{4.1.2}$$

The bending moment m_n is thus

$$m_n = m_x\cos^2\varphi + m_y\sin^2\varphi - 2m_{xy}\sin\varphi\cos\varphi =$$

$$m_{Fx}\cos^2\varphi + m_{Fy}\sin^2\varphi \tag{4.1.3}$$

which is the bending moment in a positive yield line.

In a similar way it is possible to calculate the twisting moment in a yield line. One finds

$$m_{nt} = \frac{1}{2}(m_{Fx} - m_{Fy})\sin 2\varphi \tag{4.1.4}$$

The formulas express the significant result that the bending and twisting moments in a yield line can be calculated as if the principal moments were found in sections coinciding with the directions of the reinforcement, which is naturally not the case at other points of the yield surface, than those corresponding to $m_{xy} = 0$.

These are the formulas intuitively proposed by Johansen, [43.1], which are thus consistent with the yield conditions developed later.

In the special case of an isotropic slab where $m_{Fx} = m_{Fy} = m_F$, we get

$$m_n = m_F \tag{4.1.5}$$

$$m_{nt} = 0 \tag{4.1.6}$$

i.e. the bending moment is independent of the angle φ, and the twisting moment is zero.

Similar expressions are of course valid for a negative yield line.

The dissipation D along the yield lines having the discontinuities θ_n in the angular deflection rates and the arc length ds, is

$$D = \int |m_n||\theta_n| d s \tag{4.1.7}$$

For practical purposes, however, it is simpler to calculate the work done by the external and internal forces on each slab part and thereafter summing over all slab parts. As the work done by an arbitrary system of forces, when it is rotated, is equal to the moment about the axis of rotation times the angle of rotation, the work equation may be written

$$\sum_j M_{ej}\omega_j = \sum_j M_{ij}\omega_j \tag{4.1.8}$$

where M_{ej} is the moment about the axis of rotation of the external load acting on the j'th slab part, M_{ij} is the corresponding moment with opposite sign of the bending and twisting moments along the yield lines, and ω_j is the rotation rate of the j'th slab part.

Even though only the bending moment perform work in a yield line, it is naturally possible to include the work done by the twisting moments, since their contribution vanishes by summation over all the slab parts, see [64.1].

It must be strongly emphasized that although the dissipation in a yield line is always positive, the terms on the left hand side of (4.1.8) may not all be positive.

For a continuous curvature rate field, the dissipation can be found by means of the flow rule and the expression for the internal work.

In the special case of an isotropic slab, the result is

$$D = \frac{1}{2}\iint[\frac{1}{2}(m_F+m_F')(|\kappa_1|+|\kappa_2|) + \frac{1}{2}(m_F-m_F')(\kappa_1+\kappa_2)]dx\, dy \qquad (4.1.9)$$

A simple example is a circular "fan", where a circular area or a part of a circular area is deformed to a cone with vertex in the center. There is a yield line along the limiting circle.

The dissipation in the more general case of a "fan" , where the negative yield line is an arbitrary curve , was derived by Mansfield, [57.2][60.1] and in a more direct way by the author [64.1]. For the case of nonpolar fans, see [67.4].

4.2 Upper bound solutions by equilibrium methods

Instead of using the work equation on yield line patterns Ingerslev [21.1] and Johansen [43.1] formulated an alternative approach based on equilibrium equations for the individual slab parts formed by the yield lines.

The main advantage of the equilibrium method is that the minimizing process in the work equation method is avoided. Using the equilibrium method the necessary algebra is often reduced a great deal compared to the work equation method. Furthermore by the equilibrium method, information is often gained for instance about column reactions and support reactions, information which cannot be delivered by the work equation method. Finally equilibrium equations may also show, how an estimated yield line pattern has to be changed in order to furnish a better result.

Ingerslev simply proposed to establish the necessary number of equilibrium equations by assuming, for an isotrop slab, that the shear forces and the twisting moments in the yield lines were zero, and that the bending moment for a homogeneously reinforced slab was constant. He demonstrated the technique in several examples, for instance the rectangular slab with uniform load. Johansen found that Ingerslev's solutions were in agreement with the minimized value obtained by the work equation. However cases were also found, where the two methods were not in agreement. Such a case is shown in Figure 4.2.1, where a yield line pattern consisting of one positive yield line, originating from a corner, is considered in a rectangular slab with two adjacent edges simply supported and the other two edges free.

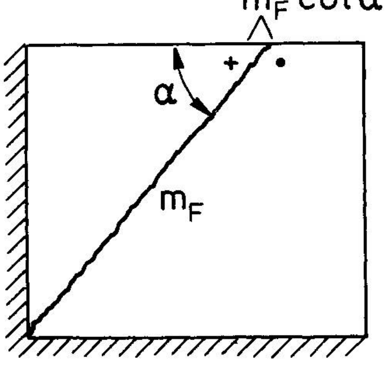


Fig. 4.2.1

The discrepancy was according to Johansen due to the fact that the shear forces and twisting moments are not allways zero in a yield line, and he proceeded to determine the statical equivalence of the shear forces and twisting moments in the form of concentrated forces at the ends of the yield lines, the so-called nodal forces. One important assumption in his calculations was that the bending moment has a stationary value in the yield line.

In the isotropic case considered in Figure 4.2.1, the nodal forces were found to be two numerically equal but opposite directed forces $m_F \cot\alpha$ at the point, where the yield line intersects the free boundary.

The nodal force theory of Johansen was not too convincing, and several attempts were made to improve the theory.

The author, [64.1][65.4], suggested to distinguish between nodal forces, which are simply the usual Kirchhoff boundary forces, and nodal forces, which are the statical equivalence of shear forces in internal yield lines. For isotropic slabs Johansen's theory and the author's gave identical results, while this was not the case for orthotropic slabs.

A sufficient condition for finding identical results by the work equation method and the equilibrium method is, according to the author's theory, that the equilibrium equations for each slab part that has been formed by the yield lines, are satisfied in such a way that a so-called stationary moment field may be found in each slab part. A stationary moment field is a statically admissible, but not necessarily safe, moment field for which, in the isotropic case, the shear forces and twisting moments are zero along all internal yield lines.

There are many cases, for which it is impossible to find a stationary moment field, and in all these cases, it has been found that the nodal forces cannot be determined by means of general formulas.

Some important examples are slabs, for which the number of geometrical parameters are not sufficient to make it possible to satisfy all necessary equilibrium equations, slabs where yield lines end at corners, slabs where yield lines intersect in a statically impossible way (e.g. three positive and one negative yield line) and slabs where a yield line passes point loads.

Alternative theories explaining the limitations of the Johansen nodal force theory have been given by Nylander,[60.2][63.5], Kemp, [65.2], Morley, [65.3], Wood, [65.5], Jones, [65.6] and Møllmann, [65.7].

Nodal forces can also be derived for curved yield lines, [43.1][64.1].

A number of solutions with curved yield lines were obtained numerically by the author, [62.4][63.3].

4.3 Yield line formulas

A collection of solutions for isotropic slabs covering most of the problems met in practice has been worked out by Johansen, [49.1][72.1]. By means of the affinity theorem, see section 6, the solutions can be used for a class of orthotropic slabs too.

To deal in an approximate manner by several loading cases Johansen, [43.1], found some superposition principles, see also [63.2].

5. LOWER BOUND SOLUTIONS

5.1 Introduction

One disadvantage of the upper bound methods for designing a slab is that it is very difficult, if not impossible, to vary the reinforcement in accordance with the stresses. Particularly this disadvantage is felt strongly, when one is concerned with the problem of determining the extent of top reinforcement, which is generally not carried through the whole slab. Also the reinforcement near columns and in supporting beams may constitute a problem when dealing with upper bound methods. Sometimes it is also argued that the upper bound methods are unsafe, since they lead to an overestimation of the load carrying capacity. This is of course, theoretically, correct, but this point is more or less academic, since the membrane effect generally gives a reserve capable of compensating more than necessary for this overestimation.

Nevertheless it is quite natural to study the possibilities of approaching the load carrying capacity from below.

A lower bound solution requires the determination of a statically admissible, safe stress field.

A number of lower bound solutions exists for isotropic and homogeneously reinforced slabs, but it is much more difficult by simple means to obtain a lower bound solution than to obtain an upper bound solution for a slab with given reinforcement.

The problem to find the reinforcement in a given slab is simpler, since then only a statically admissible stress field is required. Knowing this the necessary reinforcement can be determined by means of the formulas (3.3.5), which automatically renders the solution safe.

An extremely simple, statically admissible stress field can sometimes be found using Hillerborg's strip method [56.1][59.2][68.4][68.5][74.2], where only bending in two perpendicular directions is considered.

5.2 The strip method

The idea behind the strip method is that the slab is imagined to carry the load as two sets of beams at right angles to each other. Namely, if m_{xy} is made equal to zero in the equilibrium equation, we get

$$\frac{\partial^2 m_x}{\partial x^2} + \frac{\partial^2 m_y}{\partial y^2} = - p \tag{5.2.1}$$

which is satisfied, if

$$\left.\begin{aligned} \frac{\partial^2 m_x}{\partial x^2} &= - p_x \\ \frac{\partial^2 m_y}{\partial y^2} &= - p_y \\ p_x + p_y &= p \end{aligned}\right\} \tag{5.2.2}$$

The first and second equation in (5.2.2) are simple beam equations. The sub-division of the load per unit area p into p_x and p_y is arbitrary, and need not be the same throughout the slab.

It is rather evident that this simple method will be rather uneconomical, if the slab is homogeneously reinforced. If, however, the reinforcement is varied in accordance with the moment field, the reinforcement volume can easily compete with upper bound solutions, and, as shown by Hillerborg, [74.2], even exact solutions can be obtained.

The strip method can be used for many types of slabs supported on columns, if the moment field library is supplemented by a statically admissible moment field for a rectangular slab, uniformly loaded and supported in the middle on a column. Such a moment field has been developed by Hillerborg. With this moment field for instance, the slab shown in Figure 5.2.1 can be calculated by first assuming the load to be

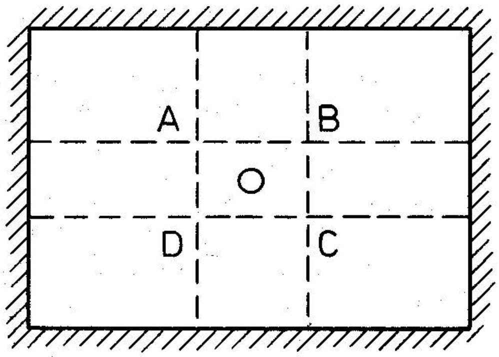


Fig. 5.2.1

transferred to the strips passing over the column. Then these strips are calculated as supported by a uniformly distributed reaction acting on the rectangular part ABCD. Finally these moment fields are superimposed on the moment field for a loading opposite to the reaction, acting on the rectangular part ABCD, which is now imagined as being supported on the column.

The strip method is not as general in its application as the yield line theory, in fact, it has to be altered and adjusted according to the various types of slabs.

5.3 Simple moment fields for rectangular slabs

The equilibrium equation for a slab can in the special case of uniform loading on rectangular slabs be satisfied if the bending moments vary as a parabolic cylindrical surface and if the twisting moments vary as a hyperbolic paraboloid. A moment field of this type was first suggested by Prager, [52.1], in his exact solution for the simply supported square slab, see also [55.1].

It has turned out that many rectangular slabs with different kinds of support conditions can be treated by the use of the above mentioned moment fields.

A number of solutions have been given by Bach and Nielsen, [78.2].

6. EXACT SOLUTIONS

6.1 Exact solutions for isotropic, homogeneously reinforced slabs

To find an exact solution one has to determine a statically admissible,safe moment field. The curvature rate field corresponding to this moment field, according to the flow rule, has to satisfy the compatibility equations and the corresponding deflexion rate has to satisfy the geometrical boundary conditions.

If the yield condition is satisfied in a zone, we might distinguish between 3 types of yield zones.

In type 1 both principal moments are equal to m_F or m_F'. It is easily shown that the equilibrium equations can only be satisfied with $p = 0$. The shear forces are similarly zero. Each section is thus a principal section.

In type 2 the principal moments are $m_1 = m_F$ and $m_2 = -m'_F$. It may be shown that in the case $p = 0$, the principal sections form a Hencky net (slip line net). This static analogy was first pointed out by Johansen, [43.1] and developed further by the author, [62.5][64.1]. A corresponding geometrical analogy was described by Johnson [69.2] and the complete analogy by Collins [71.2].

In a yield zone of type 3 there is only yielding in one principal direction. For instance we might have $m_1 = m_F$, $-m'_F \leqq m_2 \leqq m_F$, where the equals sign is only valid at certain points. Therefore

$$\left.\begin{array}{l} \kappa_1\kappa_2 = \kappa_x\kappa_y - \kappa_{xy}{}^2 = 0 \\ \kappa_1 + \kappa_2 > 0 \quad \text{or} \quad \kappa_1 + \kappa_2 < 0 \quad \text{everywhere} \end{array}\right\} \qquad (6.1.2)$$

The general solution to (6.1.2) is developable surfaces. The curves along which the principal curvature is zero are straight lines (generatrices). The possible surfaces are conical, cylindrical and tangential surfaces.

There exists a number of exact solutions for isotropic slabs, some of which are given in Figure 6.1.3.

The solutions a, b and c was given by Johansen, [43.1]. The solution b contains the well-known solution $P = 2\pi(m_F + m'_F)$, which is valid for a concentrated force acting on a circular slab with fixed or simply supported edges as special cases. As showed by Haythornthwaite and Shield [58.1], the solution is valid for an arbitrary fixed slab, g.

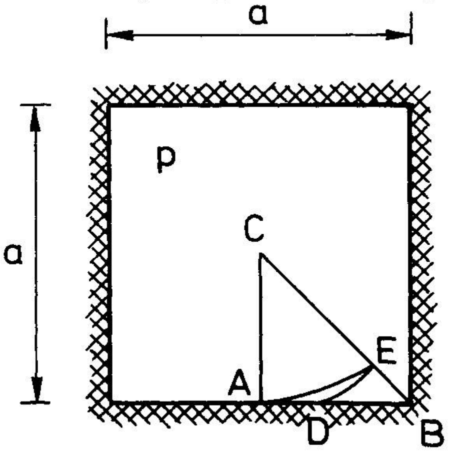
Exact solutions for circular slabs are relatively easy to obtain when the loading is rotationally symmetrical. Mention should be made of an interesting solution obtained by Nylander [59.3], for the case of a slap supported on both an exterior and an interior circular support, where two radial fields, separated by a circular yield line, do not solve the problem as could be expected.

Solution d was given by Prager, [52.1], and solutions e and f by Wood, [62.3]. Johansen gave solution f as an upper bound solution, [43.1]. Solutions h - p are the author's, [62.5][63.4][64.1].

The solution p and some other known solutions have equivalents in the slip line theory.

Ingerslev's yield line solution for the rectangular slab was shown by the author to be exact only if the negative yield moment has a certain value ranging from $m'_F = m_F$ for a square slab to $m'_F = 3m_F$ for a very long slab [64.1].

The clamped square slab for a long time denied its solution. In fact it was being claimed that the problem had no solution according to the present plastic theory, [68.3]. However in 1974 it was shown by Fox, [74.1], that the exact load carrying capacity in the case $m_F = m'_F$ is $p\,a^2/m_F = 42.851$.



$p\,a^2/m_F = 42.851$

Fig. 6.1.2

The solution turned out to be rather complicated containing one region C A E with a yield zone of type 3 and one region A E D with a yield zone of type 2. Finally there is a rigid portion E B D.

Fox, [72.2], also solved the rectangular simply supported slab with a concentrated force.

Finally a class of solutions was developed by Massonnet [67.1].

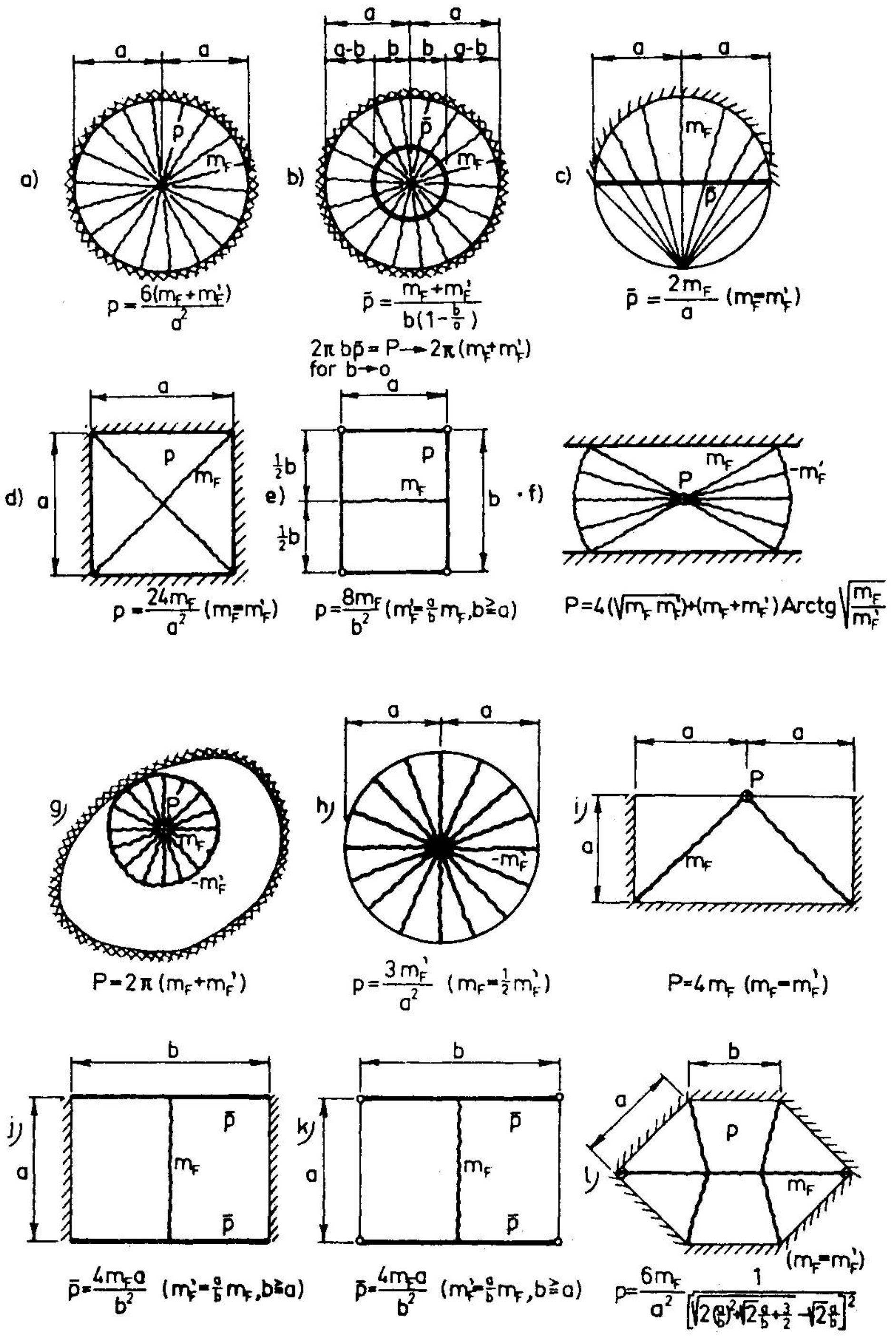


Fig. 6.1.3(continued)

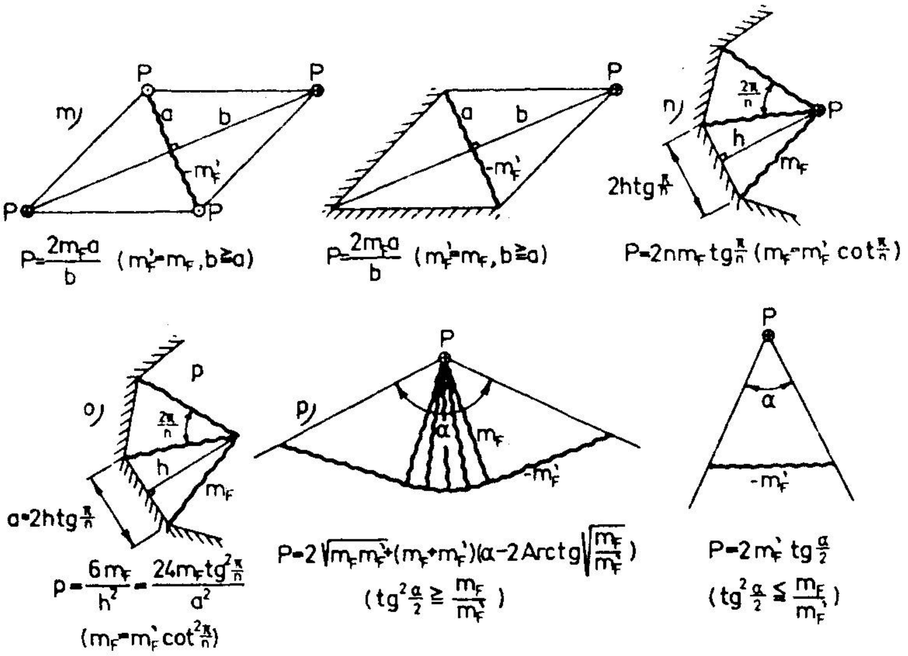


Fig. 6.1.3

6.2 The affinity theorem

For a special class of orthotropic slabs exists an interesting affinity theorem. It was developed by Johansen [43.1] for upper bound solutions and extended to lower bound solutions and exact solutions by the author [64.1].

The special class of orthotropic slabs for which the theorem is valid is characterized by $m_{Fx} = m_F$, $m'_{Fx} = m'_F$, $m_{Fy} = \mu m_F$ and $m'_{Fy} = \mu m'_F$.

The affinity theorem enables one to transform solutions for isotropic slabs to a special but rather general class of orthotropic slabs. This implies that for most practical purposes only calculations for isotropic slabs need to be performed.

7. ANALYTICAL OPTIMUM REINFORCEMENT SOLUTIONS

It is a natural task for a designer to look for one or another kind of optimal solution.

A fundamental question in the plastic theory for reinforced concrete slabs is to find the absolute minimum of the reinforcement volume for a given slab, with a prescribed load.

Considerable progress in answering this question has been gained by the work of Morley, Lowe and Melshers, Rozvany and others. A review paper containing most of the available information has been written by Rozvany and Hill [76.1], to which the reader is referred.

If the slab thickness has been given and if the variation of the compressive zones in the concrete is neglected, the Drucker-Shield criterion for minimum volume of a plastic structure, [56.2], immediately shows that one has to look for a constant principal curvature rate field throughout the slab, to which it is possible to assign a principal moment field corresponding in direction and sign to the curvature field. For many important cases the curvature rate field is the same for a wide class of load configurations on the same slab.

Morley, [66.2], gave a solution for the simply supported square slab, which is illustrated in Figure 7.1.

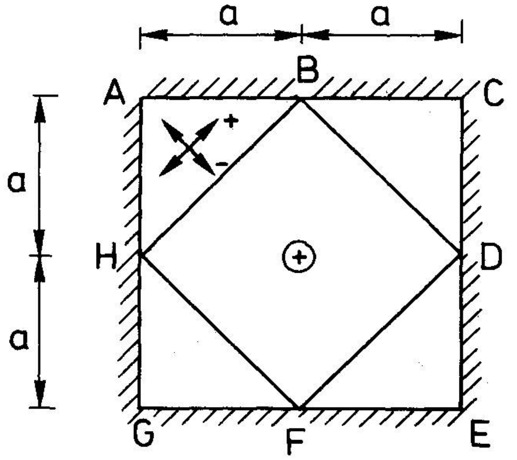


Fig. 7.1

In the region BDFH the two principal curvature rates are positive and equal. In the triangular regions the principal curvature rates are equal and have opposite signs. A load acting in the region BDFH is transferred to the infinitely narrow beams BD, DF, FH and HB by strip action. The strips can be arbitrarily selected. A load acting in the triangular regions can for instance be carried by strips lying under 45° to the edges and spanning from support to support.

A great number of solutions of this kind have been given by Rozvany anf Hill, [76.1].

It will be seen that the reinforcement has to be rather artificially arranged. A special problem is furnished if concentrated reinforcement bands in, theoretically, infinitely narrow beams is required since this might give rise to problems concerning the concrete stresses.

Anyway the optimal solutions are extremely useful as a basis for comparisons with the kind of solutions which for one reason or another are preferred by the designer.

Optimization of reinforcement with such constraints as to render the solutions more practical has also been considered. References may be found in the review paper by Rozvany and Hill, see also section 8.

8. NUMERICAL METHODS

The development of electronical computers has opened up new possibilities for finding approximate solutions to structural problems.

To find lower bound solutions in the plastic theory, one needs to create a sufficiently wide class of statically admissible stress fields and to find the one corresponding to the greatest load factor. Statically admissible stress fields can be created for instance by means of the finite element method, where the stress field within each element is expressed by a number of parameters. Equilibrium requirements within the element, continuity requirements along the element boundaries and the statical boundary conditions lead to a set of linear equations.

If the yield conditions are linearized, one gets a set of linear constraints, which together with the equilibrium equations constitutes a linear programming problem for the determination of the largest load which can be carried by the slab.

A similar method can be used in order to determine optimal reinforcement arrangements both in cases where the reinforcement is allowed to vary from point to point and in cases where the reinforcement arrangement is subject to certain geometrical constraints.

In Figure 8.1 a solution obtained by Pedersen [74.3] for the clamped square slab uniformly loaded ($m_F = m_F'$) is illustrated.

The finite element used was a rectangular element with bending moments varying as a parabolic cylindrical surface and twisting moments varying as a hyperbolic paraboloid, i.e. the load within each element was assumed to be constant.

The linearized yield conditions used were

$$\left.\begin{aligned} m_x \pm m_{xy} &\leq m_{Fx} \\ -m_x \pm m_{xy} &\leq m_{Fx}' \\ m_y \pm m_{xy} &\leq m_{Fy} \\ -m_y \pm m_{xy} &\leq m_{Fy}' \end{aligned}\right\} \qquad (8.1)$$

These equations were checked at the corners and in the middle of the element. However for the solutions obtained, the correct yield condition (3.3.1) were checked in a finer mesh, and the solution was proportioned if needed to fulfill the correct yield condition in all check points.

The figure shows the load carrying capacity obtained as a function of the mesh size. Also the total computer time is shown for some of the calculations.

As mentioned in section 6.1 the exact solution is $pa^2/m_F = 42.851$, which means that the best numerical solution deviates only a few percent from the exact one.

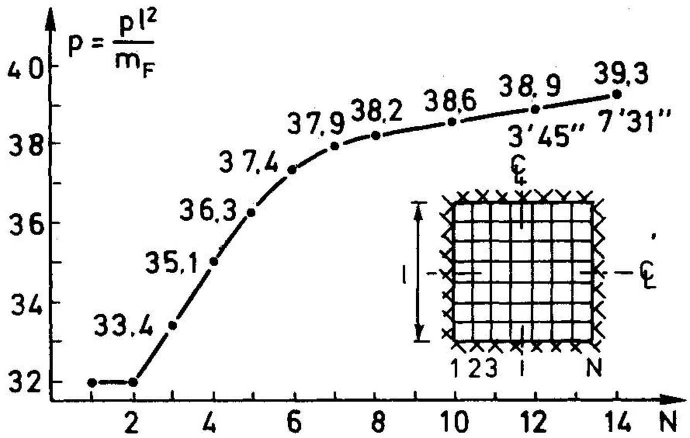


Fig. 8.1

The first calculations of this kind reported in the litterature were these of Wolfensberger, [64.2], whose procedure was very similar to that described above. Also Anderheggen, [72.4], Ceradini, [65.11], Gavarini, [66.4] and Sacchi, [66.5] have adopted such an approach.

Instead of using linear programming for the determination of the load carrying capacity of reinforced concrete slabs, Chan, [72.3], has used quadratic programming, which however led to considerably higher computer times.

Another approach has been used by Bäcklund, [73.1], who determined upper and lower bounds by following the complete behavior of the slab when the load grows from zero to the ultimate value.

Linear programming methods require a large computer and computer times far exceeding those required for linear elastic calculations. Nevertheless it is to be expected that in the near future commercial programs based on the plastic theory of reinforced concrete slabs will be in operation.

9. MEMBRANE ACTION

The theory presented neglects the fact that the strain field, corresponding to bending and twisting moments only, always results in strains in the slab middle surface, and these strains do not generally satisfy the compatibility equations and the geometrical boundary conditions. This leads to in-plane forces in the slab.

Further the rigid plastic theory in its standard formulation (1st order theory) neglects effects of changes in geometry. Since plates and reinforced concrete slabs often are rather flexible structures, the changes in geometry sometimes has a considerable effect on the load carrying capacity. These effects are often called membrane effects, and one speaks about a compressive membrane effect, which often predominates at small deflections and of a tensile membrane effect, which is dominating at larger deflections.

In plane forces arise already in the early stages of cracking.

A uniformly loaded, simply supported square slab often has a load-deflection relationship of a type shown in Figure 9.1. Instead of yielding under constant load, one

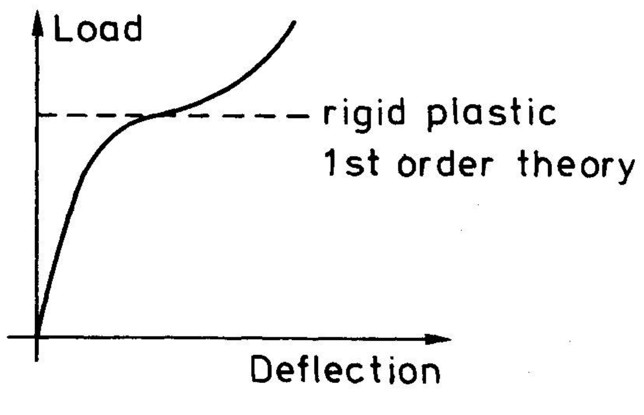


Fig. 9.1

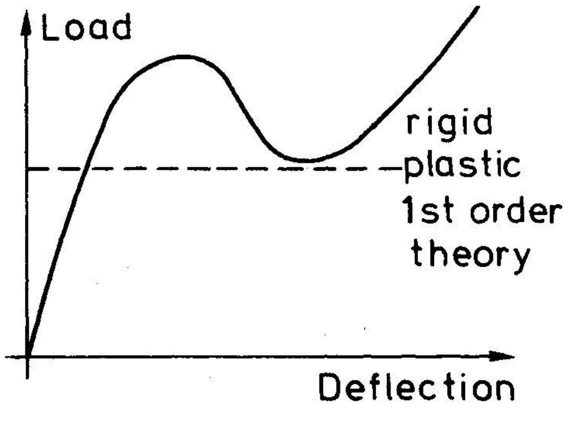


Fig. 9.2

hardly observes anything peculiar at the load corresponding to the rigid plastic 1st order theory. The real collapse load generally is somewhat higher than the rigid plastic 1st order load. Small degrees of reinforcement lead to relatively higher collapse loads compared to the rigid plastic 1st order load than higher degrees of reinforcement.

Quite different behavior is observed for a clamped slab if horizontal displacements are prevented along the edges. A typical load deflection curve is shown in Figure 9.2.

Failure is here by a snap-through action after which the load approximately reaches the rigid plastic 1st order load. Finally the load is again increased through a tensile membrane action. The maximum load may far exceed the rigid plastic 1st order load.

A theoretically correct determination of the full load deflection curve taking account of the elastic deformations, cracking of the concrete, realistic constitutive equations of the concrete until failure and the effect of changes in geometry is extremely complicated and has not yet been obtained.

Estimates of the effect of changes in geometry can however be obtained relatively simple by means of a series of upper bound calculations assuming the form of the deflected slab to be known. For instance a circular slab loaded at the center by a point load can be assumed to deflect as a cone similar to the deflection rate cone found by 1st order rigid plastic theory. Similarly a square slab can be assumed to deflect into a pyramidal form corresponding to the deflection rate form found by 1st order rigid plastic theory, too.

Having fixed the deflected form it is a relatively simple task by means of the usual upper bound technique to calculate the load corresponding to the deflected form assumed. The load carrying capacity of course turns out to be a function of the deflection.

For a simply supported slab, respectively a clamped slab, the load deflection curve obtained in this way will be of the type shown in Figure 9.3 and Figure 9.4.

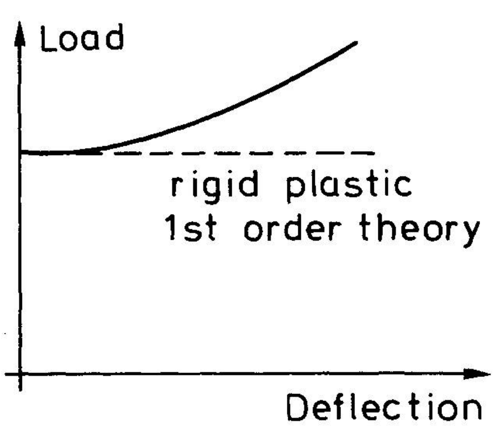


Fig. 9.3

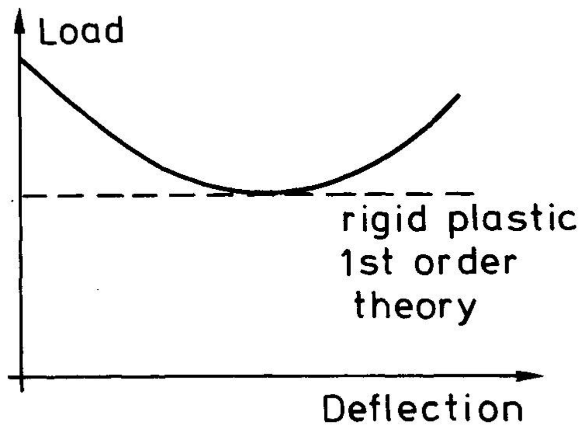


Fig. 9.4

The maximum load found for the clamped slab will not be reached in practice because of the elastic deformations neglected.

As shown by Calladine, [68.2], the calculations in several cases turn out to be very much simpler using the 3-dimensional theory instead of the 2-dimensional theory usually adopted in slab theory.

Because of the great effect of the elastic deformations on the load carrying capacity of clamped slabs, the rigid plastic theory cannot be used with confidence in practice. Since large reserves in load carrying capacity are inherent in the effect of changes in geometry, one of the most urgent needs of slab research is to create a reliable design method capable of utilizing these reserves.

Although already Johansen, [43.1], was aware of the tensile membrane action, the first to demonstrate the great effect of restrained edges was Ockleston, [55.2], who in a test series on a condemned building became aware of a break-down of the rigid plastic 1st order theory for internal slab parts. Several research workers have since that time studied the problem theoretically and experimentally, among them Wood, [62.3] and Park, [64.3]. An upper bound analysis of a type described above were among others performed by Sawczuk, [64.6][65.9], Janas and Sawczuk, [66.3], Morley, [67.6], Janas, [68.1] and, as mentioned already, by Calladine, [68.2]. A litterature survey has been performed by Bäcklund [72.5]. Concerning membrane action, see also [58.2][63.9][64.4][64.5][65.8][65.10][67.5][73.2][75.1][78.1].

NOTATIONS

⊕	Concentrated force
+	Downward-directed concentrated force
●	Upward-directed concentrated force
～～～～	Yield line
///////	Simply supported edge
xxxxxxx	Fixed edge
———	Free edge
━━━	Line load
o	Column without restraint

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Punching Shear in Concrete Slabs

Poinçonnement des dalles en béton

Durchstanzen von Betonplatten

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SUMMARY
The failure mechanism is examined, and various theories and design rules for centrical punching shear are reviewed. Based upon the classical theory of plasticity, an analytical solution is presented, describing the punching phenomenon in agreement with experimental evidence. Test results are compared with strength predictions of building codes and of plastic analysis. It is concluded that, in spite of completely different basic concepts, the two methods are not incompatible. Excentrical punching is briefly treated.

RESUME
Le mécanisme de rupture est étudié et diverses théories et règles de calcul pour le poinçonnement centrique sont évaluées. Fondée sur la théorie classique de plasticité, une solution analytique est présentée, décrivant avec fidélité le phénomène de poinçonnement. Des résultats d'essais sont comparés avec les prévisions de normes et de l'analyse plastique. Il y a lieu de constater que les deux méthodes ne sont pas incompatibles, bien que basées sur des notions complètement différentes. Le poinçonnement excentrique est enfin traité brièvement.

ZUSAMMENFASSUNG
Der Bruchmechanismus für zentrisches Durchstanzen wird untersucht, und verschiedene dafür entwickelte Theorien und Berechnungsverfahren werden besprochen. Eine auf die klassische Plastizitätstheorie sich stützende analytische Lösung wird angegeben, die das Durchstanzphänomen der Versuchserfahrung entsprechend beschreibt. Versuchsergebnisse werden mit rechnerischen Voraussagen von Bemessungsvorschriften und von plastischer Berechnung verglichen. Es wird gefolgert, dass die zwei Methoden nicht unvereinbar sind, obwohl die Grundlagen sehr verschieden sind. Exzentrisches Durchstanzen wird kurz behandelt.

1. INTRODUCTION

Punching shear failure may occur in concrete slabs - prestressed or conventionally reinforced - subjected to highly concentrated loads, e.g. impact loads or wheel loads on bridges, or at slender columns supporting flat slabs. The failure is located in a surface running through the slab from the loaded area to the opposite face (cf. Figure 1). The concrete body limited by the failure surface is simply punched out. This type of failure is not much impeded by the main reinforcement, and will therefore tend to reduce the ultimate load to a value below the flexural capacity of the slab.

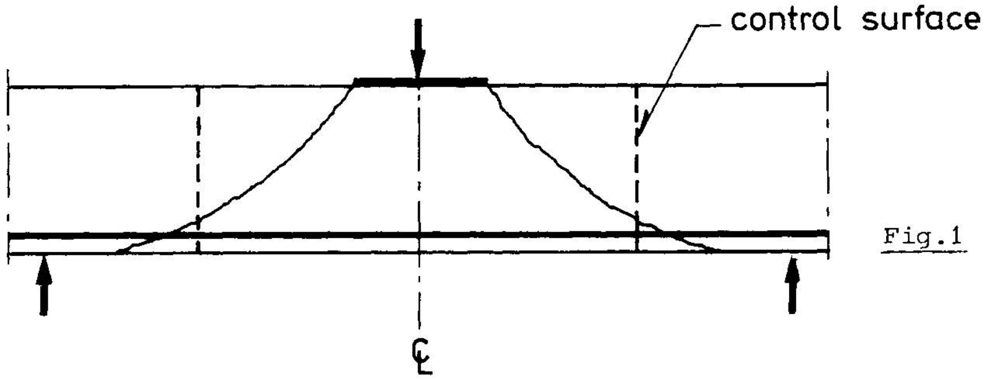


Fig.1 Punching shear failure

A decade of research has shown that the classical theory of plasticity may be used as an efficient tool in the analysis of shear problems in concrete structures, cf. NIELSEN & al. [78.3] and BRAESTRUP & al. [78.1].

In the present paper, we shall briefly review some design rules and theories for punching shear, and we shall present a theoretical approach based upon the theory of plasticity. The design rules and the predictions of plastic analysis are compared with some test results reported in the literature.

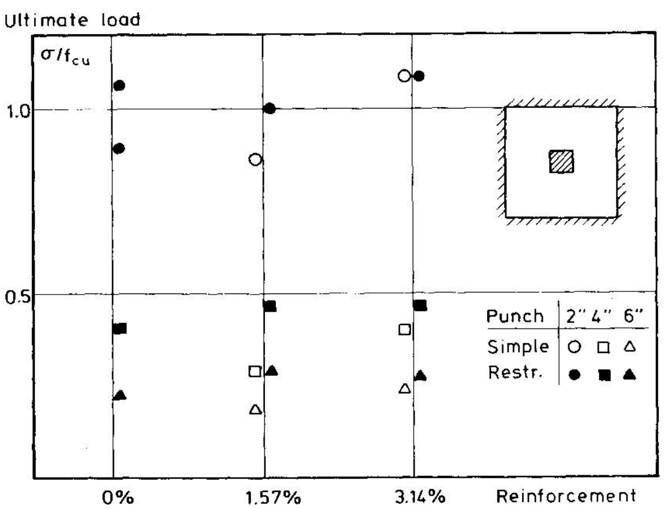


Fig.2 Punch load vs. main reinforcement. Simple and restrained slabs tested by TAYLOR & HAYES [65.2]

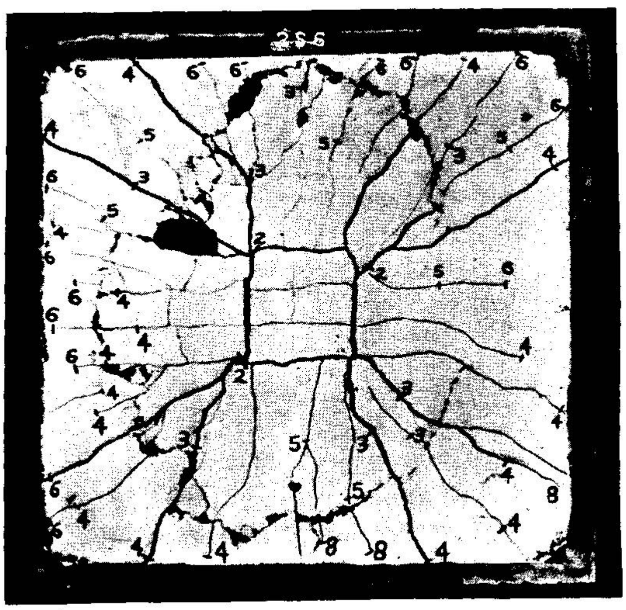


Fig.3 Failure of lightly reinforced simple slab with 6" punch. (From TAYLOR & HAYES [65.2])

2. THE MECHANISM OF FAILURE

The failure of slabs subjected to concentrated loading is very dependent upon the support conditions, especially the degree of restraint against in-plane edge movements. Thus the question of punching shear can hardly be separated from that of compressive membrane action (dome effect). This point is borne out nicely by a test series carried out by TAYLOR & HAYES [65.2].

They tested three series of square slabs, centrally loaded by square punches of varying size. The flexural reinforcement of the series corresponded to 0%, 1.57%, and 3.14%, respectively. The slabs were either simply supported or laterally restrained by a heavy welded steel frame. The unreinforced specimens, however, were tested in the restrained condition only.

The results corresponding to punches of sizes 2", 4", and 6" are shown on Figure 2. We have plotted the applied ultimate pressure σ, rendered nondimensional through division by the cube strength f_{cu}, against the percentage of reinforcement. It appears from the figure that not only are the strengths of the restrained specimens generally higher, they are also virtually independent of the amount of flexural reinforcement, which is not the case for the simply supported slabs.

In a real slab subjected to punching at an interior point, lateral movements will be restrained by the surrounding structure. Unfortunately, at most punching tests, care has not been taken to ensure similar conditions. Consequently, the ultimate load P_{test} may be expected to be approximately equal to the flexural capacity P_{flex}. Indeed, TAYLOR & HAYES [65.2] report that their unrestrained slabs with weak reinforcement were close to flexural failure.

The strength in flexure may be estimated by yield line theory, as done by GESUND & KAUSHIK [70.1]. They calculated the ratio P_{flex}/P_{test} for 106 alleged punching tests and found an average of 1.015 with a standard deviation of 0.248. Later, GESUND & DIKSHIT [71.3] developed punching shear formulas based upon yield line theory. The applicability of yield line theory was questioned by CLYDE & CARMICHAEL [74.5], who introduced considerations of the moment field (lower bound approach). The latter authors criticized the conventional test procedures, as did also CHRISWELL & HAWKINS [74.4].

The fact that so many flexural failures have been regarded as punching shear is probably due to the deceiving aspect of the collapse mode. However, one lesson to be learned from plastic analysis is that the actual appearance of the failure is not important for the strength. In fact, the ultimate load can often be predicted quite as well or even better by a completely different failure mechanism.

For weakly reinforced unrestrained slabs, the failure is accompanied by radial cracks and yielding of the main reinforcement, cf. Figure 3. The crushing of the concrete around the load and the spalling at the opposite face may be explained as secondary phenomena related to the rotational capacity in connection with the flexural failure mechanism. Consequently, failures involving yielding of the main reinforcement shall not be regarded as punching.

Heavily reinforced thin slabs may collapse before yielding of the reinforcement by a mode involving crushing of the concrete without the formation of a punching cone. Such slabs might be treated as overreinforced in flexure, and again the failure is related to lack of rotational capacity.

As a consequence, we shall consider as punching shear failures, only collapse modes characterized by the punching out of a concrete body in the direction of the loading, the remainder of the slab remaining comparatively rigid.

3. DESIGN RULES AND THEORIES

Most attempts to analyse the punching shear resistance of slabs are based upon the procedure as follows:
First a nominal shear stress is calculated, dividing the ultimate load P by the area of a socalled control surface around the loaded area, cf. Figure 1. The control surface is a cylinder, the section being a certain critical perimeter of length p, and the height being defined as either the total slab depth h, the effective depth d, or the internal moment lever arm z. The shear stress τ is then compared with a strength parameter for the concrete, usually some tensile strength measure.

The method sketched above was introduced by TALBOT [13.1] early in the century. TALBOT tested square footings, centrally loaded by square colomns of size a. As critical perimeter he used a square with the side length $a+2d$, and the depth of the control surface was taken as z. TALBOT concluded from the tests that the ultimate shear stress $\tau = P/4(a+2d)z$ was of the same order of magnitude as that of simple beams without shear reinforcement.

TALBOT's method forms the basis of most building code regulations concerning punching shear. In Section 7, we shall briefly review some examples, and compare the predictions with test results.

The code rules differ considerably in the definition of the control surface and in the choice of concrete strength measure. Some of the codes have modified the formula by introducing empirical factors depending upon the slab depth and the amount of flexural reinforcement. Many such empirical design rules have been formulated, and skall not be discussed here. References, reviews, and comparisons may be found in the reports from the ACI-ASCE Committee 326 [62.1], the Comité Européen du Béton [66.2], and the ASCE-ACI Committee 426 [74.2], as well as in the report by MOE [61.1], the paper by BERNAERT [65.1], and the thesis by ZAGHLOOL [71.5]. Further contributions are contained in the ACI Shear Symposium Volume [74.1]. Some authors, e.g. AOKI & SEKI [71.2], take account of compressive membrane effects, cf. also [74.2]. Excentrical punching is treated in Section 9.

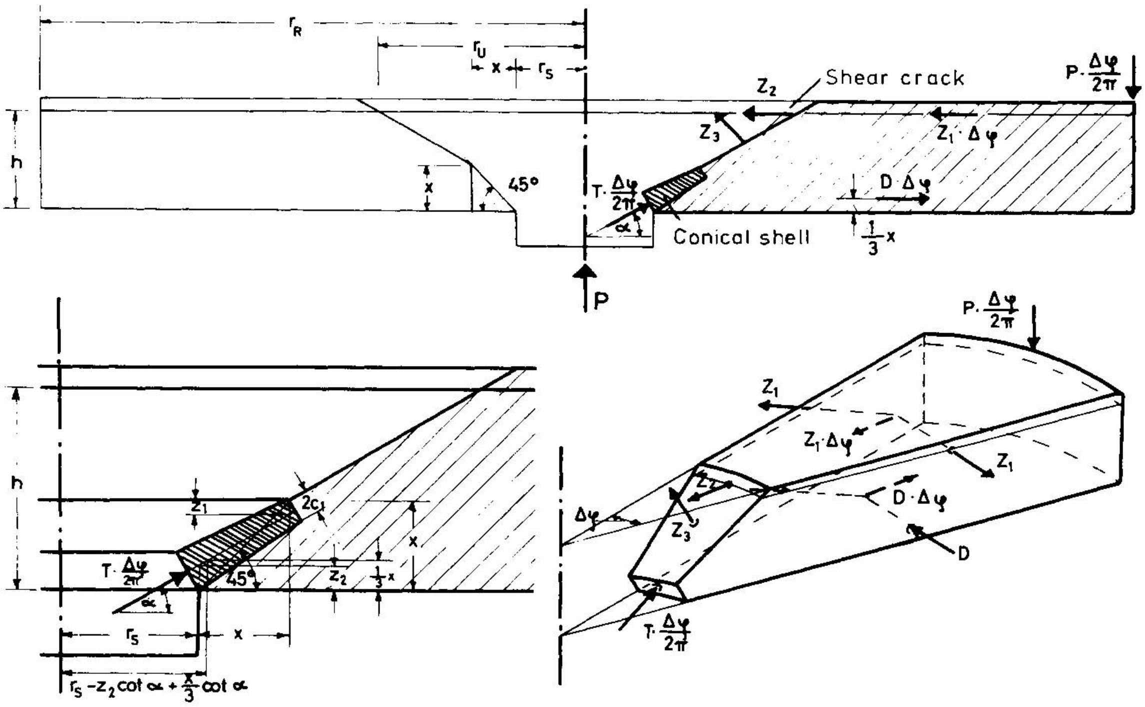


Fig.4 Mechanical model of KINNUNEN & NYLANDER. (From REIMANN [63.3])

A derivation of a punching strength formula based upon a rational mechanical model was attempted by KINNUNEN & NYLANDER [60.1]. They tested circular slabs, loaded at the free edge, and centrally supported on a circular column. The reinforcement was axisymmetrically disposed. Based upon test observations, an idealized model of the cracked state was proposed, cf. Figure 4. It consists of a rigid central truncated cone, confined by a shear crack, and segmental slab portions separated by yield lines. The segments are supposed to be carried by a compressed conical shell between the column and the root of the shear crack. Punching failure is assumed to occur when the tangential compressive concrete strain at the column face reaches a certain critical value.

The model was modified for orthogonal, two-way reinforcement by KINNUNEN [63.2], and extended to slabs with shear reinforcement by ANDERSSON [63.1]. The theory forms the basis for the Swedish code regulations [64.1], and has had a profound influence upon European code considerations as well. Thorough reviews with design examples are found in a CEB bulletin [66.2] and in the report by SCHAIDT & al. [70.3]. The analysis leads to an iterative procedure which tends to be quite complicated, and a simplified version has recently been given by NYLANDER & KINNUNEN [76.7].

The collapse mode considered by KINNUNEN & NYLANDER is basically that of flexural failure. Indeed, for the tests of [60.1], GESUND & KAUSHIK [70.1] found $P_{flex}/P_{test} = 1.075$ with a standard deviation of 0.159. This is bound up with the fact that the test slabs were provided with little or no lateral restraint. The theory has been modified by HEWITT & BATCHELOR [75.2], who introduce a "restraint factor", depending upon the boundary conditions, to take account of dome effects. The authors report excellent agreement with test results.

The model of KINNUNEN & NYLANDER was critically reviewed by REIMANN [63.3], who proposed a theory based upon a similar failure mechanism, replacing the compressed conical shell by a yield hinge around the column. As critical parameter, REIMANN considered the concrete stress (rather than the strain) in the circumferential direction at the column face.

Another theoretical approach to the punching problem is due to LONG & BOND [67.1]. The bending moments in the vicinity of the column are found by elastic analysis. Punching is considered to take place when the concrete stress reaches a critical value, corresponding to a failure envelope for concrete under biaxial compression. When various "correction factors" are introduced, the predicted punching loads agree well with test results. MASTERSON & LONG [74.9] extended the theory to cover the effect of compressive membrane action. A simplified version of the method has been presented by LONG [75.4].

A common feature of the mechanical models reviewed above, is that they involve yielding of the reinforcement, combined with crushing of the concrete around the punch (column) Thus they are related to flexural collapse.

The fact that no attempt was made to analyse a proper shear failure mechanism is probably due to the lack of a simple constitutive description of the resistance of concrete to shearing deformation. The problem was attacked by BRAESTRUP & al. [76.2], using the modified Coulomb failure criterion, introduced by CHEN & DRUCKER [69.1] and described in the section below. The analysis is reviewed in Section 5, cf. also NIELSEN & al. [78.3].

MARTI & THUERLIMANN [77.4] (see also MARTI & al. [77.5]) also apply the Coulomb failure criterion, but without a tension cut-off, cf. Sections 4 and 5.

4. CONSTITUTIVE MODEL FOR CONCRETE

A plastic analysis of the punching problem may be based upon the failure mechanism sketched on Figure 1 and discussed in Section 2: A concrete plug is punched out in the direction of the load, the rest of the slab remaining rigid. The deformations are located in a rotationally symmetrical failure surface, in which the concrete is in a state of plane strain.

To calculate the ultimate punching load, we apply the work equation, i.e. equate the external work done by the punching force with the internal work dissipated in the failure surface. To determine the internal work, we introduce a constitutive model for the concrete as follows:

> The concrete is assumed to be a rigid, perfectly plastic material with the modified Coulomb failure criterion as yield condition and the deformations governed by the associated flow rule (normality condition).

This assumption complies with the requirements of limit analysis. This means that a load found by equating the rates of external and internal work is an upper bound solution for the ultimate punching force.

The modified Coulomb failure criterion consists of two conditions:

$$\tau = c - \sigma\tan\varphi \qquad (1a),$$

and $$\sigma = f_t \qquad (1b)$$

Here τ and σ are shear and normal stress, respectively, on an arbitrary section in the material. c is the cohesion, φ is the angle of internal friction, and f_t is the uniaxial tensile strength. The uniaxial compressive strength f_c is related to the cohesion through the formula

$$f_c = 2c\sqrt{k} \qquad (2),$$

where the parameter k is determined by the angle of friction:

$$k = \frac{1+\sin\varphi}{1-\sin\varphi} \qquad (3)$$

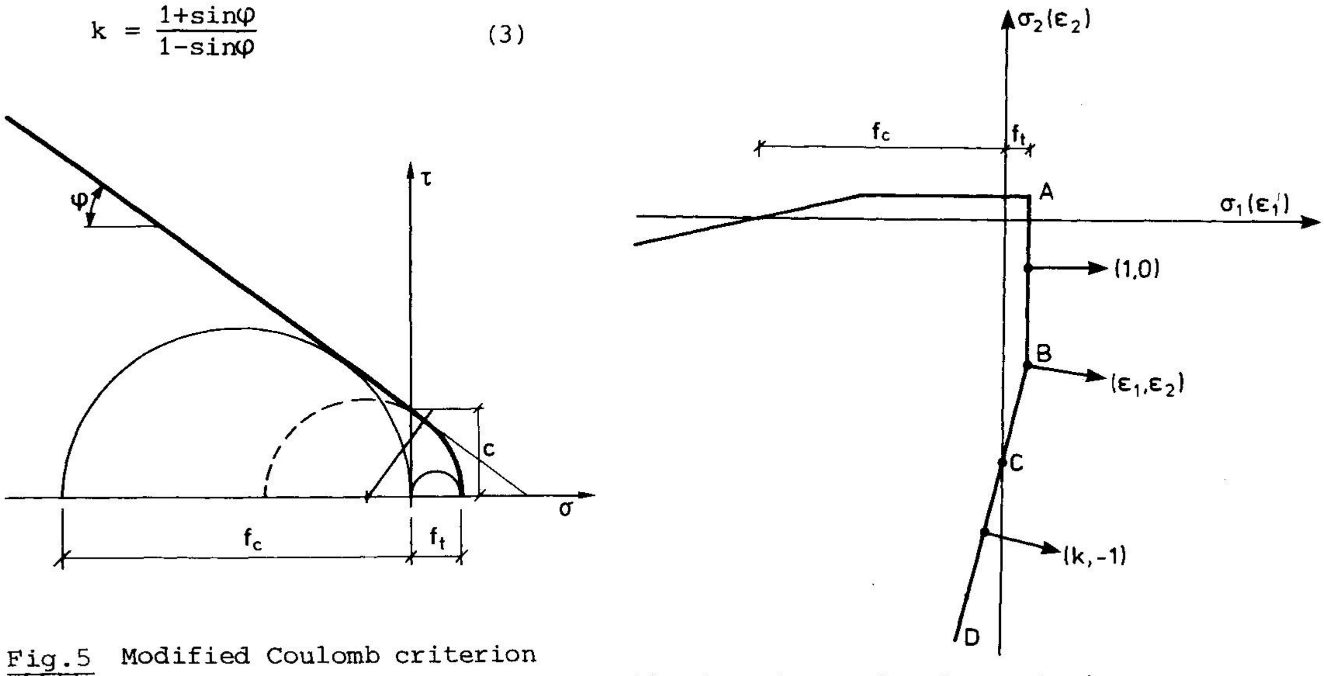


Fig.5 Modified Coulomb criterion
a) Failure envelope b) Yield locus in plane strain

The failure criterion is sketched in a σ,τ-diagram on Figure 5a. The straight line, equation (1a), corresponds to Coulomb sliding failure. The circular cut-off, representing equation (1b), is a modification introduced to take account of the limited tensile strength of concrete.

The yield locus in the case of plane strain is shown on Figure 5b. The principal stresses are σ_1 and σ_2, and the principal strain rates are termed ε_1 and ε_2. The associated flow rule requires the vector $(\varepsilon_1,\varepsilon_2)$ to be an outwards directed normal to the yield locus at the corresponding stress point (σ_1,σ_2). At a corner, the vector $(\varepsilon_1,\varepsilon_2)$ is confined by the normals to the adjacent parts of the locus.

This simple constitutive model describes the strength properties of concrete, and the deformations at failure, by means of only three material parameters: The tensile strength f_t, the compressive strength f_c, and the angle of friction φ. The elastic deformations are neglected, and unlimited ductility at failure is assumed. The latter assumption is rather unrealistic, and necessary modifications are discussed in Section 8.

The unmodified Coulomb criterion, equation (1a), used by MARTI & THUERLIMANN [77.4], contains only two parameters, e.g. the compressive and the tensile strengths. Then the angle of friction is given by equation (3) with $k = f_c/f_t$. Thus the criterion leaves no possibility of varying f_t independently of f_c without affecting the value of φ. Further, with a reasonably small tensile strength, the angle of friction must be fairly great (cf. Figure 5a), which is not realistic in the presence of hydrostatic compression. For instance, a tensile strength equal to 10% of the compressive strength corresponds to an angle of friction $\varphi \approx 55^\circ$.

Consider a kinematical discontinuity (failure surface) in the concrete. Figure 6b shows the intersection of the failure surface with the plane determined by the surface normal n and the relative velocity vector v, inclined at the angle α to the surface. The discontinuity is an idealization of a narrow region of depth Δ with a high, homogeneous strain rate v/Δ (Figure 6a). The rate of internal work W_ℓ dissipated per unit area of the failure surface is:

$$W_\ell = \Delta(\varepsilon_1\sigma_1+\varepsilon_2\sigma_2)$$

The principal strain rates are found to be:

$$\varepsilon_1 = \frac{v}{2\Delta}(1+\sin\alpha) \qquad \text{and} \qquad \varepsilon_2 = -\frac{v}{2\Delta}(1-\sin\alpha)$$

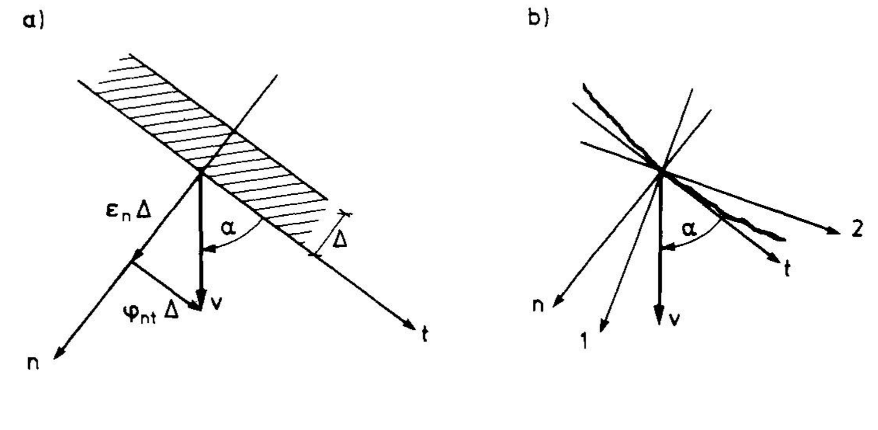


Fig.6 Failure surface in plain concrete
a) Narrow zone with high straining b) Kinematical discontinuity

According to the flow rule, the corresponding stresses are determined by point B on the yield locus, Figure 5b. Inserting, we find W_ℓ as a function of v and α:

$$W_\ell = \frac{1}{2} v f_c(\ell - m\sin\alpha) \quad \text{for} \quad \varphi \leq \alpha \leq \pi/2 \tag{4}$$

The parameters ℓ and m are defined as:

$$\ell = 1 - (k-1)f_t/f_c \quad \text{and} \quad m = 1 - (k+1)f_t/f_c \tag{5}$$,

where k is given by equation (3).

Equation (4) reduces to

$$W_\ell = v f_t \quad \text{for} \quad \alpha = \pi/2 \tag{6}$$

and $$W_\ell = \frac{1}{2} v f_c(1-\sin\varphi) \quad \text{for} \quad \alpha = \varphi \tag{7}$$

The assumed yield criterion and the associated flow rule does not allow the situation $\alpha < \varphi$. To describe such failure mechanisms,it will be necessary to introduce a more sophisticated constitutive model for concrete.

A more detailed derivation of equation (4) is given in reference [76.2] or [78.3].

5. PLASTIC ANALYSIS

Consider a concrete slab of depth h, supported along a circular perimeter with diameter D, and centrally loaded by a circular punch (or column) of diameter d_0. The applied load is P, and the slab is supposed to be reinforced in such a manner that flexural failure is prevented. We assume that the reinforcement is able to resist axial forces only, i.e. dowel action is neglected. Then the reinforcement does not contribute to the internal work, and the work equation yields:

$$P\,v = \int W_\ell \, dA \tag{8}$$

Here W_ℓ is given by equation (4), and the integral is taken over the entire failure surface, sketched on Figure 7a. The generatrix is described by the function $r = r(x)$, and we have:

$$dA = 2\pi r \frac{dx}{\cos\alpha} \quad \text{and} \quad \tan\alpha = \frac{dr}{dx} = r'$$

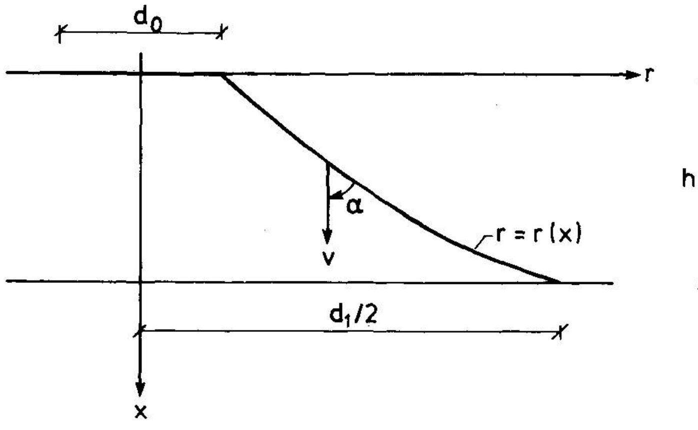


Fig.7a Failure surface

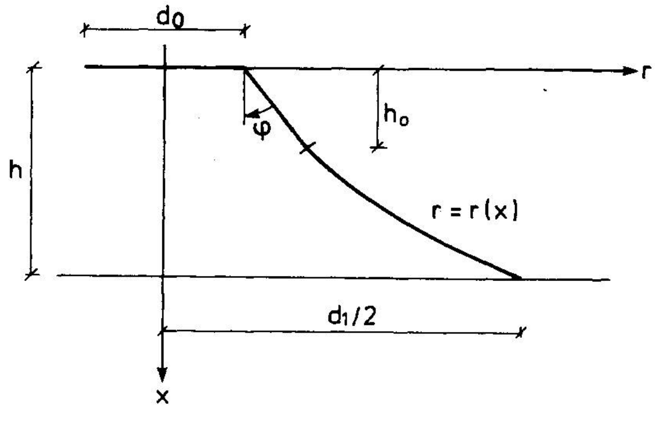


Fig.7b Optimal generatrix

Inserting into equation (8), we find the upper bound:

$$P = \pi f_c \int_0^h (\ell\sqrt{1+(r')^2} - mr')\ r\ dx \tag{9}$$

The lowest upper bound solution is found by variational calculus. The problem is to determine the function $r = r(x)$ which minimizes the functional (9), subject to the condition:

$$\alpha \geq \varphi \quad , \quad \text{i.e.} \quad r' \geq \tan\varphi \tag{10}$$

As shown by BRAESTRUP & al. [76.2], the solution is:

$$r = \frac{d_0}{2} + \tan\varphi \qquad \text{for } 0 \leq x \leq h_0 \tag{11a}$$

$$r = a\cosh\frac{x-h_0}{c} + b\sinh\frac{x-h_0}{c} \qquad \text{for } h_0 \leq x \leq h \tag{11b}$$

Here $c = \sqrt{a^2-b^2}$. The optimal generatrix is sketched on Figure 7b. It consists of a catenary curve (11b), joined by a straight line (11a) at the level $x = h_0$. The three constants a, b, and h_0 are determined by the boundary conditions:

$$a = \frac{d_0}{2} + h_0\tan\varphi \tag{12}$$

$$\frac{b}{c} = \tan\varphi \tag{13}$$

$$\frac{d_1}{2} = a\cosh\frac{h-h_0}{c} + b\sinh\frac{h-h_0}{c} \tag{14}$$

Here $d_1 \leq D$ is the diameter of the intersection of the failure surface with the bottom face of the slab (Figures 7). This diameter is determined so as to give the lowest upper bound. For certain values of d_0, d_1, and h, the optimal solution has $h_0 = 0$, i.e. the failure surface contains no conical part. In this case, equation (13) reduces to the inequality:

$$\frac{b}{c} \geq \tan\varphi \tag{15}$$

and equations (12) and (14) determine the constants a and b .

The corresponding lowest upper bound for the ultimate load is (cf.[76.2]):

$$P = \frac{1}{2}\pi f_c[h_0(d_0+h_0\tan\varphi)\frac{1-\sin\varphi}{\cos\varphi} + \ell c(h-h_0) + \ell\left(\frac{d_1}{2}\sqrt{\left(\frac{d_1}{2}\right)^2-c^2}-ab\right) - m\left(\left(\frac{d_1}{2}\right)^2 - a^2\right)] \tag{16}$$

where d_1 is given by equation (14).

In order to satisfy the condition (10), we must require that $D \geq D_0$, where

$$D_0 = d_0 + 2h\tan\varphi \tag{17}$$

If we choose $D = D_0$, the failure surface degenerates to a truncated cone, equation (11a). Then we have $\alpha = \varphi$ at all points, which means that the failure takes place as pure sliding (stress regime BD on Figure 5b). In this case, the failure load is independent of the tensile strength. Indeed, with $h_0 = h$, equation (16) yields:

$$P = \frac{1}{2}\pi f_c\ h(d_0+h\tan\varphi)\frac{1-\sin\varphi}{\cos\varphi} \tag{18}$$

Reference [76.2] describes an iterative procedure which determines the optimal upper bound solution for given geometrical quantities h, d_0, and D, and for given material properties f_t, f_c, and φ. The result is not very dependent upon the angle of friction, and the conventional value $\varphi = 37^o$ (corresponding to $\tan\varphi = 0.75$ and $k = 4$) is used throughout. The solution, however, is very sensitive to the ratio f_t/f_c. For $f_t = 0$, the lowest upper bound decreases with increasing d_1, which means that the optimal failure surface will extend all the way to the support. If we introduce a non-zero tensile strength, then the upper bound will be a minimum for a finite value of d_1. Figure 8 shows examples of optimal generatrices corresponding to various relative punch diameters d_0/h, plotted for different values of f_t/f_c. The support diameter D is chosen sufficiently great so as to not affect the solution. It appears that even a very small tensile strength results in a considerable contraction of the failure surface around the punch.

As a non-dimensional load parameter, we may take the quantity τ/f_c, where

$$\tau = \frac{P}{\pi(d_0+2h)h} \tag{19}$$

is the average shear stress on a control cylinder of depth h, circumscribing the loaded area in the distance h (cf. Section 3).

For $f_t = 0$, the theoretical load parameter is a function of the support diameter D and the punch diameter d_0. When the support diameter increases towards infinity, the ultimate load approaches zero asymptotically.

For a finite tensile strength and a sufficiently great support diameter, the failure takes place within the support ($d_1 < D$). Then the load parameter is a function of the punch diameter only. In reference [78.3], the load parameter is plotted as a function of d_0 and D for zero and non-zero tensile strength (cf. also Figure 10 of the section below).

a) $f_t = f_c/400$

b) $f_t = f_c/250$

c) $f_t = f_c/100$

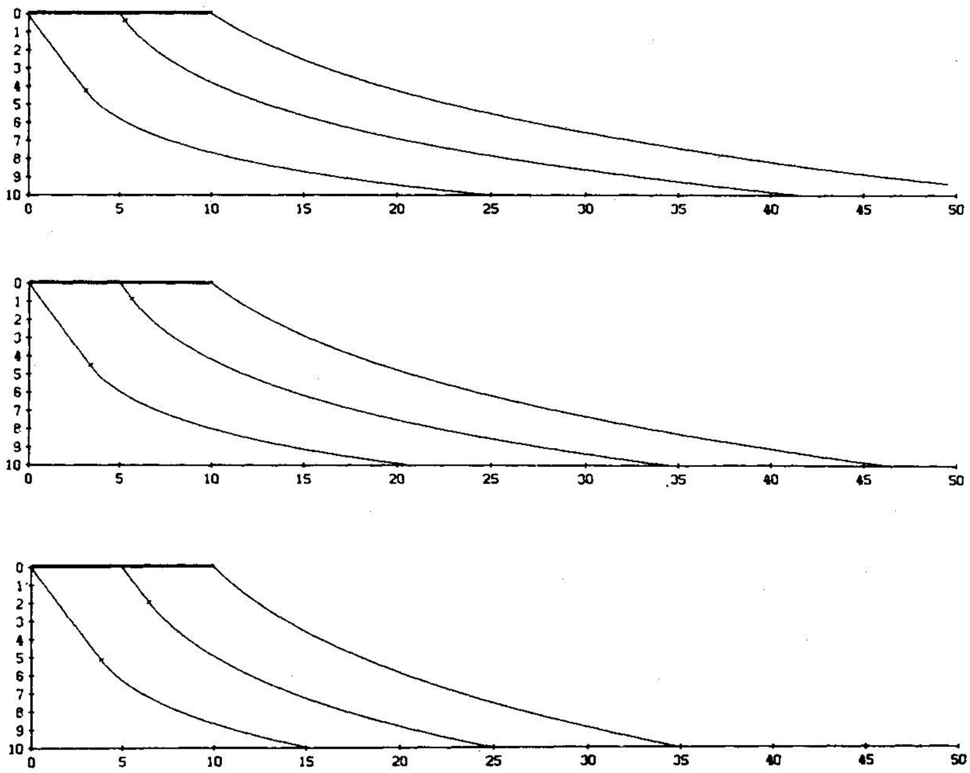


Fig.8 Optimal failure surface generatrices for $d_0/h = 0, 1$, and 2

The analysis shows that if we have some tensile strength and a support diameter which is not too close to D_0 , given by equation (17), then the load parameter is almost constant. Thus a shear stress τ , defined by an expression similar to equation (19), seems to be an appropriate choice as a design variable.

MARTI & THUERLIMANN [77.4] consider a conical failure surface with the half angle φ . They find the upper bound solution:

$$P = \frac{1}{2}\,\pi f_c h\left(\frac{d_0}{\sqrt{k}} + \frac{1}{2}h\left(1 - \frac{1}{k}\right)\right)$$

Inserting equation (3), this is seen to be equivalent with equation (18).

In reference [78.2], conical failure surfaces with half angles $\alpha > \varphi$ were analysed. The solution was improved by adding a truncated cone with half angle φ , in such a way that the generatrix becomes a broken line. Still, a substantial improvement is obtained with the optimal failure surface derived above (cf.[78.2]).

In the plastic analysis described in this section, it has been assumed that the punching load was balanced by an annular reaction only. At a column supporting a flat slab, the punching force is due to loads on the slab. A distributed counterpressure can easily be taken into account in the analysis, see [76.2]. The effect is very similar to that of a tensile strength of the same magnitude, cf. Figure 8.

The presence of a uniformly distributed shear reinforcement would have the same effect as a counterpressure. However, generally any shear reinforcement will be concentrated in one or several rings around the punch. Then it will have no influence upon the shape of the generatrix, unless a more dangerous failure surface can be found which does not activate all or part of the reinforcement. The yield force of the active shear reinforcement will simply have to be added to the punching load, equation (16).

6. EXPERIMENTAL VERIFICATION

The plastic analysis developed in the preceding section seems to offer a satisfactory description of punching failure, whether this is achieved by actual punching of slabs, or by pulling out of a disc imbedded in a concrete block. Similar phenomena are also observed at the popouts produced by internal pressure near a concrete surface, e.g. due to alkali-aggregate reactions, cf. BACHE & ISEN [68.1].

A striking feature of such failures is the extension of the failure surface, and the thin, even razor-sharp edge of the punched-out body. This is most easily appreciated at pull-out tests, where the failure surface is not disturbed by the presence of reinforcement. Figure 9a shows a failure piece produced by HESS [75.1]. The failure was obtained without any annular counterpressure, tension being applied simultaneously to two bolts imbedded in opposite faces of the specimen.

A test of this kind highlights the influence of the tensile strength. For $f_t = 0$, the theory predicts a splitting failure at the level of the imbedded disc, at an applied force equal to zero. This is obviously at variance with experience. However, as shown in reference [76.2], a tensile strength equal to only 0.25% of the compressive strength is sufficient to ensure realistic failure surfaces. This extremely low value indicates that the effectiveness of the tensile strength is very small (cf. Section 8).

Figure 9b shows the generatrix corresponding to $f_t = f_c/400$ and the same relative punch diameter ($d_0/h = 0.72$) as for the specimen of Figure 9a. The agreement between predicted and observed shape is excellent.

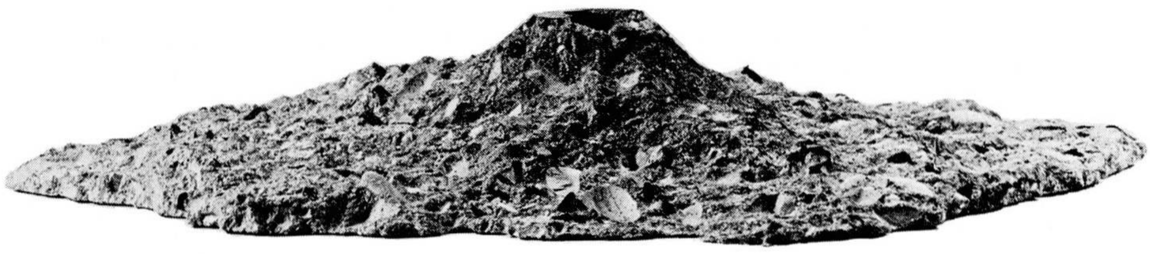
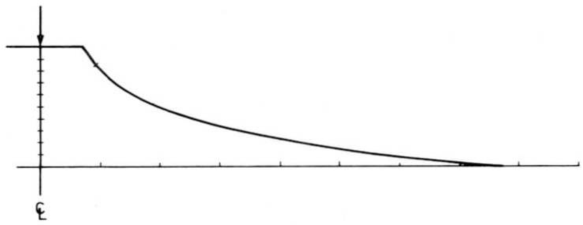


Fig.9 Pull-out test, $d_0/h = 0.72$
a) Failure piece (HESS [75.1])
b) Predicted shape, $f_t = f_c/400$

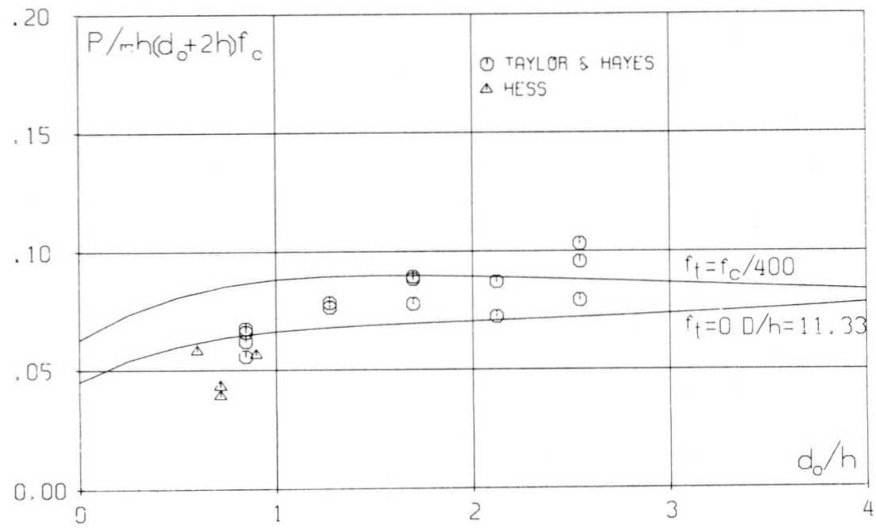


The fact that we have to introduce a diminutive tensile strength is of course not satisfactory. For the punching of slabs, where there is a well-defined maximum extension of the failure surface, we may (conservatively) neglect the tensile strength altogether. Provided the support diameter is not very great, this does not have any great influence upon the predicted ultimate load.

On Figure 10, we have calculated the load parameter for the restrained slabs tested by TAYLOR & HAYES [65.2], and plotted it against the relative punch diameter d_0/h (d_0 is taken as the diameter of the circle with the same perimeter as the square punch). For comparison are shown the theoretical curves corresponding to $f_t = f_c/400$ and to $f_t = 0$. In the former case, the tensile strength is sufficient to ensure that the failure takes place within the support. In the latter case, the support diameter D is put equal to the span of the square slabs. Figure 10 also shows the results of the pull-out tests of HESS [75.1]. For these points, the curve corresponding to $f_t = 0$ is without meaning, as the predicted load would be zero in that case.

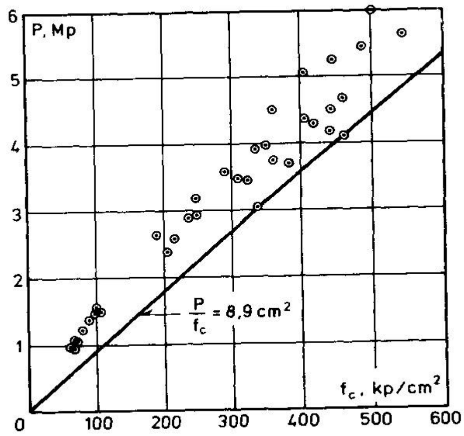
The plot shows that the load parameter does not vary much with the punch diameter, a fact which is reflected by the common design rules (cf. Section 3). What little variation there seems to be, is to some extent described by the plastic analysis.

Fig.10
Load parameter as function of punch diameter. Tests compared with theory.



If we choose the support diameter $D = D_0$, as given by equation (17), then the load is determined by equation (18), i.e. independent of the tensile strength. Consequently, it is possible to measure the compressive concrete strength by means of pull-out tests. This is the idea behind the Lok-test, developed by KIERKEGAARD-HANSEN [75.3]. The geometry of the test rig is very close to satisfying equation (17), but was designed empirically to give a good correlation between pull-out force and compressive strength. The success is demonstrated on Figure 11, showing the results of some tests carried out at the Structural Research Laboratory [74.3]. The solid line represents the relationship predicted by equation (18), if we take $\tan\varphi = 0.60$ to satisfy equation (17). As the angle of friction for concrete is slightly higher, the formula underestimates the pull-out strength somewhat (cf. JENSEN & BRAESTRUP [76.5].

Fig.11
Results of pull-out tests (Lok-strengths) compared with concrete cylinder strengths (References [74.3] and [76.5])



The applicability of the Lok-test to concrete quality control has been confirmed by field investigations, see LEKSOE & JENSEN [77.3].

The plastic analysis of the preceding section can also be applied to the radial punching of circular cylinders. A preliminary investigation of this problem indicates excellent agreement between the predicted and experimental loads, cf. HESS & al. [78.2].

7. PREDICTIONS BASED ON BUILDING CODES

Below we shall briefly review four typical codes of practice for punching design. They are all based upon the notion of a control surface (cf. Section 3).

CEB-FIP Model Code

The Comité Euro-International du Béton and the Féderation Internationale de la Précontrainte recently completed a Model Code [78.4] for reinforced and prestressed concrete structures. The design shear stress is calculated as:

$$\tau = \frac{P}{\kappa(1+50\rho)pd}$$

The critical perimeter p is defined as the length of the shortest, convex curve which nowhere is closer than $0.5\ d$ to the loaded area. The depth factor $\kappa \geq 1$ is calculated as $\kappa = 1.5 - d$, d being inserted in meters. The reinforcement factor $1+50\rho$ is determined by inserting $\rho \leq 0.008$ as the mean proportional of the reinforcement ratios in the two orthogonal reinforcement directions. The 1976 draft [76.3] for the Model Code contained the same formula, but the upper limit for the benificial influence of the reinforcement was considerably higher.

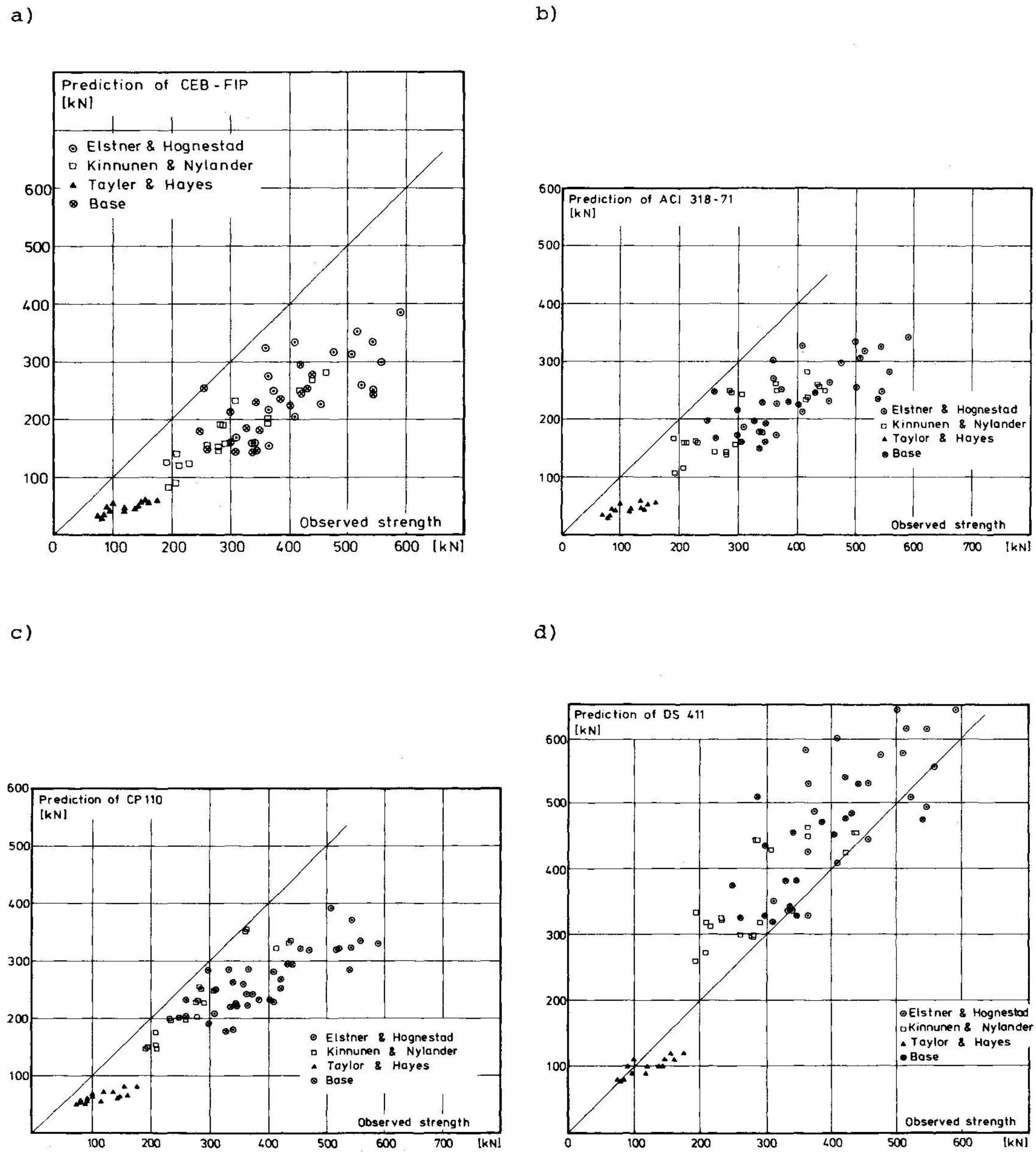


Fig.12
a) European Model Code
b) American building code
c) British code of practice
d) Danish code of practice

Punching test results compared with building code predictions

The shear stress τ is required to be inferior to $1.6\ \tau_{Rd}$, τ_{Rd} being the design concrete shear strength, tabulated as a function of the characteristic compressive strength f_{ck} (proportional to $f_{ck}^{2/3}$).

ACI 318-71

The American building code ACI 318-71 [71.1] puts:

$$\tau = \frac{P}{0.85\ pd}$$

where p is the minimum perimeter which approaches no closer than $0.5\ d$ to the loaded area. Thus obviously the critical perimeter must have rounded corners, like in the case of CP 110 (below) and the CEB-FIP Model Code (above). Nevertheless, the Commentary on the building code shows critical perimeters with sharp corners, in the manner of DS 411 (below). The comparison calculations (cf. below) are carried out using the minimum perimeter with rounded corners.

The shear stress τ must be inferior to the shear strength of the concrete, which is calculated as a function of the compressive strength f_c (proportional to $\sqrt{f_c}$).

CP 110

The British code of practice CP 110 [72.1] has

$$\tau = \frac{P}{\xi_s pd}$$

Here p is the smallest perimeter which nowhere is closer than $1.5\ h$ to the loaded area. The factor $\xi_s \geq 1$ depends upon the slab depth, according to a table in the code. The shear stress τ is required to be inferior to the concrete shear strength, which is tabulated in the code as a function of the compressive strength and of the ratio of reinforcement.

DS 411

The Danish building code DS 411 [76.4] puts

$$\tau = \frac{P}{ph} ,$$

where p is the perimeter of a figure similar to the loaded area in the distance d. The shear stress τ must be inferior to the tensile concrete strength which is tabulated as a function of the compressive strength f_c (proportional to $\sqrt{f_c}$).

As mentioned in Section 2, many tests reported in the literature as punching may just as well be described as due to flexure. HESS [77.1] has made a critical assessment of a great number of tests in order to exclude all the flexural failures. On Figures 12, some of the remaining results are compared with the strength given by the building codes described above.

The plots comprise some typical test series, viz.: ELSTNER & HOGNESTAD [56.1], KINNUNEN & NYLANDER [60.1], TAYLOR & HAYES [65.2], and BASE [66.2]. The latter series is reported on page 83 of the CEB Bulletin. In the analysis, average strengths are used, rather than characteristic values. Cube strengths f_{cu} are converted to cylinder strengths f_c by the formula $f_c = 0.8\ f_{cu}$.

The figures show that DS 411 slightly overestimates the load-carrying capacity, whereas the other codes are rather conservative, especially ACI 318-71 and the CEB-FIP Model Code. The latter represents a change from the draft [76.3], which was more liberal, cf. HESS & al. [78.2].

8. PREDICTIONS BASED ON PLASTIC ANALYSIS

The upper bound solution derived in Section 5 is based upon the simplified constitutive model introduced in Section 4, which assumes unlimited ductility of the concrete. In reality, however, concrete is not a perfectly plastic material. Particularly in tension, the behaviour is quite brittle. When applying plasticity to concrete, it is therefore prudent to neglect the tensile strength. As explained in Section 6, this leads to unrealistic results for punching (pull-out) without a support. The tensile strength which is necessary to explain the observed phenoma is very small indeed, of the order of $f_t = f_c/400$. This value is by no means indicative of the true tensile concrete strength, which is approximately 10% of the compressive strength. This shows that what tensile strength the concrete may possess, it is very little effective, due to the brittleness and possibly a "zipper" effect at failure. Consequently, we shall as a rule take the tensile concrete strength to be zero.

Also in compression, the ductility of concrete is quite limited, and we even have a falling branch on the stress-strain curve. Hence the redistribution of stresses which may be necessary to obtain the ultimate load predicted by plastic analysis can only take place at the expense of losing strength. This observation suggests that we might take account of the lack of ductility simply by reducing the concrete strength measure. Consequently, we replace the uniaxial compressive concrete strength by the effective strength f_c^*, where:

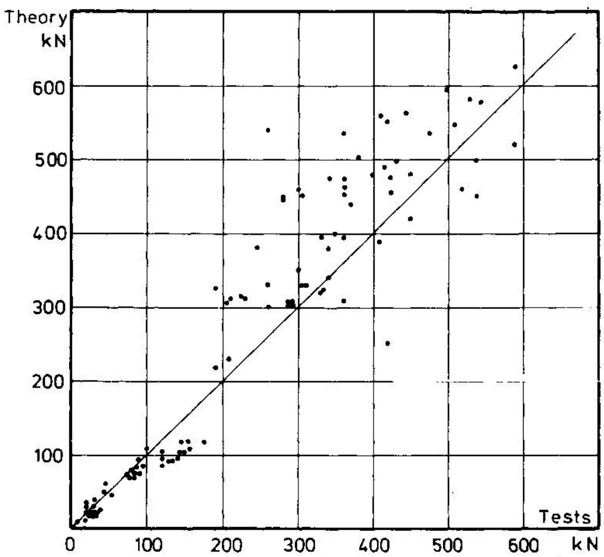
$$f_c^* = \nu f_c \tag{20}$$

Here f_c is the conventional strength, measured e.g. by the standard cylinder test, and ν is an empirical effectiveness factor describing the ductility of the concrete. The effectiveness factor is evaluated by comparison with experimental evidence.

In addition to expressing the concrete ductility, the effectiveness factor will have to describe all effects not explicitly accounted for in the theory, e.g. the influence of the neglected elastic deformations.

As seen from equation (16), the theoretical ultimate load is proportional to the concrete strength f_c. The value of ν for a given test can therefore be calculated as the ratio between the observed and the predicted strengths. The result will of course depend upon the amount of tensile strength assumed in the analysis.

Fig.13
Predictions based on plastic analysis compared with punching test results of
ELSTNER & HOGNESTAD [56.1]
BASE [66.2]
KINNUNEN & NYLANDER [60.1]
TAYLOR & HAYES [65.2]
DRAGOSAVIC & van den BEUKEL [74.6]
KAERN & JENSEN [76.6]
(References [77.1] and [78.2])



HESS [77.1] (cf. also HESS & al. [78.2]) analysed 101 punching tests carried out by ELSTNER & HOGNESTAD [56.1], BASE [66.2], KINNUNEN & NYLANDER [60.1], TAYLOR & HAYES [65.2], DRAGOSAVIC & van den BEUKEL [74.6], and KAERN & JENSEN [76.6]. Assuming $f_t = 0$ he found an average value of $\nu = 0.86$ with a coefficient of variation of 28%. For $f_t = f_c/400$, the average is $\nu = 0.69$, the coefficient of variation again being 28%.

BRAESTRUP & al. [76.2] analysed 54 tests, mainly pull-out tests reported by KIERKEGAARD-HANSEN [75.3]. Assuming $f_t = f_c/400$, the average was found to be $\nu = 0.83$ with a coefficient of variation of 16%. Thus it seems that in pull-out tests, the concrete is more effective, probably due to the greater stiffness of the specimen.

There appears to be a significant variation of ν with the concrete strength level: the stronger the concrete, the smaller the effectiveness factor. This trend is to be expected, since ν is principally a measure of ductility. The variation may be described empirically by assuming f_c^* to be proportional to $\sqrt{f_c}$. Interestingly, the same empirical relationship is often used between the tensile and compressive concrete strengths (cf. Section 7).

For all the 101 test results taken together, the best agreement is obtained with the formula $\nu = 4.22/\sqrt{f_c}$, where f_c is measured in MPa. Introducing this expression, the variance for the effectiveness factor is reduced by 12%, cf. HESS & al. [78.2]. For many of the individual test series, the variance is only half as great, indicating that much of the remaining scatter is due to difficulties in comparing different concrete strength measures.

For some of the experimental investigations, the concrete quality is given by the cube strength f_{cu}, and the results are analysed putting $f_c = 0.8\, f_{cu}$. Still, the values of f_c cover substantial variations in test procedures, regarding size of specimen, conditions of curing, and rate of loading.

On Figure 13, the predicted strengths of all the 101 test slabs are plotted against the values actually obtained. The analysis is carried out assuming $f_t = 0$ and $f_c^* = 4.22\sqrt{f_c}$. For tests with square punches, the punch diameter d_0 is taken as the diameter of the circle with the same perimeter. The support diameter for square slabs is put equal to the span length. The slab depth is inserted as the total depth h. In [78.2], similar plots are shown for the individual test series. Comparing Figure 13 with Figures 12, we note that the scatter is of the same order of magnitude as for the predictions based upon building code rules.

9. EXCENTRICAL PUNCHING

The heading actually covers two different problems: that of punching accompanied by moment transfer, and that of punching near an edge or corner of the slab.

Many attempts have been made to modify the empirical formulas based upon a control surface to take account of load excentricities. In addition to the reports cited in Section 3, reference is made to the papers by HERZOG [74.8], DRAGOSAVIC & van den BEUKEL [74.6], and van den BEUKEL [76.1]. The idea is to introduce additional shear stresses on the control surface, calculated assuming a linear variation in analogy with the normal stresses in a beam due to the bending moment. The validity of this approach is to some extent supported by analysis based upon elastic thin plate theory, cf. MAST [70.2]. Still, the method is purely formal, and any elaborate calculation of additional stresses is hardly justified, considering the rather arbitrary choice of the control surface.

Another semi-empirical method is the beam type analogy, cf. HAWKINS [74.7]. The slab sections framing into the column are idealized as beam sections, presumed capable of delivering bending moment, torque, and shear force at the control surface. A more rational approach is that of LONG [73.1], which is based upon elastic-plastic thin plate theory (cf. LONG & BOND [67.1]).

The plastic solution of Section 5 remains a valid upper bound also if the punching force at an interior slab point is accompanied by a bending moment. The question is whether or not the presence of the moment will significantly reduce the ultimate punching load. The moment is most effectively resisted by the main reinforcement, rather than by the concrete stresses in the failure surface. Therefore it would seem most reasonable to design the flexural reinforcement accordingly, and leave the punching design unaffected.

In contrast, at edges and corners the mechanism of failure is completely different. Due to the lack of symmetry, the deformation is no longer constrained to be perpendicular to the slab. Thus the main reinforcement will generally contribute to the internal work. Consequently, we would expect the resistance at edge and particularly corner columns to be governed by the flexural capacity. Indeed, ZAGHLOOL [71.5] found that punching at edge columns is a secondary phenomenon, developing after yielding of the reinforcement at the slab-column interface. Strength expressions depending primarily upon the amount of flexural steel were derived by beam type analogy. ANDERSSON [66.1] also found the shear stresses by beam type analogy and the theory of elasticity. The failure criterion was related to the theory of KINNUNEN & NYLANDER for interior columns (see Section 3), cf. also KINNUNEN [71.4].

Corner columns were investigated by INGVARSSON [77.2], who found that the shear failure was analogous to diagonal tension failure in beams. A similar failure mechanism was studied by ZAGHLOOL & de PAIVA [73.2].

Punching at edge and corner columns is the subject of a research project just started at the Structural Research Laboratory. The theoretical approach is based upon the constitutive model outlined in Section 4, and due account is taken of the flexural reinforcement, cf. above. Some tests will be carried out to complement the existing experimental evidence, which is rather meagre.

10. SUMMARY AND CONCLUSIONS

In Section 2, we define punching as a proper shear failure mechanism, distinguished from flexural collapse modes. The importance of lateral restraints is emphasized.

The commonly applied analyses of punching shear are reviewed in Section 3. Broadly speaking, two separate lines are followed. One considers the shear stress on a nominal control surface around the loaded area. This is a purely empirical method which has little relation to the actual punching phenomenon. The other approach is more rational, in the sense that it starts out from the collapse mode observed during tests. However, the considered failure mechanism is basically flexural.

Section 4 describes a constitutive model which may be used in the plastic analysis of a proper shear failure. The concrete is assumed to be a rigid, perfectly plastic material with the modified Coulomb failure criterion as yield condition and the associated flow rule.

An upper bound solution is derived in Section 5. The optimal shape of the failure surface is determined by variational calculus. The solution agrees well with experimental evidence, as shown in Section 6. The theory also explains the fact that with a suitable design of the test rig, it is possible to measure the compressive concrete strength by means of pull-out tests.

In Section 7 and 8, the results of punching tests are compared with the strength predictions of building codes and of plastic analysis. Although the latter is based upon concepts entirely different from those of the former, it confirms the applicability of the nominal shear stress on a control surface as a design variable. The main difference is that the compressive and not the tensile concrete strength is the governing material parameter. However, the effective concrete strength depends upon the cylinder strength in much the same way as does the tensile strength. In both cases, it is the ductility of the concrete which is the decisive factor. Whether this is expressed through an effective strength or through a tensile strength is to some extent a matter of taste.

Finally, Section 9 treats excentrical punching. It is suggested that moment transfer at internal columns be considered separately from the punching. On the other hand, edge and corner columns require a different approach, mainly because of the lack of lateral restraint. Indications are that plastic analysis may provide solutions to these problems as well.

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