Plastic Analysis of Reinforced Concrete Shear Walls

Théorie de plasticité appliquée aux voiles en béton armé

Anwendung der Plastizitätstheorie auf Wände aus Stahlbeton

P. MARTI
Research Associate
Swiss Federal Institute of Technology
Zurich, Switzerland

SUMMARY
A review is given on recent attemps to establish a general theory of plane stress in reinforced concrete based on the theory of plasticity. Yield criteria for reinforced concrete wall elements are discussed and some applications to reinforced concrete shear walls are presented.

RESUME
Un aperçu est donné de quelques travaux récents basés sur la théorie de la plasticité, devant conduire à une théorie générale de l'état plan de contrainte dans des éléments en béton armé. Des critères d'écoulement pour des voiles en béton armé sont discutés et quelques applications sont présentées.

ZUSAMMENFASSUNG
Neuere plastizitätstheoretische Arbeiten werden besprochen, die sich mit einer allgemeinen Theorie des ebenen Spannungszustandes in Stahlbetonelementen befassen. Fliessbedingungen für Stahlbeton-scheiben werden diskutiert, und einige Anwendungen auf Schubwände aus Stahlbeton werden angegeben.

1. INTRODUCTION

Shear walls in buildings are primarily designed to resist the effects of lateral forces due to wind or earthquake actions. Furthermore they are often also used to carry vertical loads. Apart from sufficient strength shear walls should provide appropriate stiffness for an acceptable behaviour under wind action and sufficient ductility to secure the integrity of the wall during several cycles of elastic-plastic deformation in an earthquake. The latter two problems are beyond the scope of the present paper. Only questions concerning the static strength of reinforced concrete shear walls are considered in the following. The discussion is further confined to recent applications of the theory of perfectly plastic bodies to such structural elements.

A typical example is shown in Fig. 1. Vertical and lateral loads are transfered to the shear walls by combined bending and membrane action of the floor slabs. Thus, each wall is subjected to combined bending, shear and axial forces. The individual shear walls may act together with the other structural elements. A portion of the total bending moment and axial force may be resisted by flange action of the transverse walls located at the edges of each shear wall. Acting as membranes the floor slabs may provide an external stirrup effect on the walls.

By means of simple truss models for each structural element a picture of the general mode of force transfer within and between the different elements of relatively complex systems similar to Fig. 1 can be obtained. Indeed, similar models are wellknown for applications to problems of shear transfer in reinforced concrete beams since the time of Ritter and Mörsch some seventy years ago. The extension to more general cases of plane stress conditions like those present in shear walls does not give rise to basically new problems. The truss model concept may be considered as a typical engineering approach which essentially coincides with the lower-bound method of limit analysis. Truss models, however are not able to reflect the kinematics of the collapse mechanisms of reinforced
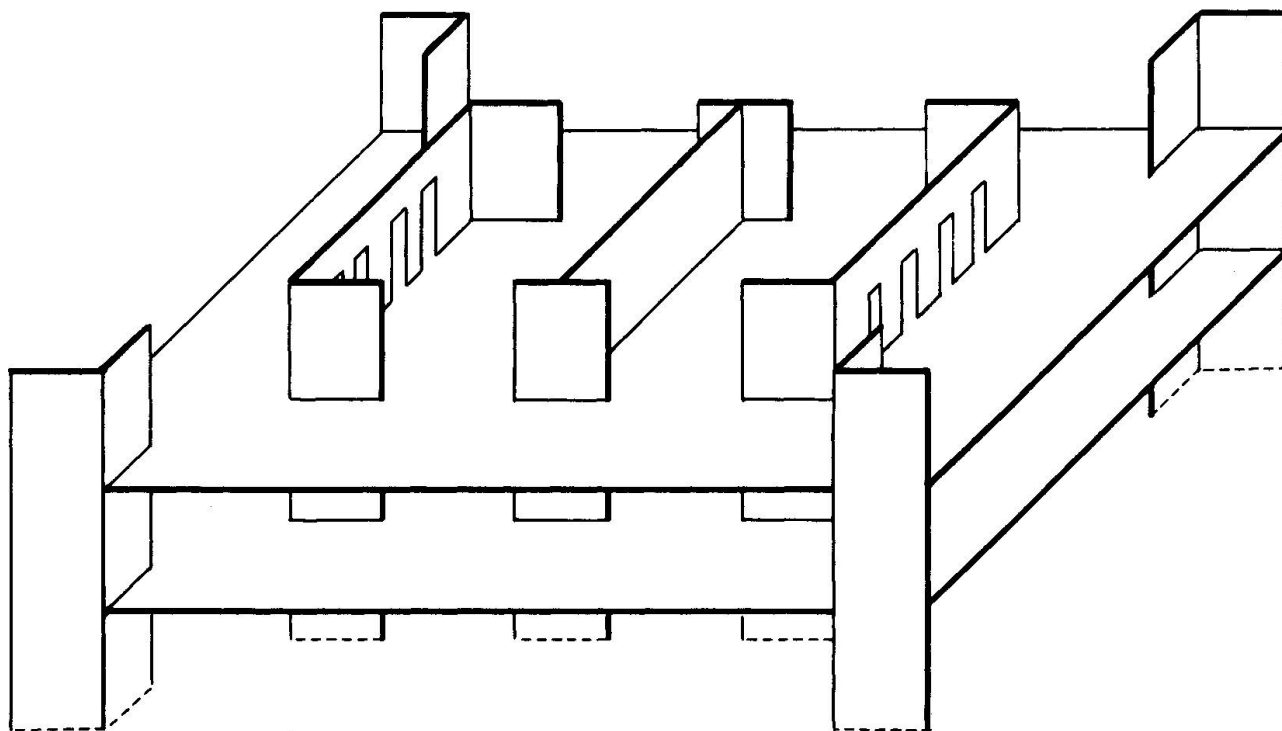

Fig. 1   Shear Walls in High-Rise Building

concrete beams or walls. Furthermore, local conditions e.g. the stress distribution at points of application of concentrated loads or reactions cannot be discussed in detail.

Recently attempts have been made to establish a general theory of plane stress in reinforced concrete based on the classical theory of plasticity. The purpose of the present paper is to review these developments. Yield criteria for reinforced concrete wall elements are discussed and some applications to reinforced concrete shear walls are presented. For simple problems exact solutions for the collapse load are obtained. In more general cases the methods of limit analysis allow to calculate lower and upper bounds for the collapse loads. Again, simple truss models prove to be helpful for constructing statically admissible stress fields according to the lower-bound method.

2. YIELD CRITERIA FOR REINFORCED CONCRETE WALL ELEMENTS

2.1 Assumptions

The investigation is based on the theory of perfectly plastic bodies in particular on the concept of plastic potential and the theorems of limit analysis.

The following assumptions are made:

1. The concrete is in a state of plane stress. It is a rigid-perfectly plastic material governed by a modified Coulomb yield criterion with associated flow rule.

2. The reinforcement is idealized as rigid-perfectly plastic with yield stresses $\pm f_y$. The bars carry forces in axial directions only. The distribution of the reinforcement is such that its action may be described by average stresses over the thickness of the wall (smeared reinforcement).

3. Local and bond failures are excluded.

2.2 Concrete

Coulomb's law of failure states that plastic flow occurs if the shear stress τ and the normal stress σ acting on any element in the material satisfy the linear equation

$$\tau + \sigma \cdot \tan\varphi - c = 0 . \tag{1}$$

c and φ denote the cohesion and the angle of internal friction, respectively. The ratio between the uniaxial tensile and compressive strengths is

$$\zeta = \frac{1-\sin\varphi}{1+\sin\varphi} .$$

As shown by Shield [1] Coulomb's yield surface is an irregular hexagonal pyramid in principal stress space. The boundary planes are determined by the equations

$$\sigma_i \cdot (1+\sin\varphi) - \sigma_j \cdot (1-\sin\varphi) - 2 \cdot c \cdot \cos\varphi = 0 , \qquad i \neq j. \tag{2}$$

For problems of bearing capacity of concrete blocks or rock Chen and Drucker [2] introduced a modified Coulomb yield criterion by assuming a small tensile strength

$$\sigma = f_t \geq 0 . \tag{3}$$

This modified criterion consisting of the two conditions (1) and (3), i.e. a material description with three constants, is taken as a basis for the further treatment. In Fig. 2 the yield criterion is represented in the stress plane (a) and in the principal stress space (b) for $\sigma_3 = 0$ (plane stress).

For the application of limit analysis concrete is often assumed to have zero tensile and finite compressive strength. According to this assumption the yield locus DEFGHI in plane stress, Fig. 2 (b), is reduced to the wellknown square yield criterion ODEF. Nielsen [3], [4], Müller [5] and Clyde [6] treated the problem of plane stress in reinforced concrete wall elements using the square yield criterion. Considering the brittleness and the actual strain-softening property of concrete in compression the assumption of perfectly plastic behaviour is somewhat questionable. Nevertheless, this drastic idealization may well be taken as a basis for the calculation of collapse loads of reinforced concrete beams and walls. It is only necessary to take the limited ductility of concrete into account by choosing an effective value f_c which is reduced in comparison with the ordinary compressive strength measured e.g. on cylinders, [5], [7], [8], [9], [10]. The value f_c of the so-called effective concrete compressive strength may be interpreted as the average failure stress in a particular strain range depending on the problem under consideration. The magnitude of f_c must be determined by tests. The concept of an effective strength or yield stress level in connection with the application of the theory of plasticity is not new. In fact it is widely used in soil mechanics [17].

For some applications of limit analysis to plain and reinforced concrete elements the disregard of the concrete tensile strength is too restrictive. The square

$$f_c = 2 \cdot c \cdot \tan(\frac{\pi}{4} + \frac{\varphi}{2}) \, f_t$$

(a)   Stress Plane
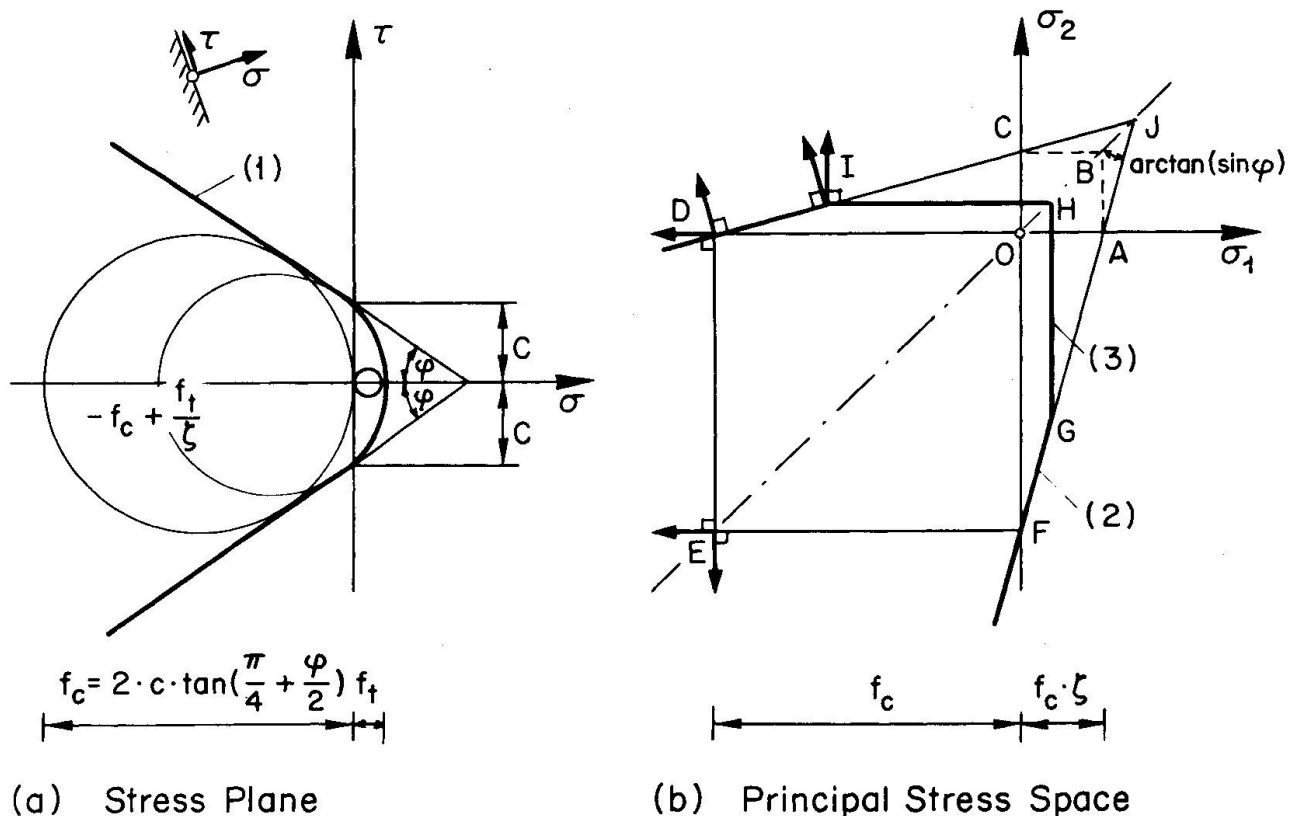
(b)   Principal Stress Space

Fig. 2   Coulomb's Yield Criterion

yield criterion is then replaced by a more general one, e.g. the modified
Coulomb yield criterion applicable to plane strain as well as general stress
states. Jensen [11] used the modified Coulomb yield criterion to analyse select-
ed problems of plain and reinforced concrete. Braestrup et. al. [12] used the
same criterion to determine upper bounds for the load producing axisymmetric
punching of a reinforced concrete slab. Marti and Thürlimann [13] treated prob-
lems of plane stress and plane strain in reinforced concrete by using the unmodi-
fied Coulomb yield criterion, i.e. two material constants for the concrete only.
As mentioned in [13], too high deviatoric stresses are obtained for higher hydro-
static pressures whereas the idealization might be reasonable e.g. in the tension-
compression domains for plane stress condition. To avoid these deficiencies the
use of the modified Coulomb yield criterion along with a realistic value for the
angle of internal friction φ as a third material constant was suggested.

2.3 Reinforcement

A set of reinforcing bars parallel to axis η inclined at angle ϑ to the x-axis
is considered, Fig. 3 (a). According to assumption 2 the action of the reinforce-
ment can be described by the average stress

$$z_\eta = \mu_\eta \cdot \sigma_s \quad , \quad |z_\eta| \leq \mu_\eta \cdot f_y \quad . \tag{4}$$

μ_η and σ_s denote the reinforcement ratio and the stress in the bars, respectively.
Transformation of the uniaxial stress state Eq. (4) to the orthogonal coordinates
x and y yields the average stresses

$$z_x = z_\eta \cdot \cos^2\vartheta \quad , \quad z_y = z_\eta \cdot \sin^2\vartheta \quad , \quad z_{xy} = z_\eta \cdot \sin\vartheta \cdot \cos\vartheta \quad . \tag{5}$$

These stresses are taken as vector components in the stress space Fig. 3 (b). By
linear combination of all possible stress states in the ξ- and η-reinforcements
the parallelogram-shaped domain BDFH for the skew mesh is obtained. The extension
to cases with more than two reinforcement directions is obvious. If the contri-
bution of the compression reinforcement is neglected, e.g. [5], [10], the shaded
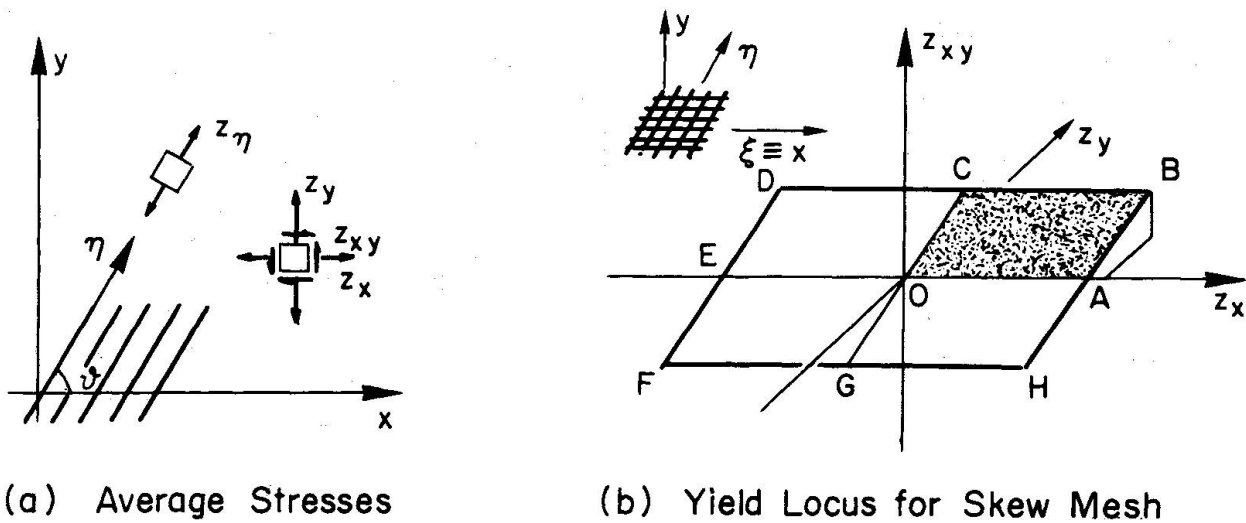domain OABC results.

(a) Average Stresses   (b) Yield Locus for Skew Mesh

Fig. 3   Yield Criterion for Reinforcement

2.4 Reinforced Concrete

Consider a differential wall element of reinforced concrete, Fig. 4. The thickness is taken equal to unity. The element is subjected to the membrane stresses N_x, N_y and N_{xy}. According to assumption 3 the bond between reinforcement and concrete is preserved until collapse. Therefore, the plastic strain increments or strain rates at collapse are equal for reinforcement and concrete. Hence, the yield surface in the stress space $\{N_x, N_y, N_{xy}\}$ is the envelope of all linear combinations

$$
\begin{aligned}
N_x &= n_x + z_x, \\
N_y &= n_y + z_y, \\
N_{xy} &= n_{xy} + z_{xy},
\end{aligned}
\tag{6}
$$

of stresses $\{n_x, n_y, n_{xy}\}$ and $\{z_x, z_y, z_{xy}\}$ which do not violate the yield criteria for the concrete and the reinforcement, respectively.

Fig. 4 Differential Wall Element

The modified Coulomb yield surface assumed for concrete is represented in Fig. 5. It consists of the three elliptical cones

$$
\begin{array}{lll}
\text{I} & : & n_{xy}^2 - (f_t - n_x) \cdot (f_t - n_y) = 0, \\[2mm]
\text{II} & : & n_{xy}^2 - [\zeta \cdot (f_c + n_y) - n_x] \cdot [\zeta \cdot (f_c + n_x) - n_y] \cdot \dfrac{1}{(1+\zeta)^2} = 0, \\[2mm]
\text{III} & : & n_{xy}^2 - (f_c + n_x) \cdot (f_c + n_y) = 0.
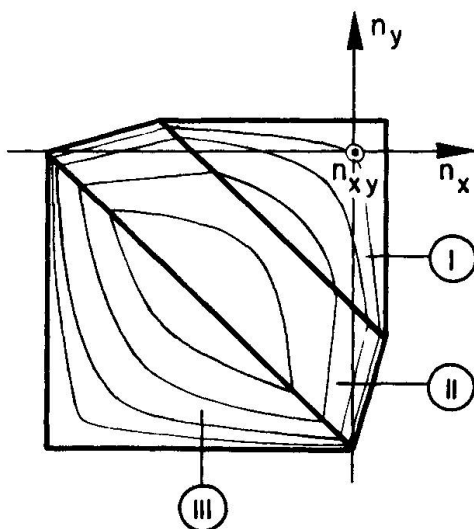\end{array}
\tag{7}
$$

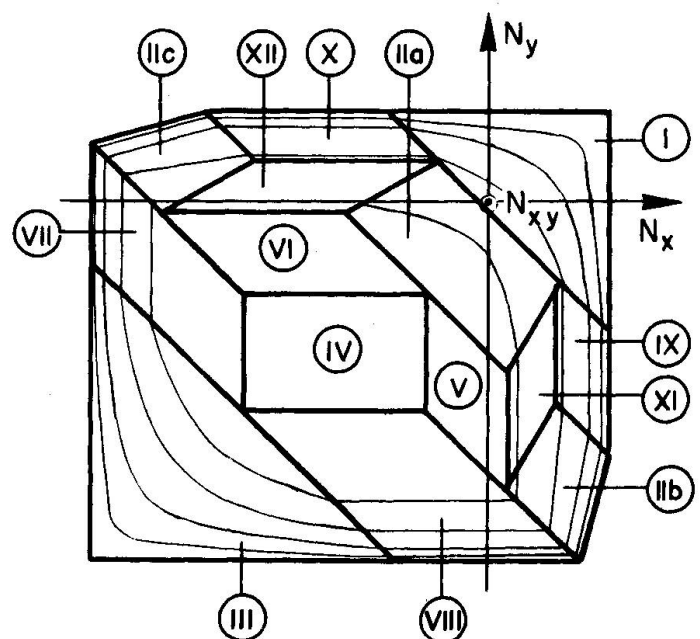Fig. 5 Yield Surface for Concrete in Plane Stress

Fig. 6 Yield Surface for Orthogonally Reinforced Wall Element

According to Eq. (6) the yield surface for the reinforced concrete wall element
may be obtained by simple translation of the yield surface for concrete, Eqs. (7),
with the origin moved within the yield surface for the reinforcement. In Fig. 6
the resulting yield surface is represented for an element reinforced in the two
orthogonal directions x and y. The cone II is split into three pieces. The two
cones I and III are simply shifted. New yield regimes are created, viz. the
planes IV, XI and XII and the circular cylinders V-X. The anisotropy of the yield
criterion is obvious.

The stress states in the concrete and in the reinforcement for each yield regime
may easily be visualized by means of simple geometrical considerations. As indi-
cated in [13], where Coulomb's yield criterion was assumed for concrete, the de-
tailed description of the different yield regimes given in that paper can be ex-
tended to the case of Fig. 6 without further difficulties. It is to be noted
that for the yield regimes V-XII the reinforcements in one and for regime IV the
reinforcements in both directions do not yield.

Except for several special cases the yield surface of Fig. 6 is of rather theo-
retical interest. In particular it should not be applied uncritically to plain
concrete. In fact for most practical applications in reinforced concrete it will
be judicious to neglect the concrete tensile strength and to provide a well dis-
tributed minimum reinforcement even in areas, where theoretically no reinforce-
ment is necessary.

If the concrete tensile strength is neglected the yield regimes II and IX-XII
disappear. In this form the yield criterion was derived for the first time by
Nielsen [3] in the case of isotropic reinforcement. For low degrees of reinforce-
ment say $\mu \cdot f_y / f_c < 0.1$ he suggested the use of an approximate yield surface com-
posed of the two cones I and III, where f_t in Eq. (7) is replaced by $\mu \cdot f_y$. The
kinematic aspects for isotropic walls governed by this yield criterion and the
associated flow rule were discussed in [4]. In this thesis various exact solu-
tions for isotropic walls (deep beams) and methods for the technical calcula-
tion of walls were presented. Müller [5] treated the yield criterion under the
assumption of zero concrete tensile strength for arbitrary reinforcement and in-
vestigated the general stress and velocity fields in walls which are possible for
the different yield regimes. For orthogonally reinforced elements the same yield
criterion was also given by Clyde [6] in an attempt to reconcile recent develop-
ments such as the space truss model and the skew bending approaches in the theo-
ry of shear, torsion and bending of reinforced concrete beams. In theoretical
accordance with the aforementioned references Clark [14] presented design equa-
tions for proportioning skew or orthogonal reinforcement to resist given membrane
forces and suggested limits to the applicability of mild and high-strength steel
as compression reinforcement.

2.5 Flow Rule and Velocity Discontinuities

According to the concept of plastic potential the plastic strain rates ε_x, ε_y
and γ_{xy} are given by the flow rule

$$\varepsilon_x = \sum_{k=1}^{n} \frac{\partial \Phi_k}{\partial N_x} \cdot \lambda_k \ , \qquad \varepsilon_y = \sum_{k=1}^{n} \frac{\partial \Phi_k}{\partial N_y} \cdot \lambda_k \ , \qquad \gamma_{xy} = \sum_{k=1}^{n} \frac{\partial \Phi_k}{\partial N_{xy}} \cdot \lambda_k \ , \qquad (8)$$

$$\lambda_k = 0 \text{ if } \Phi_k < 0, \qquad \lambda_k \geq 0 \text{ if } \Phi_k = 0 \ ,$$

where the yield criterion is determined by n functions $\Phi_k(N_x, N_y, N_{xy}) \leq 0$.

Application of Eqs. (8) to the different yield regimes of Fig. 6 shows that one of the two strain rates ε_x and ε_y vanishes for the regimes V-XII and that $\varepsilon_x = \varepsilon_y = 0$ for regime IV. This is the kinematic interpretation of the above statement that one or both reinforcements do not yield. It is further to be noted that the principal directions for the plastic strain rates coincide with those of the stress state in the concrete but deviate in general from those of the membrane stresses $\{N_x, N_y, N_{xy}\}$.

For many actual collapse mechanisms the velocity fields are found to be discontinuous. Furthermore lines of discontinuity for velocities often prove to be convenient in deriving upper bounds for the collapse load. Consider a narrow zone of homogeneous deformation with thickness $d \to 0$ between two rigid parts, Fig. 7(a). The direction of the relative velocity v between the two parts forms the angle α with the deforming zone. The principal strain rates indicated in Mohr's circle, Fig. 7(b), are

$$\varepsilon_1 = \frac{v}{2 \cdot d} \cdot (1+\sin\alpha), \quad \varepsilon_2 = -\frac{v}{2 \cdot d} \cdot (1-\sin\alpha) \quad . \tag{9}$$

The principal directions 1 and 2 bisect the angles between the parallel to the discontinuity, I, and the normal to the velocity direction, II. In these so-called slip-line directions pure shearing strain rates occur.

The kinematic conditions for the yield regimes I $(\varepsilon_1 > 0, \varepsilon_2 = 0)$, II $(\varepsilon_2/\varepsilon_1 = -\zeta)$ and III $(\varepsilon_1 = 0, \varepsilon_2 < 0)$ restrict the possibilities for discontinuities to $\alpha = \pi/2$, φ and $-\pi/2$, respectively. All other cases $\pi/2 > \alpha > -\pi/2$ can be related to one of the remaining regimes and in particular to the lines of intersection of different regimes.

Discontinuity lines for velocities according to the present theory should not be confused with the actual cracks observed in tests. The directions of the latter

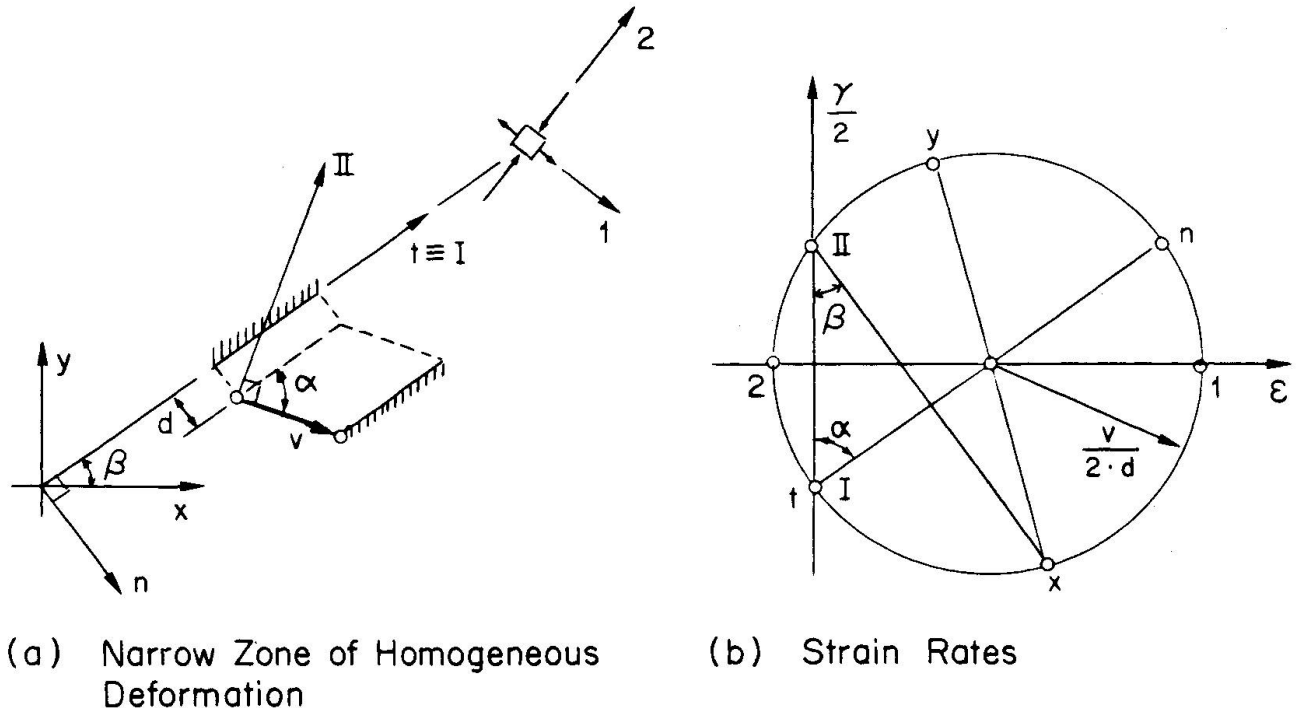(a)   Narrow Zone of Homogeneous   (b)   Strain Rates
      Deformation

Fig. 7   Line of Discontinuity for Velocities

tend towards the directions of the concrete compression trajectories at collapse. In contrast to this discontinuity lines or first slip lines may form angles up to $\pi/4$ with the principal directions in the concrete depending on the yield regime considered. Only for regime I ($\alpha = \pi/2$) a simple separation parallel to the concrete compression trajectories occurs. For this special case with coinciding slip lines the term "collapse crack" was introduced by Müller [5], [15].

2.6 Experimental Evidence

The redistribution of stresses predicted by plastic analysis is well confirmed experimentally for conditions corresponding to yield regime I of the presented yield criterion. Nielsen [4] gave a detailed review on available tests on reinforced concrete walls and deep beams. For reinforced concrete beams with thin-walled closed cross-section subjected to constant torsion and bending the existence of failure mechanisms which are compatible with the stress states assumed in space truss models was demonstrated by Müller [15], [5]. It can be deduced that the space truss models implicitely use yield regime I. Consequently, further confirmation of yield regime I is rendered by the generally observed good agreement between theory and experiments.

The experimental verification is not yet as well developed for the remaining yield regimes. It seems that presently the most evident indications can be obtained from the interpretation of shear tests on reinforced concrete beams, where the failure was initiated by web crushing. Recent investigations clearly demonstrate that the collapse loads and the failure mechanisms of such beams may well be predicted by plastic analysis provided that an appropriate concrete compressive strength f_c is assumed in the calculations [5], [7-10]. Müller [5] also found good agreement of his general theory of plane stress with tests on shear wall coupling beams reported by Paulay [16].

Much remains to do, however. The validity of the yield criterion should further be investigated. Minimum requirements for the detailing of the reinforcement should be established along with a better determination of the appropriate yield stress level or effective strength parameters.

3. APPLICATIONS TO SHEAR WALLS

3.1 Static and Kinematic Conditions, Exact Solutions

Referred to a Cartesian coordinate system a statically admissible stress field must satisfy the equilibrium conditions

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + p_x = 0, \qquad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + p_y = 0 \qquad (10)$$

and the statical boundary conditions

$$t_x = N_x \cdot \cos\alpha + N_{xy} \cdot \sin\alpha, \qquad t_y = N_{xy} \cdot \cos\alpha + N_y \cdot \sin\alpha . \qquad (11)$$

p_x and p_y are the components of the volume forces and t_x and t_y denote the components of the stresses on the boundary with its normal at an angle α to the x-axis.

If a strain rate field satisfies the compatibility condition

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \qquad (12)$$

continuous velocities with components u and v exist for simply connected regions. The strain rates are determined by the equations

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad , \qquad \varepsilon_y = \frac{\partial v}{\partial y} \quad , \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad (13)$$

and the velocities are obtained by integration. Apart from continuous fields discontinuities have often to be considered for exact as well as for upper-bound solutions, cf. section 2.5 above. Strain rate and velocity fields satisfying the compatibility conditions and the kinematic boundary conditions are termed kinematically admissible.

A stress field is said to be compatible with a strain rate field if the yield criterion and the flow rule are satisfied. For an exact solution a kinematically admissible strain rate field and a compatible, statically admissible stress field must be specified.

For each regime of the yield criterion, Fig. 6, and for all transitions in between the static and kinematic consequences for reinforced concrete walls can be treated separately. As an example the case of yield regime I is considered assuming zero concrete tensile strength. If volume forces p_x and p_y are neglected and the average stresses z_x and z_y due to the orthogonal reinforcement do not vary along the bars,

$$\frac{\partial z_x}{\partial x} = \frac{\partial z_y}{\partial y} = 0 \quad ,$$

the equilibrium conditions Eqs. (10) reduce to

$$\frac{\partial n_x}{\partial x} + \frac{\partial n_{xy}}{\partial y} = 0 \quad , \qquad \frac{\partial n_{xy}}{\partial x} + \frac{\partial n_y}{\partial y} = 0 \quad . \qquad (14)$$

By using the stress function F,

$$n_x = -\frac{\partial^2 F}{\partial y^2} \quad , \qquad n_y = -\frac{\partial^2 F}{\partial x^2} \quad , \qquad n_{xy} = \frac{\partial^2 F}{\partial x \partial y} \quad , \qquad (15)$$

Eq. (7) for yield regime I ($f_t = 0$) renders the differential equation

$$\frac{\partial^2 F}{\partial x^2} \cdot \frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 = 0 \qquad (16)$$

for all developable surfaces. Except for special cases (plane, cylindrical and conical surfaces) developable surfaces can be described by tangents to a space curve. Projection on the x-y-plane leads to a one-parametric set of straight lines and to their envelope. These straight lines, being the projected directions of zero principal curvature of F, are the compression trajectories of the concrete. Generally they form non-centered fans. The plastic deformation at collapse is a uniaxial straining directed at right angles to the concrete compression trajectories.

A detailed discussion covering all regimes for $f_t = 0$ is given in [5]. In particular it is found that the first slip lines are again straight and tangential to

an envelope for the regimes V-VIII for which the reinforcements in one direction
do not yield, $\varepsilon_x = 0$ or $\varepsilon_y = 0$.

Few exact solutions have been developed so far. Nielsen [4] treated several ex-
amples of isotropic walls (deep beams). A direct application of the different
regimes of the presented yield criterion is often obtained by idealizing the com-
pression zone and the longitudinal reinforcement of reinforced concrete beams as
stringers, i.e. bars with vanishing diameter and finite uniaxial strength. The
webs, bounded by the stringers, are governed by the conditions of the appropriate
yield regime. In this way coinciding lower- and upper-bound solutions have been
derived for some problems [5], [9]. However, many questions require further clar-
ification among others the conditions at points of application of concentrated
loads or reactions.

Consider the rectangular wall element ACDF with the thickness taken equal to
unity, Fig. 8 (a). Along AC and DF the element is bounded by two rigid parts.
The element is subjected to the shear force V. The bending moment vanishes at
midspan, x = a/2. The concrete is assumed to have zero tensile strength. Rein-
forcement is provided in the x-direction only ($\mu_x > 0$, $\mu_y = 0$). It is supposed
to be placed symmetrically with respect to the straight line y = 1/2, e.g. con-

(a) Stress Field   (b) Collapse Mechanism
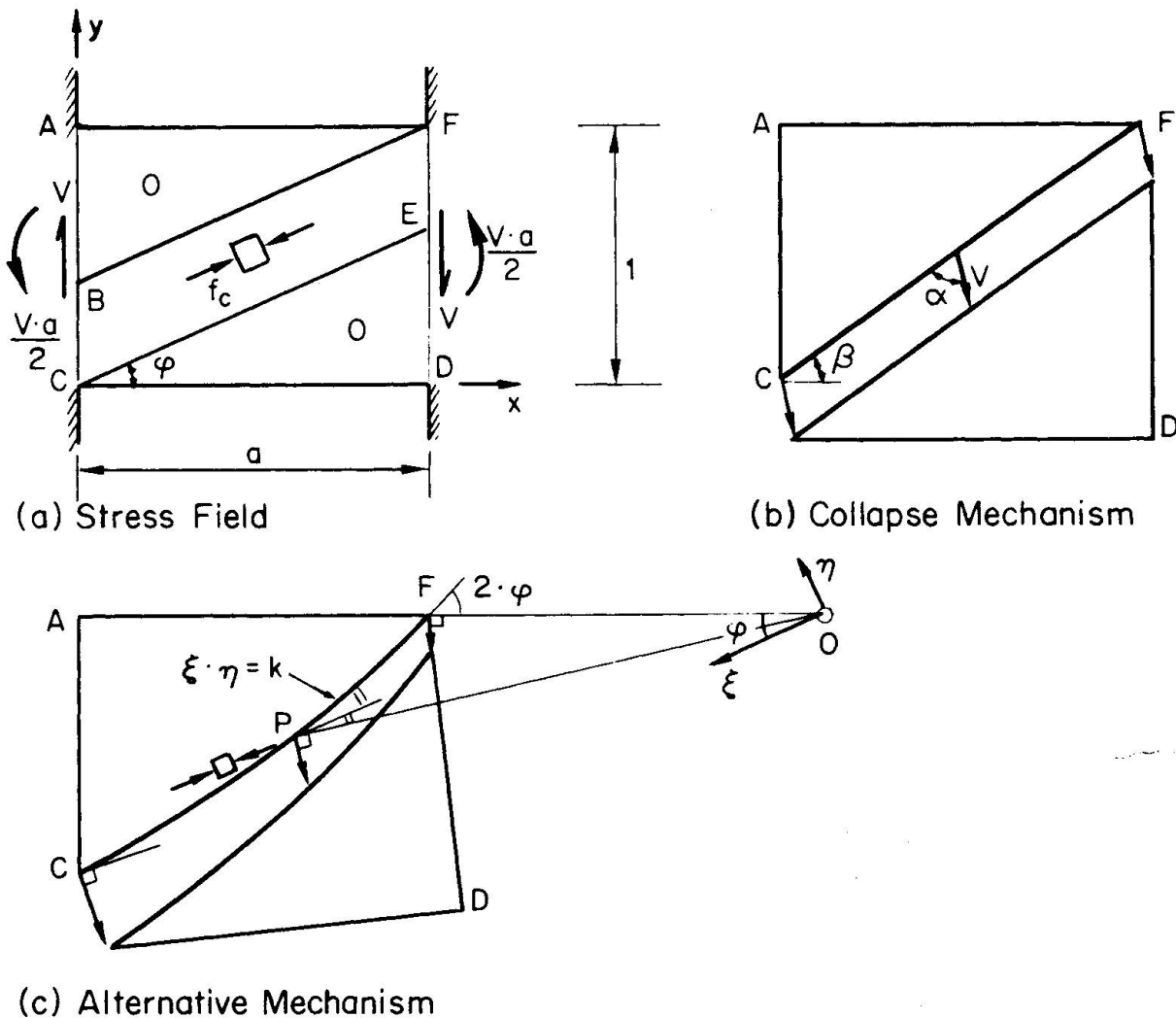

(c) Alternative Mechanism

Fig. 8   Shear Transfer by Strut Action - No Shear Reinforcement

centrated at the edges AF and CD or/and uniformly distributed. With $\omega_x = \mu_x \cdot f_y/f_c$ the resistance of the reinforcement is related to the yield stress level of the concrete. The shear force is transfered by the concrete compression strut BCEF, inclined at angle φ to the x-axis. The lines BF and CE are discontinuity lines for the stress field. They separate the stress-free regions ABF and CDE from BCEF. The equilibrium conditions render

$$V = f_c \cdot \sin\varphi \cdot \cos\varphi \cdot (1-a \cdot \tan\varphi) \tag{17}$$

and with $\beta = \text{arccota}$, Fig. 8 (b), for yielding reinforcement

$$\sin(2 \cdot \varphi - \beta) = (1-2 \cdot \omega_x) \cdot \sin\beta \quad . \tag{18}$$

For $\omega_x \geqslant .5$ the reinforcement does not yield and the angle φ and the shear force V remain constant:

$$2 \cdot \varphi = \beta \quad , \qquad V = V_{max} = \frac{f_c}{2} \cdot \tan(\frac{\beta}{2}) \quad .$$

A kinematically admissible mechanism is shown in Fig. 8 (b). It is characterized by a jump of the velocity along the diagonal CF of the wall element. The work equation renders the least upper bound

$$V = \frac{f_c}{2} \cdot [\sqrt{4 \cdot \omega_x \cdot (1-\omega_x) + a^2} - a] \tag{19}$$

for an angle α determined by

$$\cos(\alpha+\beta) = -(1-2 \cdot \omega_x) \cdot \sin\beta \quad . \tag{20}$$

The upper-bound solution, Eq. (19), coincides with the lower-bound solution, Eq. (17). The direction of the concrete compression strut BCEF bisects the angle between the discontinuity line CF and the normal to the velocity:
$2 \cdot \varphi = \alpha + 2 \cdot \beta - \pi/2$. For $\omega_x \geqslant .5$ the velocity v is parallel to the y-axis, $\alpha + \beta = \pi/2$, i.e. perpendicular to the non-yielding reinforcement. Fig. 8 (c) reflects the fact that the collapse mechanisms are not uniquely determined in general. An alternative mechanism is shown involving separation of the element along the hyperbola CPF and rotation of the rigid body CDF around the center of rotation O. As indicated in Fig. 8 (c) the direction of the concrete compression field bisects the angle between the tangent to the discontinuity line and the normal to the jump direction at an arbitrary point P. For the same collapse load, Eq. (19), an infinite number of mechanisms are possible with discontinuity lines lying between the limiting cases of Figs. 8 (b) and (c) and centers of rotation on the right-hand side of the element.

For the determination of the collapse load of reinforced concrete beams without shear reinforcement the stress field Fig. 8 (a) was already given by Drucker [18]. Nielsen et. al. [9] gave the complete lower- and upper-bound analyses leading to Eq. (19) for beams with rectangular cross-section subjected to a constant shear force V and reinforced in the longitudinal direction only. The mechanism Fig. 8 (c) is due to Müller [5]. He also investigated stress fields for combined shear transfer by strut action and shear reinforcement ($\mu_y > 0$) and found good agreement with tests on shear wall coupling beams reported by Paulay [16].

3.2 Lower-Bound Solutions

The static or lower-bound method of limit analysis requires the determination of a statically admissible stress field which nowhere violates the yield criterion. The corresponding loads are lower bounds for the actual collapse load.

The fundamental significance of the lower-bound theorem for design is evident. Based on equilibrium and yield considerations alone a safe structure can be designed. If a configuration is found to transfer given forces it is only necessary to keep the stresses everywhere below yield. An adequate proportioning of the structure as a whole and of its parts is achieved if the adopted force transfer system is consistently followed.

In principle, statically admissible stress fields for walls can be found by introducing a stress function F for the membrane stresses N_x, N_y and N_{xy} similar to that used in Eqs. (15) for the concrete stresses n_x, n_y and n_{xy}, viz.

$$N_x = -\frac{\partial^2 F}{\partial y^2} - \int p_x d_x, \qquad N_y = -\frac{\partial^2 F}{\partial x^2} - \int p_y d_y, \qquad N_{xy} = \frac{\partial^2 F}{\partial x \partial y} \ . \tag{21}$$

Herewith the equilibrium conditions, Eqs. (10), are identically satisfied. The statical boundary conditions, Eqs. (11), determine the values of F and its first derivatives along the boundary of the wall. Thus, any function F satisfying the statical boundary conditions may be used as a stress function. Suitable numerical methods can be applied for the determination of the stresses. The reinforcement is then proportioned according to the principles outlined in section 2. Finally, the design may be improved by adjusting the stress distribution.

Although the described procedure is feasible and relatively simple its application will seldom be justified for practical design purposes. Still, for particular problems the use of simple continuous stress fields might be advantageous. For many practical problems, however, an approach using discontinuous stress fields will be sufficient. Therefore, two further methods of lower-bound analysis are indicated in the following, viz. the use of truss models and the construction of discontinuous stress fields.

From an engineering point of view it is often desirable to gain a simple picture of a discrete stress distribution within the continuous structure. Truss models of any kind are simple and adaptable systems for this purpose. Even for unusually complex geometrical and/or loading conditions a clear physical view of the mode of force transfer can be obtained. The forces acting in the truss members must be in equilibrium within the truss and with the external forces. If these members have sufficient strength all requirements for the application of the lower-bound theorem are fulfilled on the chosen level of idealization. Therefore, truss models render a sound basis for a safe design of structures.

A two stories high shear wall with I-shaped cross-section cut out of a building is shown in Fig. 9 together with the intermediate floor slab. Vertical and horizontal loads are acting on each story as indicated. Truss models are given for all structural elements. The resistance of the concrete and of the reinforcement is represented by compression and tension members, respectively. Distributed shear and normal forces acting in the web of the shear wall are transfered by simple strut action from story to story. Along the joint of the internal struts with the slab, the forces in the struts above and below are in equilibrium with the external forces transfered to these points by bending and membrane action of the slab. At the base of the outermost strut on the left side the vertical and the horizontal components of the strut force are taken by the compression flange
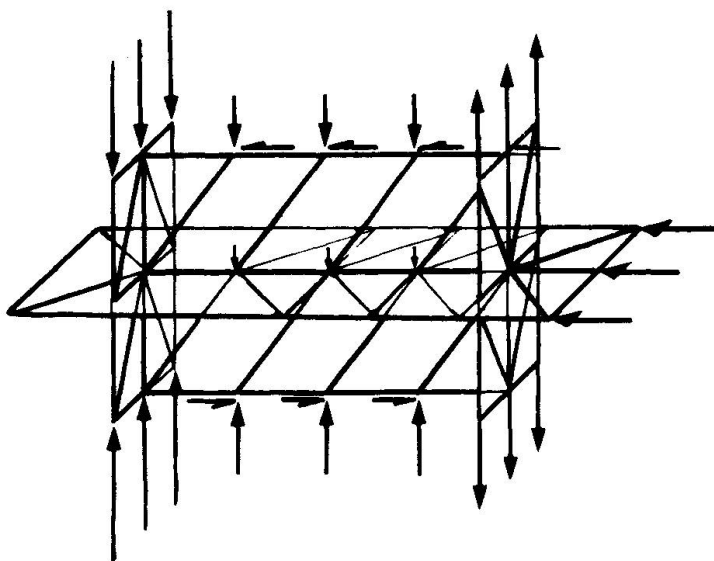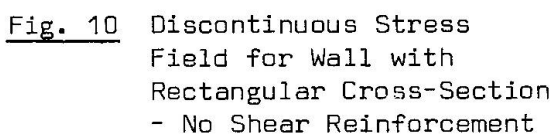
and the slab on each story. Simi-
larly, the reactions at the top of
the outermost struts on the right
side are taken by the tension flange
and the floor slabs, respectively. The
horizontal components of the outer-
most strut forces in the story above
(left side) and below (right side)
act on a truss in the plane of the
floor slab. This truss consists of
four tension members along the con-
tour of the slab and of the com-
pression diagonals connecting the
corners of the slab with the points
of load application at the junctions
of web, flanges and slabs. From this
truss action in the floor slabs an
external stirrup effect results for
the web of the shear wall. The trans-

Fig. 9 Truss Models for Floor Slab and
       Flanged Shear Wall

fer of the horizontal loads acting on the slab boundary is also indicated. Com-
pression diagonals which meet at the connections of the internal struts in the web
with the slab are tied together at their ends with transverse reinforcement in the
slab. Similar considerations for both flanges allow to proportion their cross-
sections and the required londitudinal and transverse reinforcements.

Local problems can be investigated using improved truss models with additional
members. As an example the transfer of the vertical components of the strut forces
in the web to the flanges in Fig. 9 is considered. In reality these forces will
tend to be transfered continuously over the height of a story. Admissible stress
distributions can be found. The favourable external stirrup effect of the slabs
can again be used if a local reinforcement is provided over a certain width of the
web along the flanges.

If the average axial stresses over appropriate cross-sections or widths of each
member of the truss are kept below yield the dimensions of both concrete and rein-
forcement can be determined. Whereas generally this requirement is not too diffi-
cult to meet the conditions at the joints of the truss members require special
examination. There, different uniaxial stress fields overlap. At the boundaries
of the overlapping regions stress discontinuities occur with jumps of the normal
stresses parallel to the discontinuity lines.

General configurations of discontinuous stress fields can be constructed if the
widths of the truss members are gradually increased. Details of this technique
are given elsewhere, e.g. [17]. As illustration a shear wall with rectangular
cross-section is considered, Fig. 10. Along AD and EH the wall is bounded by
floor slabs. The vertical and the horizontal loads transfered by the slab EH to
the wall are acting uniformly distributed along EH and EG, respectively. Two uni-
axial compression fields are assumed for the concrete, viz. the regions ABGH and
ACEG. The overlapping region in the triangle ABG is bounded by the discontinuity
lines AG and BG. Again, the floor slabs act as external stirrups. For example,
the slab AD equilibrates the horizontal shear forces resulting along AB and CD
in the walls above and below the slab, respectively. Only in the tension zone
CDEF a uniformly distributed reinforcement is necessary. The average vertical
stress due to this reinforcement is constant within the region CDEF between the

Fig. 10 Discontinuous Stress
Field for Wall with
Rectangular Cross-Section
- No Shear Reinforcement

Reinforcement

slabs. Along EF it changes by an amount
equal to the vertical stress component of
the inclined concrete compression field
ACEG.

Lower-bound solutions using discontinuous
stress fields were given by Nielsen [4] for
rectangular walls with various loading and
support conditions. Other frequently occur-
ing types of walls, e.g. walls with open-
ings, should similarly be investigated. The
results should be checked by experiments
and by complementary upper-bound analyses.

As for the examples considered above, theo-
retically no reinforcement will often be
necessary in different areas of walls.
However, the use of a well distributed mini-
mum reinforcement throughout the walls is
highly recommended in order to ensure the
desired redistribution of stresses. Of course, no simple rule covering all possi-
ble cases can be given for the necessary amount of minimum reinforcement. In any
case it will be of such an amount that a considerable contribution to the strength
of the wall results. For economical reasons this extra strength should generally
be taken into account.

The choice of the membrane stresses N_x, N_y and N_{xy} as generalized stresses implies
some theoretical consequences so far not mentioned. The statical boundary condi-
tions, Eq. (11), are satisfied in the form of generalized stresses, i.e. only
for the sums of the stresses in the concrete and of the average stresses due to
the smeared reinforcement, Eq. (6). Similarly, along a line of discontinuity only
the sum of the normal and shear stresses acting on elements parallel to the dis-
continuity line must be continuous. The stresses in the concrete and in the rein-
forcement may have jumps which cancel each other. In both cases infinitely large
bond stresses are theoretically required. According to assumption 3 in section
2.1 above such situations are not excluded. Actually, high stress concentrations
may often occur. However, using a cautious design approach they will not endanger
the overall strength in most practical applications. Still, more detailed informa-
tion may be desirable. Therefore, problems of local stress distributions around
the discrete reinforcing bars and problems of bond and local failures in general
should further be investigated. It is felt that considerable progress can be ob-
tained from a limit analysis approach using a suitable yield criterion for the
concrete, e.g. a modified Coulomb criterion with appropriate yield stress levels.

3.3 Upper-Bound Solutions

Upper bounds for the actual collapse load are obtained from the work equation for
an arbitrary, kinematically admissible velocity field. The work equation equates
the external rate of work to the internal rate of dissipation computed from the
flow rule, Eq. (8).

Compared with lower-bound solutions upper-bound analyses do not have the same
direct applicability in practical design. They should rather be used to check the
chosen dimensions. On the other hand, starting from the observed modes of failure,
the upper-bound method may render excellent services for the interpretation of

tests.

Only one typical example is considered in the following. The rectangular wall element ABCD in Fig. 11(a) represents a shear wall with rectangular cross-section. As part of a building it is bounded by floor slabs along AB and CD which act as rigid external stirrups. The element is subjected to the bending moment M, the axial force N and the shear force V. The thickness of the wall is taken equal to unity.

A mechanism with a velocity discontinuity along the diagonal BD of the element is analysed. The relative velocity v between the two parts ABD and BCD is inclined at angle α to the discontinuity line. Obviously, this mechanism corresponds to that of Fig. 8 (b). Since an external axial compression force has the same effect on the concrete as an internal longitudinal reinforcement the shear force V in the unreinforced element is determined by Eq. (19) if the reinforcement ratio ω_x is replaced by $-N/f_c$. The corresponding interaction curve in Fig. 11 (c) is the circular line OAB. For the reinforced element the concept of linear combination outlined in section 2.4 can again be applied. In fact, the contributions of the reinforcements in x- and y-direction correspond in the work equation to those of the forces N and V, respectively. Thus, the curve CDEF is obtained by simple translation $\pm\,\omega_x$ in N-direction and the curve GHIJ if the action $\omega_y{\cdot}a$ of the shear reinforcement is added.

An unlimited increase of the ratio of shear reinforcement, ω_y, is of no use. For the mechanism Fig. 11 (b) the shear reinforcement does not yield, $\varepsilon_y = 0$. This mechanism may be considered as the special case a = 0 of the mechanism Fig. 11(a). Its interaction curve in Fig. 11 (c) for the unreinforced element is the half-circle OKB, which may also be interpreted as the projection parallel to the axis n_y of the yield surface for concrete ($f_t = 0$), Fig. 5, on the plane $\{n_x,\ n_{xy}\}$. Addition of longitudinal reinforcement gives the line CNLKMPF, the projection of the yield surface for reinforced concrete, Fig. 6, on the plane $\{N_x,\ N_{xy}\}$. For the chosen parameters a and ω_y the final interaction curve CNHIPF results.

Since only upper bounds for the collapse load were derived the possibility of a failure under combined bending and axial force should also be checked. In Fig. 11 (d) the exact interaction curve for the uniaxial stress state due to bending and axial force alone is again obtained by linear combination of the admissible stress states for concrete and steel. The yield locus for the unreinforced concrete is the parabola OAB with the equation

$$\frac{M}{f_c/2} + \left(\frac{N}{f_c} + \frac{1}{2}\right)^2 = \frac{1}{4}\ .$$

If the longitudinal reinforcement is concentrated in halves along both edges of the element the yield locus CDE for the reinforcement results. This is the half of a square with diagonals of length $2{\cdot}\omega_x$. If the reinforcement is uniformly distributed the inscribed parabola CFE with the equation

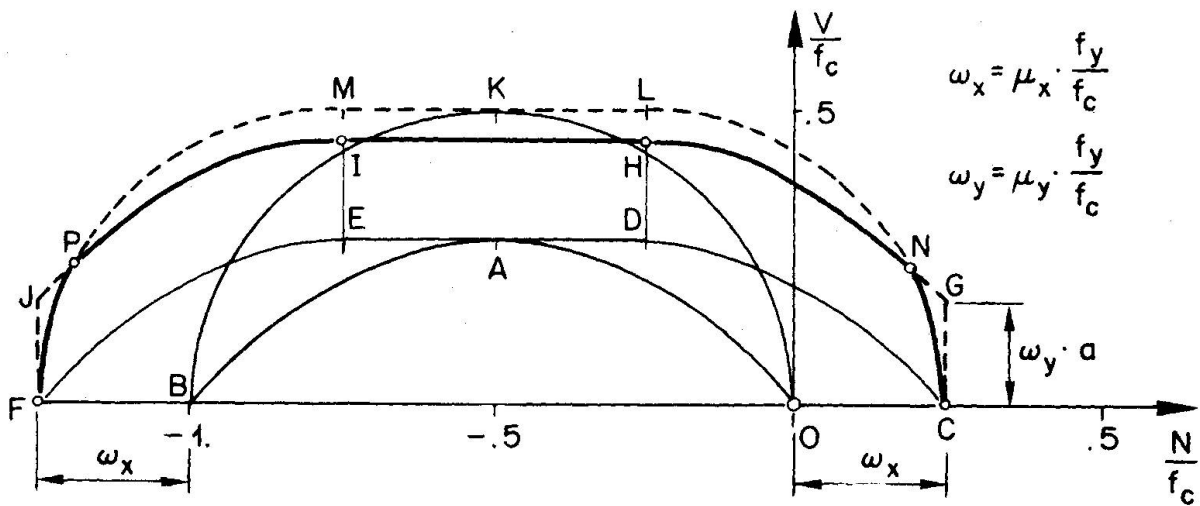$$\frac{M}{f_c/2}{\cdot}2{\cdot}\omega_x + \left(\frac{N}{f_c}\right)^2 = \omega_x^2$$

is obtained. Linear combination of the yield locii OAB and CDE renders the final interaction curve CDGHI.
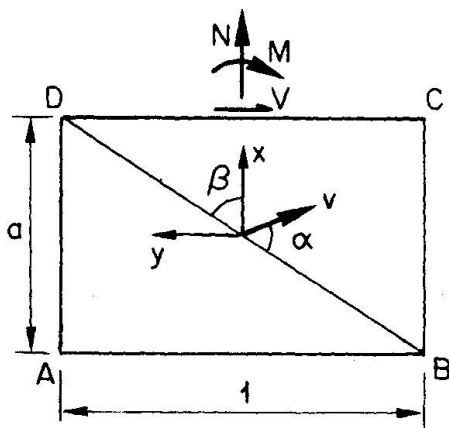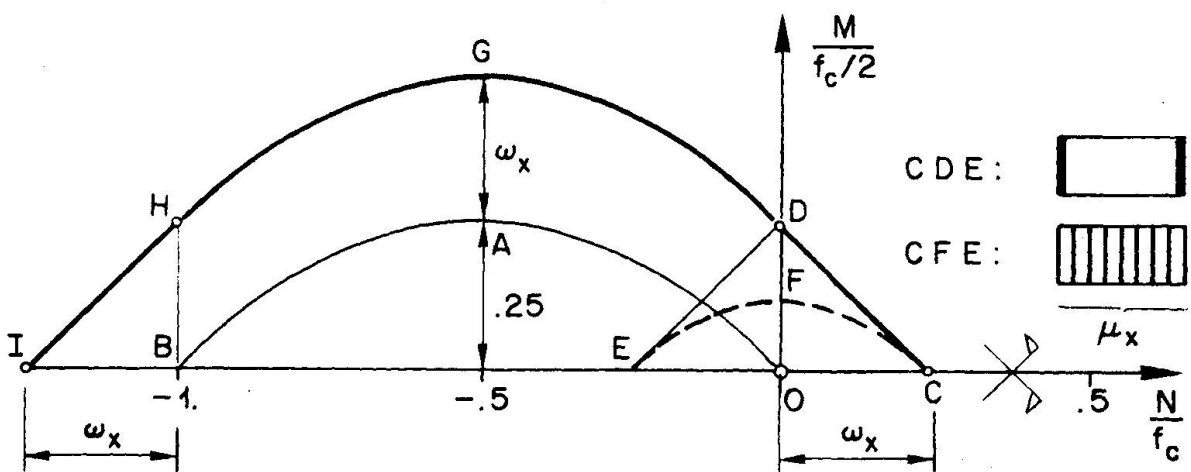
(a) Velocity Discontinuity
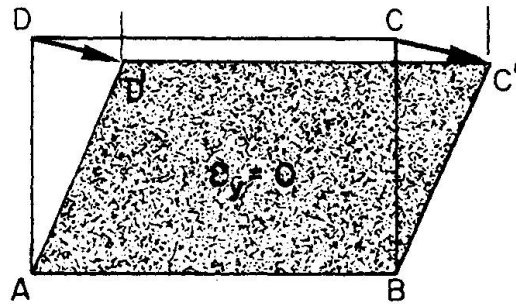
(b) Homogeneous Deformation

(c) Interaction Shear–Axial Force

(d) Interaction Bending–Axial Force

Fig. 11   Upper-Bound Analysis for Shear Wall Element

REFERENCES

1   Shield, R.T.: "On Coulombs Law of Failure in Soils", Journal of the
        Mechanics and Physics of Solids, Vol. 4, 1955, pp. 10-16.

2   Chen, W.F. and Drucker, D.C.: "Bearing Capacity Problems of Concrete
        Blocks and Rock", Journal of the Engineering Mechanics Division, ASCE,
        Vol. 95, No. EM4, Aug., 1969, pp. 955-978.

3   Nielsen, M.P.: "Yield Conditions for Reinforced Concrete Shells in the
        Membrane State", Non-classical Shell Problems, Proc. IASS Symposium,
        Warsaw 1963. Amsterdam: North-Holland Publishing Company, pp. 1030-1040.

4   Nielsen, M.P.: "On the Strength of Reinforced Concrete Discs", Acta Poly-
        technica Scandinavica, Civil Engineering and Building Construction Series
        No. 70, 1971, p. 261.

5   Müller, P.: "Plastische Berechnung von Stahlbetonscheiben und -balken",
        Dissertation Nr. 6083, ETH Zürich, Institut für Baustatik und Konstruktion,
        ETH Zürich, Bericht Nr. 83, Birkhäuser Verlag Basel und Stuttgart, 1978.

6   Clyde, D.H.: "A General Theory for Reinforced Concrete Elements",
        Australasian Conference on the Mechanics of Structures and Materials,
        Christchurch New Zealand, Aug. 1977.

7   Braestrup, M.W.: "Plastic Analysis of Shear in Reinforced Concrete",
        Magazine of Concrete Research, Vol. 26, No. 89, Dec. 1974, pp. 221-228.

8   Nielsen, M.P. and Braestrup, M.W.: "Plastic Shear Strength of Reinforced
        Concrete Beams", Bygningsstatiske Meddelelser, Vol. 46, No. 3, Sept. 1975,
        pp. 61-99.

9   Nielsen, M.P., Braestrup, M.W., Jensen, B.C. and Bach, F.: "Concrete
        Plasticity", Danish Society for Structural Science and Engineering,
        Special Publication, Preliminary Manuscript, Dec. 1976.

10   Braestrup, M.W., Nielsen, M.P. and Bach, F.: "Plastic Analysis of Shear
        in Concrete", Danish Center for Applied Mathematics and Mechanics, No. 120,
        May 1977.

11   Jensen, B.C.: "Lines of Discontinuity for Displacements in the Theory of
        Plasticity of Plain and Reinforced Concrete", Magazine of Concrete Research,
        Vol. 27, No. 92, Sept. 1975, pp. 143-150.

12   Braestrup, M.W., Nielsen, M.P., Jensen, B.C. and Bach, F.: "Axisymmetric
        Punching of Plain and Reinforced Concrete", Technical University of Den-
        mark, Structural Research Laboratory, Report R 75, 1976, p. 33.

13   Marti, P. and Thürlimann, B.: "Fliessbedingung für Stahlbeton mit Berück-
        sichtigung der Betonzugfestigkeit", Beton- und Stahlbetonbau, Vol. 72, No. 1,
        Jan. 1977, pp. 7-12.

14   Clark, L.A.: "The Provision of Tension- and Compression Reinforcement to
        Resist In-Plane Forces", Magazine of Concrete Research, Vol. 28, No. 94,
        March 1976, pp. 3-12.

15   Müller, P.: "Failure Mechanisms for Reinforced Concrete Beams in Torsion
        and Bending", Institut für Baustatik und Konstruktion, ETH Zürich, Re-
        port No. 65, Reprint from IABSE Memoires, Vol. 36-II, Zürich 1976.
        Birkhäuser Verlag Basel und Stuttgart, 1976.

16   Paulay, T.: "The Coupling of Shear Walls", Thesis Univ. of Canterbury,
        Christchurch, New Zealand, 1969.

17   Chen, W.F.: "Limit Analysis and Soil Plasticity", Developments in Geo-
        technical Engineering Vol. 7, Elsevier, New York, 1975, p. 638.

18   Drucker, D.C.: "On Structural Concrete and the Theorems of Limit Analysis",
        IABSE Publications, Vol. 21, 1961, pp. 49-59.

Leere Seite

Blank page

Page vide