

Zeitschrift: IABSE reports of the working commissions = Rapports des commissions de travail AIPC = IVBH Berichte der Arbeitskommissionen
Band: 28 (1979)
Artikel: Plastic analysis of reinforced concrete shear walls
Autor: Marti, P.
DOI: <https://doi.org/10.5169/seals-22900>

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Plastic Analysis of Reinforced Concrete Shear Walls

Théorie de plasticité appliquée aux voiles en béton armé

Anwendung der Plastizitätstheorie auf Wände aus Stahlbeton

P. MARTI

Research Associate

Swiss Federal Institute of Technology

Zurich, Switzerland

SUMMARY

A review is given on recent attempts to establish a general theory of plane stress in reinforced concrete based on the theory of plasticity. Yield criteria for reinforced concrete wall elements are discussed and some applications to reinforced concrete shear walls are presented.

RESUME

Un aperçu est donné de quelques travaux récents basés sur la théorie de la plasticité, devant conduire à une théorie générale de l'état plan de contrainte dans des éléments en béton armé. Des critères d'écoulement pour des voiles en béton armé sont discutés et quelques applications sont présentées.

ZUSAMMENFASSUNG

Neuere plastizitätstheoretische Arbeiten werden besprochen, die sich mit einer allgemeinen Theorie des ebenen Spannungszustandes in Stahlbetonelementen befassen. Fliessbedingungen für Stahlbetonscheiben werden diskutiert, und einige Anwendungen auf Schubwände aus Stahlbeton werden angegeben.



1. INTRODUCTION

Shear walls in buildings are primarily designed to resist the effects of lateral forces due to wind or earthquake actions. Furthermore they are often also used to carry vertical loads. Apart from sufficient strength shear walls should provide appropriate stiffness for an acceptable behaviour under wind action and sufficient ductility to secure the integrity of the wall during several cycles of elastic-plastic deformation in an earthquake. The latter two problems are beyond the scope of the present paper. Only questions concerning the static strength of reinforced concrete shear walls are considered in the following. The discussion is further confined to recent applications of the theory of perfectly plastic bodies to such structural elements.

A typical example is shown in Fig. 1. Vertical and lateral loads are transferred to the shear walls by combined bending and membrane action of the floor slabs. Thus, each wall is subjected to combined bending, shear and axial forces. The individual shear walls may act together with the other structural elements. A portion of the total bending moment and axial force may be resisted by flange action of the transverse walls located at the edges of each shear wall. Acting as membranes the floor slabs may provide an external stirrup effect on the walls.

By means of simple truss models for each structural element a picture of the general mode of force transfer within and between the different elements of relatively complex systems similar to Fig. 1 can be obtained. Indeed, similar models are wellknown for applications to problems of shear transfer in reinforced concrete beams since the time of Ritter and Mörsch some seventy years ago. The extension to more general cases of plane stress conditions like those present in shear walls does not give rise to basically new problems. The truss model concept may be considered as a typical engineering approach which essentially coincides with the lower-bound method of limit analysis. Truss models, however are not able to reflect the kinematics of the collapse mechanisms of reinforced

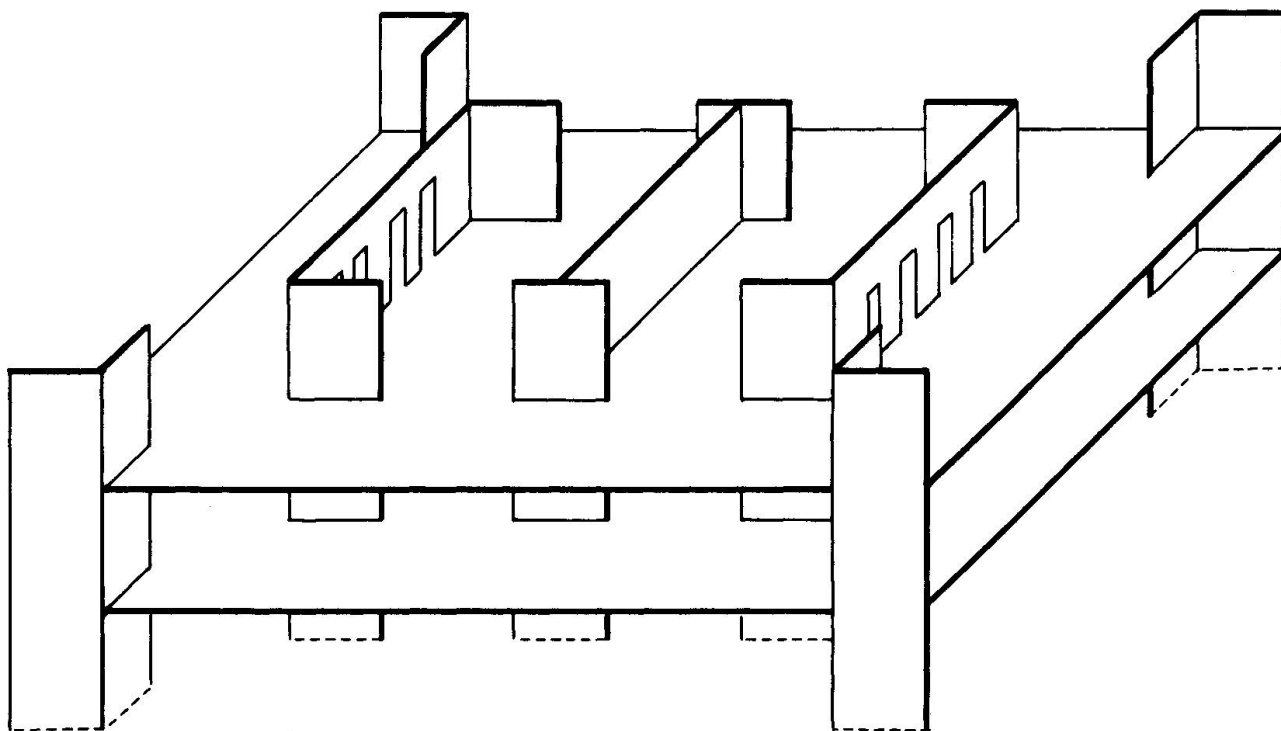


Fig. 1 Shear Walls in High-Rise Building

concrete beams or walls. Furthermore, local conditions e.g. the stress distribution at points of application of concentrated loads or reactions cannot be discussed in detail.

Recently attempts have been made to establish a general theory of plane stress in reinforced concrete based on the classical theory of plasticity. The purpose of the present paper is to review these developments. Yield criteria for reinforced concrete wall elements are discussed and some applications to reinforced concrete shear walls are presented. For simple problems exact solutions for the collapse load are obtained. In more general cases the methods of limit analysis allow to calculate lower and upper bounds for the collapse loads. Again, simple truss models prove to be helpful for constructing statically admissible stress fields according to the lower-bound method.

2. YIELD CRITERIA FOR REINFORCED CONCRETE WALL ELEMENTS

2.1 Assumptions

The investigation is based on the theory of perfectly plastic bodies in particular on the concept of plastic potential and the theorems of limit analysis.

The following assumptions are made:

1. The concrete is in a state of plane stress. It is a rigid-perfectly plastic material governed by a modified Coulomb yield criterion with associated flow rule.
2. The reinforcement is idealized as rigid-perfectly plastic with yield stresses $\pm f_y$. The bars carry forces in axial directions only. The distribution of the reinforcement is such that its action may be described by average stresses over the thickness of the wall (smeared reinforcement).
3. Local and bond failures are excluded.

2.2 Concrete

Coulomb's law of failure states that plastic flow occurs if the shear stress τ and the normal stress σ acting on any element in the material satisfy the linear equation

$$\tau + \sigma \cdot \tan \phi - c = 0 \quad (1)$$

c and ϕ denote the cohesion and the angle of internal friction, respectively. The ratio between the uniaxial tensile and compressive strengths is

$$\zeta = \frac{1 - \sin \phi}{1 + \sin \phi} \quad .$$

As shown by Shield [1] Coulomb's yield surface is an irregular hexagonal pyramid in principal stress space. The boundary planes are determined by the equations

$$\sigma_i \cdot (1 + \sin \phi) - \sigma_j \cdot (1 - \sin \phi) - 2 \cdot c \cdot \cos \phi = 0 \quad , \quad i \neq j \quad (2)$$

For problems of bearing capacity of concrete blocks or rock Chen and Drucker [2] introduced a modified Coulomb yield criterion by assuming a small tensile strength

$$\sigma = f_t \geq 0 \quad (3)$$



This modified criterion consisting of the two conditions (1) and (3), i.e. a material description with three constants, is taken as a basis for the further treatment. In Fig. 2 the yield criterion is represented in the stress plane (a) and in the principal stress space (b) for $\sigma_3 = 0$ (plane stress).

For the application of limit analysis concrete is often assumed to have zero tensile and finite compressive strength. According to this assumption the yield locus DEFGHI in plane stress, Fig. 2 (b), is reduced to the wellknown square yield criterion ODEF. Nielsen [3], [4], Müller [5] and Clyde [6] treated the problem of plane stress in reinforced concrete wall elements using the square yield criterion. Considering the brittleness and the actual strain-softening property of concrete in compression the assumption of perfectly plastic behaviour is somewhat questionable. Nevertheless, this drastic idealization may well be taken as a basis for the calculation of collapse loads of reinforced concrete beams and walls. It is only necessary to take the limited ductility of concrete into account by choosing an effective value f_c which is reduced in comparison with the ordinary compressive strength measured e.g. on cylinders, [5], [7], [8], [9], [10]. The value f_c of the so-called effective concrete compressive strength may be interpreted as the average failure stress in a particular strain range depending on the problem under consideration. The magnitude of f_c must be determined by tests. The concept of an effective strength or yield stress level in connection with the application of the theory of plasticity is not new. In fact it is widely used in soil mechanics [17].

For some applications of limit analysis to plain and reinforced concrete elements the disregard of the concrete tensile strength is too restrictive. The square

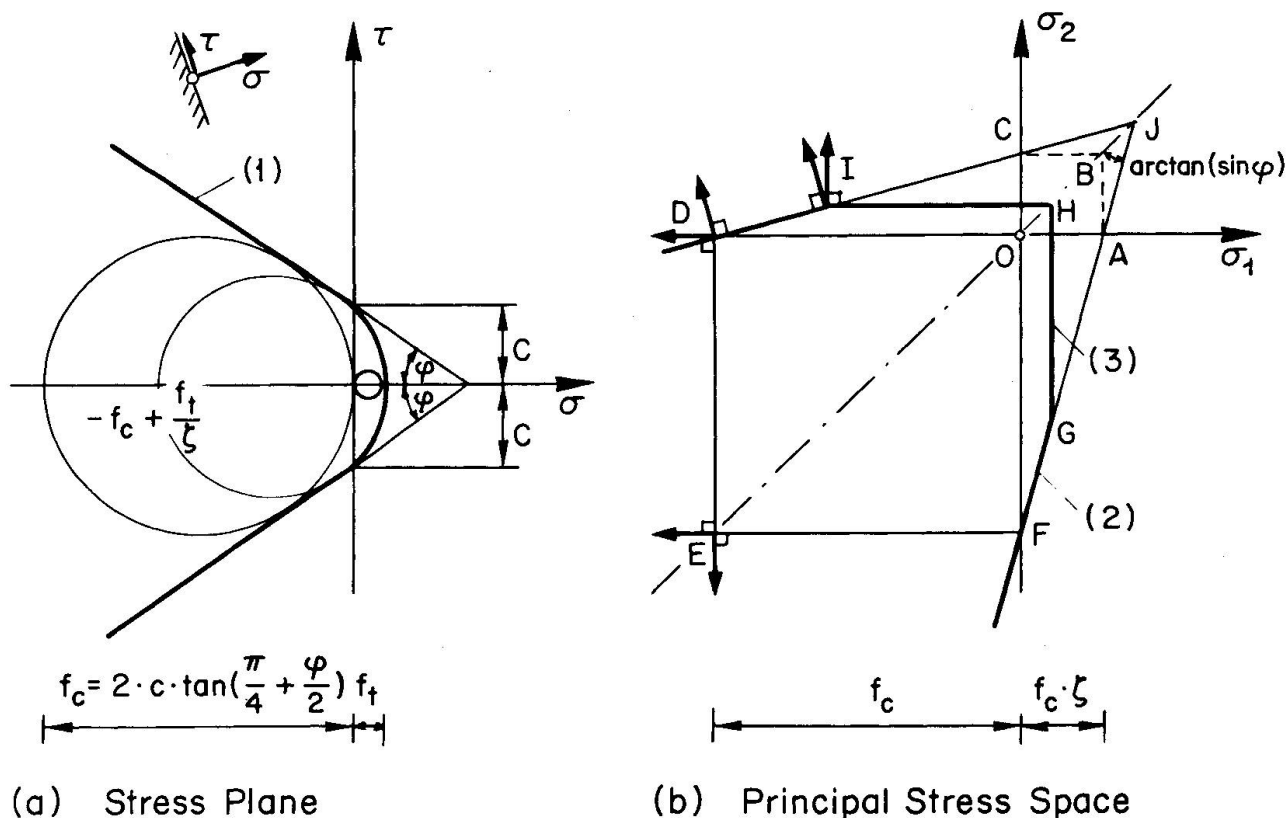


Fig. 2 Coulomb's Yield Criterion

yield criterion is then replaced by a more general one, e.g. the modified Coulomb yield criterion applicable to plane strain as well as general stress states. Jensen [11] used the modified Coulomb yield criterion to analyse selected problems of plain and reinforced concrete. Braestrup et. al. [12] used the same criterion to determine upper bounds for the load producing axisymmetric punching of a reinforced concrete slab. Marti and Thürlimann [13] treated problems of plane stress and plane strain in reinforced concrete by using the unmodified Coulomb yield criterion, i.e. two material constants for the concrete only. As mentioned in [13], too high deviatoric stresses are obtained for higher hydrostatic pressures whereas the idealization might be reasonable e.g. in the tension-compression domains for plane stress condition. To avoid these deficiencies the use of the modified Coulomb yield criterion along with a realistic value for the angle of internal friction ϕ as a third material constant was suggested.

2.3 Reinforcement

A set of reinforcing bars parallel to axis η inclined at angle ϑ to the x -axis is considered, Fig. 3 (a). According to assumption 2 the action of the reinforcement can be described by the average stress

$$z_{\eta} = \mu_{\eta} \cdot \sigma_s \quad , \quad |z_{\eta}| \leq \mu_{\eta} \cdot f_y \quad . \quad (4)$$

μ_{η} and σ_s denote the reinforcement ratio and the stress in the bars, respectively. Transformation of the uniaxial stress state Eq. (4) to the orthogonal coordinates x and y yields the average stresses

$$z_x = z_{\eta} \cdot \cos^2 \vartheta \quad , \quad z_y = z_{\eta} \cdot \sin^2 \vartheta \quad , \quad z_{xy} = z_{\eta} \cdot \sin \vartheta \cdot \cos \vartheta \quad . \quad (5)$$

These stresses are taken as vector components in the stress space Fig. 3 (b). By linear combination of all possible stress states in the ξ - and η -reinforcements the parallelogram-shaped domain BDFH for the skew mesh is obtained. The extension to cases with more than two reinforcement directions is obvious. If the contribution of the compression reinforcement is neglected, e.g. [5], [10], the shaded domain OABC results.

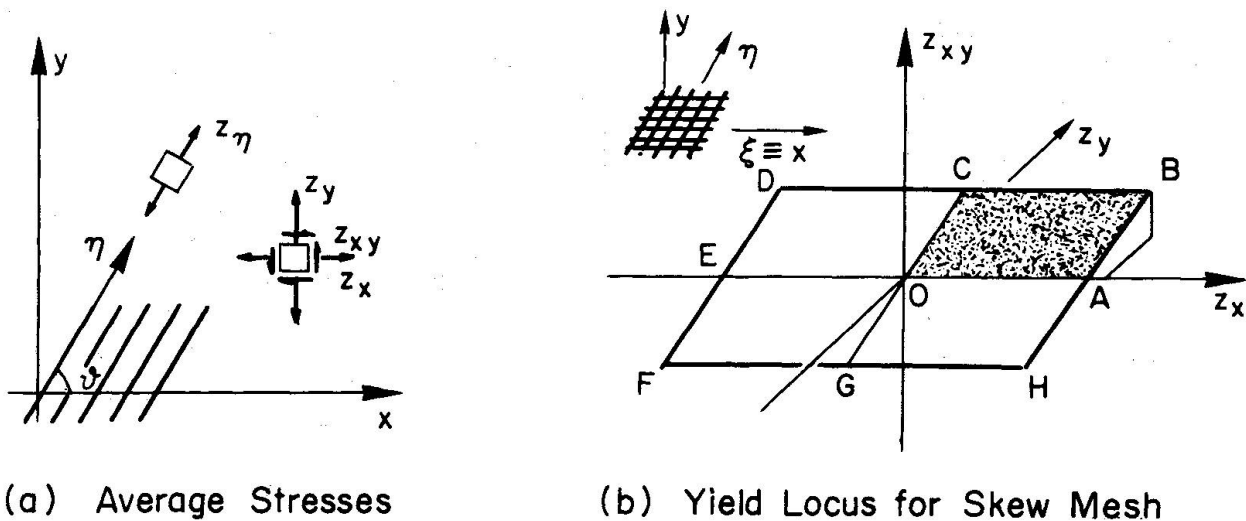


Fig. 3 Yield Criterion for Reinforcement



2.4 Reinforced Concrete

Consider a differential wall element of reinforced concrete, Fig. 4. The thickness is taken equal to unity. The element is subjected to the membrane stresses N_x , N_y and N_{xy} . According to assumption 3 the bond between reinforcement and concrete is preserved until collapse. Therefore, the plastic strain increments or strain rates at collapse are equal for reinforcement and concrete. Hence, the yield surface in the stress space $\{N_x, N_y, N_{xy}\}$ is the envelope of all linear combinations

$$\begin{aligned} N_x &= n_x + z_x, \\ N_y &= n_y + z_y, \\ N_{xy} &= n_{xy} + z_{xy}, \end{aligned} \quad (6)$$

of stresses $\{n_x, n_y, n_{xy}\}$ and $\{z_x, z_y, z_{xy}\}$ which do not violate the yield criteria for the concrete and the reinforcement, respectively.

Fig. 4 Differential Wall Element

The modified Coulomb yield surface assumed for concrete is represented in Fig. 5. It consists of the three elliptical cones

$$\begin{aligned} \text{I} &: n_{xy}^2 - (f_t - n_x)(f_t - n_y) = 0, \\ \text{II} &: n_{xy}^2 - [\zeta \cdot (f_c + n_y) - n_x] \cdot [\zeta \cdot (f_c + n_x) - n_y] \cdot \frac{1}{(1+\zeta)^2} = 0, \\ \text{III} &: n_{xy}^2 - (f_c + n_x)(f_c + n_y) = 0. \end{aligned} \quad (7)$$

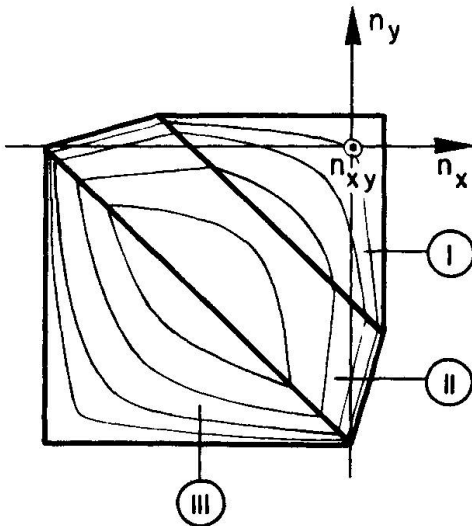


Fig. 5 Yield Surface for Concrete in Plane Stress

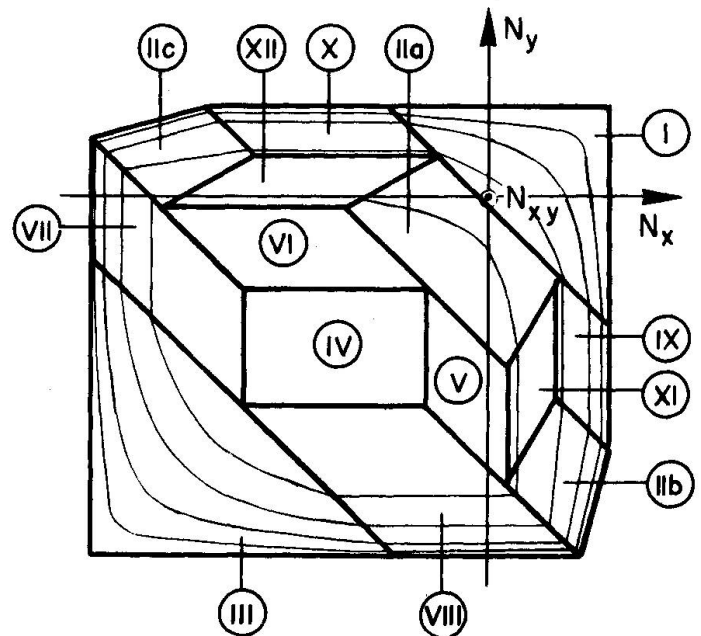


Fig. 6 Yield Surface for Orthogonally Reinforced Wall Element

According to Eq. (6) the yield surface for the reinforced concrete wall element may be obtained by simple translation of the yield surface for concrete, Eqs. (7), with the origin moved within the yield surface for the reinforcement. In Fig. 6 the resulting yield surface is represented for an element reinforced in the two orthogonal directions x and y . The cone II is split into three pieces. The two cones I and III are simply shifted. New yield regimes are created, viz. the planes IV, XI and XII and the circular cylinders V-X. The anisotropy of the yield criterion is obvious.

The stress states in the concrete and in the reinforcement for each yield regime may easily be visualized by means of simple geometrical considerations. As indicated in [13], where Coulomb's yield criterion was assumed for concrete, the detailed description of the different yield regimes given in that paper can be extended to the case of Fig. 6 without further difficulties. It is to be noted that for the yield regimes V-XII the reinforcements in one and for regime IV the reinforcements in both directions do not yield.

Except for several special cases the yield surface of Fig. 6 is of rather theoretical interest. In particular it should not be applied uncritically to plain concrete. In fact for most practical applications in reinforced concrete it will be judicious to neglect the concrete tensile strength and to provide a well distributed minimum reinforcement even in areas, where theoretically no reinforcement is necessary.

If the concrete tensile strength is neglected the yield regimes II and IX-XII disappear. In this form the yield criterion was derived for the first time by Nielsen [3] in the case of isotropic reinforcement. For low degrees of reinforcement say $\mu \cdot f_y / f_c < 0.1$ he suggested the use of an approximate yield surface composed of the two cones I and III, where f_t in Eq. (7) is replaced by $\mu \cdot f_y$. The kinematic aspects for isotropic walls governed by this yield criterion and the associated flow rule were discussed in [4]. In this thesis various exact solutions for isotropic walls (deep beams) and methods for the technical calculation of walls were presented. Müller [5] treated the yield criterion under the assumption of zero concrete tensile strength for arbitrary reinforcement and investigated the general stress and velocity fields in walls which are possible for the different yield regimes. For orthogonally reinforced elements the same yield criterion was also given by Clyde [6] in an attempt to reconcile recent developments such as the space truss model and the skew bending approaches in the theory of shear, torsion and bending of reinforced concrete beams. In theoretical accordance with the aforementioned references Clark [14] presented design equations for proportioning skew or orthogonal reinforcement to resist given membrane forces and suggested limits to the applicability of mild and high-strength steel as compression reinforcement.

2.5 Flow Rule and Velocity Discontinuities

According to the concept of plastic potential the plastic strain rates ϵ_x , ϵ_y and γ_{xy} are given by the flow rule

$$\epsilon_x = \sum_{k=1}^n \frac{\partial \Phi_k}{\partial N_x} \cdot \lambda_k, \quad \epsilon_y = \sum_{k=1}^n \frac{\partial \Phi_k}{\partial N_y} \cdot \lambda_k, \quad \gamma_{xy} = \sum_{k=1}^n \frac{\partial \Phi_k}{\partial N_{xy}} \cdot \lambda_k, \quad (8)$$

$$\lambda_k = 0 \text{ if } \Phi_k < 0, \quad \lambda_k \geq 0 \text{ if } \Phi_k = 0,$$

where the yield criterion is determined by n functions $\Phi_k(N_x, N_y, N_{xy}) \leq 0$.



Application of Eqs. (8) to the different yield regimes of Fig. 6 shows that one of the two strain rates ϵ_x and ϵ_y vanishes for the regimes V-XII and that $\epsilon_x = \epsilon_y = 0$ for regime IV. This is the kinematic interpretation of the above statement that one or both reinforcements do not yield. It is further to be noted that the principal directions for the plastic strain rates coincide with those of the stress state in the concrete but deviate in general from those of the membrane stresses $\{N_x, N_y, N_{xy}\}$.

For many actual collapse mechanisms the velocity fields are found to be discontinuous. Furthermore lines of discontinuity for velocities often prove to be convenient in deriving upper bounds for the collapse load. Consider a narrow zone of homogeneous deformation with thickness $d \rightarrow 0$ between two rigid parts, Fig. 7(a). The direction of the relative velocity v between the two parts forms the angle α with the deforming zone. The principal strain rates indicated in Mohr's circle, Fig. 7(b), are

$$\epsilon_1 = \frac{v}{2 \cdot d} \cdot (1 + \sin \alpha), \quad \epsilon_2 = -\frac{v}{2 \cdot d} \cdot (1 - \sin \alpha) \quad (9)$$

The principal directions 1 and 2 bisect the angles between the parallel to the discontinuity, I, and the normal to the velocity direction, II. In these so-called slip-line directions pure shearing strain rates occur.

The kinematic conditions for the yield regimes I ($\epsilon_1 > 0, \epsilon_2 = 0$), II ($\epsilon_2/\epsilon_1 = -\zeta$) and III ($\epsilon_1 = 0, \epsilon_2 < 0$) restrict the possibilities for discontinuities to $\alpha = \pi/2$, φ and $-\pi/2$, respectively. All other cases $\pi/2 > \alpha > -\pi/2$ can be related to one of the remaining regimes and in particular to the lines of intersection of different regimes.

Discontinuity lines for velocities according to the present theory should not be confused with the actual cracks observed in tests. The directions of the latter

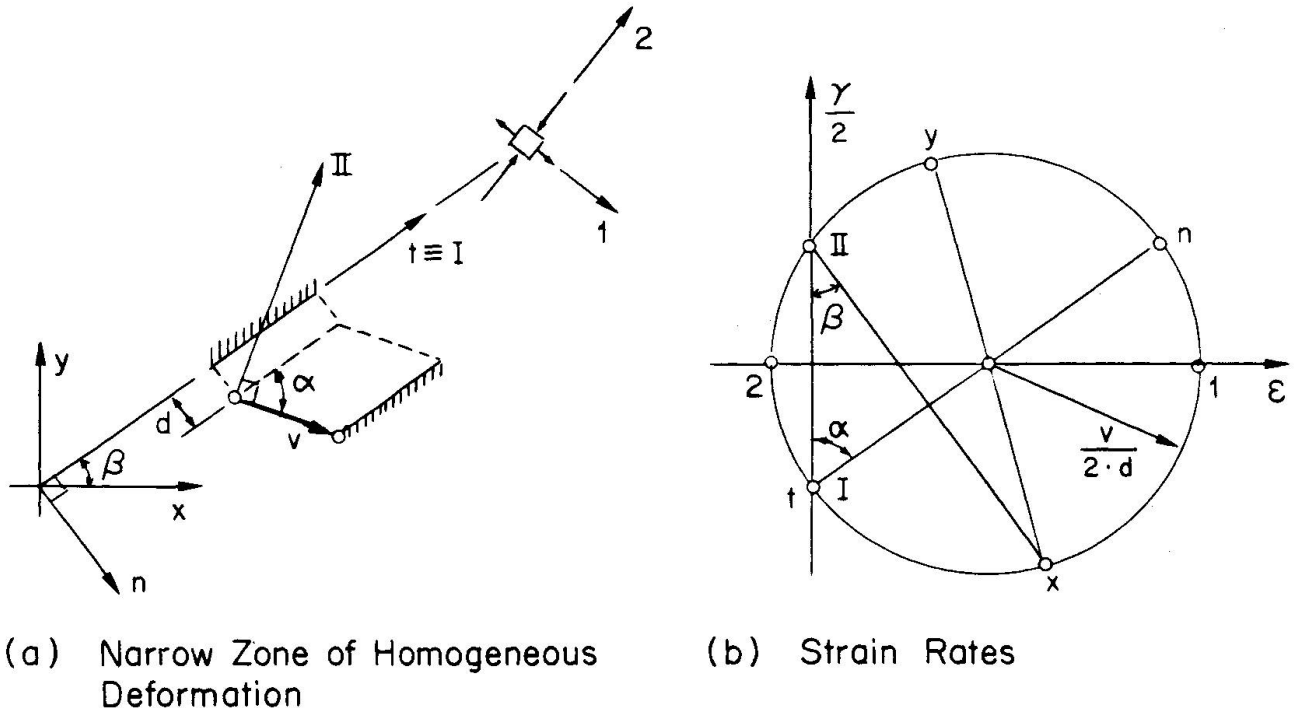


Fig. 7 Line of Discontinuity for Velocities

tend towards the directions of the concrete compression trajectories at collapse. In contrast to this discontinuity lines or first slip lines may form angles up to $\pi/4$ with the principal directions in the concrete depending on the yield regime considered. Only for regime I ($\alpha = \pi/2$) a simple separation parallel to the concrete compression trajectories occurs. For this special case with coinciding slip lines the term "collapse crack" was introduced by Müller [5], [15].

2.6 Experimental Evidence

The redistribution of stresses predicted by plastic analysis is well confirmed experimentally for conditions corresponding to yield regime I of the presented yield criterion. Nielsen [4] gave a detailed review on available tests on reinforced concrete walls and deep beams. For reinforced concrete beams with thin-walled closed cross-section subjected to constant torsion and bending the existence of failure mechanisms which are compatible with the stress states assumed in space truss models was demonstrated by Müller [15], [5]. It can be deduced that the space truss models implicitly use yield regime I. Consequently, further confirmation of yield regime I is rendered by the generally observed good agreement between theory and experiments.

The experimental verification is not yet as well developed for the remaining yield regimes. It seems that presently the most evident indications can be obtained from the interpretation of shear tests on reinforced concrete beams, where the failure was initiated by web crushing. Recent investigations clearly demonstrate that the collapse loads and the failure mechanisms of such beams may well be predicted by plastic analysis provided that an appropriate concrete compressive strength f_c is assumed in the calculations [5], [7-10]. Müller [5] also found good agreement of his general theory of plane stress with tests on shear wall coupling beams reported by Paulay [16].

Much remains to do, however. The validity of the yield criterion should further be investigated. Minimum requirements for the detailing of the reinforcement should be established along with a better determination of the appropriate yield stress level or effective strength parameters.

3. APPLICATIONS TO SHEAR WALLS

3.1 Static and Kinematic Conditions, Exact Solutions

Referred to a Cartesian coordinate system a statically admissible stress field must satisfy the equilibrium conditions

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + p_x = 0, \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + p_y = 0 \quad (10)$$

and the statical boundary conditions

$$t_x = N_x \cdot \cos \alpha + N_{xy} \cdot \sin \alpha, \quad t_y = N_{xy} \cdot \cos \alpha + N_y \cdot \sin \alpha. \quad (11)$$

p_x and p_y are the components of the volume forces and t_x and t_y denote the components of the stresses on the boundary with its normal at an angle α to the x-axis.

If a strain rate field satisfies the compatibility condition



$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \quad (12)$$

continuous velocities with components u and v exist for simply connected regions. The strain rates are determined by the equations

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (13)$$

and the velocities are obtained by integration. Apart from continuous fields discontinuities have often to be considered for exact as well as for upper-bound solutions, cf. section 2.5 above. Strain rate and velocity fields satisfying the compatibility conditions and the kinematic boundary conditions are termed kinematically admissible.

A stress field is said to be compatible with a strain rate field if the yield criterion and the flow rule are satisfied. For an exact solution a kinematically admissible strain rate field and a compatible, statically admissible stress field must be specified.

For each regime of the yield criterion, Fig. 6, and for all transitions in between the static and kinematic consequences for reinforced concrete walls can be treated separately. As an example the case of yield regime I is considered assuming zero concrete tensile strength. If volume forces p_x and p_y are neglected and the average stresses z_x and z_y due to the orthogonal reinforcement do not vary along the bars,

$$\frac{\partial z_x}{\partial x} = \frac{\partial z_y}{\partial y} = 0,$$

the equilibrium conditions Eqs. (10) reduce to

$$\frac{\partial n_x}{\partial x} + \frac{\partial n_{xy}}{\partial y} = 0, \quad \frac{\partial n_{xy}}{\partial x} + \frac{\partial n_y}{\partial y} = 0. \quad (14)$$

By using the stress function F ,

$$n_x = -\frac{\partial^2 F}{\partial y^2}, \quad n_y = -\frac{\partial^2 F}{\partial x^2}, \quad n_{xy} = \frac{\partial^2 F}{\partial x \partial y}, \quad (15)$$

Eq. (7) for yield regime I ($f_t = 0$) renders the differential equation

$$\frac{\partial^2 F}{\partial x^2} \cdot \frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x \partial y} \right)^2 = 0 \quad (16)$$

for all developable surfaces. Except for special cases (plane, cylindrical and conical surfaces) developable surfaces can be described by tangents to a space curve. Projection on the x - y -plane leads to a one-parametric set of straight lines and to their envelope. These straight lines, being the projected directions of zero principal curvature of F , are the compression trajectories of the concrete. Generally they form non-centered fans. The plastic deformation at collapse is a uniaxial straining directed at right angles to the concrete compression trajectories.

A detailed discussion covering all regimes for $f_t = 0$ is given in [5]. In particular it is found that the first slip lines are again straight and tangential to

an envelope for the regimes V-VIII for which the reinforcements in one direction do not yield, $\epsilon_x = 0$ or $\epsilon_y = 0$.

Few exact solutions have been developed so far. Nielsen [4] treated several examples of isotropic walls (deep beams). A direct application of the different regimes of the presented yield criterion is often obtained by idealizing the compression zone and the longitudinal reinforcement of reinforced concrete beams as stringers, i.e. bars with vanishing diameter and finite uniaxial strength. The webs, bounded by the stringers, are governed by the conditions of the appropriate yield regime. In this way coinciding lower- and upper-bound solutions have been derived for some problems [5], [9]. However, many questions require further clarification among others the conditions at points of application of concentrated loads or reactions.

Consider the rectangular wall element ACDF with the thickness taken equal to unity, Fig. 8 (a). Along AC and DF the element is bounded by two rigid parts. The element is subjected to the shear force V . The bending moment vanishes at midspan, $x = a/2$. The concrete is assumed to have zero tensile strength. Reinforcement is provided in the x -direction only ($\mu_x > 0$, $\mu_y = 0$). It is supposed to be placed symmetrically with respect to the straight line $y = 1/2$, e.g. con-

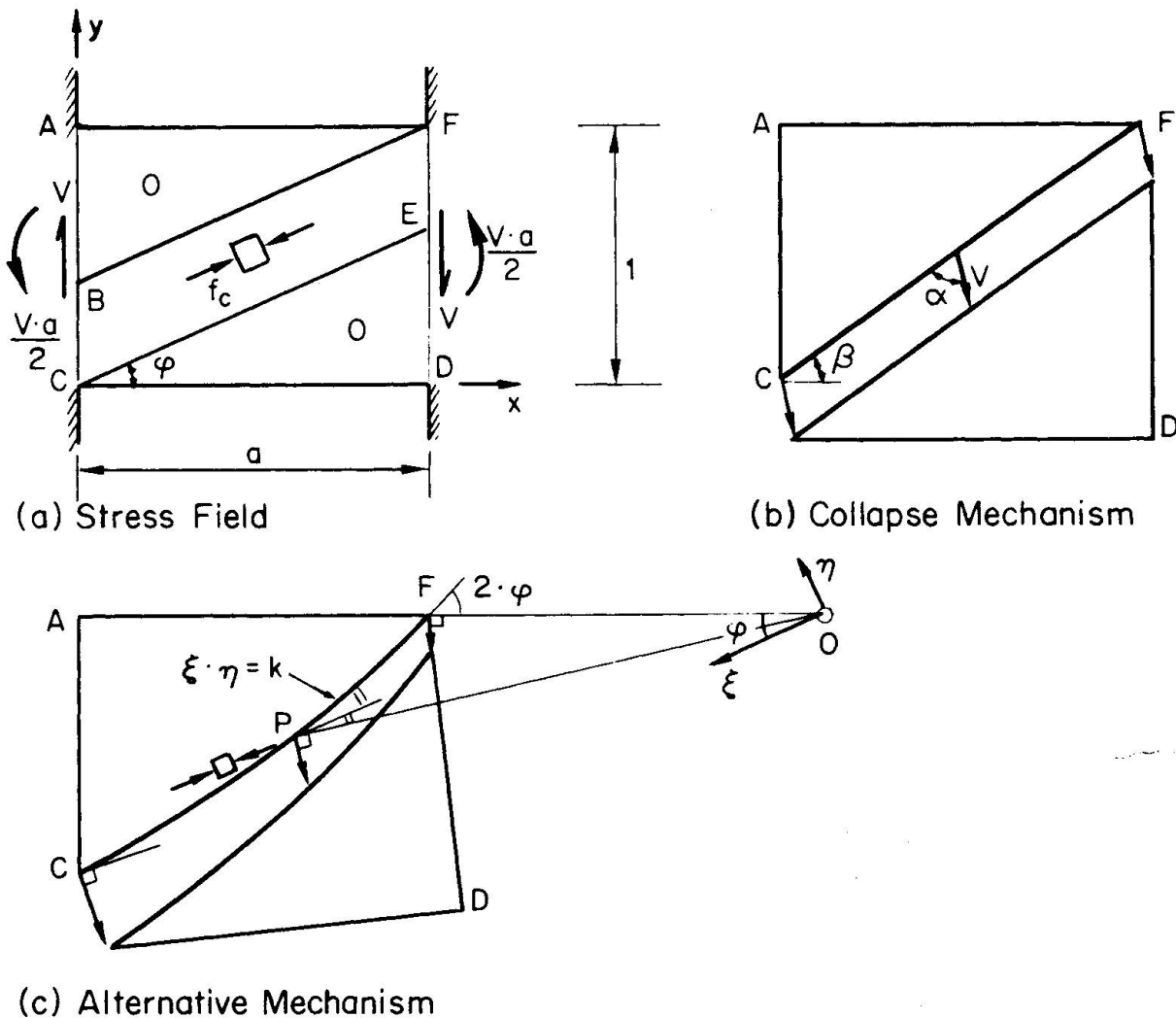


Fig. 8 Shear Transfer by Strut Action - No Shear Reinforcement



centrated at the edges AF and CD or/and uniformly distributed. With $\omega_x = \mu_x \cdot f_y / f_c$ the resistance of the reinforcement is related to the yield stress level of the concrete. The shear force is transferred by the concrete compression strut BCEF, inclined at angle ϕ to the x-axis. The lines BF and CE are discontinuity lines for the stress field. They separate the stress-free regions ABF and CDE from BCEF. The equilibrium conditions render

$$V = f_c \cdot \sin\phi \cdot \cos\phi \cdot (1 - a \cdot \tan\phi) \quad (17)$$

and with $\beta = \text{arccota}$, Fig. 8 (b), for yielding reinforcement

$$\sin(2\phi - \beta) = (1 - 2\omega_x) \cdot \sin\beta \quad (18)$$

For $\omega_x \geq .5$ the reinforcement does not yield and the angle ϕ and the shear force V remain constant:

$$2\phi = \beta, \quad V = V_{\max} = \frac{f_c}{2} \cdot \tan\left(\frac{\beta}{2}\right)$$

A kinematically admissible mechanism is shown in Fig. 8 (b). It is characterized by a jump of the velocity along the diagonal CF of the wall element. The work equation renders the least upper bound

$$V = \frac{f_c}{2} \cdot [\sqrt{4\omega_x \cdot (1 - \omega_x) + a^2} - a] \quad (19)$$

for an angle α determined by

$$\cos(\alpha + \beta) = -(1 - 2\omega_x) \cdot \sin\beta \quad (20)$$

The upper-bound solution, Eq. (19), coincides with the lower-bound solution, Eq. (17). The direction of the concrete compression strut BCEF bisects the angle between the discontinuity line CF and the normal to the velocity:

$2\phi = \alpha + 2\beta - \pi/2$. For $\omega_x \geq .5$ the velocity v is parallel to the y-axis, $\alpha + \beta = \pi/2$, i.e. perpendicular to the non-yielding reinforcement. Fig. 8 (c) reflects the fact that the collapse mechanisms are not uniquely determined in general. An alternative mechanism is shown involving separation of the element along the hyperbola CPF and rotation of the rigid body CDF around the center of rotation O. As indicated in Fig. 8 (c) the direction of the concrete compression field bisects the angle between the tangent to the discontinuity line and the normal to the jump direction at an arbitrary point P. For the same collapse load, Eq. (19), an infinite number of mechanisms are possible with discontinuity lines lying between the limiting cases of Figs. 8 (b) and (c) and centers of rotation on the right-hand side of the element.

For the determination of the collapse load of reinforced concrete beams without shear reinforcement the stress field Fig. 8 (a) was already given by Drucker [18]. Nielsen et. al. [9] gave the complete lower- and upper-bound analyses leading to Eq. (19) for beams with rectangular cross-section subjected to a constant shear force V and reinforced in the longitudinal direction only. The mechanism Fig. 8 (c) is due to Müller [5]. He also investigated stress fields for combined shear transfer by strut action and shear reinforcement ($\mu_y > 0$) and found good agreement with tests on shear wall coupling beams reported by Paulay [16].

3.2 Lower-Bound Solutions

The static or lower-bound method of limit analysis requires the determination of a statically admissible stress field which nowhere violates the yield criterion. The corresponding loads are lower bounds for the actual collapse load.

The fundamental significance of the lower-bound theorem for design is evident. Based on equilibrium and yield considerations alone a safe structure can be designed. If a configuration is found to transfer given forces it is only necessary to keep the stresses everywhere below yield. An adequate proportioning of the structure as a whole and of its parts is achieved if the adopted force transfer system is consistently followed.

In principle, statically admissible stress fields for walls can be found by introducing a stress function F for the membrane stresses N_x , N_y and N_{xy} similar to that used in Eqs. (15) for the concrete stresses n_x , n_y and n_{xy} , viz.

$$N_x = -\frac{\partial^2 F}{\partial y^2} - \int p_x dx, \quad N_y = -\frac{\partial^2 F}{\partial x^2} - \int p_y dy, \quad N_{xy} = \frac{\partial^2 F}{\partial x \partial y} \quad (21)$$

Herewith the equilibrium conditions, Eqs. (10), are identically satisfied. The statical boundary conditions, Eqs. (11), determine the values of F and its first derivatives along the boundary of the wall. Thus, any function F satisfying the statical boundary conditions may be used as a stress function. Suitable numerical methods can be applied for the determination of the stresses. The reinforcement is then proportioned according to the principles outlined in section 2. Finally, the design may be improved by adjusting the stress distribution.

Although the described procedure is feasible and relatively simple its application will seldom be justified for practical design purposes. Still, for particular problems the use of simple continuous stress fields might be advantageous. For many practical problems, however, an approach using discontinuous stress fields will be sufficient. Therefore, two further methods of lower-bound analysis are indicated in the following, viz. the use of truss models and the construction of discontinuous stress fields.

From an engineering point of view it is often desirable to gain a simple picture of a discrete stress distribution within the continuous structure. Truss models of any kind are simple and adaptable systems for this purpose. Even for unusually complex geometrical and/or loading conditions a clear physical view of the mode of force transfer can be obtained. The forces acting in the truss members must be in equilibrium within the truss and with the external forces. If these members have sufficient strength all requirements for the application of the lower-bound theorem are fulfilled on the chosen level of idealization. Therefore, truss models render a sound basis for a safe design of structures.

A two stories high shear wall with I-shaped cross-section cut out of a building is shown in Fig. 9 together with the intermediate floor slab. Vertical and horizontal loads are acting on each story as indicated. Truss models are given for all structural elements. The resistance of the concrete and of the reinforcement is represented by compression and tension members, respectively. Distributed shear and normal forces acting in the web of the shear wall are transferred by simple strut action from story to story. Along the joint of the internal struts with the slab, the forces in the struts above and below are in equilibrium with the external forces transferred to these points by bending and membrane action of the slab. At the base of the outermost strut on the left side the vertical and the horizontal components of the strut force are taken by the compression flange

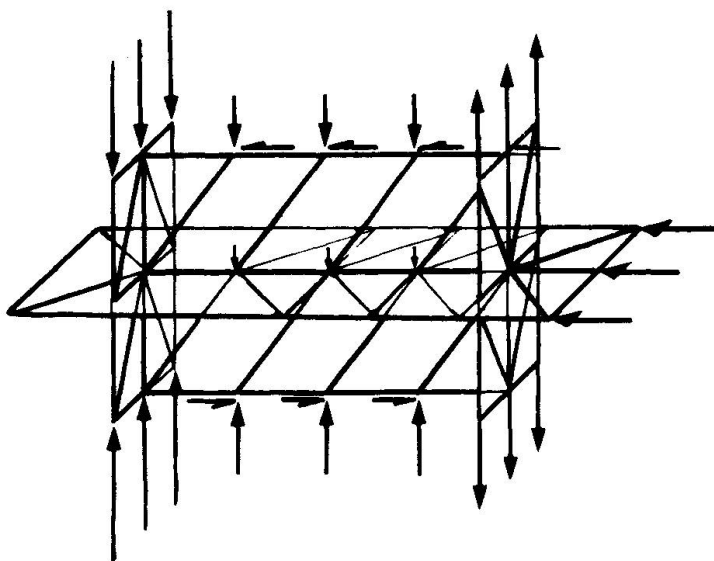


Fig. 9 Truss Models for Floor Slab and Flanged Shear Wall

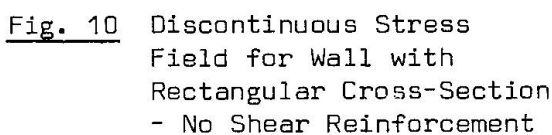
and the slab on each story. Similarly, the reactions at the top of the outermost struts on the right side are taken by the tension flange and the floor slabs, respectively. The horizontal components of the outermost strut forces in the story above (left side) and below (right side) act on a truss in the plane of the floor slab. This truss consists of four tension members along the contour of the slab and of the compression diagonals connecting the corners of the slab with the points of load application at the junctions of web, flanges and slabs. From this truss action in the floor slabs an external stirrup effect results for the web of the shear wall. The trans-

fer of the horizontal loads acting on the slab boundary is also indicated. Compression diagonals which meet at the connections of the internal struts in the web with the slab are tied together at their ends with transverse reinforcement in the slab. Similar considerations for both flanges allow to proportion their cross-sections and the required longitudinal and transverse reinforcements.

Local problems can be investigated using improved truss models with additional members. As an example the transfer of the vertical components of the strut forces in the web to the flanges in Fig. 9 is considered. In reality these forces will tend to be transferred continuously over the height of a story. Admissible stress distributions can be found. The favourable external stirrup effect of the slabs can again be used if a local reinforcement is provided over a certain width of the web along the flanges.

If the average axial stresses over appropriate cross-sections or widths of each member of the truss are kept below yield the dimensions of both concrete and reinforcement can be determined. Whereas generally this requirement is not too difficult to meet the conditions at the joints of the truss members require special examination. There, different uniaxial stress fields overlap. At the boundaries of the overlapping regions stress discontinuities occur with jumps of the normal stresses parallel to the discontinuity lines.

General configurations of discontinuous stress fields can be constructed if the widths of the truss members are gradually increased. Details of this technique are given elsewhere, e.g. [17]. As illustration a shear wall with rectangular cross-section is considered, Fig. 10. Along AD and EH the wall is bounded by floor slabs. The vertical and the horizontal loads transferred by the slab EH to the wall are acting uniformly distributed along EH and EG, respectively. Two uniaxial compression fields are assumed for the concrete, viz. the regions ABGH and ACEG. The overlapping region in the triangle ABG is bounded by the discontinuity lines AG and BG. Again, the floor slabs act as external stirrups. For example, the slab AD equilibrates the horizontal shear forces resulting along AB and CD in the walls above and below the slab, respectively. Only in the tension zone CDEF a uniformly distributed reinforcement is necessary. The average vertical stress due to this reinforcement is constant within the region CDEF between the



Lower-bound solutions using discontinuous stress fields were given by Nielsen [4] for rectangular walls with various loading and support conditions. Other frequently occurring types of walls, e.g. walls with openings, should similarly be investigated. The results should be checked by experiments and by complementary upper-bound analyses.

desired redistribution of stresses. Of course, no simple rule covering all possible cases can be given for the necessary amount of minimum reinforcement. In any case it will be of such an amount that a considerable contribution to the strength of the wall results. For economical reasons this extra strength should generally be taken into account.

The choice of the membrane stresses N_x , N_y and N_{xy} as generalized stresses implies some theoretical consequences so far not mentioned. The statical boundary conditions, Eq. (11), are satisfied in the form of generalized stresses, i.e. only for the sums of the stresses in the concrete and of the average stresses due to the smeared reinforcement, Eq. (6). Similarly, along a line of discontinuity only the sum of the normal and shear stresses acting on elements parallel to the discontinuity line must be continuous. The stresses in the concrete and in the reinforcement may have jumps which cancel each other. In both cases infinitely large bond stresses are theoretically required. According to assumption 3 in section 2.1 above such situations are not excluded. Actually, high stress concentrations may often occur. However, using a cautious design approach they will not endanger the overall strength in most practical applications. Still, more detailed information may be desirable. Therefore, problems of local stress distributions around the discrete reinforcing bars and problems of bond and local failures in general should further be investigated. It is felt that considerable progress can be obtained from a limit analysis approach using a suitable yield criterion for the concrete, e.g. a modified Coulomb criterion with appropriate yield stress levels.

3.3 Upper-Bound Solutions

Compared with lower-bound solutions upper-bound analyses do not have the same direct applicability in practical design. They should rather be used to check the chosen dimensions. On the other hand, starting from the observed modes of failure, the upper-bound method may render excellent services for the interpretation of



tests.

Only one typical example is considered in the following. The rectangular wall element ABCD in Fig. 11(a) represents a shear wall with rectangular cross-section. As part of a building it is bounded by floor slabs along AB and CD which act as rigid external stirrups. The element is subjected to the bending moment M , the axial force N and the shear force V . The thickness of the wall is taken equal to unity.

A mechanism with a velocity discontinuity along the diagonal BD of the element is analysed. The relative velocity v between the two parts ABD and BCD is inclined at angle α to the discontinuity line. Obviously, this mechanism corresponds to that of Fig. 8 (b). Since an external axial compression force has the same effect on the concrete as an internal longitudinal reinforcement the shear force V in the unreinforced element is determined by Eq. (19) if the reinforcement ratio ω_x is replaced by $-N/f_c$. The corresponding interaction curve in Fig. 11 (c) is the circular line OAB. For the reinforced element the concept of linear combination outlined in section 2.4 can again be applied. In fact, the contributions of the reinforcements in x- and y-direction correspond in the work equation to those of the forces N and V , respectively. Thus, the curve CDEF is obtained by simple translation $\pm \omega_x$ in N-direction and the curve GHIJ if the action $\omega_y \cdot a$ of the shear reinforcement is added.

An unlimited increase of the ratio of shear reinforcement, ω_y , is of no use. For the mechanism Fig. 11 (b) the shear reinforcement does not yield, $\epsilon_y = 0$. This mechanism may be considered as the special case $a = 0$ of the mechanism Fig. 11(a). Its interaction curve in Fig. 11 (c) for the unreinforced element is the half-circle OKB, which may also be interpreted as the projection parallel to the axis n_y of the yield surface for concrete ($f_t = 0$), Fig. 5, on the plane $\{n_x, n_{xy}\}$. Addition of longitudinal reinforcement gives the line CNLKMPF, the projection of the yield surface for reinforced concrete, Fig. 6, on the plane $\{N_x, N_{xy}\}$. For the chosen parameters a and ω_y the final interaction curve CNHIPF results.

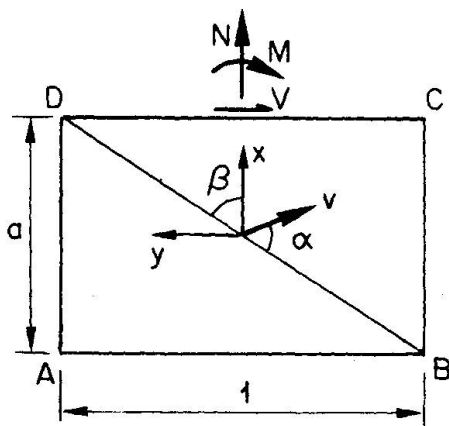
Since only upper bounds for the collapse load were derived the possibility of a failure under combined bending and axial force should also be checked. In Fig. 11 (d) the exact interaction curve for the uniaxial stress state due to bending and axial force alone is again obtained by linear combination of the admissible stress states for concrete and steel. The yield locus for the unreinforced concrete is the parabola OAB with the equation

$$\frac{M}{f_c/2} + \left(\frac{N}{f_c} + \frac{1}{2} \right)^2 = \frac{1}{4} \quad .$$

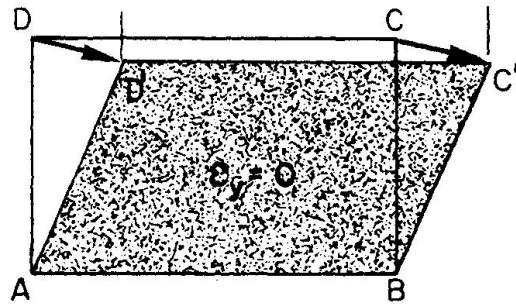
If the longitudinal reinforcement is concentrated in halves along both edges of the element the yield locus CDE for the reinforcement results. This is the half of a square with diagonals of length $2 \cdot \omega_x$. If the reinforcement is uniformly distributed the inscribed parabola CFE with the equation

$$\frac{M}{f_c/2} \cdot 2 \cdot \omega_x + \left(\frac{N}{f_c} \right)^2 = \omega_x^2$$

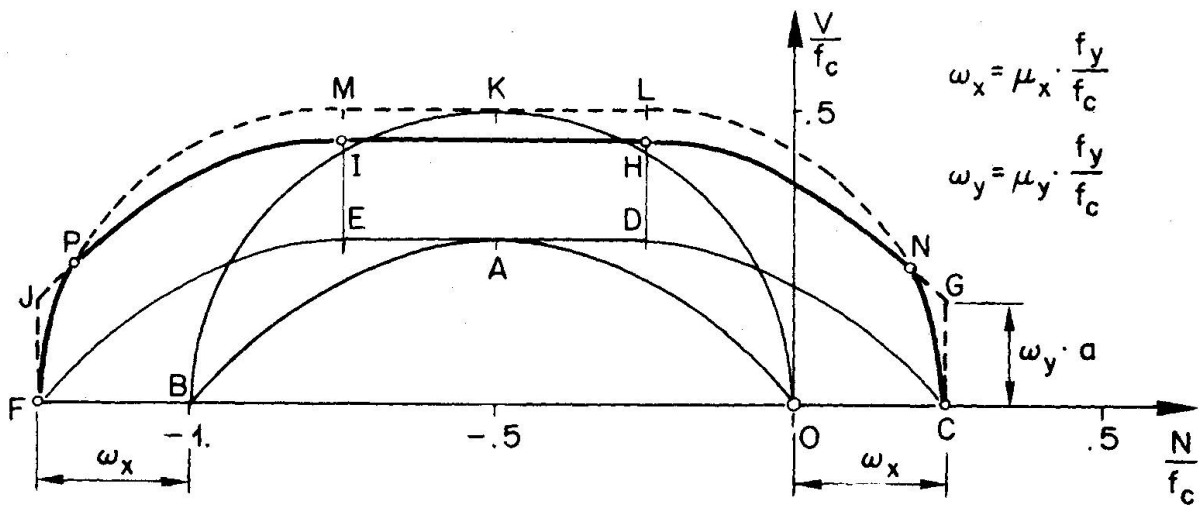
is obtained. Linear combination of the yield loci OAB and CDE renders the final interaction curve CDGHI.



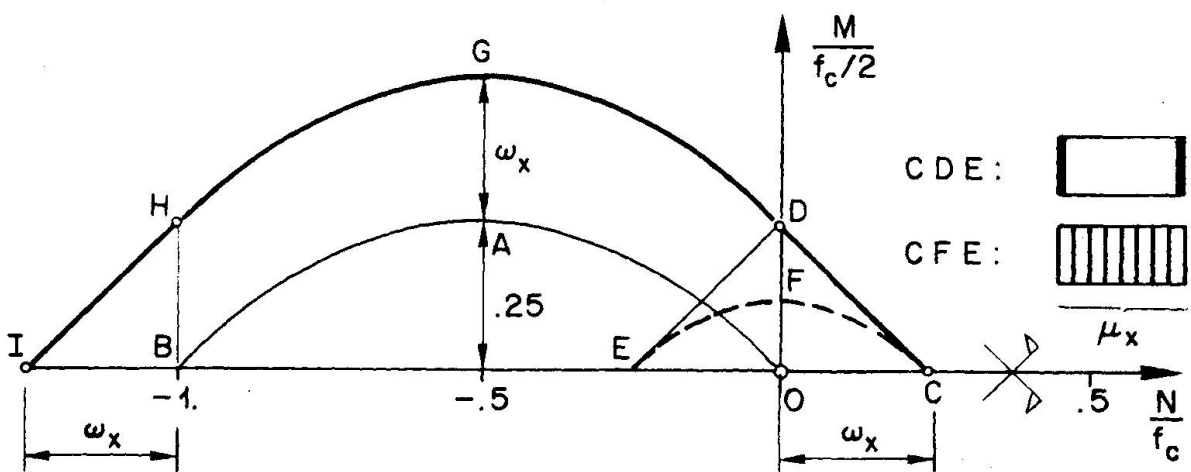
(a) Velocity Discontinuity



(b) Homogeneous Deformation



(c) Interaction Shear-Axial Force



(d) Interaction Bending-Axial Force

Fig. 11 Upper-Bound Analysis for Shear Wall Element



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