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PHENOMENE DE SAUT DANS LE PROBLEME DE FLAMBEMENT EULERIEN

S.D. Leites
Gosstroy
Moscow
USSR

ABSTRACT

A more exact solution of non-linear buckling of columns is obtained, based on the governing differential equation whose curvature expressions are modified to correspond to ten real values. The non-linear differential equations are solved numerically and the phenomenon of "jump" (the discontinuity of the two states of equilibrium) is analysed as an Eulerian problem. The rigidity of the column is observed to be reduced when the geometric non-linearity is taken into account. The results obtained show the relation of the phenomenon of "jump" and the level of bifurcation.

1. INTRODUCTION

Il n'existe de système idéal (c'est-à-dire dépourvu de toute imperfection) ni dans la nature, ni dans le domaine technique. C'est pourquoi il ne faut envisager un système idéal que comme un résultat de la transition limite à partir du système non idéal.

De ce point de vue le problème classique Euler concernant la bifurcation d'équilibre d'une barre élastique comprimée axialement présente un cas limite du problème plus général, celui du comportement de la barre comprimée et fléchie.

Le présent exposé donne une analyse de grands déplacements d'une barre élastique biarticulée soumise à une charge de compression appliquée avec des excentricités inégales, cette analyse étant faite par voie de l'intégration de l'équation différentielle de flexion qui comprend une expression précise de courbure.

Plusieurs études ont été consacrées aux problèmes de grands déplacements des barres élastiques sollicitées aussi bien en compression simple qu'en compression avec flexion. C'étaient J.-L. Lagrange [18], L. Saalschütz [20], G.H. Halphen [16], C.J. Kriemler [17], M. Born [12], L. Malkin [19] et beaucoup d'autres qui s'en occupaient.

Une différence essentielle entre le présent exposé et les études précédentes basées d'ailleurs sur le même mécanisme mathématique, consiste en une plus large interprétation des relations analytiques obtenues. Pour une barre comprimée et fléchie (système non idéal) il existe au moins deux séries de solutions des équations de calcul. La première série reflète des états primaires d'équilibre d'une barre correspondant à une croissance naturelle de la compression. La deuxième série de solutions que les auteurs mentionnés ci-dessus passent sous silence, reflète des états secondaires d'équilibre d'une barre qui ne peuvent pas avoir lieu au cours de cette croissance naturelle de la compression. Il n'est possible de "jeter" une barre à un état secondaire d'équilibre qu'à l'aide du saut, c'est-à-dire d'un procédé de transition à travers les états d'équilibre adjacents. Les états primaires d'équilibre sont toujours stables, tandis que ceux secondaires se divisent en stables et instables. Une valeur minima de la force de compression qui rend possibles des états secondaires d'équilibre est nommée transcritique [5]. Sur la courbe de comportement d'une barre (déformation caractéristique - force de compression) le point transcritique sert de limite pour les états secondaires de l'équilibre stable et instable.

Lorsque le paramètre d'une non-idéalité tend vers zéro, on obtient une solution du système idéal (barre soumise à une compression simple ou bien à une charge de compression appliquée à ses extrémités avec des excentricités égales mais diamétralement opposées). La forme d'équilibre du système idéal à un état non-troublé (avant-critique) est orthogonale à sa forme primaire, c'est-à-dire à la courbe de flambement. Dès qu'une force de compression gagne sa valeur critique, le système idéal subit un flambement au sens eulérien, et alors une bifurcation d'équilibre a lieu. Une analyse montre qu'il faut considérer cette bifurcation comme un cas limite du phénomène de saut, et notamment, comme un saut de longueur nulle.

Une telle interprétation du problème Euler a été illustrée [5] grâce à une analyse des états primaires et secondaires d'équilibre d'une barre comprimée et fléchie par une charge appliquée à ses extrémités avec des excentricités égales. La transition limite vers une compression simple a mis en lumière une corrélation mentionnée ci-dessus entre le phénomène de bifurcation et celui de saut. Avec cela, il faut considérer la force critique eulérienne comme une limite inférieure de la force transcritique réalisable au fur et à mesure de la réduction de l'excentricité au zéro.

Quelques résultats de l'étude [5] ont été reproduits dans l'exposé de T. Renzulli [21], deux années plus tard.

2. BARRE COMPRIMEE ET FLECHIE CHARGEE AUX EXTREMITES AVEC DES EXCENTRICITES INEGALES

Nous allons étudier des déplacements de flexion dans le plan des forces appliquées à une barre initialement rectiligne (fig. 1) tout en négligeant des déformations de compression axiale. On désigne par :

- x - l'abscisse d'un point situé sur la ligne élastique de la barre et mesurée à partir de son appui gauche fixe,
- y - l'ordonnée d'un point situé sur la ligne élastique de la barre et mesurée à partir de l'axe initial rectiligne,
une direction de la flèche vers le bas est estimée comme positive,
- s - la longueur de l'arc de la ligne élastique de la barre mesurée à partir de l'appui gauche,
- θ - L'angle de rotation de la section autour de l'appui gauche, la rotation au sens des aiguilles de montre est estimée comme positive,
- l - la longueur de la barre,
- x_1 - la distance entre les extrémités de la barre déformée,
- EI - la rigidité de la barre lorsque celle-ci fléchit dans le plan de l'application des forces,
- N - la force normale de compression,
- H - la réaction d'appui perpendiculaire à l'axe non-déformé de la barre,
- a - l'excentricité de la force de compression appliquée à l'appui gauche,
- κa - l'excentricité de la force de compression appliquée à l'appui droit ($-1 \leq \kappa \leq 1$).

L'équation différentielle de flexion de la barre est du type suivant :

$$EI \frac{d^2 y}{ds^2} \left[1 - \left(\frac{dy}{ds} \right)^2 \right]^{-\frac{1}{2}} + N(a+y) - Hx = 0 \quad (1)$$

Cette équation comprend une expression de courbure proposée par F.S. Iassinski [11] .

En posant $\alpha^2 = N/EI$, $\beta = H/N$ (2)

on peut écrire la solution de l'équation (1) sous la forme paramétrique :

$$\left. \begin{aligned} x &= \frac{1}{\alpha(1+\beta^2)^{3/4}} \left[2K\beta (\cos \varphi_0 - \cos \varphi) + \int_{\varphi_0}^{\varphi} \frac{1-2k^2 \sin^2 \varphi}{\sqrt{1-k^2 \sin^2 \varphi}} d\varphi \right] \\ y &= \frac{2K}{\alpha} (1+\beta^2)^{1/4} \cos \varphi + \beta x - \alpha \\ s &= \frac{1}{\alpha(1+\beta^2)^{1/4}} \int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} \end{aligned} \right\} \quad (3)$$

où : φ est une variable auxiliaire (paramètre) et φ_0 est une amplitude (la valeur de cette variable pour $x = 0$, $s = 0$ et le module κ ($|\kappa| < 1$) jouent le rôle de constantes d'intégration).

Il est possible de transformer des intégrales elliptiques définies par la solution (3) en une forme normale de Legendre en se basant sur les relations :

$$\left. \begin{aligned} \int_0^{\varphi} \frac{dt}{\sqrt{1-k^2 \sin^2 t}} &= F(\varphi, k), \quad \int_0^{\varphi} \sqrt{1-k^2 \sin^2 t} dt = E(\varphi, k) \\ \int_0^{\varphi} \frac{1-2k^2 \sin^2 t}{\sqrt{1-k^2 \sin^2 t}} dt &= 2E(\varphi, k) - F(\varphi, k) \end{aligned} \right\} \quad (4)$$

On peut écrire l'équation des tangentes des angles de rotation :

$$\frac{dy}{dx} = \beta - \frac{1+\beta^2}{\beta + \phi} \quad (5)$$

où :

$$\Phi = \frac{1/2 - k^2 \sin^2 \varphi}{k \sin \varphi \sqrt{1 - k^2 \sin^2 \varphi}} \quad (6)$$

l'angle de rotation de la section autour de l'appui gauche est déterminé par :

$$\operatorname{tg} \theta = \left(\frac{dy}{dx} \right)_0 = \beta - \frac{1 + \beta^2}{\beta + \Phi_0} \quad (7)$$

où la valeur Φ_0 est obtenue à l'aide de la substitution de φ_0 par φ dans l'expression (6)

Lorsque le total de moments de toutes les forces appliquées à l'extrémité droite de la barre est égal à zéro, on obtient :

$$\beta x_1 = \alpha (1 - \chi) \quad (8)$$

A partir des conditions limites :

$$\left. \begin{array}{l} y=0 \text{ pour } \chi=0, \quad s=0, \quad \varphi=\varphi_0 \\ y=0 \text{ pour } \chi=\chi_1, \quad s=l, \quad \varphi=\varphi_1 \end{array} \right\} \quad (9)$$

et de la relation (8) on peut trouver :

$$\cos \varphi_0 = \frac{\alpha \alpha}{2k(1+\beta^2)^{1/4}}, \quad \cos \varphi_1 = \frac{\chi \alpha \alpha}{2k(1+\beta^2)^{1/4}} \quad (10)$$

La distance entre les extrémités de la barre déformée est égale à :

$$x_1 = \frac{2}{\alpha(1+\beta^2)^{3/4}} \left[k\beta (\cos \varphi_0 - \cos \varphi_1) - 2U \right] \quad (11)$$

La longueur de la barre est présentée par une relation ci-après :

$$l = \frac{2T}{\alpha(1+\beta^2)^{1/4}} \quad (12)$$

Pour les intégrales elliptiques dont les valeurs de φ_0 et φ_1 servant de limite, on a admis des notations suivantes :

$$\left. \begin{array}{l} T = \frac{1}{2} \int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \\ U = \frac{1}{2} \int_{\varphi_0}^{\varphi} \frac{1/2 - k^2 \sin^2 \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} d\varphi \end{array} \right\} \quad (13)$$

En éliminant la valeur X_1 , des équations (8) et (11) on a :

$$\beta = \frac{(1-\alpha) k \cos \varphi_0}{2U} \quad (14)$$

Si nous introduisons un paramètre sans dimension :

$$u = \alpha \ell = \sqrt{N/EI} \ell, \quad (15)$$

nous obtiendrons une expression pour la force de compression :

$$N = \frac{u^2 EI}{\ell^2} \quad (16)$$

L'expression (14) donne :

$$u = \frac{T}{(1+\beta^2)^{1/4}} \quad (17)$$

Les relations (10) et (12) mènent au système des équations de calcul :

$$\left. \begin{aligned} \cos \varphi_1 &= X \cos \varphi_0, \\ k \cos \varphi_0 &= \frac{\alpha}{\ell} \frac{T}{\sqrt{1+\beta^2}} \end{aligned} \right\} \quad (18)$$

3. ETATS PRIMAIRES ET SECONDAIRES D'EQUILIBRE D'UNE BARRE

Puisque les fonctions trigonométriques inverses sont multiformes, le système d'équations (18) a un nombre infini de solutions. Nous ne considérons ici que des solutions pour lesquelles $u < 2\pi$ et, par conséquent, la force de compression N est inférieure à la deuxième valeur critique.

Si une valeur de l'excentricité et celle de k sont constantes, les trois valeurs k , φ_0 et φ_1 satisfaisant aux équations (18) détermineront un certain état d'équilibre de la barre. Les trois valeurs en question forment la solution du système (20).

Dans le plan des coordonnées (K, φ) chacune de ces solutions est présentée par deux points (fig. 2). Un ensemble de solutions du système (18) présenté sur la figure 2 à l'aide de deux courbes $\varphi_0(K)$ et $\varphi_1(K)$ forme une série de solutions.

Pour $u \leq 2\pi$ le système (18) n'aura que deux séries de solutions. La première série correspondant à un rapport des amplitudes $\varphi_1 = \arccos(X \cos \varphi_0)$ détermine des états primaires d'équilibre qui surgissent au cours d'une croissance naturelle de la force de compression N à partir de sa valeur initiale, nulle. Le symbole $\arccos$ correspond ici à la valeur principale de la fonction trigonométrique inverse.

La deuxième série de solutions satisfaisant à la relation $\varphi_1 = 2\pi - \arccos(X \cos \varphi_0)$ interprète des états secondaires d'équilibre qui ne peuvent pas être

réalisés au cours de cette croissance naturelle de la force de compression N . Il n'est possible de "jeter" une barre en un état secondaire d'équilibre que par un moyen artificiel.

Si les trois valeurs k, φ_0, φ_1 présentent une solution du système (18), les valeurs

$$\bar{k} = -k, \quad \bar{\varphi}_0 = \varphi_0 - \pi, \quad \bar{\varphi}_1 = \varphi_1 - \pi \quad (19)$$

correspondant à leur tour aux équations du système (18) et reflèteront le même état d'équilibre de la barre.

L'algorithme, forme commode pour réaliser un calcul à l'ordinateur, suppose une recherche des radicaux du système d'équations, une détermination des paramètres de l'état d'équilibre de la barre et un traçage de son axe courbe. On a pour points de départ des valeurs telles que $a/l, \chi, \varphi_0$. Les valeurs de φ_1 sont calculées d'après la formule :

$$\varphi_1 = \left| \arccos (\chi \cos \varphi_0) - \varphi \right| \quad (20)$$

avec $\varphi = 0$ ($\varphi = 2\pi$) pour la première (deuxième) série de solutions ; une valeur du module k correspondant à la deuxième équation du système (18) est basée sur un procédé itératif. Ces valeurs une fois définies, on passe à la détermination du paramètre de charge z égal au rapport de la force de compression N à sa valeur eulérienne $N_E = \pi^2 EI / \ell^2$,

$$z = \frac{N}{N_E} = \left(\frac{u}{\pi} \right)^2 = \frac{4 T^2}{\pi^2 \sqrt{1+\beta^2}} \quad (21)$$

L'angle de rotation autour de l'appui gauche θ est déterminé d'après l'équation (7).

Pour tracer l'axe courbe de la barre on utilise des relations suivantes :

$$\left. \begin{aligned} \frac{x}{\ell} &= \frac{W(\varphi) - W(\varphi_0)}{T(1+\beta^2)} \\ \frac{y}{\ell} &= \frac{\sqrt{1+\beta^2}}{T} k \cos \varphi + \beta \frac{x}{\ell} - \frac{a}{\ell} \end{aligned} \right\} \quad (22)$$

Une désignation admise ci-dessus :

$$W(\varphi) = \int_0^\varphi \frac{1/2 - k^2 \sin^2 t}{\sqrt{1 - k^2 \sin^2 t}} dt - k\beta \cos \varphi \quad (23)$$

L'algorithme décrit est utilisé dans un programme pour ordinateur. Les intégrales elliptiques sont calculées d'après la formule d'intégration numérique de Simpson.

4. BARRES IDEALES ET NON-IDEALES

On a analysé, à l'aide d'un ordinateur, des états d'équilibre d'une barre représentés par sept schémas qui existent pour des valeurs particulières a/l et χ .

Schéma a, $a > 0$, $\chi = 1$ (fig. 3a). Les excentricités aux extrémités de la barre sont égales et diamétralement opposées.

Schéma b, $a > 0$, $\chi = 0,5$ (fig. 3b). Les excentricités aux extrémités de la barre ont une même direction et des valeurs absolues différentes.

Schéma c, $a > 0$, $\chi = 0$ (fig. 3c). L'excentricité à l'extrémité droite de la barre est égale à zéro.

Schéma d, $a > 0$, $\chi = -0,5$ (fig. 3d). Les excentricités aux extrémités de la barre ont des directions opposées et des valeurs différentes.

Schéma e, $a > 0$, $\chi = -0,95$ (fig. 3e). Les excentricités aux extrémités de la barre ont des directions opposées tandis que leurs valeurs absolues sont presque voisines.

Schéma f, $a > 0$, $\chi = -1$ (fig. 3f). Les excentricités aux extrémités de la barre sont égales, mais diamétralement opposées (charge antisymétrique).

Schéma g, $a = 0$ (fig. 3g). Barre soumise à une compression simple.

Pour le calcul numérique se rapportant aux schémas a à f il est admis que $a = 0,01l$.

La notion d'un schéma idéal suppose une forme géométrique parfaite (dans notre cas il s'agit d'un axe rigoureusement rectiligne avant l'application d'une charge) et une charge exactement centrée (une charge de compression appliquée strictement suivant l'axe de la barre).

D'une façon plus générale, nous allons nommer idéale une barre dont la ligne élastique dès le commencement de l'application d'une charge est orthogonale à sa propre forme originale (à une courbe de flambement).

Les barres dont l'axe à un état non-troublé (avant-critique) ne correspond pas à une condition d'orthogonalité stipulée ci-dessus sont nommées non-idéales.

Dans une barre biarticulée (dont l'axe initial est rectiligne, la rigidité longitudinale constante et sa propre forme originale symétrique par rapport à la section moyenne de la barre) les schémas ci-après peuvent être rapportés aux systèmes idéaux :

Schéma f, $\chi = -1$ (fig. 3f). L'axe courbe de la barre est antisymétrique par rapport à la section moyenne de cette barre, dès le commencement de l'application d'une charge.

Schéma g, $a = 0$ (fig. 3g). La forme rectiligne d'équilibre reste constante pour n'importe quelle charge de compression dont la valeur est inférieure à celle d'Euler.

Dans tous les autres cas ($\alpha > 0$, $k \neq -1$) la barre est non-idéale.

5. ANALYSE DU COMPORTEMENT DES BARRES NON-IDEALES

En utilisant le programme d'ordinateur mentionné ci-dessus, nous avons obtenu les résultats numériques pour un grand nombre de problèmes particuliers et tracé les courbes du comportement pour les sept schémas indiqués sur la figure 3. Les figures 4, 5 montrent les courbes de comportement pour les schémas a et e. Ces courbes correspondent à une allure du paramètre de charge $Z = N/N_E$ fonction du déplacement caractéristique (angle de rotation autour de l'appui gauche).

La courbe OLM sur ces abaque se rapporte aux états primaires de l'équilibre qui est stable pour $N < N_E$ ainsi que pour $N > N_E$.

Des états secondaires d'équilibre sont représentés par la courbe RTS, la branche RT y correspondant à l'équilibre stable et la branche TS à l'équilibre instable de la barre.

Le point T sert de limite entre des états secondaires de l'équilibre stable et instable. Une force de compression spécifique de cette limite est nommée transcritique [5] :

$$N_t = \frac{U_t^2 EI}{l^2} = Z_t N_E \quad (24)$$

La force transcritique est une valeur minima de la force de compression rendant possible l'apparition des états secondaires d'équilibre, pour une valeur donnée de l'excentricité a/l . Une étude théorique de cette force transcritique est basée sur la condition de stabilité ci-après :

$$\frac{dN}{d\theta} = 0 \quad (25)$$

Une interprétation développée de ce critère pour une barre à une extrémité encastrée et une autre libre est donnée dans un autre exposé [5] .

A chaque valeur de la force de compression $N < N_t$ ne correspond qu'un seul état d'équilibre de la barre, et notamment, l'état primaire, stable. A chaque valeur de la force de compression $N > N_t$ correspondent trois états d'équilibre de la barre, et notamment un état primaire, stable, et deux autres secondaires dont l'un est stable (celui qui est caractérisé par des déplacements de valeur absolue plus grande) et l'autre instable (celui qui est caractérisé par des déplacements de valeur absolue plus petite).

La figure 6 représente une allure de la force transcritique N_t en fonction du rapport des excentricités χ dont une (a/l) est fixée d'avance. Aux environs du point $\chi = -1$ la force transcritique N_t est un peu inférieure à la valeur eulérienne N_E l'ordre de grandeur de la différence $N_E - N_t$ (pour des élancements et des excentricités ordinaires) ne dépassant que quelques centièmes du pour cent.

Sur la fig. 7 est rapporté un tracé de la force transcritique N_t en fonction de l'excentricité a/l pour $\chi = 1$ (les excentricités aux extrémités de la barre sont égales).

6. ANALYSE DU COMPORTEMENT ET DE LA STABILITE D'UNE BARRE CHARGÉE ANTISYMETRIQUEMENT

Une barre comprimée et fléchie par une charge appliquée à ses extrémités avec des excentricités égales et diamétralement opposées (schéma e, $a > 0$, $\alpha = -1$, fig. 3f) se rapporte aux barres idéales.

Il est assez difficile d'analyser ce cas, une interprétation contradictoire de ce problème dans la littérature spéciale en étant la preuve. G. Bürgermeister et H. Steup [13], auteurs d'une monographie volumineuse concernant la théorie de stabilité, se réfèrent à une étude de W. Cornelius et estiment qu'une bifurcation d'équilibre a lieu dès le commencement de l'application d'une charge, c'est-à-dire pour une force nulle de compression. Le point de vue de G.O. Iarochévitch [10] est tout à fait contraire puisqu'il nie complètement la possibilité d'un flambement dans une barre antisymétriquement chargée.

E.G. Kossykh dans son exposé [2] donnerait une solution véritable de ce phénomène : il a constaté que la force critique devient inférieure à celle d'Euler. E.G. Kossykh a passé sous silence les problèmes des états secondaires d'équilibre et du mécanisme du flambement.

Sur la base du calcul sur ordinateur pour $\alpha = 0, 0,1, \dots$ on a établi un abaque de comportement d'une barre chargée antisymétriquement (fig. 8) où le point k correspond à la force critique N_* et la courbe OKS à des états non troublés de l'équilibre qui est stable pour $N < N_*$ (branche OK) et instable pour $N > N_*$ (branche KS).

Des états troublés d'équilibre de la barre représentés sur l'abaque en question par la courbe RKM sont possibles pour $N < N_*$ et peuvent être considérés comme ceux de l'équilibre stable. A chaque valeur de la force de compression $N > N_*$ correspond un état non-troublé de l'équilibre instable et deux états non troublés de l'équilibre stable pour lesquels une symétrie inverse de la ligne élastique initiale (sous la forme de S) est transformée par la superposition d'une courbe de flambement symétrique (par rapport à la section moyenne de la barre).

De petites courbes aux environs du point critique K (fig. 8) représentent des états d'équilibre d'une barre non-idéale à paramètre $\alpha = -1 + \varepsilon$ où ε est une quantité infime. Il est à noter d'ailleurs, que la courbe inférieure correspond aux états primaires et la courbe supérieure aux états secondaires d'équilibre.

Si on passe maintenant à une transition limite avec ε se visant au zéro, ces deux courbes séparées se rapprocheront successivement et pour $\varepsilon = 0$ elles se rencontreront au point K qui caractérise une bifurcation d'équilibre dans une barre idéale chargée antisymétriquement.

En admettant que $\varphi_0 = 0$, $\varphi_1 = \pi$, (état critique) et en se référant à l'équation (7) on peut déduire une relation précise :

$$\operatorname{tg} \theta_* = \beta = 2\alpha / \chi, \quad (26)$$

Au moment du flambement les tangentes aux extrémités de l'axe courbe de la barre deviennent parallèles à l'axe des excentricités, c'est-à-dire à une ligne reliant les points d'application des forces de compression (fig. 9).

Ce critère géométrique du flambement d'une barre chargée antisymétriquement peut être utilisé pour la solution de diverses problèmes. En particulier, E. Chwalla l'avait introduit dans son étude n'embrassant que de faibles déplacements de la barre.

En analysant la relation (17) pour $\varphi_0 = 0, \varphi_1 = \pi$ on constate que la valeur critique u_* est inférieure à π , d'où :

$$N_* = \frac{u_*^2 EI}{\rho^2} < N_E \quad (27)$$

Cette constatation a un sens théorique plutôt que pratique, la diminution de N_* par rapport à N_E étant très faible. Le module K étant sensiblement inférieur à l'unité, on obtient une relation approximative :

$$u_* \approx \pi \left[1 - 0,383 \left(\frac{a}{b} \right)^2 \right] \quad (28)$$

Le calcul d'après cette formule et l'étude de l'abaque à échelle réduite (fig. 8) montrent qu'une réduction de la force critique pour $a = 0,011$ n'atteint même pas un centième de pour cent.

Une non-linéarité géométrique prise en compte, la valeur de la force critique peut être précisée tandis qu'un problème géométrique linéaire ne donne qu'une valeur approximative de $N_* \approx N_E$. Dans le cas considéré un état non-troublé (avant-critique) de la barre est celui de déformation, c'est pourquoi une valeur réelle de la force critique dépend de l'excentricité a/l . Afin que la valeur de la force critique soit rigoureusement précisée, une analyse exacte de l'état avant-critique de contrainte est aussi nécessaire.

Il est à noter que dans le cadre du problème envisagé la prise en compte d'une non-linéarité géométrique est équivalente à une réduction de la rigidité de flexion de la barre.

Une analyse de la transition limite à partir d'une barre non idéale à paramètre $\kappa = -1 \pm \varepsilon$ pour $\varepsilon \rightarrow 0$, montre que la force critique N_* appliquée à une barre chargée antisymétriquement présente une valeur limite de la force transcritique N_t :

$$N_* = \lim_{\varepsilon \rightarrow 0} N_t \quad (29)$$

7. ANALYSE DU COMPORTEMENT ET DE LA STABILITE D'UNE BARRE SOUMISE A UNE COMPRESSION SIMPLE

Une barre comprimée axialement est une barre idéale (schéma g, $a=0$, fig. 3g) Les équations de calcul (18) deviennent du type :

$$\varphi_1 = |\varphi_0 - \varphi|, \quad k \cos \varphi_0 = 0 \quad (30)$$

La fig. 10 reflète un comportement de la barre suivant lequel pour la force critique eulérienne N_E il existe une bifurcation (point K). Une analyse complète de l'abaque analogue est donnée dans un autre exposé [5] de l'auteur.

Une confrontation avec les courbes de comportement d'une barre non-idéale (schéma a, $a > 0$, $\kappa = 1$ fig. 4) montre que la courbe OKM de la fig. 10 n'est qu'une position limite (au fur et à mesure de la réduction de l'excentricité a au zéro) de la courbe OLM des états primaires d'équilibre tracée sur la figure 4. La courbe RKS de la figure 10 présente une position limite de la courbe RTS des états secondaires d'équilibre tracée sur la figure 4.

La configuration des courbes (fig. 10) aux environs du point K est poussée à l'extrême pour montrer d'une façon plus nette la nature du point de bifurcation.

L'excentricité a tendant vers zéro, le point T s'approche du point K (point de bifurcation) et ils coïncident pour $a = 0$. Il s'en suit que la force critique eulérienne appliquée à une barre le long de son axe présente une limite inférieure de la force transcritique :

$$N_{\varepsilon} = \lim_{\varepsilon \rightarrow 0} N_t \quad (31)$$

Soit, à une valeur quelconque de la force de compression $N > N_t$, correspondent trois états d'équilibre d'une barre non-idéale désignés sur la figure 4 par des caractères K_1, K_2, K_3 . Une barre se trouvant dans un état primaire d'équilibre K_1 ne peut être mise en état K_2 , puis en état K_3 qu'à l'aide du saut, c'est-à-dire à l'aide d'un procédé dynamique de transition à travers des états adjacents non équilibrés.

Ce phénomène de saut est représenté sur la figure 4 par un tracé discontinu. La longueur de saut minima d est égale à l'écart des points K_1 u K_2 . Au cours de la transition limite à partir de la barre non-idéale ($d > 0$) vers la barre idéale ($d = 0$) cette longueur de saut d décroît successivement et sa valeur limite devient nulle :

$$\lim_{\varepsilon \rightarrow 0} d = 0 \quad (32)$$

Cette analyse montre qu'une bifurcation d'équilibre présente un cas limite (dégradé) du phénomène de saut (saut à longueur nulle).

Pour rendre plus claires des particularités essentielles du comportement des barres élastiques soumises à une compression simple et à une compression avec flexion il est recommandé d'utiliser un prototype constitué d'un système à l'unique degré de liberté.

En étudiant le comportement de ce prototype on constate que toutes les lois établies sont inhérentes au problème en question, pour n'importe quel nombre de degrés de liberté, et elles ne sont pas liées (comme on pouvait le supposer) au caractère compliqué des fonctions transcendantes déterminant de grands déplacements de la barre élastique.

Ce prototype est décrit dans une monographie de l'auteur [4], l'analyse de son comportement étant donnée dans un autre article [5].

8. CONCLUSIONS

Les résultats obtenus servent à établir une corrélation entre le phénomène de bifurcation et celui de saut ainsi qu'à mettre au jour la nature de la force critique et le mécanisme du flambement dans le domaine élastique.

Les lois établies gardent leur importance pour une gamme assez large de problèmes de stabilité élastique liés au phénomène de bifurcation.

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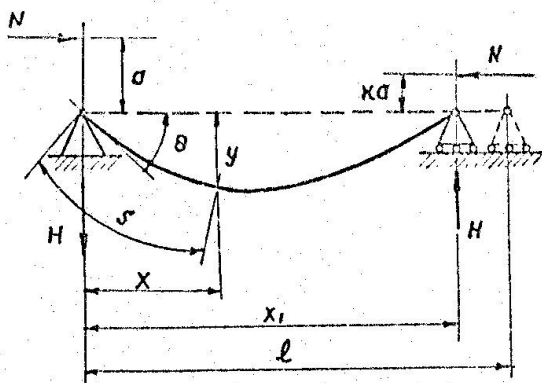


Fig. 1. Barre soumise à des charges de compression appliquées à ses extrémités avec des excentricités inégales.

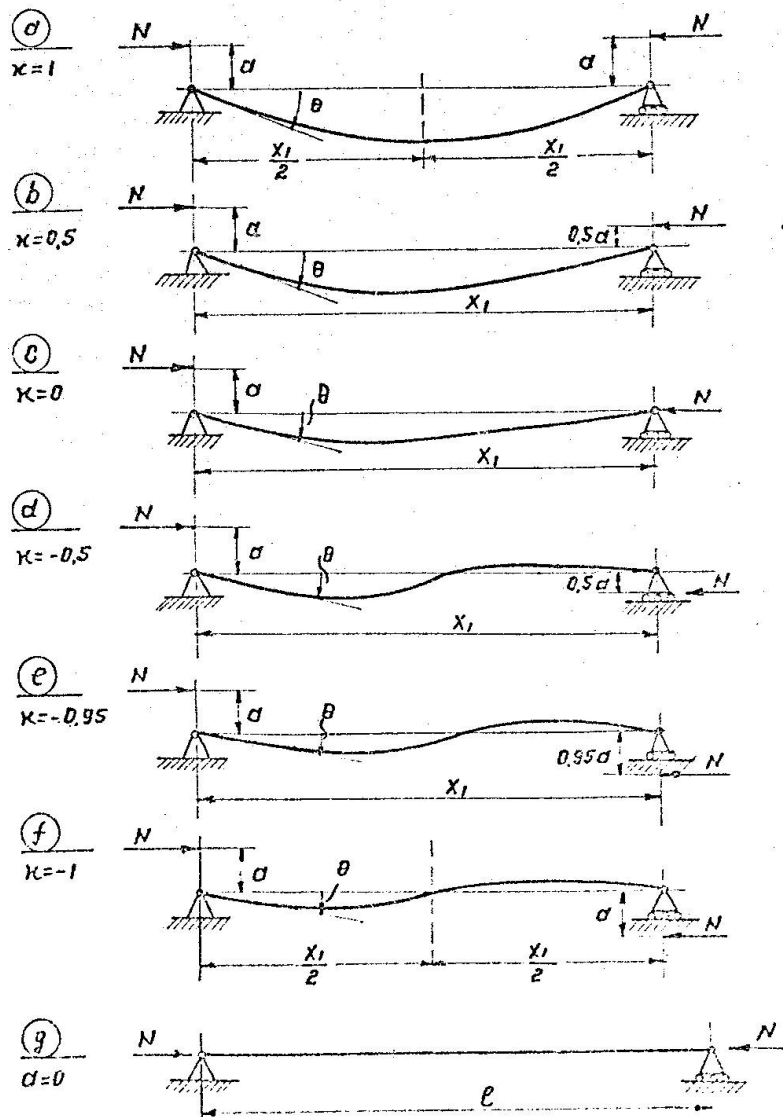


Fig. 3. Sept schémas de la barre comprimée et fléchie

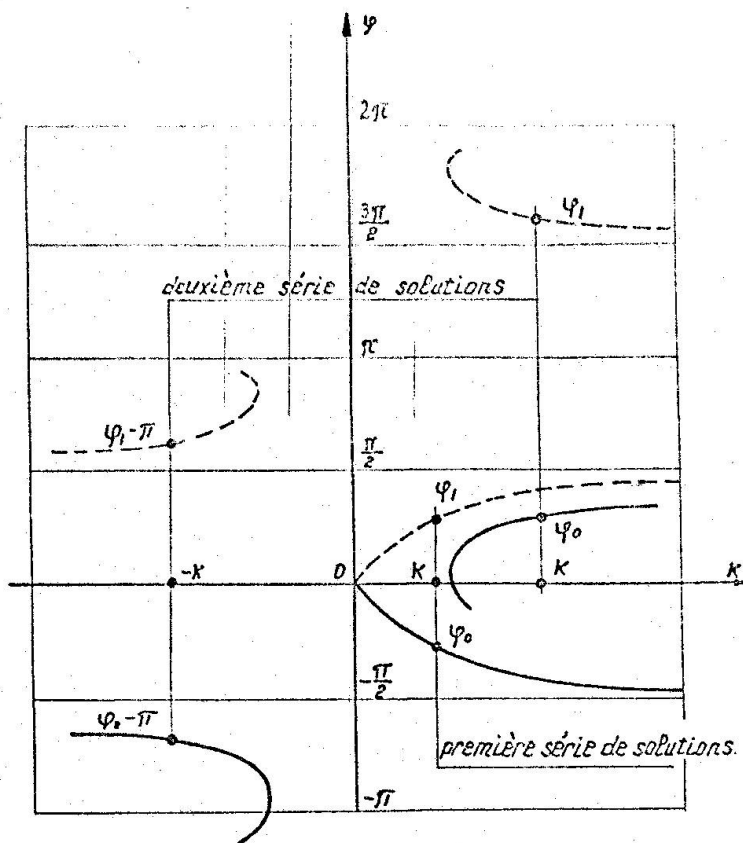


Fig. 2. Amplitudes ψ_0 et ψ_1 en fonction du module κ (abaque schématisique)

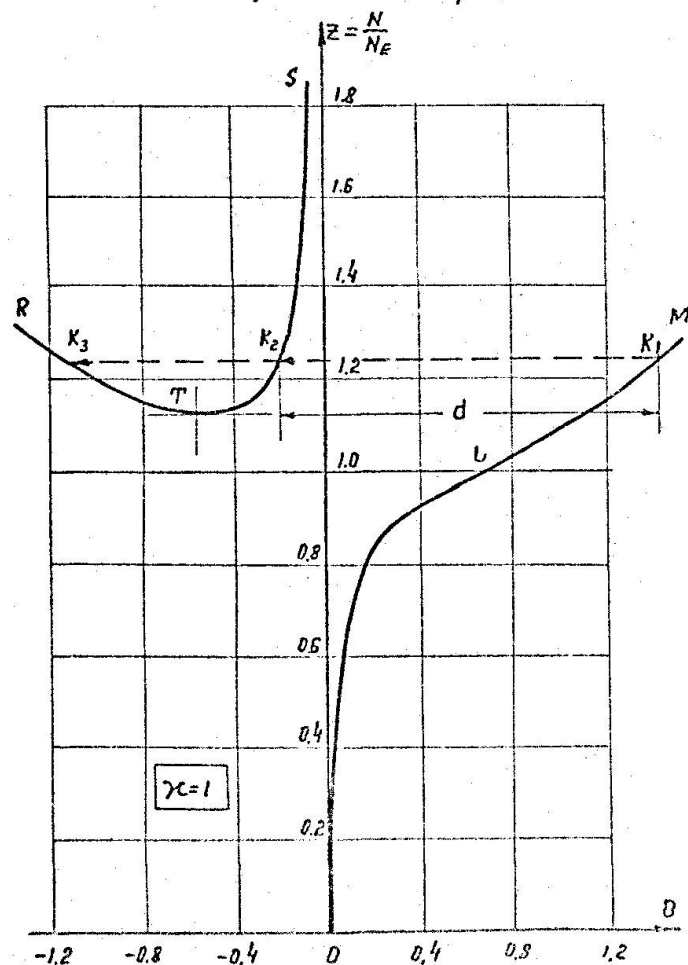


Fig. 4. Comportement de la barre conforme au schéma a

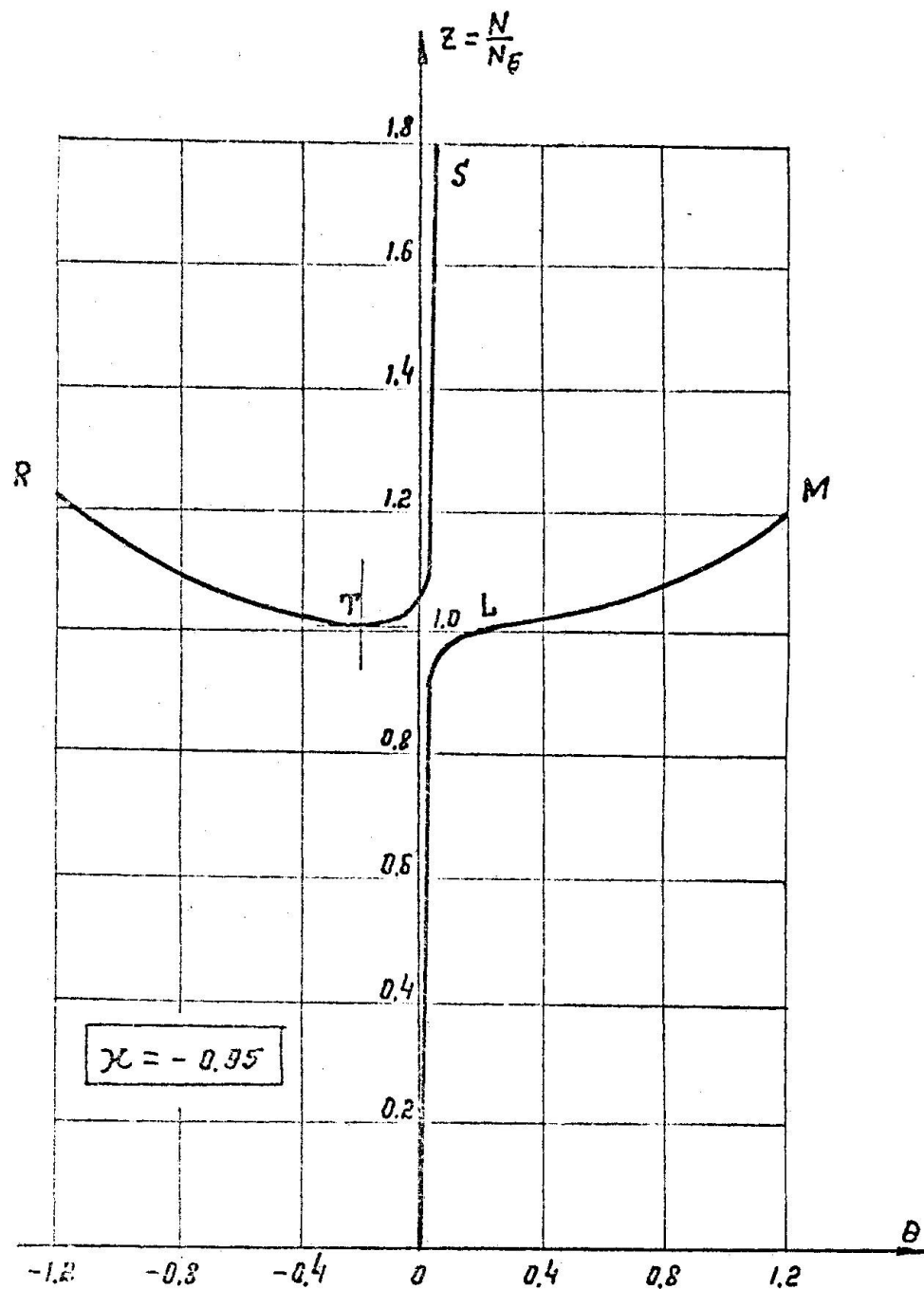


Fig. 5. Comportement de la barre conforme au schéma e .

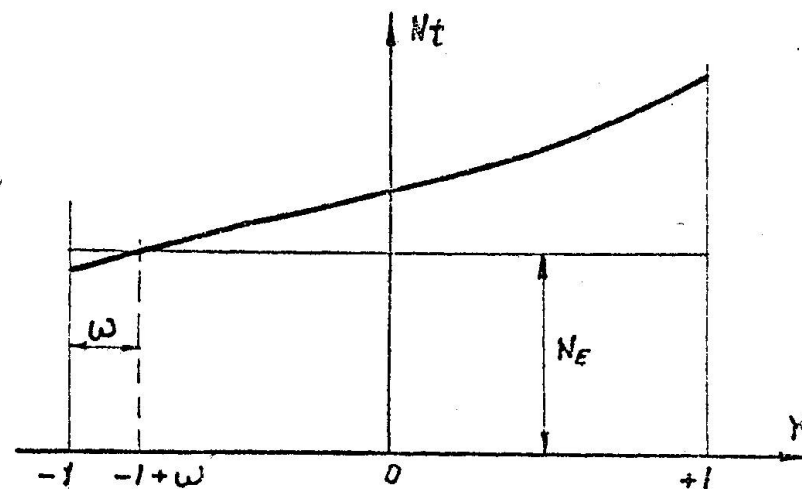


Fig. 6. Courbe schématique de $N_t(x)$.

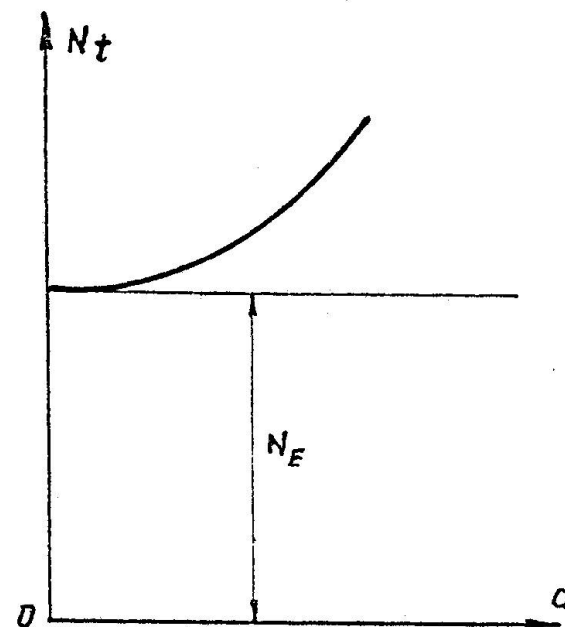


Fig. 7. Courbe schématique de $N_t(\alpha)$ pour $x=1$.

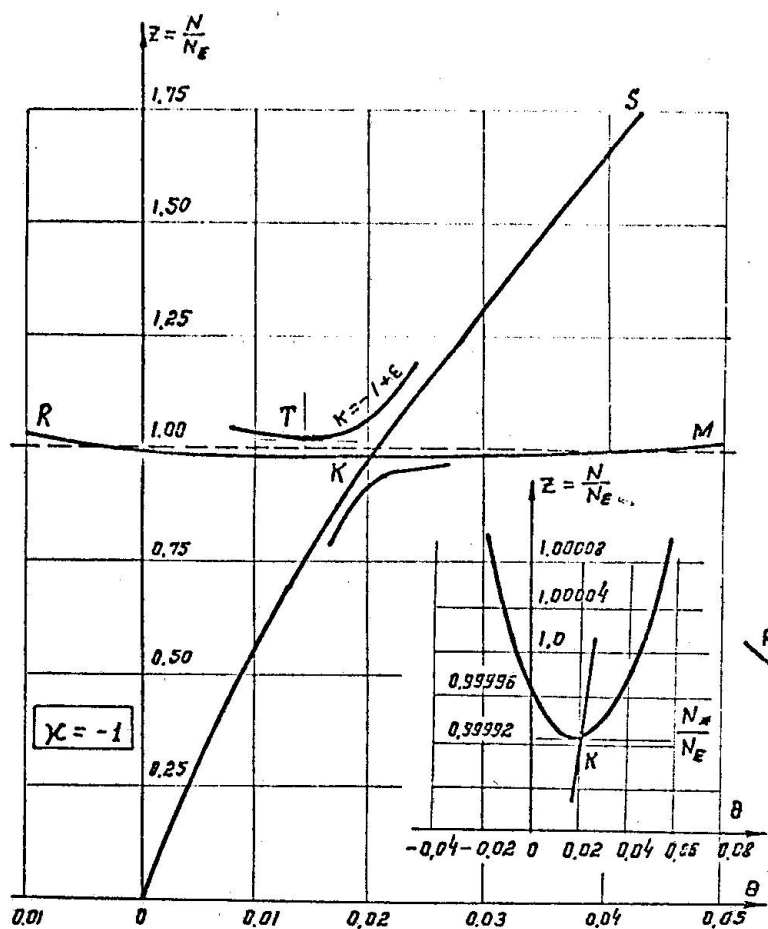


Fig. 8. Comportement de la barre et abaque à échelle réduite pour le schéma f .

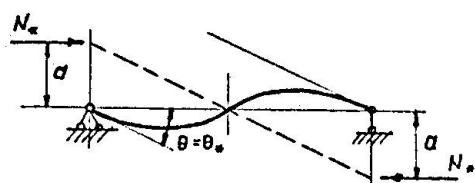


Fig. 9. Barre chargée d'une façon antisymétrique au moment du flambement.

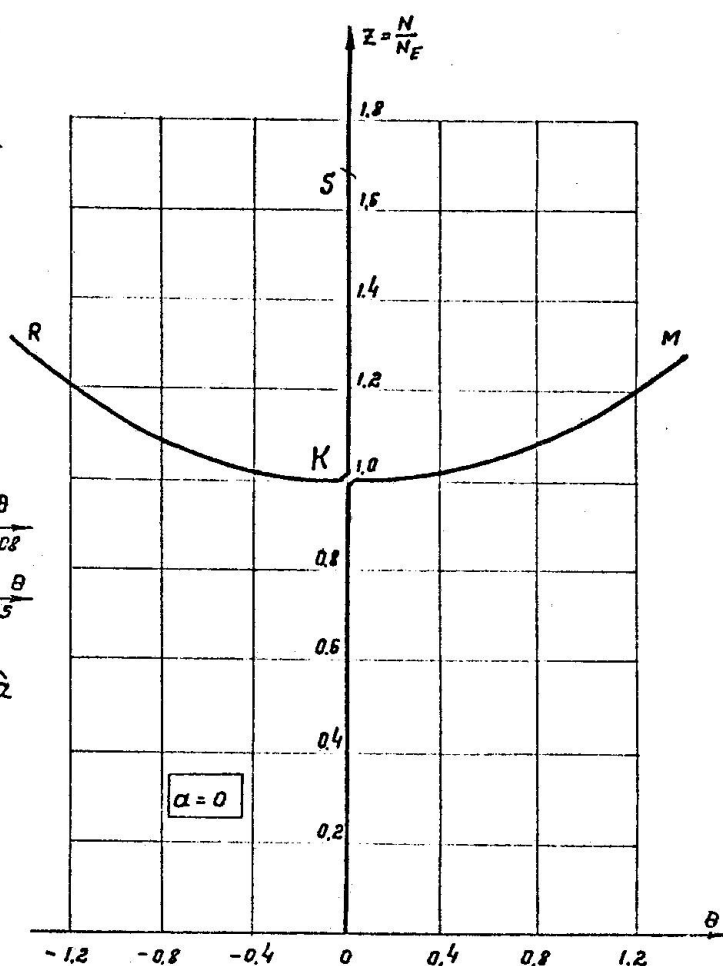


Fig. 10. Comportement de la barre conforme au schéma g .

COMPRESSIVE STRENGTH OF COLUMNS WITH INITIAL DEFLECTIONS

Yuzuru Fujita
Professor
Department of Naval Architecture
University of Tokyo

Koichiro Yoshida
Associate Professor
Department of Naval Architecture
University of Tokyo

ABSTRACT

In this paper, the effects of initial deflections of columns on the compressive maximum loads are investigated, where columns are assumed to be under the combined states of axial compression and uni-axial bending.

The compression tests of 20 columns of rectangular section and 8 columns of H-type section with sine half wave type initial deflections were conducted.

The theoretical solutions were computed by using incremental Finite Element Method. The accordance of both results was satisfactory. Moreover, the maximum compressive loads of columns with irregular type of initial deflections and of eccentrically compressed columns with initial deflections were calculated theoretically.

1. INTRODUCTION

Structural elements of actual structures usually have several kinds of imperfections ; initial deflections, residual stresses, unexpected discontinuities of structural components and so on.

These imperfections may severely weaken the strength of structures designed based on the ideal states of structures. It is necessary to clarify the effects of imperfections not only qualitatively but also quantitatively. The theoretical analysis of the structures with imperfections, however, will not be easy in many cases partly because linear problems will become non-linear ones and in some cases two dimensional problems will become three dimensional ones and partly because it will be difficult to consider a suitable analytical models for so small amount of imperfections.

In this paper, the effects of initial deflections of columns on the compressive maximum loads are investigated theoretically and experimentally, where columns are assumed to behave under the combined states of axial compression and uni-axial bending. This is one of the most basic problems and studied by K. Jezek (1). However, as the comprehensive study on the effects of shapes and magnitudes of initial deflections on the compressive strength of columns seemed to be not enough, in this paper, authors applied the Finite Element Method to calculate the compressive maximum loads of columns with arbitrary shape of initial deflection and compared the results with the experiments in several cases.

2. COMPRESSION TESTS AND THEORETICAL CALCULATIONS OF COMPRESSIVE STRENGTH OF COLUMNS WITH INITIAL DEFLECTIONS

2.1. Compression Tests of Columns of Rectangular Section and H-type Section

The sectional dimensions of columns of rectangular section (hereafter, abbreviated to R-section) and of columns of H-type section (hereafter, abbreviated to H-section) were kept constant throughout specimens (see fig. 1) and the slenderness ratios were varied as follows ; 40, 60, 80, 100 and 120 for columns of R-section and 60 and 80 for columns of H-section. Initial deflections of specimens were formed by four points bending into the shape closely similar to half wave mode of sine curve and then stress relief was performed. (Hereafter, this type of initial deflections is called as a regular one). Four kinds of magnitudes of initial deflections were given for each slenderness ratio. Therefore, the total number of specimens was 28 pieces.

In the Fig. 2, the examples of measured initial deflections are shown. In the compression tests, the specimens were compressed by screw-type universal testing machine so that they might be bent about their weak axes with no rotational restraint at both ends.

These end conditions were attained by using end fixtures shown in the fig. 1.

2.2. Theoretical Calculation

Theoretical calculation was performed by the Finite Element Method because it is easy to deal with arbitrary shapes of initial deflections and with different kinds of end conditions. A column is divided into a number of beam elements and stiffness matrix of an element is calculated by assuming beam theory. To calculate the maximum load of a compressed column, it is necessary to trace its behaviour up to the stage of unloading after the maximum load attained. Therefore, the magnitude of both lateral deflection and axial strain of the column may become

considerably large.

In other words, both geometrical and material nonlinearity have to be simultaneously included in the theory. These two kinds of nonlinearity can be overcome approximately by the incremental load method which assumes piecewise linearity.

The errors due to this assumption are compensated approximately in this paper by modifying each load increment by the unbalance force at the previous step of configuration.

In the following, basic formulations for numerical calculations are given by using the general three dimensional finite displacement theory (2). When a body is deformed from some step of configuration to the adjacent step of configuration, the following relation may be assured from the principle of virtual work.

$$\begin{aligned} & \frac{1}{2} \iiint_V \delta(\Delta \sigma_{\lambda\mu} \Delta \epsilon_{\lambda\mu}) dV + \frac{1}{2} \iiint_V \sigma_{\lambda\mu} \delta(\Delta u_{k,\lambda} \Delta u_{k,\mu}) dV - \iiint_V \Delta \bar{X}_\lambda \delta \Delta u_\lambda dV \\ & - \iint_{S_1} \Delta \bar{P}_\lambda \delta \Delta u_\lambda dS + \left(\iiint_V \sigma_{\lambda\mu} \delta \Delta \epsilon_{\lambda\mu} dV - \iiint_V \bar{X}_\lambda \delta \Delta u_\lambda dV - \iint_{S_1} \bar{P}_\lambda \delta \Delta u_\lambda dS \right) = 0 \quad (1) \end{aligned}$$

where Δ is a symbol to express an increment and $\bar{X}$ and $\bar{P}$ are body force and external force applied on force boundary S_1 , respectively. $\Delta \epsilon_{\lambda\mu}$ is a linear term of a strain increment expressed by finite displacement increments.

Assumption that external force, internal force and their increments are not varied during the variation with respect to displacement increment $\delta \Delta u_\lambda$ gives

$$\delta(\Delta \Pi) = 0 \quad (2)$$

where

$$\begin{aligned} \Delta \Pi = & \frac{1}{2} \iiint_V \Delta \sigma_{\lambda\mu} \Delta \epsilon_{\lambda\mu} dV + \frac{1}{2} \iiint_V \sigma_{\lambda\mu} (\Delta u_{k,\lambda} \Delta u_{k,\mu}) dV - \iiint_V \Delta \bar{X}_\lambda \Delta u_\lambda dV \\ & - \iint_{S_1} \Delta \bar{P}_\lambda \Delta u_\lambda dS + \left(\iiint_V \sigma_{\lambda\mu} \Delta \epsilon_{\lambda\mu} dV - \iiint_V \bar{X}_\lambda \Delta u_\lambda dV - \iint_{S_1} \bar{P}_\lambda \Delta u_\lambda dS \right) \quad (3) \end{aligned}$$

In the Finite Element Method, $\Delta \Pi$ for a global system is defined as summation of $\Delta \Pi_E$ for each element, where $\Delta \Pi_E$ is calculated by using variables with respect to the local coordinates of the element.

From (3),

$$\begin{aligned} \Delta \Pi_E = & \frac{1}{2} \Delta \delta^T \left(\iiint_{V_E} \mathbf{B}^T \mathbf{D} \mathbf{B} dV \right) \Delta \delta + \frac{1}{2} \Delta \delta^T \left(\iiint_{V_E} \sigma_{\lambda\mu} \mathbf{G}_{,\lambda}^T \mathbf{G}_{,\mu} dV \right) \Delta \delta \\ & - \Delta \mathbf{Q}^T \Delta \delta + \left(\iiint_{V_E} \sigma_{\lambda\mu}^T \mathbf{B} dV \right) \Delta \delta - \mathbf{Q}^T \Delta \delta \quad (4) \end{aligned}$$

where

$$\Delta \sigma = \mathbf{D} \Delta \epsilon, \quad \Delta \epsilon = \mathbf{B} \Delta \delta, \quad \Delta u = \mathbf{G} \Delta \delta, \quad \Delta \mathbf{Q} = \iiint_{V_E} \mathbf{G}^T \bar{\mathbf{X}} dV + \iint_{S_E} \bar{\mathbf{G}} \Delta \bar{\mathbf{P}} dS$$

$\Delta \delta$: incremental nodal displacement vector with respect to local coordinates.

Performing the coordinates transformation of variables from the local to the global and then summing $\Delta \Pi_E$, $\Delta \Pi$ for the global system is obtained as

$$\Delta \Pi = \frac{1}{2} \Delta \delta_s^T K \Delta \delta_s + \frac{1}{2} \Delta \delta_s^T K_G \Delta \delta_s - \Delta Q_s^T \Delta \delta_s + (R^T \Delta \delta_s - Q_s^T \Delta \delta_s) \quad (5)$$

where

$$K = \sum \pi_E^T \left(\iiint_{V_E} B^T D B dV \right) \pi_E, \quad K_G = \sum \pi_E^T \left(\iiint_{V_E} \sigma_{\lambda\mu} \epsilon_{\lambda}^T \epsilon_{\mu} dV \right) \pi_E$$

$$R = \sum \pi_E \left(\iiint_{V_E} B^T \sigma dV \right)$$

π_E : coordinate transformation matrix

$\Delta \delta_s$: incremental nodal displacement vector with respect to global coordinates

From the principle of stationary,

$$(K + K_G) \Delta \delta_s = \Delta Q_s - (R - Q_s) \quad (6)$$

The second term of the right hand side of the above equation can be considered to be caused by the incompleteness of the equilibrium conditions between external and internal forces at the previous step of configuration.

The problem treated here is combination of axial compression and uniaxial bending of columns and so, a first and a third order polynomials with respect to an axial coordinate are used as an axial and a lateral displacement function, respectively.

The stress-strain relation was assumed referring to results of material tests, to be perfectly plastic for the columns of R-sections and to be strain-hardening ($E_t = 1/100 E = \text{constant}$) for the columns of H-sections (see fig. 3). The theoretical calculations of the maximum loads of the compression test specimens were performed by assuming symmetry of configuration about the midpoint of column length. The examples of subdivision of elements are shown in the fig. 4.

2.3. Comparison of Experiments with Calculations

Several examples of the relation between compressive load and central deflection are shown in figures 5 to 7, where the deflection is the additional one to the original configuration and the magnitude is measured by using a scale below the abscissa.

In the case of the column with small slenderness ratio as well as small initial deflection, the relation between compressive load and additional deflection is almost linear and the inclination is steep before initial yield load is attained. This tendency may be caused by the fact that the axial force component of the sectional forces is comparatively large. The compressive load decreases suddenly after the maximum load is attained (see R 4005 in figure 5).

While, in the case of the column with large slenderness ratio as well as large initial deflection, the relation between compressive load and additional deflection deviates already in the early stage considerably from linearity and reduction of the compressive load after the maximum load attained is hardly recognized (see R 12020 in figure 6).

The experimental and calculated maximum loads for all specimens are listed in table 1. The variation of the maximum loads due to the magnitudes of initial deflections is shown in figures 8 and 9 to take slenderness ratios as a parameter. As to the calculations in these figures, for the tested specimens, the measured initial deflections were, of course, used as in-put data but otherwise the regular initial deflections were assumed.

Therefore, even if the slenderness ratios are kept constant, the calculated results are, strictly speaking, not necessary to be able to be connected by one continuous fair curve but in these figures, this attempt was done because the shapes of the initial deflections of all the specimens resembled closely the half wave mode of sine curve. According to these figures, almost all the calculated points are connected by each curve without excessive unnaturalness. In fig. 8, the Euler critical stress for the elastic range and the yield stress for the plastic range were taken respectively as an intersection of each curve with the ordinate and in figure 9, the same as above for the elastic range but natural extension of a curve for the plastic range were taken, respectively.

3. COLUMN CURVES OF COLUMNS WITH REGULAR INITIAL DEFLECTIONS

By using the curves in figures 8 and 9, column curves in figure 10 and figure 11 can be made where a/r is taken as a parameter. Figure 12 shows comparison of present calculations with Jezek's ones and in this case, a/l is taken as a parameter. Although Jezek's curves were obtained by assuming parabolic shapes of initial deflection symmetrical about the midpoint, both curves coincide well because Jezek's parabola is similar to the half wave mode of sine curve.

4. THE MAXIMUM COMPRESSIVE LOADS OF COLUMNS WITH IRREGULAR INITIAL DEFLECTIONS

It may be presumed that columns with regular initial deflections and with no rotational restraint at both ends are the weakest. However, it is necessary to know quantitative effects of shapes of initial deflections or of end restraints on the maximum loads of compressed columns. In this section, these kinds of problems are treated. The calculated results for columns with four kinds of irregular initial deflections and with fully rotational restraints at both ends are shown in table 2. In this table, the ratios of the maximum load of each column to the one of the column 8021, which is the column with regular initial deflection, are shown too.

According to this table, the compressive strength of the column 8022 and 8023 increases by only few percents, but as to the column 8024, the increase is considerably large. However, it may be true that in an actual compression test of a column like 8024, mode change will occur at some intermediate stage of loading and the compressive load will not reach up to the value given in table 2.

In figure 13, the aspect of gradual variation of configuration of the column R 8025 is shown, where the shaded area indicates spread of the plastic zone and in figure 14, the same situation for the column R 8026 is shown.

5. THE MAXIMUM LOADS OF ECCENTRICALLY COMPRESSED COLUMNS WITH INITIAL DEFLECTIONS

In table 3, the effects of eccentricity of applied compressive forces on the maximum loads of straight columns and of columns with initial deflections are shown. The sectional shapes of columns are rectangular and their dimensions and material constants are the same as in table 2. Initial deflections are assumed to be the same regular shapes as before. From these results, it can be seen that even if the magnitude of eccentricity is the same as the one of initial deflection, the effect is different and also that when both eccentricity and initial deflection exist, the effect of these two kinds of imperfections cannot be obtained as the summation of the individual effect.

6. CONCLUSIONS

- (1) The columns of R-section and H-section with initial deflections whose shapes are similar to the half wave mode of sine curve were compressed and the measured maximum loads were compared with the theoretical calculations by using the FEM. The experiments coincided well with the calculations with respect to both the load-deflection curves and the values of the maximum compressive loads.
- (2) Column curves of R-section and H-section were made by using the calculated maximum loads.
- (3) The maximum compressive loads of columns with several kinds of shapes of initial deflections were calculated and also the case of fixed end conditions was treated.
- (4) The maximum loads of eccentrically compressed straight columns and of eccentrically compressed columns with initial deflections were calculated.

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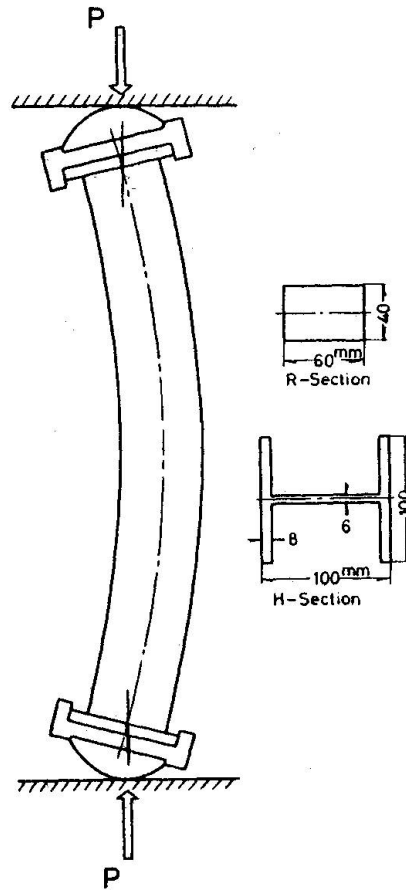


Fig.1 Compression Test Arrangement

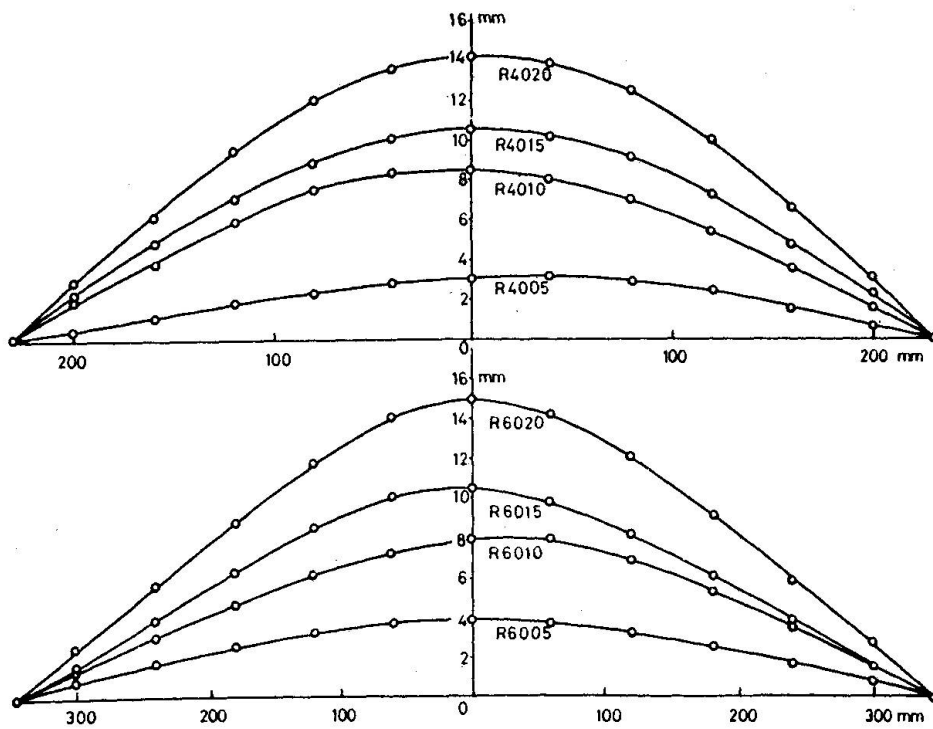
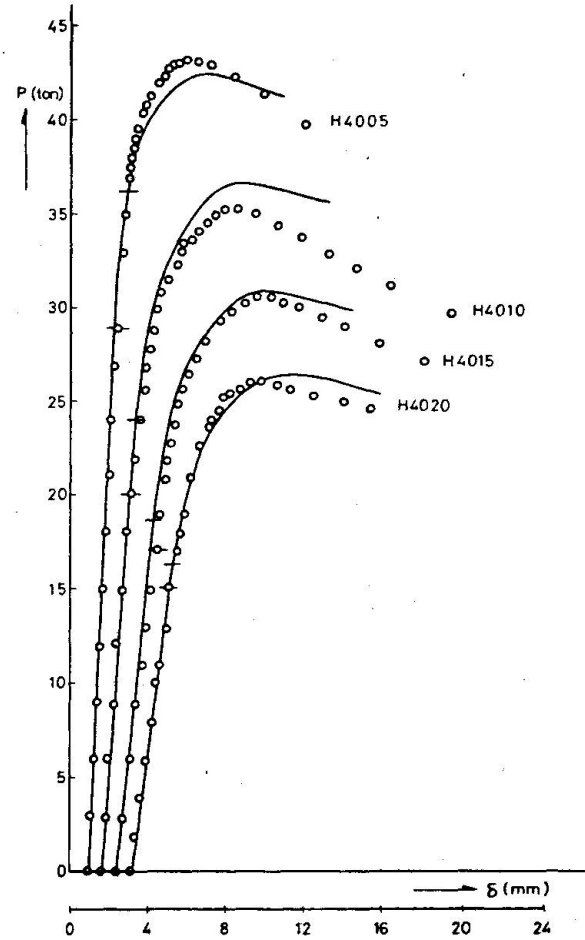
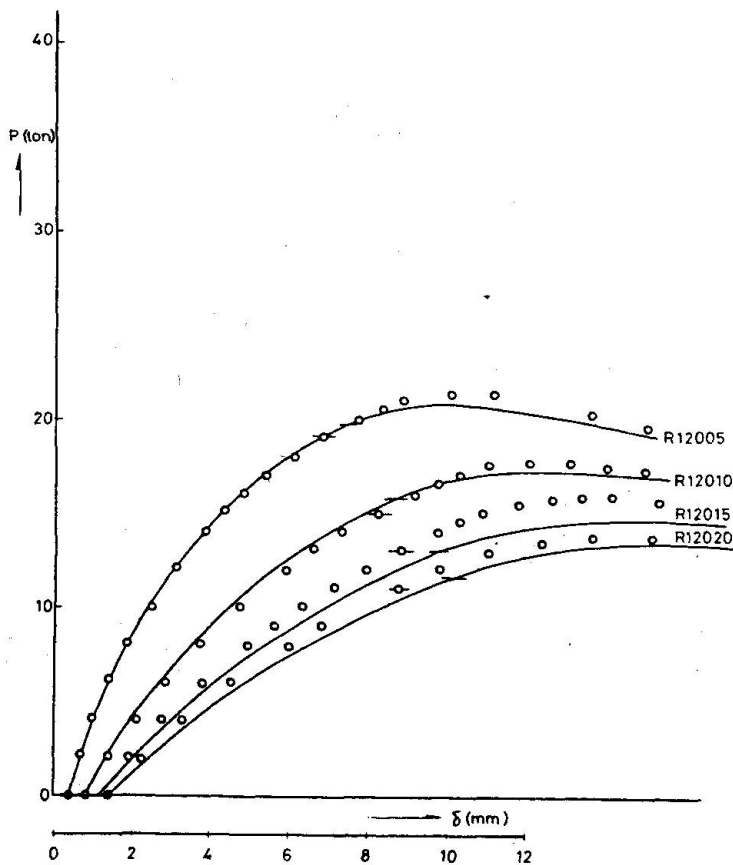
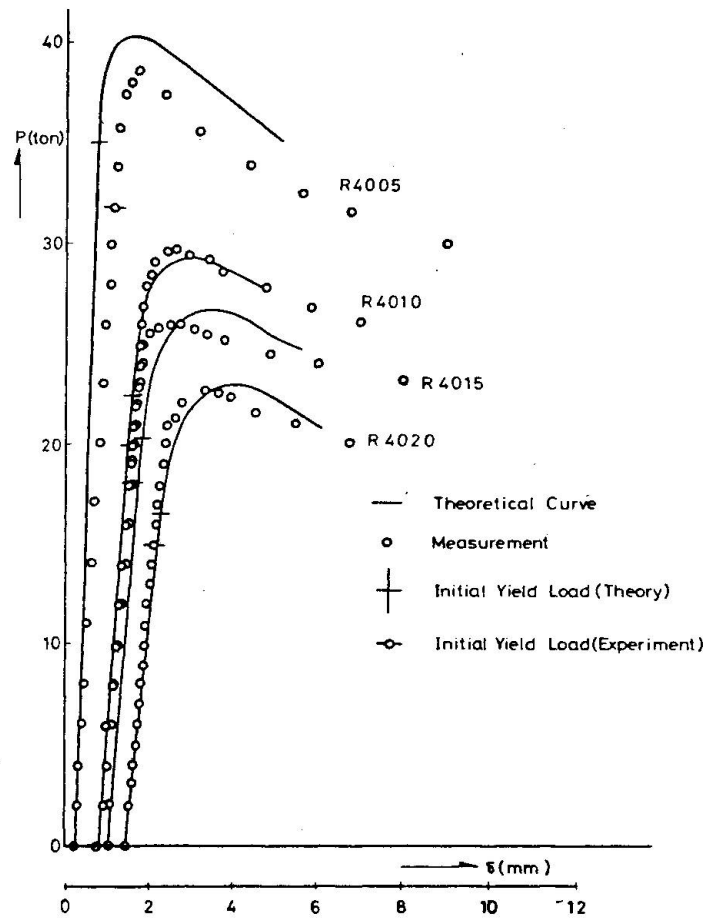
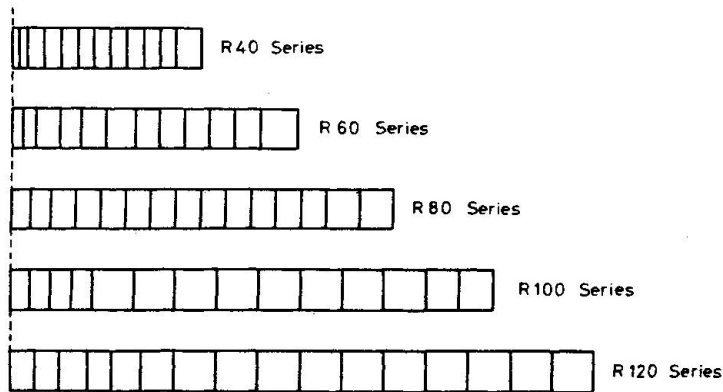
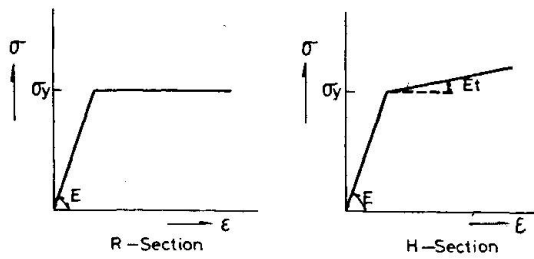


Fig.2 Initial Deformation



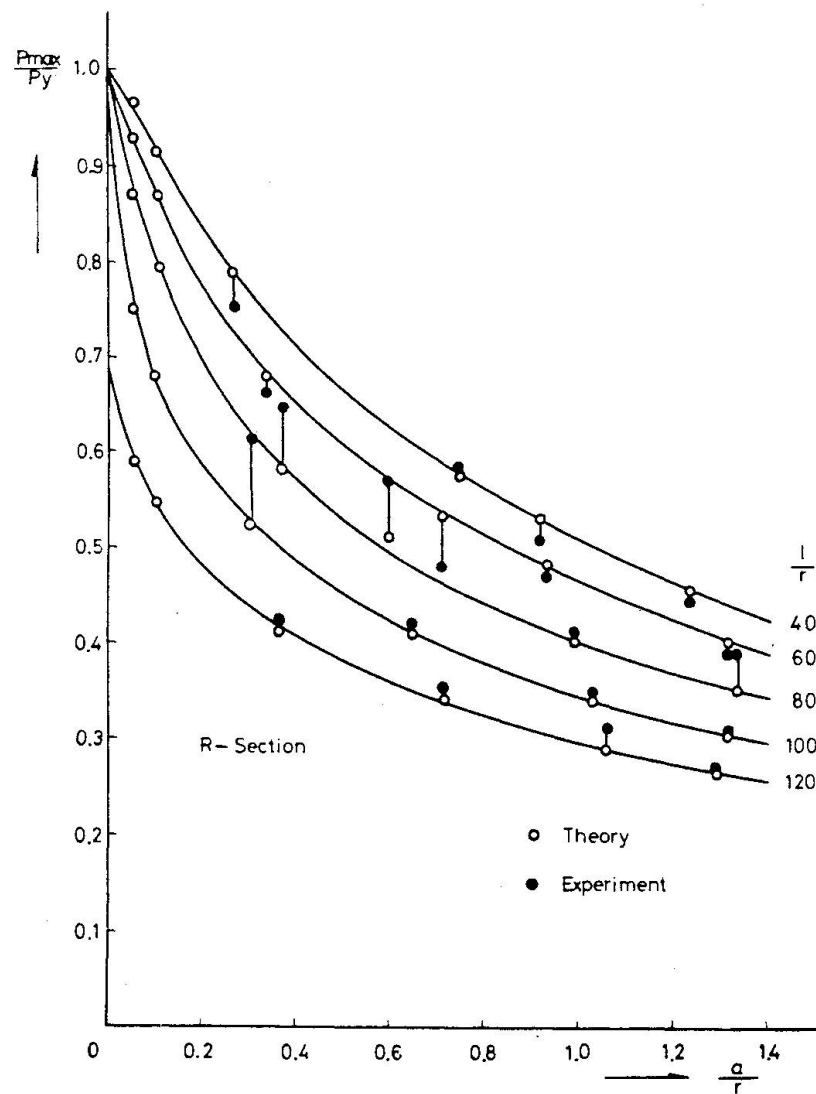


Fig.8 Relation between Maximum Loads and Initial Deflections of Rectangular Sections

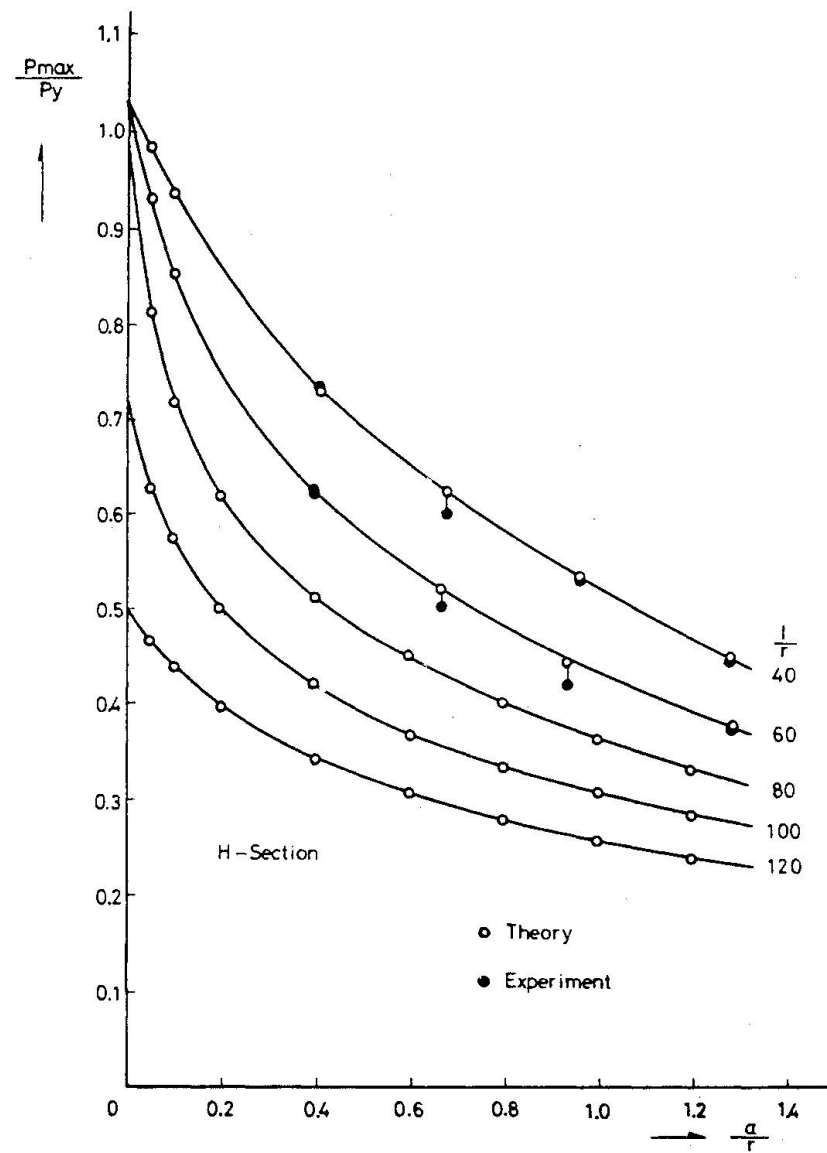


Fig.9 Relation between Maximum Loads and Initial Deflections of H-Sections

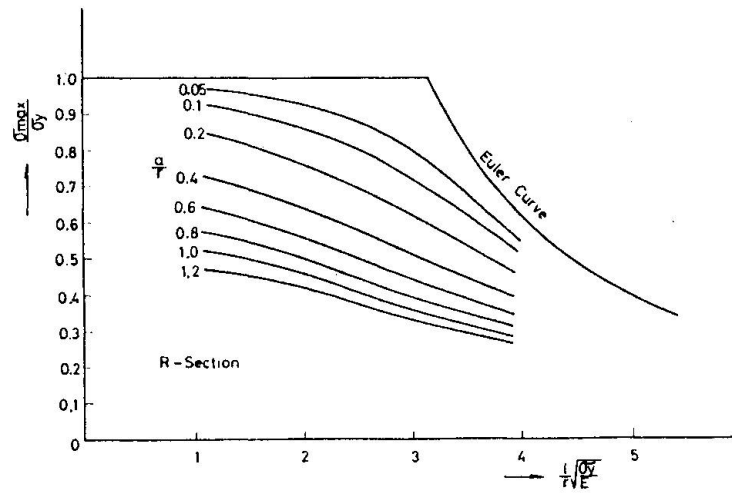


Fig.10 Column Curves of Rectangular Sections

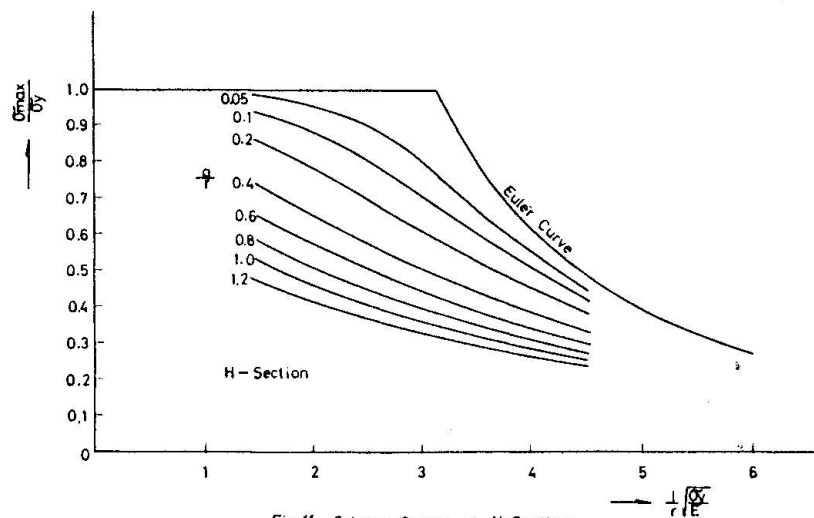


Fig.11 Column Curves of H-Section

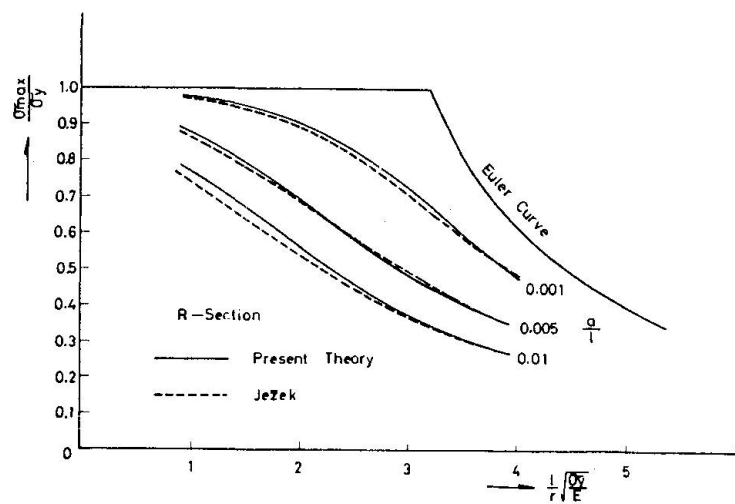


Fig.12 Comparison of Present Theory with Jeřek

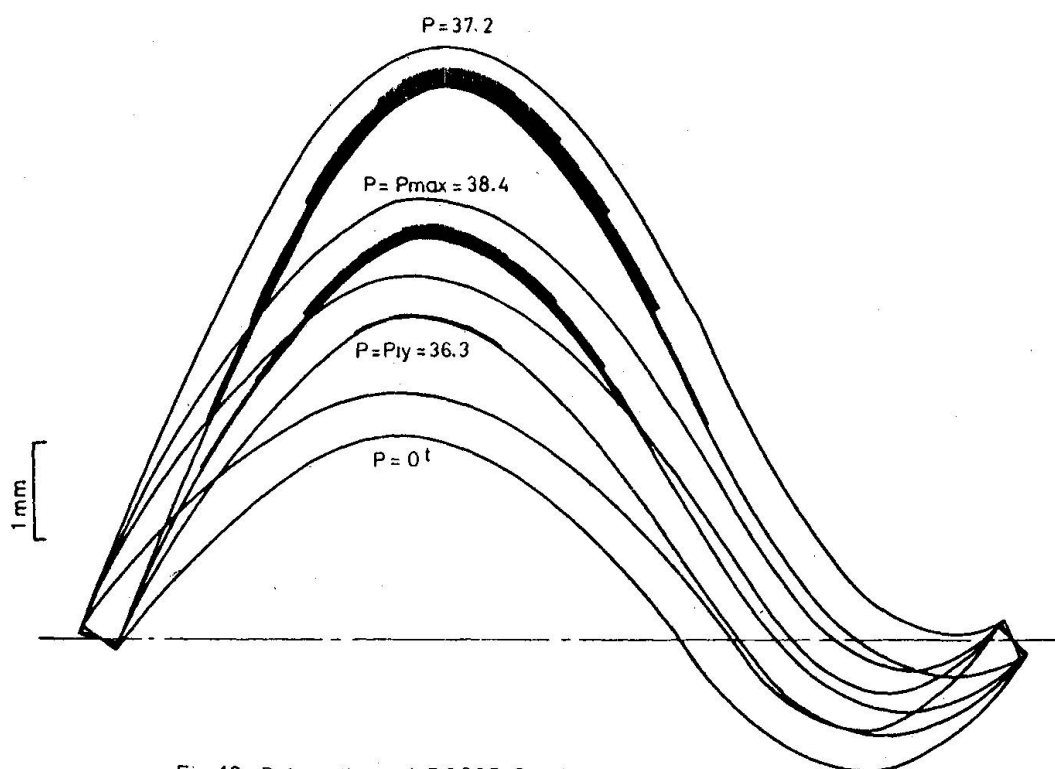


Fig.13 Deformations of R8025 Specimen

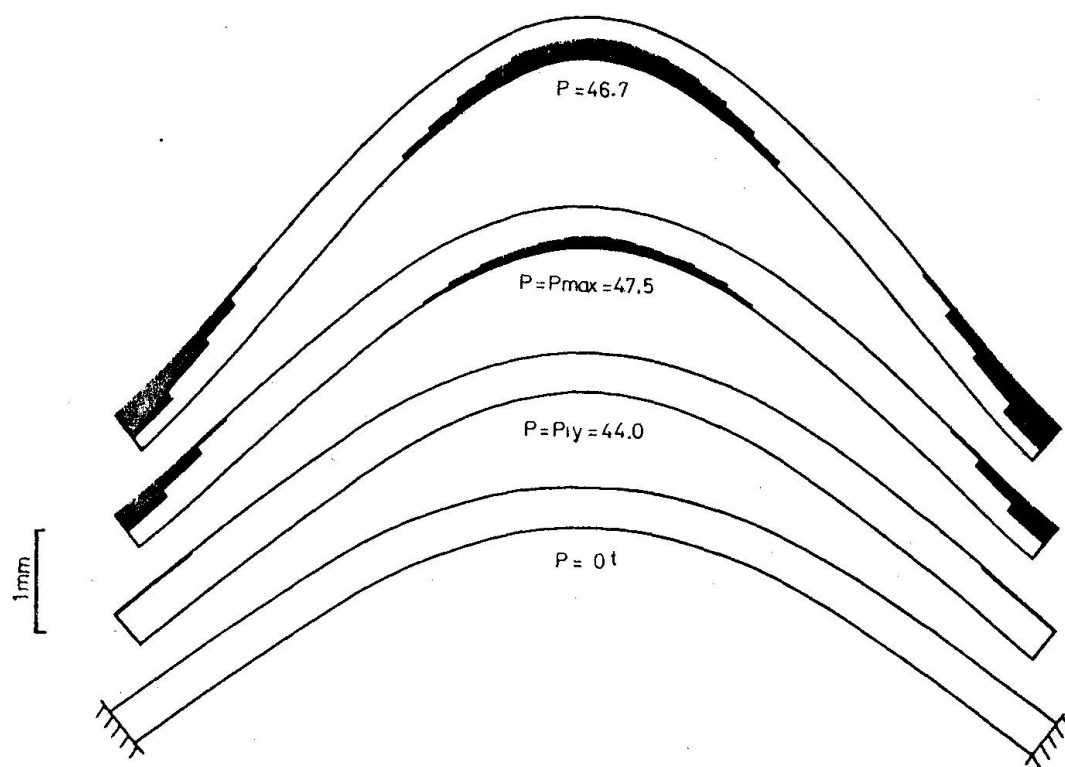


Fig.14 Deformations of R8026 Specimen

Table 1 Experimental & Calculated Results

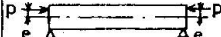
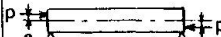
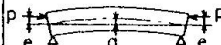
Specimen	Slenderness Ratio l/r	Initial Deflection Ratio		Experimental Maximum Load		Theoretical Maximum Load		Error % $\frac{(P_{max})_t - (P_{max})_e \times 100}{(P_{max})_e}$	
		a/r	a/l	$(P_{max})_e$	$(P_{max})_e/P_y$	$(P_{max})_t$	$(P_{max})_t/P_y$		
R4005	40	0.262	0.00655	38.6	0.757	40.3	0.791	4.3	$\bar{\sigma}_y = 21.5 \text{ kg/mm}^2$ for R-Section $= 28.6 \text{ kg/mm}^2$ for H-Section
R4010		0.742	0.0186	29.8	0.585	29.4	0.577	-1.2	
R4015		0.912	0.0228	25.9	0.510	27.0	0.531	4.2	
R4020		1.232	0.0308	22.6	0.444	23.2	0.456	2.5	
R6005	60	0.341	0.00568	33.7	0.664	34.6	0.682	3.0	$E = 2.1 \times 10^4 \text{ kg/mm}^2$ for both Section
R6010		0.707	0.0118	24.5	0.482	27.2	0.535	11.2	
R6015		0.931	0.0155	23.8	0.470	24.5	0.484	2.2	
R6020		1.311	0.0218	19.8	0.391	20.4	0.403	2.9	
R8005	80	0.368	0.00460	32.9	0.648	29.6	0.583	-10.2	$E_t = 0 \text{ kg/mm}^2$ for R-Section $= 2.1 \times 10^2 \text{ kg/mm}^2$ for H-Section
R8010		0.592	0.00740	28.9	0.570	25.9	0.511	-10.4	
R8015		0.993	0.0122	20.9	0.412	20.4	0.402	-2.5	
R8020		1.332	0.0167	19.8	0.391	17.7	0.350	-10.7	
R10005	100	0.299	0.00299	31.2	0.617	26.5	0.524	-15.2	l : length r : radius of gyration a : initial deflection at midpoint
R10010		0.646	0.00646	21.3	0.421	20.7	0.409	-2.9	
R10015		1.027	0.0103	17.6	0.348	17.2	0.340	-2.3	
R10020		1.318	0.0132	15.7	0.310	15.3	0.303	-2.6	
R12005	120	0.365	0.00304	21.5	0.425	20.8	0.411	-3.1	P_{max} : maximum load P_y : compressive yield load
R12010		0.711	0.00593	17.8	0.352	17.2	0.341	-2.6	
R12015		1.058	0.00882	15.9	0.312	14.7	0.289	-7.5	
R12020		1.283	0.0107	13.8	0.273	13.4	0.266	-2.6	
H4005	40	0.410	0.0103	43.1	0.739	42.6	0.731	-1.1	
H4010		0.679	0.0170	35.4	0.599	36.9	0.624	4.2	
H4015		0.962	0.0241	30.7	0.525	31.1	0.531	1.2	
H4020		1.283	0.0321	26.1	0.440	26.5	0.446	1.5	
H6005	60	0.400	0.00667	36.3	0.620	36.5	0.624	0.7	
H6010		0.669	0.0112	30.5	0.502	31.6	0.520	3.5	
H6015		0.939	0.0157	25.1	0.418	26.6	0.443	5.9	
H6020		1.291	0.0215	21.8	0.373	22.1	0.378	1.3	

Table 2. Calculated Results for Various Initial Deformations

Specimen	Initial Deformation	Slenderness Ratio l/r	Initial Deflection Ratio q/r	Maximum Load		Maximum Load Ratio
				P_{max}	P_{max}/P_y	
R 8021		80	0.2	35.7 ¹	0.692	1.00
H 8021				37.2	0.618	1.00
R 8022				36.9	0.702	1.03
H 8022				38.8	0.645	1.04
R 8023				36.4	0.705	1.02
H 8023				38.3	0.637	1.03
R 8024				43.2	0.836	1.21
H 8024				48.7	0.809	1.31
R 8025				38.4	0.743	1.08
H 8025				40.5	0.674	1.08
R 8026				47.5	0.921	1.33
H 8026				55.4	0.921	1.49

$$\begin{aligned} \sigma_y &= 21.5 \text{ kg/mm}^2 & E &= 2.1 \times 10^4 \text{ kg/mm}^2 & E_t &= 0 \text{ kg/mm}^2 \text{ for R-Section} \\ &= 28.6 \text{ kg/mm}^2 & &= 2.1 \times 10^4 \text{ kg/mm}^2 & &= 2.1 \times 10^2 \text{ kg/mm}^2 \text{ for H-Section} \end{aligned}$$

Table 3. Calculated Maximum Loads of Eccentrically Compressed Columns with Initial Deformations

Imperfection	Specimen	Calculated Condition	Slenderness Ratio l/r	Eccentricity or Initial Deflection	Maximum Load		
					P_{max}	P_{max}/P_y	
Eccentricity	R 8027		80	$e = 0.2^{\text{mm}}$	48.8 ¹	0.948	
	R 8028			$e = 1.0$	41.1	0.795	
	R 8029			$e = 4.0$	29.9	0.580	
	R 8030			$e = 0.2$	51.6	1.000	
	R 8031			$e = 1.0$	49.6	0.958	
	R 8032			$e = 4.0$	41.8	0.811	
Both Eccentricity and Initial Deflection	R 8033			$e = 1.0$ $a = 1.0$	36.5	0.708	
	R 8034			$e = 1.0$ $a = 1.0$	48.8	0.948	
	R 8035			$e = 1.0$ $a = 1.0$	41.6	0.807	

$$\sigma_y = 21.5 \text{ kg/mm}^2 \quad E = 2.1 \times 10^4 \text{ kg/mm}^2 \quad E_t = 0 \text{ kg/mm}^2$$

FLAMBEMENT PAR FLEXION-TORSION DES COLONNES COMPRIMEES
CENTRIQUEMENT, APPUYEES ELASTIQUEMENT AUX EXTREMITES

D. Mateescu

E. Cuteanu

Institutul Politehnic "Traian Vuia"
Catedra de Constructii Metalice
Timisoara, Roumanie

ABSTRACT

The determination of the critical load on a column of a mono-symmetric section is presented herein for the general case of end restraints. The column, whose centroid does not necessarily coincide with the shear centre, is represented by a system of eight first-order differential equations. Since ten analytical treatments of the problem poses great difficulties, ten solutions of the differential equations are obtained numerically by applying the method of Runge-Kutta-Gill. A general program was prepared and ten numerical computations are carried out in the IRIS-50 electronic digital computer. The influence of the end conditions on the value of the critical load in flexural-torsional buckling are examined using several numerical examples. A few conclusions are drawn on the influence of the geometric parameters, related to flexural, torsional, and bimoment restraints, on the buckling behaviour.

1. INTRODUCTION

Le calcul de stabilité qui reflète le plus fidèlement le comportement réel d'une structure, c'est le calcul à la divergence d'équilibre. Un tel calcul, effectué sur l'ensemble de la structure, est extrêmement laborieux et -jusqu'à présent- difficile à aborder. Toutefois, du fait que l'instabilité de la structure par divergence peut être causée par l'affaîssement d'un seul élément composant, on a proposé des méthodes de calcul basées sur l'étude de l'instabilité d'une seule barre de la structure. En isolant donc la barre, mais en tenant compte de l'influence du reste de la structure, on arrive à des simplifications essentielles qui permettent d'aborder le problème. Evidemment, l'exactitude de cette substitution reste dépendante de la correction avec laquelle on détermine les paramètres de connexion entre l'élément examiné et le reste de la structure. Comme exemples d'application de ce procédé on peut citer /1/, /2/ pour les cadres et /3/, /4/ pour les poutres à treillis constituées de barres à parois minces formées à froid.

Les colonnes des cadres sont généralement comprimées et fléchies, avec ou sans charges transversales appliquées entre les noeuds. Pour tenir compte des effets de charges transversales et des imperfections, on dispose de relations d'interaction, connues dans la littérature et adoptées par les règlements /5/, /6/, /7/.

2. DEFINITION DU PROBLEME

L'un des paramètres essentiels, dans les relations d'interaction mentionnées ci-dessus, c'est la valeur critique de l'effort de compression centrique pour la barre idéale (sans aucune imperfection).

Si la barre comprimée présente un seul plan de symétrie longitudinal, ce qui arrive bien souvent en pratique (Fig. 1), les centres de gravité et de torsion ne coïncident pas ; de ce fait l'instabilité se produit généralement par flexion transversale et torsion. Dans le but d'une utilisation plus généralisée des relations d'interaction, il est nécessaire de connaître l'effort critique du flambement centrique par flexion transversale et torsion, dans l'hypothèse d'appuis élastiques.

Le système (1-2) est remplacé par un système équivalent de huit équations différentielles du premier ordre, ce qui permet de réduire les erreurs dans le cadre du procès d'intégration numérique /12/. Le système est écrit dans une forme non dimensionnelle :

$$\left\{ \begin{array}{l} \tilde{u}' = \varrho \\ \varrho' = \beta \\ \beta' = \delta \\ \delta = g_3 n \alpha - \beta m \\ \psi' = \gamma \\ \gamma' = \alpha \\ \alpha' = \gamma \\ \gamma' = \left(g_1 - \frac{n}{g_2} \right) \alpha + \frac{g_3}{g_2} \beta \end{array} \right.$$

Pour cette forme d'instabilité, on connaît une formule généralisée par un paramètre tenant compte des conditions d'appui /8/. Il faut toutefois souligner que la formule mentionnée et respectivement les valeurs du paramètre ne concernent pas les appuis élastiques ; même dans les autres situations, les résultats ne sont exacts que si les déformées de flexion et de torsion de l'équilibre indifférent sont affiniées. (Dans l'article /9/, on donne une justification théorique de cette observation, par l'utilisation des fonctions fondamentales du flambement par flexion). Pour autres modes d'appui, la formule peut conduire à des valeurs plus ou moins satisfaisantes ; dans certains cas, les résultats peuvent devenir inutilisables (nombres imaginaires), ce qui nous impose une grande réserve.

Le but de l'article présent est de déterminer cet effort critique des barres à section monosymétrique, pour le cas général d'appui. Puisque un traitement analytique aurait présenté des difficultés on a fait appel à un procédé de calcul numérique, en intégrant le système d'équations différentielles du problème par la méthode Runge-Kutta-Gill.

3. EQUATIONS DU PROBLEME

Par rapport au système de référence intrinsèque ayant l'origine dans le centre de torsion (Fig. 2) le système d'équations différentielles est /10/, /11/ :

$$\begin{cases} B\bar{u}'''' + N\bar{u}'' - N\bar{y}_G\psi'' = 0 \\ C_\omega\psi'''' - (C - Ni_c^2)\psi'' - N\bar{y}_G\bar{u}'' = 0 \end{cases} \quad (1...2)$$

où :

- $\bar{u}$ est le déplacement du centre de torsion selon la direction de l'axe $\bar{x}$
- ψ rotation de la section transversale autour du centre de torsion C
- N effort de compression centrique, dont on calcule la valeur
- $\bar{y}_G$ ordonnée du centre de gravité G par rapport au centre de torsion C
- i_c rayon d'inertie polaire par rapport au centre de torsion C

et :

$$B = EI_y, \quad C = GI_t, \quad C_\omega = EI_\omega \quad (3...5)$$

sont les modules de rigidité à la flexion dans un plan perpendiculaire au plan de symétrie, à la torsion pure (du type Saint-Venant) et respectivement au gauchissement.

On a utilisé les notations suivantes dans (6-13) :

$$\tilde{u} = \frac{\bar{u}}{i_c} \quad , \quad m = \frac{N l^2}{B} \quad (6...13)$$

$$g_1 = \frac{C l^2}{C_w} \quad , \quad g_2 = \frac{C_w}{B i_c^2} \quad , \quad g_3 = \frac{\bar{J}_G}{i_c}$$

où l , représente la longueur de la barre.

On peut mentionner que les mêmes paramètres géométriques de la barre (16-18) sont mis en évidence si l'on procède à l'intégration du système (1-2), ce qui prouve qu'ils sont intrinsèques au phénomène et facilitent l'étude de celui-ci.

4. CONDITIONS AUX LIMITES

Les conditions aux limites nécessaires à l'intégration du système d'équations différentielles (6-13) s'obtiennent en écrivant les équations d'équilibre des éléments différentiels découpés de la barre déformée dans l'état d'équilibre indifférent, à proximité des appuis. Outre l'effort de compression, aux extrémités de la barre agissent des réactions proportionnelles aux déplacements des appuis, notamment : une force transversale $k (\bar{u} - \bar{a} \varphi')$, un moment fléchissant $K (\bar{u} - \bar{b} \varphi')$, un bimoment $\chi \varphi''$ et un moment de torsion $\mathcal{K} \varphi$ (Fig. 3...6), où $k/\text{daN/cm}$, $K/\text{daNcm/rad}$, $\chi/\text{daNcm}^2/\text{cm}^{-1}$ et $\mathcal{K}/\text{daNcm/rad}$ représentent les rigidités de la structure correspondant respectivement à un déplacement de l'extrémité de la barre en question selon la direction $\bar{x}$, à une rotation de cette section autour de l'axe $\bar{y}$, à un gauchissement de la ligne médiane de la section et à une rotation autour de l'axe $\bar{z}$ (Fig. 3...6). En vue d'une généralisation, on a considéré que l'appui élastique utilisé contre le déplacement latéral est appliqué à une distance $\bar{a}$ du centre de torsion et que le centre de gravité de l'appui utilisé contre la rotation autour de l'axe de symétrie se trouve à une distance $\bar{b}$ (Fig. 3...6). Les sens positifs des déformations $\bar{u}$ et φ sont indiqués sur la figure 3. Sur les figures 3...6, on a indiqué séparément toutes les réactions qui interviennent dans les équations d'équilibre.

Ces équations sont, pour $\bar{x} = 0$:

$$\left\{ \begin{array}{l} k_0(\bar{u}_0 - \bar{a}_0 \psi_0) + N_{cr}(\bar{u}'_0 - \bar{y}_G \psi'_0) + \bar{u}'''_0 B = 0 \\ K_0(\bar{u}'_0 - \bar{b}_0 \psi'_0) - \bar{u}''_0 B = 0 \\ \gamma_0 \psi'_0 - \bar{b}_0 K_0(\bar{u}'_0 - \bar{b}_0 \psi'_0) - \psi''_0 C_\omega = 0 \\ -C_\omega \psi'''_0 + C \psi'_0 - \gamma_0 \psi_0 + k_0(\bar{u}_0 - \bar{a}_0 \psi_0) \bar{a}_0 - N_{cr}(\psi'_0 i_c^2 - \bar{u}'_0 \bar{y}_G) = 0 \end{array} \right. \quad (19...22)$$

D'une manière similaire, on écrit les conditions pour l'appui $\bar{x} = l$, en observant les mêmes conventions de signe et en affectant de l'indice l , les valeurs pour cette section.

Le dernier terme de l'équation (22) a été obtenu en intégrant sur la section transversale l'effet des contraintes normales σ_{cr} , parce que le principe de Saint-Venant n'est pas applicable à la torsion des barres à parois minces.

En utilisant les notations (14-18) et les rigidités non dimensionnelles :

$$\tilde{k} = \frac{k l^3}{B}, \quad \tilde{K} = \frac{K l}{B}, \quad \tilde{\gamma} = \frac{\gamma l}{C_\omega}, \quad \tilde{\mathcal{K}} = \frac{\mathcal{K} l^3}{C_\omega} \quad (23...26)$$

et les cotes non dimensionnelles des niveaux d'appui :

$$\tilde{a} = \frac{\bar{a}}{i_c}, \quad \tilde{b} = \frac{\bar{b}}{i_c} \quad (27...28)$$

Les conditions (19...22) pour $\psi = \frac{\bar{x}}{l}$ 0 deviennent :

$$\left\{ \begin{array}{l} \tilde{k}_0(\tilde{u}_0 - \tilde{a}_0 \psi_0) + n(\tilde{u}'_0 - g_3 \psi'_0) + \tilde{u}'''_0 = 0 \\ \tilde{K}_0(\tilde{u}'_0 - \tilde{b}_0 \psi'_0) - \tilde{u}''_0 = 0 \\ \tilde{\gamma}_0 \psi'_0 - \frac{\tilde{b}_0}{g_2} \tilde{K}_0(\tilde{u}'_0 - \tilde{b}_0 \psi'_0) - \psi''_0 = 0 \\ -\psi'''_0 + g_1 \psi'_0 - \tilde{\mathcal{K}}_0 \psi_0 + \frac{\tilde{a}_0}{g_2} \tilde{k}_0(\tilde{u}_0 - \tilde{a}_0 \psi_0) - \frac{n}{g_2}(\psi'_0 - \tilde{u}'_0 g_3) = 0 \end{array} \right. \quad (19'...22')$$

De même on écrit les conditions pour $\varphi = \frac{\tilde{z}}{l} = 1$.

Les conditions aux limites pour des modes d'appui particuliers /8/ peuvent être obtenues en rendant les rigidités nulles ou infinies (selon le cas). Les calculs numériques, effectués à l'aide d'un ordinateur IRIS-50 selon le programme élaboré par les auteurs (en FORTRAN IV), ont traité les modes d'appui indiqués dans le tableau 1.

TABLEAU 1

Nr.	TYPE D'APPUI $\varphi = 0$
1	
2	
3	

Nr.	TYPE D'APPUI $\varphi = 1$	Nr.	TYPE D'APPUI $\varphi = 1$
1		6	
2		7	
3		8	
4		9	
5		10	

5. METHODE DE CALCUL

L'intégration du système d'équations différentielles a été effectuée en traitant le problème de stabilité non pas comme un problème à conditions marginales, mais comme un problème à valeurs initiales. Ce procédé est utilisé dans de divers domaines /12/, /13/, grâce aux possibilités offertes par les ordinateurs. Donc, les solutions du système (6-13) sont écrites sous la forme :

$$\begin{aligned}\tilde{u} &= C_1 \tilde{u}_1^* + C_2 \tilde{u}_2^* + C_3 \tilde{u}_3^* + C_4 \tilde{u}_4^* \\ \varphi &= C_1 \varphi_1^* + C_2 \varphi_2^* + C_3 \varphi_3^* + C_4 \varphi_4^* \\ &\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ \delta &= C_1 \delta_1^* + C_2 \delta_2^* + C_3 \delta_3^* + C_4 \delta_4^*\end{aligned}\tag{29...36}$$

où $(\tilde{u}_i^*, \varphi_i^*, \dots, \delta_i^*)$, $i = 1, 2, 3, 4$ sont quatre groupes de solutions particulières linéairement indépendantes qui satisfont aux conditions initiales pour $\mathcal{Y} = 0$ et au système d'équations différentielles. Ces solutions particulières sont obtenues numériquement à l'aide de la méthode Runge-Kutta-Gill.

Les conditions initiales nécessaires dans ce procédé de calcul sont établies en précisant les vecteurs linéairement indépendants $(\tilde{u}_0, \tilde{u}_0', \dots, \varphi_0, \varphi_0')$ pour $\mathcal{Y} = 0$. Ainsi, par exemple, dans le cas des conditions d'appui (19'-22') qui correspondent au type 1 du tableau 1, les quantités $\tilde{u}_0''', \tilde{u}_0'', \varphi_0''$, et φ_0''' pouvant être exprimées en fonction de $\tilde{u}_0, \tilde{u}_0', \varphi_0$ et φ_0' , les quatre solutions particulières linéairement indépendantes sont établies à l'aide des vecteurs linéairement indépendants suivants :

$$\text{I.} \quad \tilde{u}_0 = 1, \varphi_0 = 0, \tilde{u}_0' = 0, \varphi_0' = 0$$

et les autres valeurs conformément à (19'-22'),

$$\text{II.} \quad \tilde{u}_0 = 0, \varphi_0 = 1, \tilde{u}_0' = 0, \varphi_0' = 0$$

et les autres valeurs conformément à (19'-22'),

$$\text{III.} \quad \tilde{u}_0 = 0, \varphi_0 = 0, \tilde{u}_0' = 1, \varphi_0' = 0$$

et les autres valeurs conformément à (19'-22'),

$$\text{IV.} \quad \tilde{u}_0 = 0, \varphi_0 = 0, \tilde{u}_0' = 0, \varphi_0' = 1$$

et les autres valeurs conformément à (19'-22').

En effectuant les quatre intégrations numériques, les valeurs des fonctions (29-36) pour $\mathcal{Y} = 1$ sont :

$$\tilde{u}_1 = C_1 a_{11} + C_2 a_{12} + C_3 a_{13} + C_4 a_{14}\tag{37...44}$$

$$g_1 = C_1 a_{21} + C_2 a_{22} + C_3 a_{23} + C_4 a_{24}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

(37...44)
suite

$$g_4 = C_1 a_{81} + C_2 a_{82} + C_3 a_{83} + C_4 a_{84}$$

Avec les valeurs (37-44) on impose la satisfaction aux conditions de l'autre extrémité ($\psi = 1$) et l'on obtient un système d'équations homogènes ayant pour inconnues les constantes d'intégration C_1, C_2, C_3, C_4 . L'élimination de la solution banale (qui correspondrait à la forme non déformée de la barre) prétend l'annulation du déterminant formé par les coefficients de ces inconnues ; il en résulte la valeur critique de l'effort de compression. Cette opération est effectuée par un calcul itératif, à partir d'une valeur appréciée n_0 , qui est modifiée successivement jusqu'à ce que le déterminant change de signe. La valeur n_{cr} recherchée se trouvera située entre deux valeurs consécutives de n , qui correspondent à des signes différents du déterminant. En appliquant la méthode de la corde et en diminuant chaque fois le pas Δn , comme il résulte de la figure 7, on peut considérer l'itération terminée lorsque pour une modification $\Delta n \leq \epsilon$ de n , le déterminant change de signe.

Dans certains cas, il est souhaitable de connaître aussi les allures des déformées $\tilde{u}$ et ψ qui correspondent à l'équilibre indifférent. Dans ce but, les valeurs de solutions particulières $(\tilde{u}_i^*, \psi_i^*)$, $i = 1, 2, 3, 4$ en 9 points intermédiaires équidistants et aux extrémités sont retenus dans la subroutine d'intégration du système d'équations différentielles. Avec ces valeurs, la subroutine DEFØRM (Fig.8) calcule les valeurs de trois constantes d'intégration du système d'équations différentielles en fonction de la quatrième constante (que l'on prend égale à 1) et construit les déformées à l'aide de (33) et (37).

6. DOMAINE DE VARIATION DES PARAMETRES g_1, g_2, g_3 .

Pour établir les domaines de variation des trois paramètres géométriques, on a d'abord construit la section typique pour une colonne d'un hall industriel à pont roulant (Fig. 9). En variant le paramètre $\gamma = b_2/b_1$ entre 1 et 0,5, on a obtenu pour une longueur $l = 10 h$ les valeurs suivantes :

γ	1,00	0,75	0,50
g_1	3,41	5,88	15,60
g_2	1,19	0,79	0,30
g_3	0,214	0,25	0,505

Ce domaine a été élargi par la considération des barres à sections transversales $\square$ et Π étant donné leur sensibilité aiguë au flambement par flexion-torsion mais aussi leur utilisation fréquente comme éléments sollicités axialement dans certaines structures planes et spatiales. En supposant, pour le calcul des paramètres, des longueurs de barres entre 0,5 et 4,0 m, les valeurs obtenues ont été situées entre les limites suivantes :

$$2 \leq g_1 \leq 120 \quad 0,02 \leq g_2 \leq 1,2 \quad 0,1 \leq g_3 \leq 1,0$$

qui ont été retrouvées lors de la vérification numérique de certains cas concrets.

7. CALCUL DE L'EFFORT CRITIQUE ET INTERPRETATION DES RESULTATS

Afin d'étudier l'influence des conditions d'appui sur la valeur de l'effort critique du flambement par flexion-torsion on a calculé plusieurs exemples numériques. Les valeurs des paramètres géométriques ont été choisies de sorte qu'elles correspondent pour une barre sensible à ce mode d'instabilité, comme le sont par exemple les barres à parois minces formées à froid, à section Π . Ainsi, les valeurs $g_2 = 0,07$ et $g_3 = 0,8$ correspondent à un profil Π de dimensions moyennes de la gamme des profils laminés couramment ; pour g_1 , qui dépend aussi de la longueur de la barre, on a considéré quatre valeurs : 2, 25, 50 et 100.

Les conditions d'appui ont été choisies pour modéliser une colonne faisant partie d'un cadre et qui, à l'extrémité supérieure, est liée de diverses façons au reste de la structure. Afin d'avoir un critérium pour le choix des rigidités de l'appui supérieur, on a défini trois degrés d'encastrement, notamment :

$$g_k = \frac{M_i}{m_i} \quad , \quad g_n = \frac{B}{\beta} \quad , \quad g_x = \frac{M_t}{m_t} \quad (45...47)$$

où les numérateurs représentent respectivement le moment de flexion, le bimoment et le moment de torsion engendrés par le flambement de flexion-torsion à l'extrémité appuyée élastiquement ; le dénominateur de chaque fraction représente la valeur de la sollicitation au cas où la déformation afférente serait complètement annulée par l'appui. Dans les calculs effectués, on a considéré les valeurs extrêmes des degrés d'encastrement (c'est-à-dire 0 et 1) et la valeur 0,5. Avec les définitions (46-47), les quantités $\tilde{\kappa}$ et $\tilde{\chi}$ ont été exprimées en fonction du paramètre g_1 et du degré d'encastrement respectif. Les diagrammes tracés (en hypothèse $\bar{a} = \bar{b} = 0$) montrent les effets des trois composantes $\tilde{\kappa}$, $\tilde{\chi}$ et $\tilde{\chi}$ qui définissent l'ensemble de la rigidité de l'appui (Fig. 10... 14).

Afin d'obtenir des conclusions directes de ces diagrammes, vu que le paramètre g_1 dépend (outre les caractéristiques géométriques de la section transversale) du carré de la longueur (16), les diagrammes de la figure 12 ont été retracés, de sorte que seule la longueur géométrique de la barre a été maintenue comme variable. On observe que l'effet de variation du degré d'encastrement est maximum dans le domaine $g_1 \leq 25$ (Fig. 15).

En examinant les diagrammes représentés dans les figures 10, 13, 14, on peut observer que pour $g_1 \geq 25$ un degré d'encastrement $\beta\chi = 0,5$ rapproche la valeur du paramètre critique n de celle qui correspond à un degré d'encastrement limite $\beta\chi = 1$; plus le degré d'encastrement au gauchissement $\beta\chi$ est grand, plus cet effet est marqué. Par conséquent, les diagrammes montrent l'inutilité des trop grandes rigidités de ces liaisons dans le domaine des grandes valeurs g_1 .

En effet, une analyse comparative des diagrammes des figures 10, 11, 12 d'une part, respectivement 10, 13, 14 de l'autre, montre que dans les situations où l'on opère des modifications sur les liaisons qui empêchent la rotation due à la flexion ($\tilde{u}$), le taux d'augmentation du paramètre critique n est plus grand dans le domaine des grandes valeurs g_1 , où le flambement par flexion est prédominant ; en ce qui concerne les modifications qui affectent les liaisons concernant l'autre composante du phénomène, c'est-à-dire la rotation et le gauchissement dus à la torsion (Ψ, Ψ'), le taux d'augmentation du paramètre critique est plus grand dans le domaine des petites valeurs g_1 .

En modifiant aussi les deux autres paramètres g_2 et g_3 de la barre, il résulte que pour l'augmentation de g_3 , la valeur du paramètre critique n diminue (Fig. 16). L'explication réside dans la signification géométrique de ce paramètre, qui est proportionnel à la distance entre les deux centres de la section transversale. Si cette distance est plus grande, l'effort de compression centrique produit (par sa composante transversale dans l'état déformé de la barre) un effet défavorable plus marqué. Certaines grandes valeurs de ce paramètre peuvent être aussi interprétées comme une réduction de la rigidité de la barre à la torsion, de sorte qu'une liaison à l'extrémité, plus rigide du point de vue de la torsion, ne saurait influencer la valeur du paramètre critique, d'une manière décisive.

L'influence du paramètre géométrique g_2 est inverse à celle de g_3 le paramètre critique n augmentant pour les grandes valeurs de g_2 . Toutefois, son importance pour l'analyse phénoménologique du flambement par flexion-torsion est moindre.

L'influence des rigidités des appuis a été suivie aussi à l'aide de diagrammes du type de celui de la figure 17, construits pour plusieurs valeurs constantes des paramètres géométriques, respectivement par l'interprétation des allures déformées ($\tilde{u}$ et Ψ) correspondant à l'équilibre indifférent. Ainsi sur la figure 18, sont représentées les allures des déformations d'équilibre indifférent (pour deux modes d'appui) d'une barre avec $g_1 = 10$, $g_2 = 0,07$ et $g_3 = 0,8$, où l'influence favorable de l'empêchement du gauchissement de la ligne médiane de la section transversale à l'extrémité est illustrée aussi par la géométrie des allures déformées.

En ce qui concerne l'influence des niveaux d'appui $\bar{a}$ et $\bar{b}$ les calculs effectués ont mis en évidence l'existence de niveaux optimums d'appui en fonction de l'élasticité de l'appui.

Ainsi, par exemple si le degré d'encastrement $\beta\chi$ (Fig. 17) tombe au-dessous de la valeur 0,75, pour obtenir une charge critique maximale (correspondant au degré d'encastrement $\beta\chi$ respectif) l'appui simple supérieur devra être disposé à une distance $\bar{a}$ de plus en plus importante du centre de torsion, cette distance tendant vers la valeur +1. (Fig. 17-18)

Les auteurs ont l'intention de poursuivre leurs recherches concernant les barres comprimées et fléchies.

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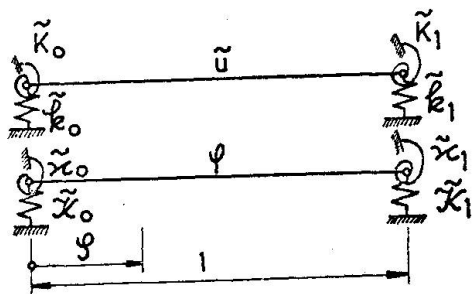


FIGURE 1

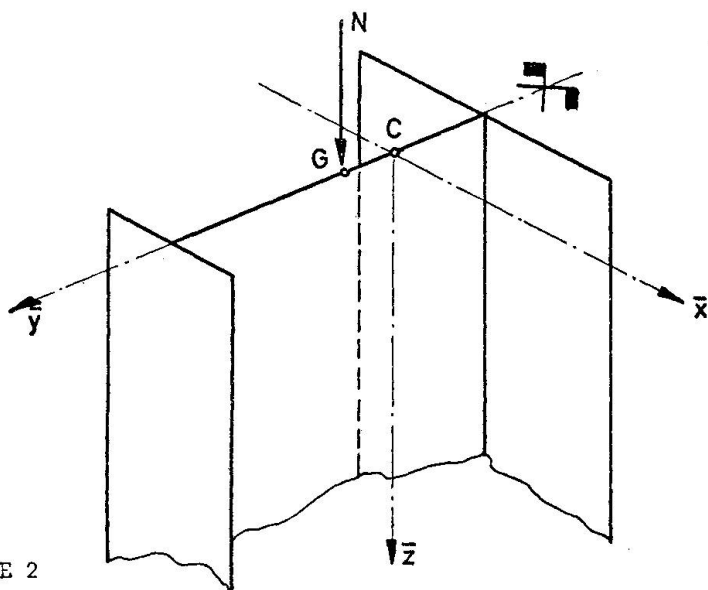
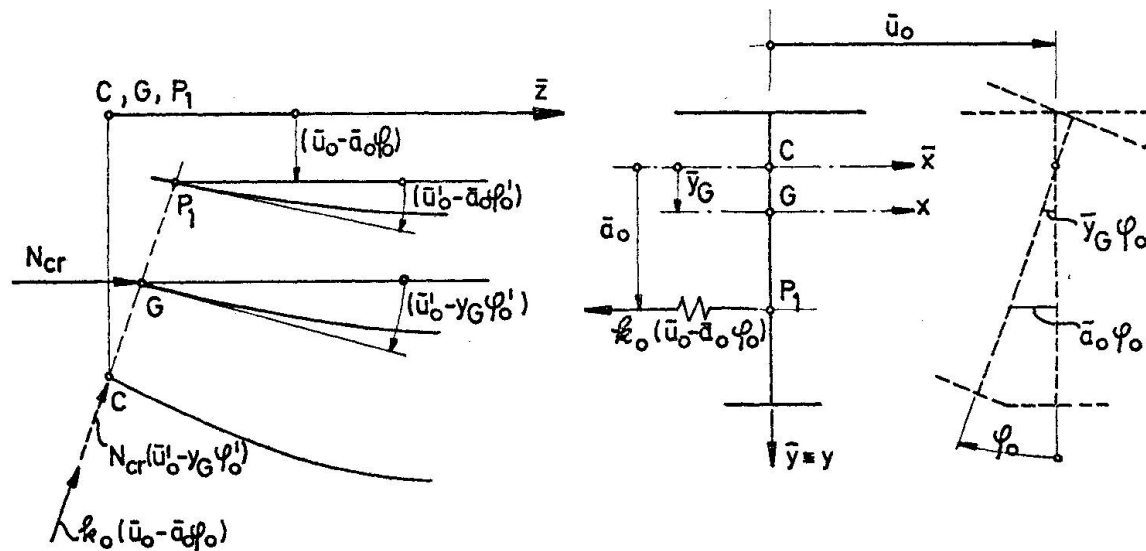


FIGURE 2

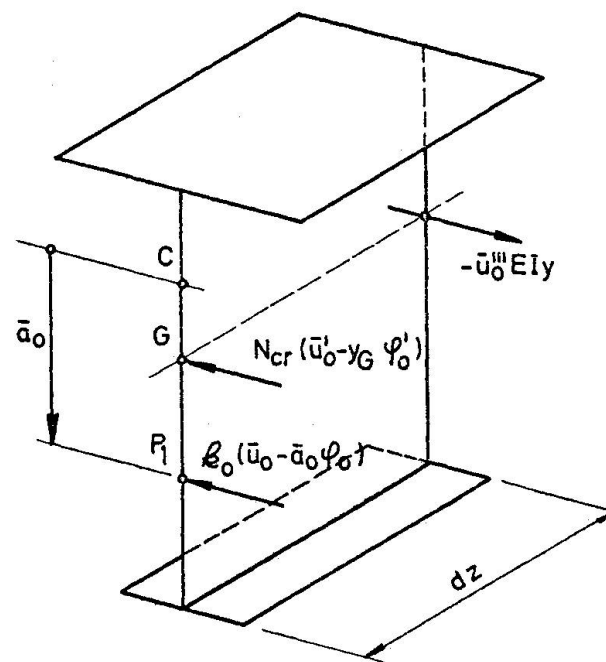


FIGURE 3

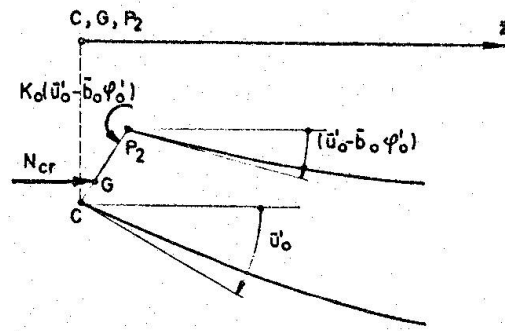


FIGURE 4

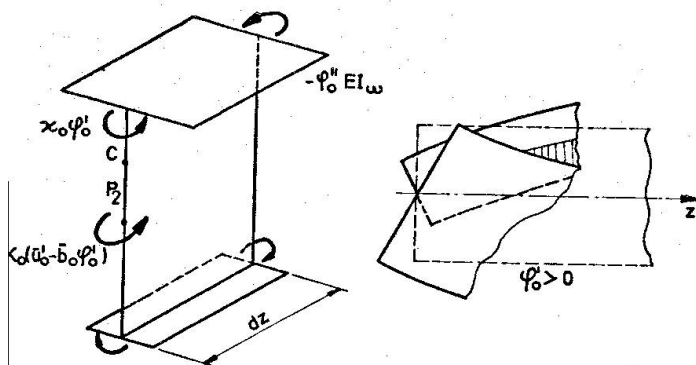
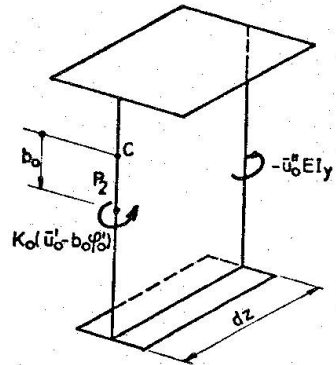


FIGURE 5

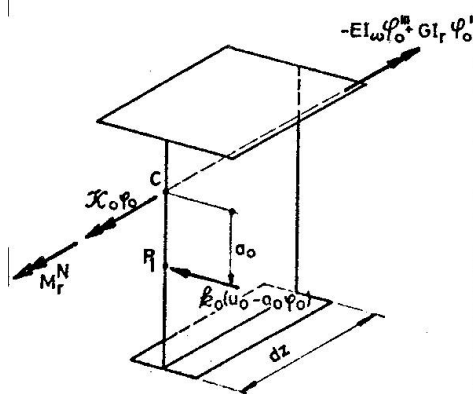


FIGURE 6

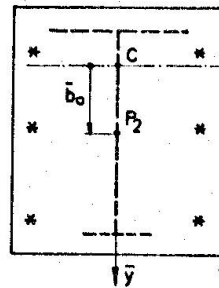


FIGURE 7

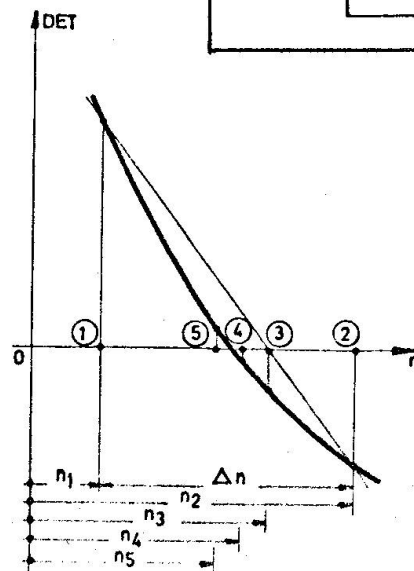


FIGURE 7

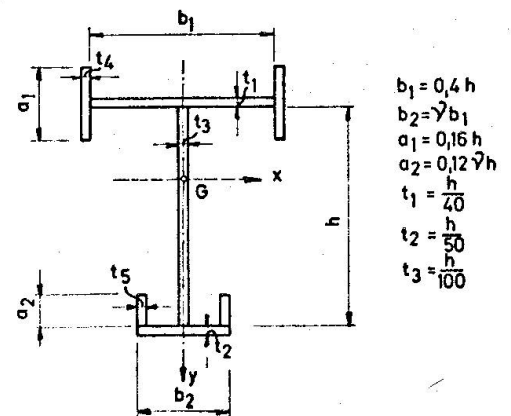


FIGURE 9

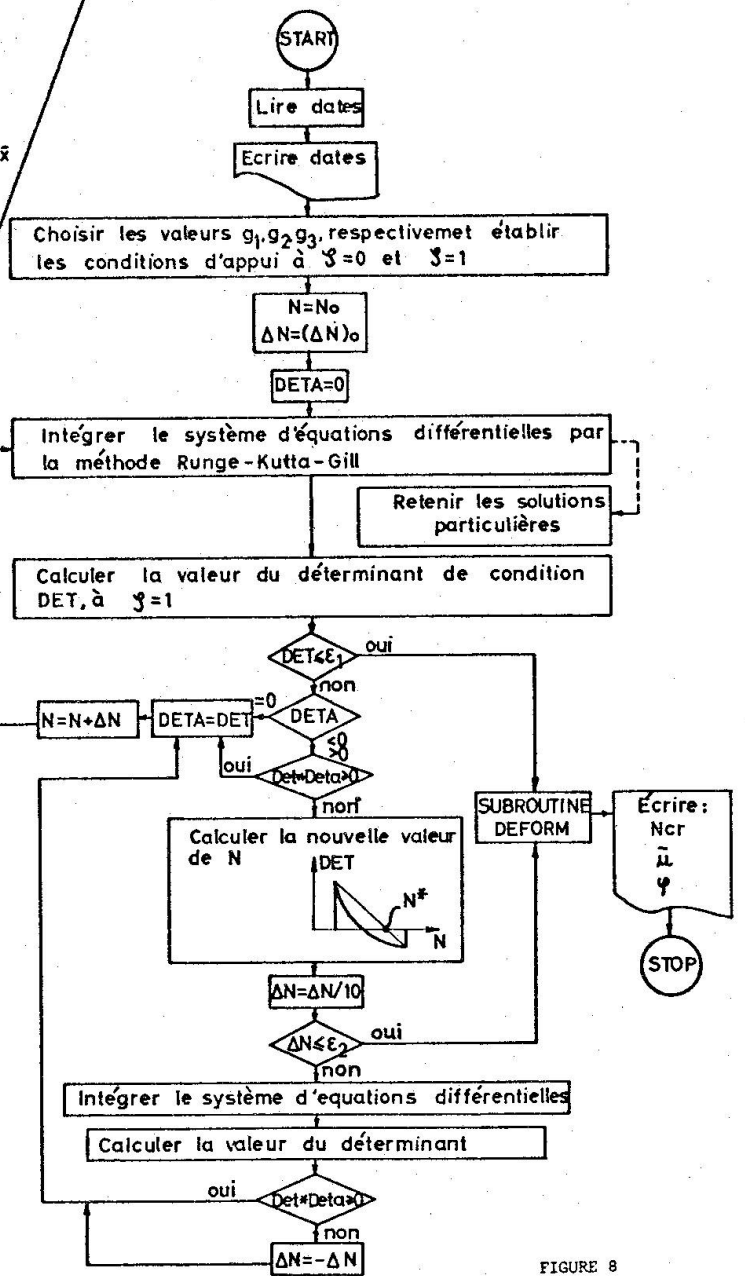


FIGURE 8

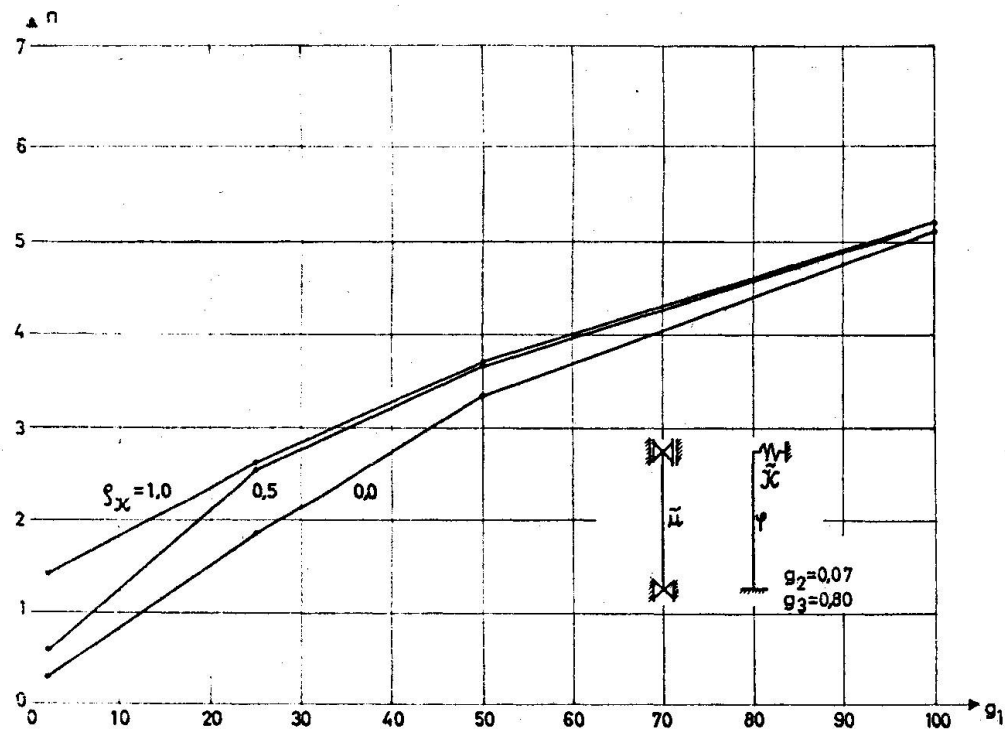


FIGURE 10

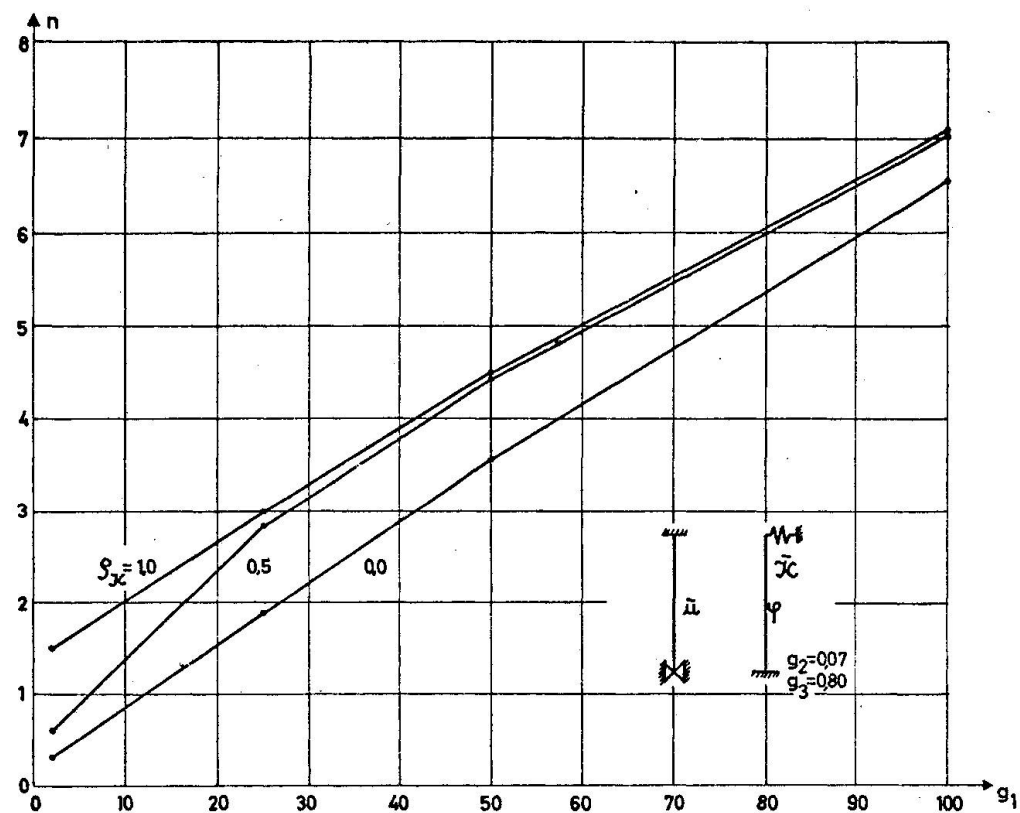
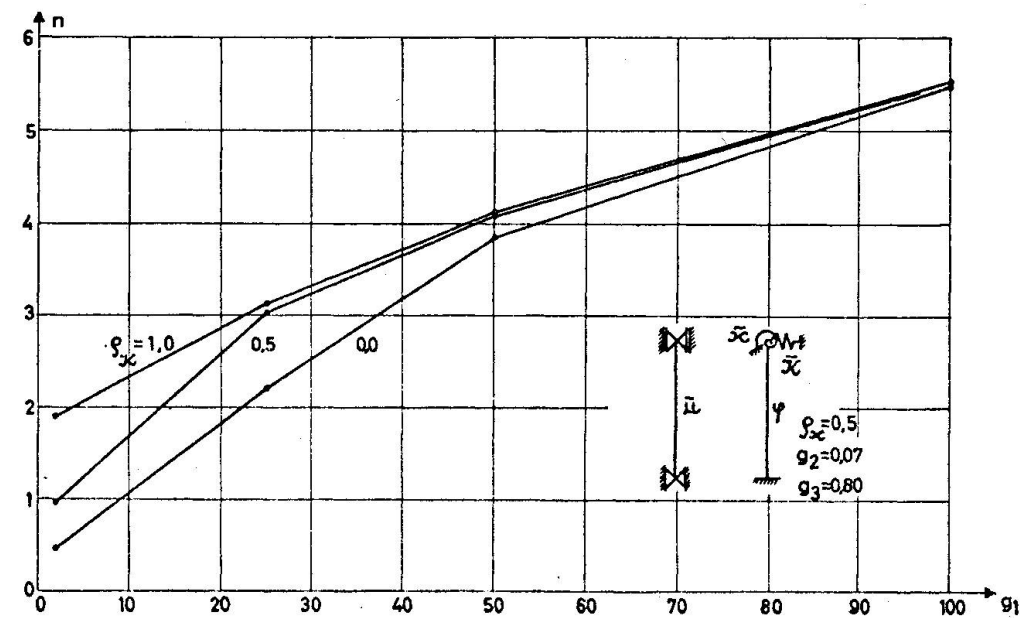
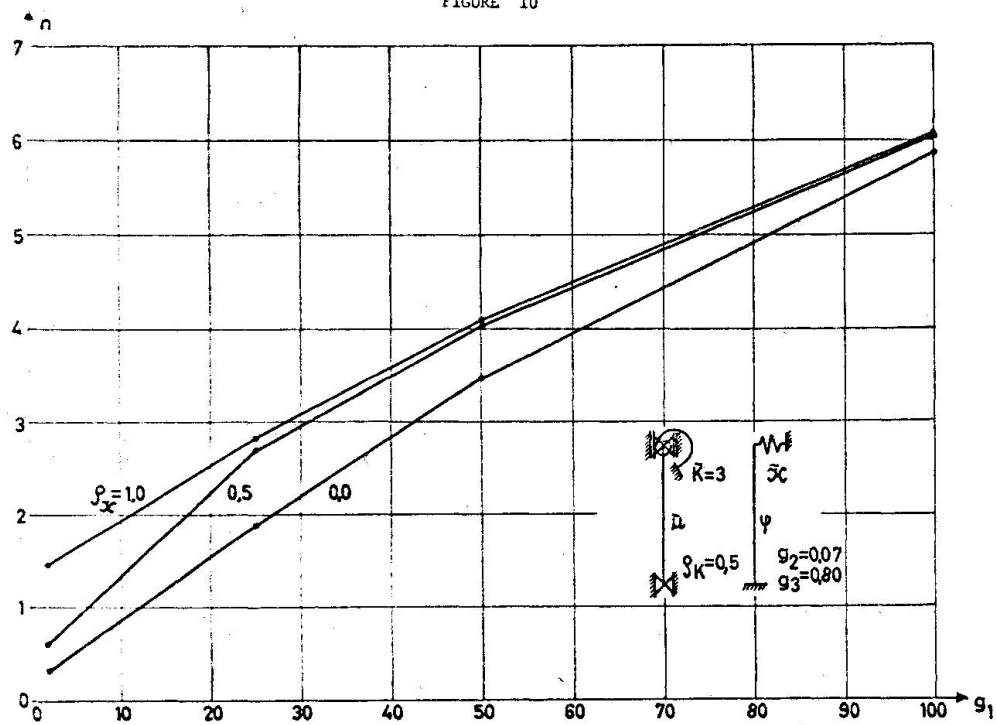


FIGURE 12



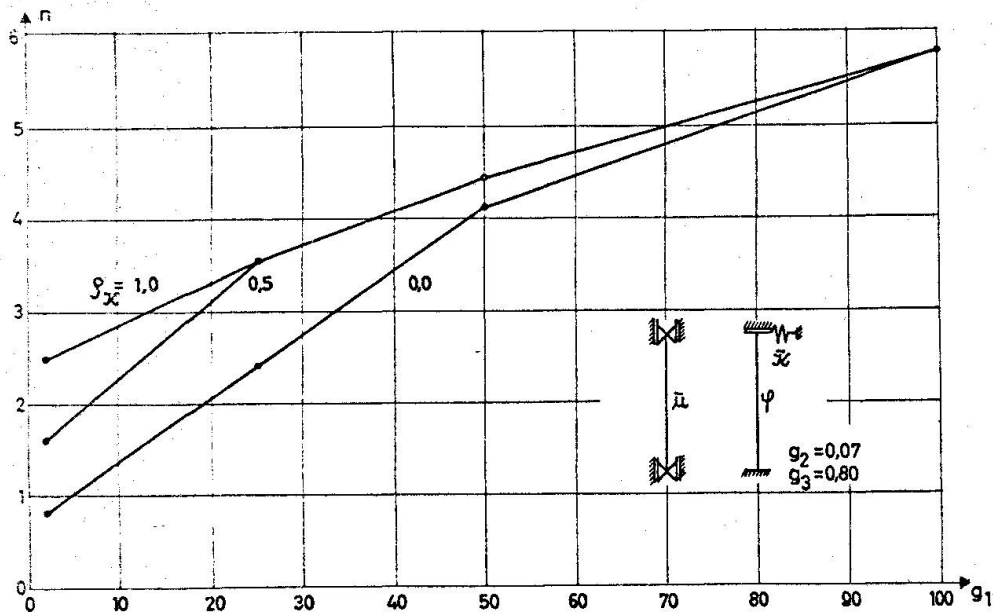


FIGURE 14

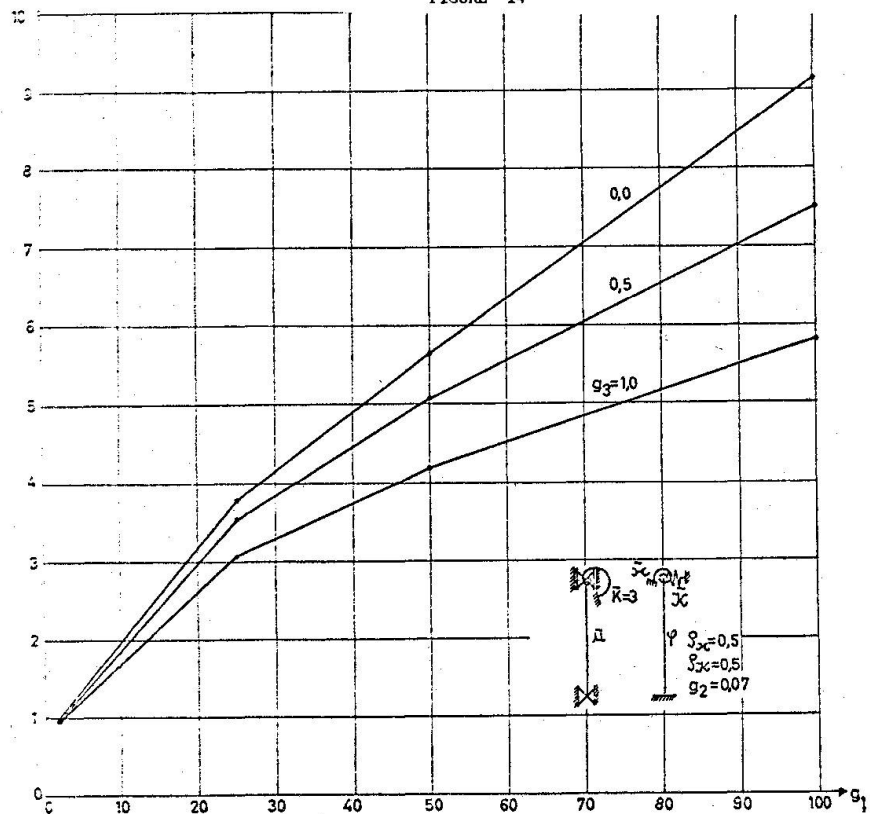


FIGURE 16

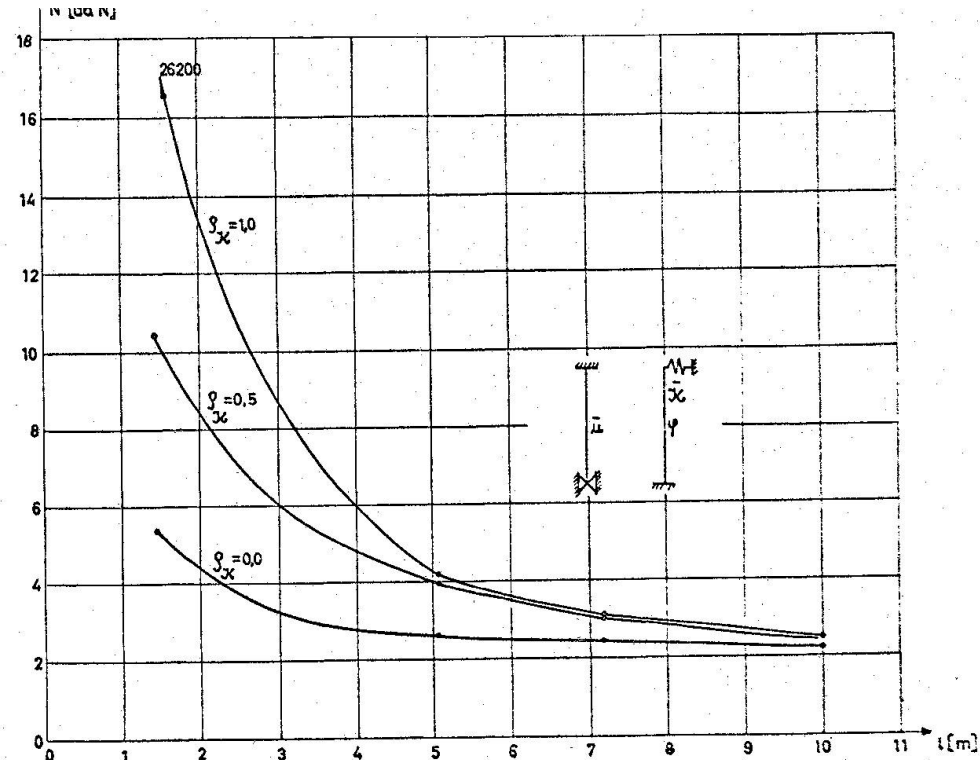


FIGURE 15

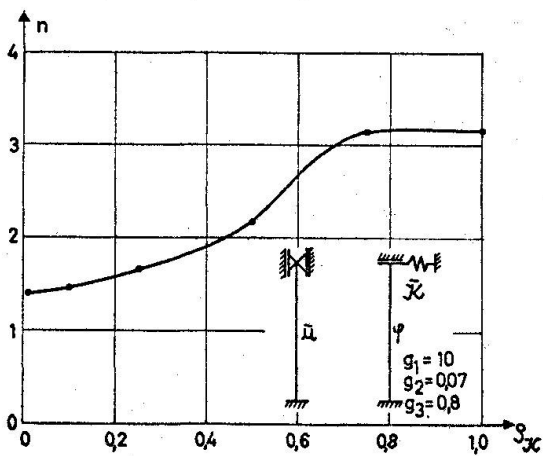


FIGURE 17

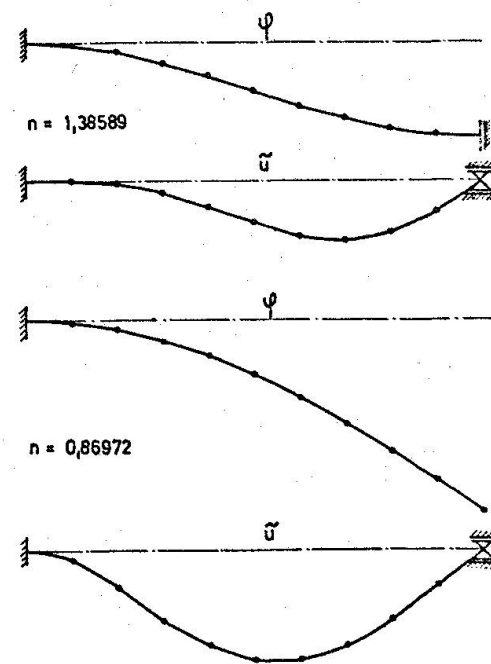


FIGURE 18

THE ANALYSIS OF COLUMN BUCKLING BEHAVIOUR

B. W. Young
Lecturer in Structural Engineering
School of Applied Sciences
University of Sussex
Brighton, England

ABSTRACT

A new treatment of the axially loaded column problem is presented which has led to design proposals for the revised versions of British Structural Steel Codes. The analytical procedure takes account of initial curvature and residual stresses due to hot-rolling and welding. Column strength calculations using non-linear moment-curvature properties based on residual stress distributions are reported for a range of typical structural sections. Comparisons are made with existing British column curves and with recent European proposals.

1. INTRODUCTION

Revisions of the two major British structural codes (BS 153 "Steel Bridges" and BS 449 "Structural Steel in Buildings") at present underway, have encouraged renewed study of a number of basic design problems. One specific problem has been the prediction of the maximum strength of steel columns (1,2) and as an introduction to this work, the present paper describes the treatment of the axially loaded column with pinned ends which is free from both applied terminal moments and lateral loading. The paper deals with the effect of imperfections on overall buckling. Local buckling and its interaction with overall failure is not discussed.

The imperfections in structural members examined here are residual stresses, (which might be due to hot-rolling, welding or flame-cutting operations) and initial lack of straightness. These imperfections have been incorporated in the analysis of a range of sections by means of computer simulated column tests. The results have permitted a rational study of the separate effects of imperfections, method of manufacture, variations in section shape, and material yield stress on column strength. The need to provide a proper treatment for all these parameters has led to the recommendation of a number of column curves to suit different classes of section.

A similar investigation commissioned by the European Convention of Steelwork Associations was proceeding concurrently with the present study. This work provides the basis for a set of European column curves. The analytical method is very similar to that discussed here and there is close correspondence between the resulting curves.

2. HISTORICAL SURVEY

The history of the axially loaded column problem is well known and it is not necessary to enter into great detail here except to note the main events in order to place the present study in perspective.

The major modification of Euler's original work (3) on elastic columns occurred at the end of the nineteenth century when Engesser proposed his tangent and reduced modulus theories (4,5) for initially straight inelastic columns. Later (in 1947) Shanley (6) was able to show that the tangent and reduced modulus loads gave lower and upper bounds respectively for the collapse of such columns. The tangent modulus approach has since found favour with American researchers (7, 8) wishing to investigate the effect of residual stresses on

column strength. In practice, the relation between the tangent modulus and the axial load in the member is found from the stress-strain curve obtained during a stub-column test. The concept of a constant tangent modulus for a particular axial load is only applicable to straight columns and is therefore an unsatisfactory means of dealing with real members.

An alternative method of predicting real member strength is the assumption of an initial curvature of sufficient magnitude to allow for all imperfections. Robertson (9) evaluated empirically an initial curvature factor in the Perry formula. This formula, which relies on the attainment of yield to define limiting column strength has provided a design method in British codes for many years. The similar Perry-Duthiel formula is used in France.

A much more satisfactory way of dealing with the real, initially curved column is to determine its maximum strength. As with the Perry approach, an initial curvature is assumed so that there is a continuous development of column deflexions without bifurcation of equilibrium. Unlike Perry, the criterion of failure is now the attainment of a maximum in the axial load-lateral deflexion curve for a column of given slenderness ratio. Residual stresses can be accounted for when the load-deflexion curve is determined for a particular cross-section.

Little previous work on the strength of steel columns has allowed for the simultaneous action of initial curvature and residual stress. Batterman and Johnston (10) have considered the simplified case of I-section columns in which only the flanges are assumed to be load-carrying. Hall and Stup (11) have made further developments of this work. The recent European study of the maximum strength of real columns is due to Beer and Schulz (12).

3. THE TREATMENT OF IMPERFECTIONS

The magnitude and distribution of hot-rolled and welded residual stresses assumed for the present analysis have been discussed elsewhere (14,15,16). From a knowledge of the residual stresses it is possible to compute moment-thrust-curvature ($M-P-\phi$) curves for the range of column sections shown in Table I. This information is then fed into the computer when the full column analysis program is in operation.

The initial centreline of the column takes the shape of a half sine-wave with an amplitude standardised at $1/1000$ of the length. In view of the general uncertainty about the allowance for initial curvature, provision is made in the calculations

for varying the magnitude of the initial bow, although keeping it proportional to the column length. The effect of initial bow on column strength is discussed in Ref. 1. Beer and Schulz also choose $1/1000$ of the column length as their initial bow.

4. THE MAXIMUM STRENGTH OF AXIALLY LOADED COLUMNS

4.1 The Basic Method

The criterion of failure which determines the maximum strength of a column has already been described (Section 2) as the attainment of a maximum in the load-deflexion curve for a column of fixed slenderness ratio. However, in view of the way in which the moment-curvature properties have been calculated it is much more convenient to carry out column calculations for fixed axial load so that the criterion of failure must be restated as the attainment of zero slope in the slenderness-ratio versus central deflexion curve for constant axial load.

The procedure for determining this criterion is first to select a column length of appropriate initial curvature which is stable under the given axial load. A numerical method which allows for the presence of residual stresses is then used to determine the deflexions which satisfy equilibrium requirements for this column. As the column length is increased, new deflected shapes are determined for each increment until finally a length is reached for which it is impossible to satisfy the conditions of equilibrium, a plastic hinge having formed at the middle of the column.

The increments in length are initially equal to the radius of gyration (r) of the column section about the chosen axis of bending. When the limiting length is reached, the calculation returns to the last stable length and proceeds to determine column deflexions for increments in length of $0.1r$ until a new (shorter) limiting length is reached.

The computation process reflects the sudden failure typical of column buckling. At the limiting, just stable length for a given axial load the maximum moment in the column is often appreciably less than the plastic moment, yet an increase in length of less than $0.1r$ is sufficient to allow a hinge to develop at the centre of length and deflexions to increase indefinitely to collapse.

4.2 Calculations of Column Deflexions

The basis for the calculation of column deflexions is simple. All that is necessary is the satisfaction of the condition of equilibrium between internal and external

moments at every point in the column length. Column deflexions may then be obtained by a relationship between the internal moment and the curvature. The problem becomes complicated when, due to the presence of residual stresses in the cross-section, this relationship is non-linear. For this reason an iterative process to determine the deflexions must be used.

For equilibrium of moments at any point in the column,

$$M_i = M_e = P (w+y) \quad (1)$$

where M_e and M_i are the external and internal column moments respectively

P is the axial load

y is the initial deflexion before axial load is applied

and w is the additional deflexion induced by the load.

In non-dimensional terms, this equilibrium equation becomes

$$\frac{M_i}{M_Y} = \frac{P}{P_Y} (W+Y) \quad (2)$$

where $W = w/c_e$, $Y = y/c_e$

c_e is the section core ($=Z/A$)

M_Y is the first yield moment with zero axial load

and P_Y is the squash load.

The internal moment is related to the curvature by the equation

$$\frac{M_i}{M_Y} = i \frac{\phi}{\phi_Y} \quad (3)$$

In a cross-section which remains elastic, i is a constant equal to unity and the calculation of column deflexions involves substitution for M_i/M_Y into equation (2) followed by simple integration.

We are considering here the behaviour of elasto-plastic columns for which i is a function of the internal moment and the axial load. Moment-curvature curves for partially plastic sections have been computed (16) and it is from these curves that we are able to obtain the function i . The loading path followed assumes no elastic unloading of previously yielded regions of the cross-section.

The curvature is related to small column deflexions by the familiar expression

$$\phi = -\frac{d^2 w}{dx^2}$$

Due to the non-linear nature of the problem, the deflexions in the expression above are differentiated numerically to obtain the curvatures. The differential coefficient is therefore replaced by a finite difference expression for the curvature at a typical node 0, thus

$$\phi_0 = - \left(\frac{w_1 + w_3 - 2w_0}{h^2} \right) \quad (4)$$

where h is the width of the interval between nodes.

The initial, unloaded deflected shape of the column is assumed to be given by a half-sine curve, thus

$$y = Y c_e = \alpha L \sin \frac{\pi x}{L}$$

where L is the column length
and α is normally set equal to $1/1000$.

At the node 0
$$Y_0 = \frac{\alpha (L/r)}{(c_e/r)} \sin \pi \left(\frac{a_0}{a_n} \right) \quad (5)$$

where a_0 is the number of intervals of width h from one end of the column and a_n is the number of intervals into which the whole column length is divided ($=L/h$). The ratio of section core to radius of gyration (c_e/r) can be obtained (16) in terms of the ratio of web to flange area, A_W/A_F .

From equations (2), (3), (4) and (5) we now have the following non-dimensional equilibrium equation at a typical column node,

$$W_1 + W_3 - 2W_0 + \frac{z_2}{i_0} \cdot \left(\frac{P}{P_Y} \right) \left[W_0 + z_1 \sin \frac{\pi a_0}{a_n} \right] = 0 \quad (6)$$

where $z_1 = \frac{\alpha (L/r)}{(c_e/r)}$

and $z_2 = \frac{(L/r)^2}{a_n^2} \left(\frac{\sigma_Y}{E} \right)$

A computer program employing a relaxation method was used to give a set of column deflexions which satisfy equation (6). Sufficient accuracy was obtained with the column divided into twenty intervals and an out-of-balance between external and internal moments of not more than 0.1%.

5. COMPUTED COLUMN CURVES

The computer program determined critical column lengths for discrete values of axial load between 10% and 95% of the squash load. A number of typical column members were treated in order to examine the influence on column strength of cross-section shape, method of manufacture, material yield stress and distribution of residual stress. The various members (coded M1 to M12) are identified in Table I.

5.1 Universal Sections (Major Axis Bending)

The computed column curves for hot rolled I-sections bending about the major axis (members M1 to M4) appear in Figure 1 in which P/P_y is plotted against the slenderness, λ_{cx} .

Note that the slenderness is defined by

$$\lambda = \frac{L}{r} \cdot \frac{1}{\pi} \sqrt{\frac{\sigma_Y}{E}}$$

thus λ is unity when the Euler critical load is equal to the squash load, (P_y).

Table I

Moment-curvature properties and column buckling strength curves were computed for the following typical structural members:

a) Hot-rolled sections

Member No.	Description	Axis of Bending	$\frac{A_w}{A_f}$	σ_Y MN m ⁻²
M1	Universal Column Shape	Major	0.3	250
M2	Medium Universal Beam Shape	"	0.75	"
M3	Extreme Universal Beam Shape	"	1.2	"
M4	As M1	"	0.3	450
M5	As M1	Minor	0.3	250
M6	As M2	"	0.75	"
M7	As M3	"	1.2	"
M8	As M1	"	0.3	450

b) Welded Sections

Member No.	Description	Axis of Bending	$\frac{A_w}{A_f}$	S_f a)	S_w a)	t/T b)
M9	Moderately Welded H-Column Shape	Major	0.3	0.2	0.2	0.6
M10	Moderately Welded Square Box	-	1.0	0.2	0.2	1.0
M11	Lightly Welded Square Box	-	1.0	0.05	0.05	1.0
M12	Moderately Welded H-Column Shape	Minor	0.3	0.2	0.2	0.6

a) S_f and S_w indicate respectively the proportion of flange and web occupied by weld tension block.

b) T = flange thickness, t = web thickness

As we have seen in Ref. 14, the section geometry determines the residual stress. This effect is illustrated by the three curves for members M1, M2 and M3 in which the ratio A_W/A_F takes the values 0.3, 0.75 and 1.2. The yield stress for the three members is 250 MN m^{-2} . It is evident from these curves that the geometry does not have a significant effect for major axis buckling of Universal sections, it would therefore be reasonable to assume a single design curve for major axis buckling of all hot-rolled I-shapes in a particular grade of steel.

Member M4 has a yield stress of 450 MN m^{-2} and $A_W/A_F = 0.3$. The reduced significance of the residual stress apparent here is due to the fact that these stresses are independent of the material yield stress. Where the residual stresses in the flange toes of member M1 were 50% of the yield stress, in member M4 the same absolute value is now only 28%.

5.2 Universal Sections (Minor Axis Bending)

The variation of cross-section geometry and the resulting distributions of residual stress have a slightly more pronounced effect on the minor axis buckling of hot-rolled Universal sections. The column curves for these members appear in Figure 2. The ratio A_W/A_F takes the values 0.3, 0.75 and 1.2 for members M5, M6 and M7 which have a common yield stress of 250 MN m^{-2} . The results for member M8 which has a yield stress of 450 MN m^{-2} and $A_W/A_F = 0.3$ (compare with curve for M5) can barely be distinguished from those for member M7. The strength of these two members is therefore shown by a single curve in Figure 2. Comparison of curves for M1 and M4 (Figure 1) and for M5 and M8 (Figure 2) indicates that an increase in material yield stress has approximately the same proportional effect on column strength in minor axis bending as in major axis bending.

5.3 Welded Sections

Column curves for welded sections are shown in Figure 3. Due to the variation in strength of a particular section induced by differing amounts of welding it is convenient to plot these curves on the basis of a fictitious reduced yield stress, σ_{YR} .

For box sections and I-sections bending about the major axis,

$$\sigma_{YR} = \sigma_Y - \frac{1}{2} \sigma_r$$

where σ_r is the residual compressive stress induced by welding. For I-sections bending about the minor axis;

$$\sigma_{YR} = \sigma_Y - \sigma_r$$

A virtue of using this form of presentation for welded members is that all column curves tend to fall close to a single design curve (shown dashed in Figure 3).

It must be noted that the strength of welded members predicted in this study is likely to be pessimistic because of the severe assumption made for the distribution of compressive residual stresses.

6. BASIC COLUMN DESIGN CURVES

6.1 Very Short Columns

It is possible to dispose of all very short columns regardless of their subsequent behaviour by allowing them to reach the squash load, P_y . This is not unreasonable since otherwise it would not be possible to perform a stub column test. In these circumstances, strain-hardening begins to have an effect and an estimate of the limiting slenderness for the development of the squash load can be obtained by replacing the elastic modulus in Euler's equation with the strain-hardening modulus $E_s (\approx \frac{E}{30})$, then at the squash load, λ is approximately 0.20.

This figure is not adhered to exactly for all design curves but is used as a general guide for the squash length of a stocky column. The concept represents an improvement on the Perry-Robertson curves of the existing British Codes which only allow the development of the squash load at zero column length.

6.2 Buckling Curves for Hot-rolled and Welded Sections

It is clear from the discussion in Section 5 that a single column buckling curve is insufficient to specify the range of column strengths dictated by section geometry, method of manufacture, axis of bending and material yield stress. It is therefore suggested that for column design a number of basic curves be provided to cover the band of results obtained during this study.

The curves derived for the hot-rolled sections M1, 2,3, 7 and 8 fall into a group for which a single design curve would suffice. We will refer to this as Curve B. Curves A and C are chosen to coincide with those derived for members M4 and M5. As Figure 3 shows, Curve B is also satisfactory for welded members provided the appropriate reduced yield stress is used.

6.3 Allocation of Sections to Design Curves

We are now in a position to relate members to their appropriate design curve. The chart in Figure 4 permits the selection of column curves for a wide range of structural sections including some which have not been examined specifically.

To demonstrate the use of the chart in Figure 4, consider the allocation of a column design curve to a Universal Beam section bending about the minor axis. The section has a web to flange area ratio (A_w/A_f) of 0.75 and a flange thickness of 22 mm. The steel is Grade 50, thus σ_y is 350 MN m^{-2} . From the type of section, axis of bending, area ratio and yield stress, the point 1 is found in the lower half of the chart. Projecting upwards from point 1, the intersection (point 2) with the horizontal line corresponding to the flange thickness is obtained in the upper half of the chart. Point 2 lies in the region labelled B, thus column design curve B should be used for the member.

The study of hot-rolled sections which led to the establishment of the design curves has already been discussed. This work was principally concerned with sections for which the flange thickness was less than about 25 mm. Although this restriction includes a high proportion of the total tonnage of Universal sections rolled in Great Britain, design rules must also be provided for thicker sections which suffer from more severe residual stresses (14). A further basic design curve, (Curve D) is therefore provided for these members.

Hot finished tubes and rectangular hollow sections have been placed on the right-hand side of the chart in Figure 4 in view of the generally small residual stresses expected in these sections (17). The higher stresses which might occur with increased wall thickness are automatically allowed for. Further study of the stresses in these sections is needed.

Hot-rolled channel and angle sections have received little attention in the past, but O'Connor (18) has reported large residual stress measurements in a few channel and angle sections and as a consequence these members appear on the left-hand side of the selection chart. More information on residual stress distributions in these sections is required.

Although the thickness of components is allowed for when the residual stress is calculated, welded sections are usually made up from Universal Mill flats which already contain residual stresses from the hot-rolling operation. These additional stresses have not been included in the present analysis but this is not a serious objection for thin plates since the design curves for welded members tend to be conservative. The use of thick plates which could contain considerable cooling residual stresses requires a lower design curve. The column selection chart in Figure 4 allows for this. More research on the problem is needed, together with the treatment of flame-cut flanges.

6.4 Code Presentation

To promote the use of computers in structural design, the column curves have been specified by simple formulae. The general expression for the limiting slenderness is then,

$$\lambda_c^2 = \sum_{n=1}^{n=4} c_n \left(\frac{P}{P_Y} \right)^{n-2} \quad (7)$$

where values of the coefficients for all the basic design curves are given in Table II.

Table II

Coefficients in column formulae (equation 7)

Design Curve	C ₁	C ₂	C ₃	C ₄
A	+1.07	-1.15	+2.97	-2.83
B	+0.97	-0.46	+0.84	-1.30
C	+0.92	-0.08	-1.14	+0.34
D	+0.87	0	-1.71	+0.87

As formulae are too cumbersome for direct use in codes, it is proposed that column strength data should consist of four separate charts (derived from the above formulae) which correspond to the basic design curves. Each chart will have families of curves for different values of yield stress and will show maximum column stress against slenderness ratio.

7. EXPERIMENTAL RESULTS AND OTHER COLUMN DESIGN CURVES

7.1 Experimental Results

The interpretation of test results for nominally axially loaded pin-ended columns is generally unsatisfactory since small amounts of restraint or eccentricity of loading are inevitably present and can have an appreciable effect on the column strength.

Some measure of experimental comparison with the present work can be made with results reported by Beedle (19) on a large number of sections tested at Lehigh University. These tests confirm the wide range of column strengths due to variations in section geometry and method of manufacture. Particularly noticeable from this series are the low values of critical axial load obtained for welded members, a conclusion already reached in the present study.

A series of tests on high yield strength steel sections conducted by Strymowicz and Horsley (20) have been compared with Basic Curves A and B in Ref.1. A large number of the results in this series lay above the Euler hyperbola which suggests that some restraint may have developed in the support bearings. If the test points are moved to the left so that they all lie below the Euler curve, there is good agreement with the Basic Curves.

7.2 Comparison with Perry-Robertson Curves

Curves A, B, C and D are compared with existing BS 449 and BS 153 column curves in Figure 5. A and B lie above the code curves for all values of load, while the C and D curves are below the code curves for values of λ in the range 0.8 to 1.2. The BS 449 curve falls rapidly as the slenderness increases and lies below curve D for values of thrust less than $0.3 P_y$.

It is clear from this comparison that the single Perry-Robertson curve cannot cater adequately with all types of member. While it can be argued that these curves have been satisfactory for design hitherto, it is now possible to make a much more favourable allowance for many classes of section.

7.3 Comparison with the European Column Curves

The late Prof. Beer and Dr. Schulz (12) have recommended three column curves to Commission 8 of the European Convention of Steelwork Associations which take initial curvature and residual stresses into account. The curves are derived analytically and are supported by an extensive test programme. The design information is presented for code use in the form of tables of critical stress. The three European curves a, b, c are compared with basic curves A, B, C and D in Figure 6, where it should be noted that they fall away immediately from the squash load. The types of member to which each curve applies are shown in tabular form.

Points of difference between the two approaches to axially loaded column design have been the subject of recent discussions. It is hoped that the final European column curves will contain the best elements of both versions.

8. CONCLUSIONS

a) Four column design curves based on real member imperfections have been found necessary to specify the strength of a wide range of structural steel sections. The single Perry-Robertson curves of the British Codes (BS 153 and 449) are considered inadequate to deal efficiently with all sections.

b) Residual stresses due to hot-rolling or welding have a marked effect on column strength. The magnitudes and distributions of these stresses assumed in the calculation of moment-curvature properties are pessimistic, thus the computed column curves may be viewed as lower bounds to true collapse loads.

c) The initial central bow in the column was chosen to be $L/1000$. This figure is considered to be a satisfactory allowance for practical columns. Increase in size of bow above this figure begins to have a considerable effect on column strength.

d) Comparison with current European column design curves has shown a similarity in form to those presented here. Some disagreement with the allocation of members to these curves exists and the situation is at present under review by the European study group.

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NOTATION

- A - cross-sectional area of member
- A_F - total flange area
- A_W - web area
- a_0 - number of intervals from one end of column to node 0
- a_n - total number of intervals into which the column is divided (L/h)
- C_n - column formulae coefficients (equation 7)
- c_e - section core ($=Z/A$)
- E - Modulus of Elasticity
- h - interval between nodes
- I - second moment of area
- i - effective stiffness function from moment curvature relation
- L - column length
- M - general moment
- M_e - external moment
- M_i - internal moment
- M_Y - yield moment ($=Z\sigma_Y$)
- P - axial load
- P_Y - squash load ($=A\sigma_Y$)
- P_{YR} - reduced squash load ($=A\sigma_{YR}$)
- r - radius of gyration
- t - plate thickness
- W - non-dimensional deflexion ($=w/c_e$)
- w - column deflexion induced by load
- x - distance along column
- Y - non-dimensional deflexion ($=y/c_e$)
- y - initial column deflexions under no load
- Z - elastic section modulus
- α - initial column bow/column length
- ϕ - curvature
- ϕ_Y - yield curvature ($=M_Y/EI$)
- λ - slenderness ($= \frac{L/r}{\pi} \frac{\sigma_Y}{E}$)
- σ_Y - yield stress
- σ_{YR} - reduced yield stress
- σ_r - compressive residual stress due to welding

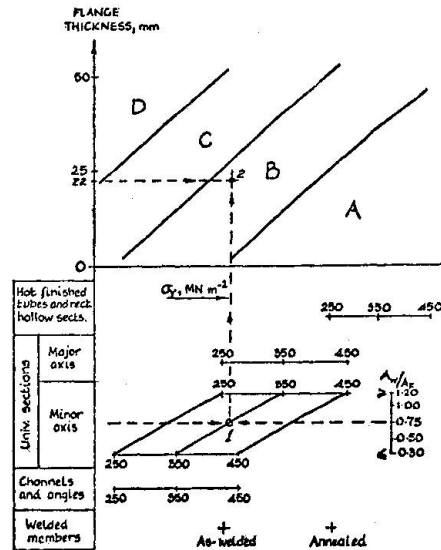
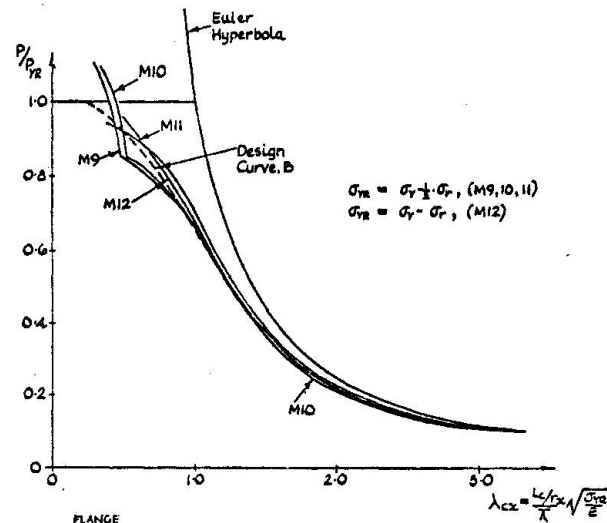
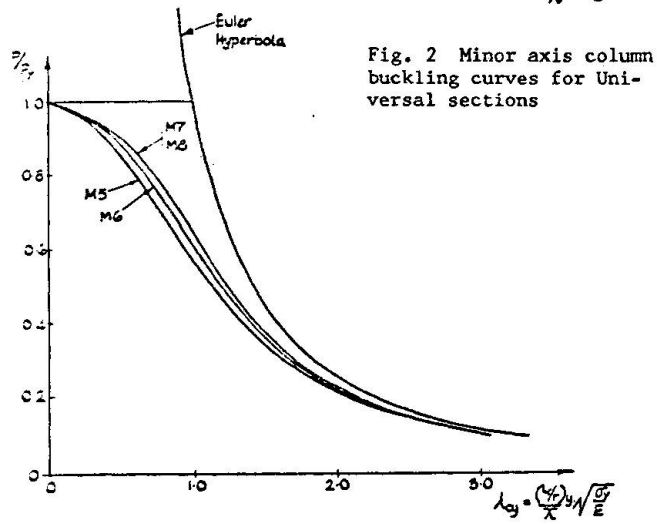
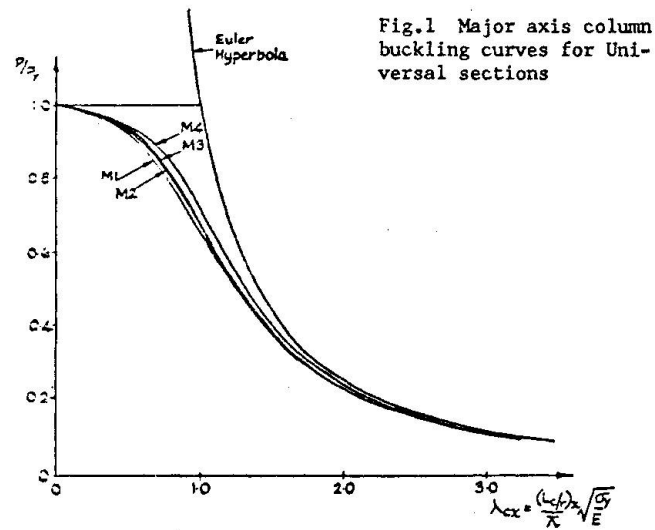
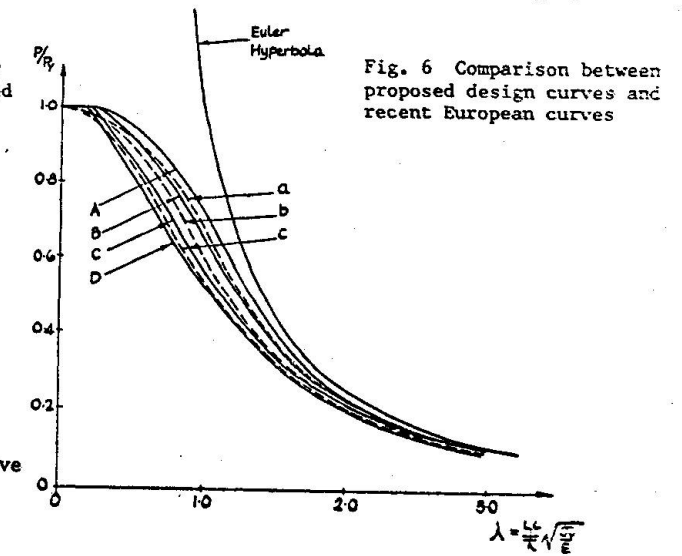
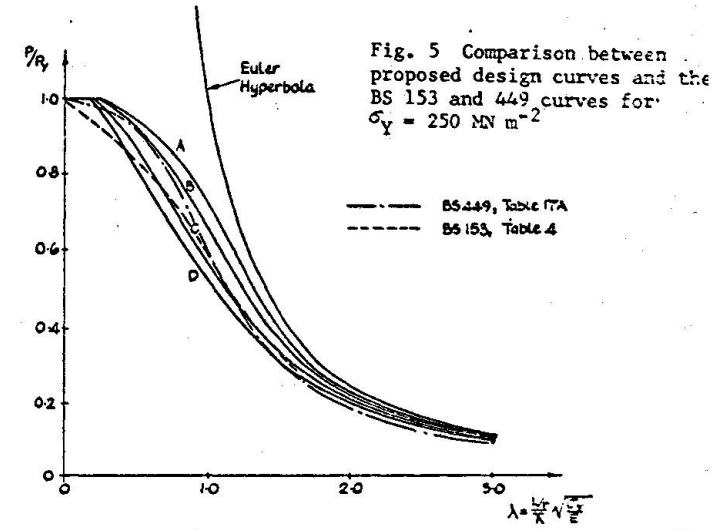


Fig. 3 Adjusted column curves for buckling of welded members

Fig. 4 Column curve selection chart



BUCKLING CURVES OF HOT-ROLLED STEEL SHAPES
WITH STRUCTURAL IMPERFECTIONS

Federico M. Mazzolani
Professor of Civil Engineering
University of Naples
Italy

ABSTRACT

Theoretical and experimental researches directed to study the main buckling problems of metal members are carried out at the Istituto di Tecnica delle Costruzioni of the University of Naples (see bibliography 1,2,3,4,5,6,7,8).

The modern approach to the buckling problems requires to take into account structural imperfections, which characterize the actual bars and whose determination must be made in experimental way.

The principal results of experimental investigations on Italian hot-rolled steel shapes are here briefly summarized. Structural patterns of residual stresses have been obtained on the basis of such experimental results. Simulation calculations have provided the buckling curves for the examined profiles with structural imperfections. The simulation curves have been compared to the CECM-ECCS ones (9).

EXPERIMENTAL INVESTIGATION

The hot-rolled I shapes in structural steel Fe 42C, which have been examined, are:

IPE 200; HEA 100; HEB 100; HEM 100;
HEA 200; HEB 200; HEM 200;
HEA 300; HEB 300; HEM 300;

The following groups of tests have been made:

- a) Distribution on the cross section of the mechanical characteristics: yield point, ultimate strength, elongation, toughness, hardness; the principal scatters of the yield point distribution are shown in fig. 1.

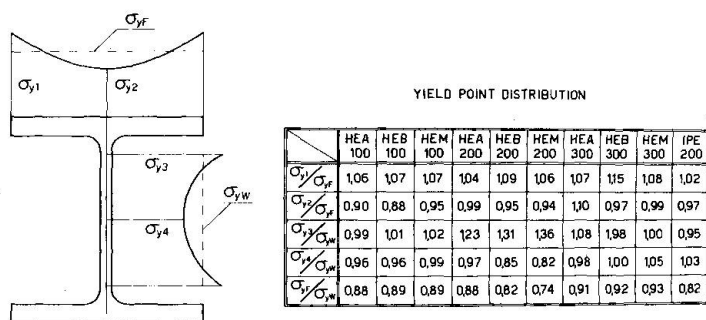


fig. 1

- b) Stub column tests; the lowering of the proportional limits appears from fig. 2

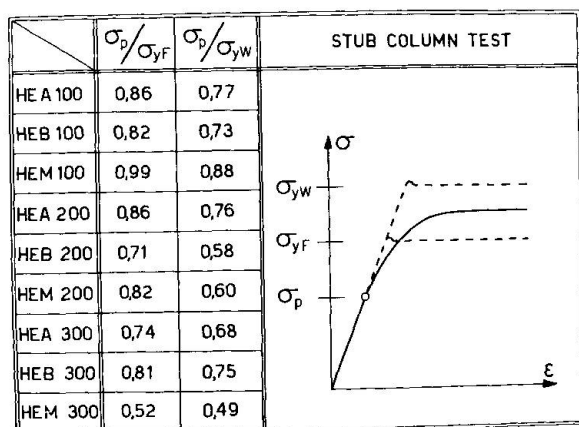


fig. 2

- c) Evaluation of residual stresses by means of sectioning test; the interpretation of the experimental results leads to the pattern of fig. 3.

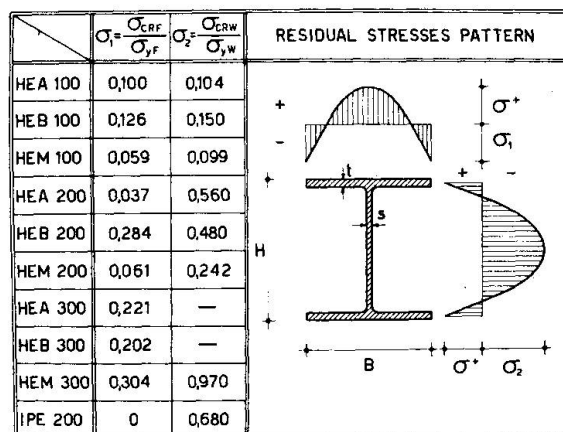


fig. 3

More details on the testing procedures and on the number of specimens are reported in (1), (7).

RESIDUAL STRESSES PATTERNS

The diagrams of residual stresses experimentally obtained have shown that the distributions are deeply influenced by the rotorizing procedure. Residual stresses in the flanges of the same bar, exhibit quite different distributions (fig. 4).

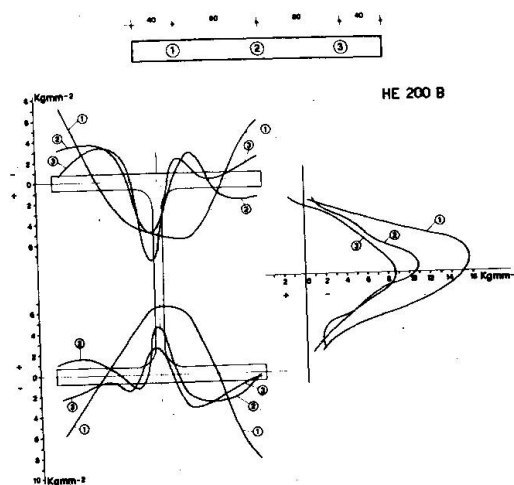


fig. 4

This fact makes impossible the statistical interpretation of the results and out of meaning the theoretical prediction of the thermal forming of residual stresses.

The collection of experimental data for a certain number of specimens makes possible the definition of bands which contain residual stresses for the examined shapes (fig. 5).

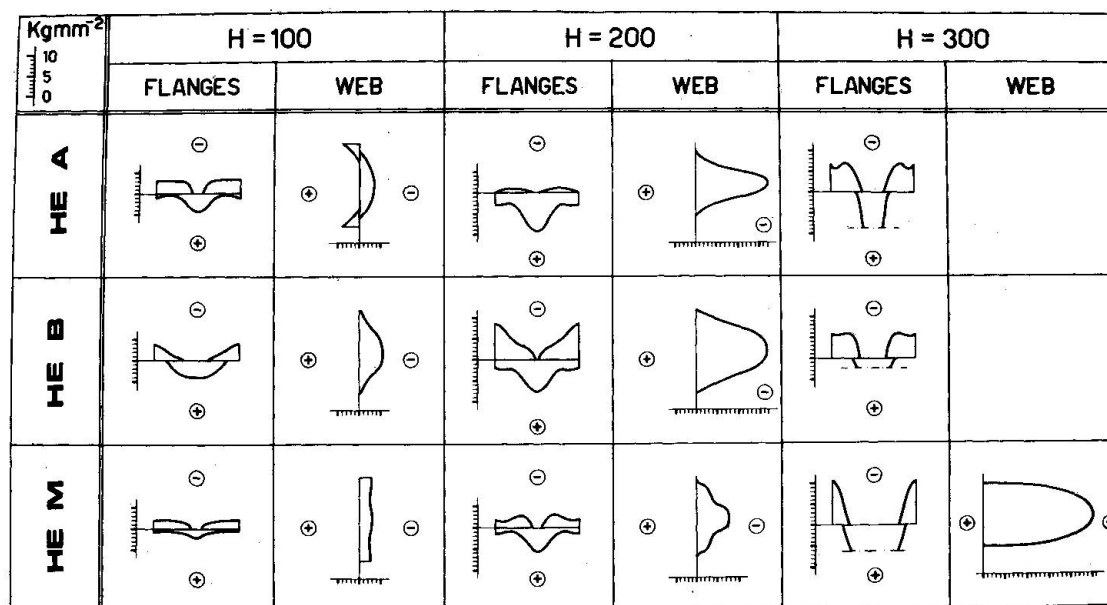


fig. 5

Starting from these data a structural pattern of residual stresses has been proposed, which utilizes the maximum compressive stresses at the edges of the flanges and at the center of the web. The condition that this pattern is in equilibrium leads to the residual stresses distributions shown in fig. 6 (1,5,6,7,8 cases). The cases 2 and 4 correspond to an average distribution and the case 3 considers the effect of the rotorizing straightening.

BUCKLING CURVES

The effect of the residual stresses distributions on the carrying capacity of simply compression columns has been examined by means of a simulation calculation. The residual stresses patterns and the average yield points used in the buckling calculations are shown in fig. 6.

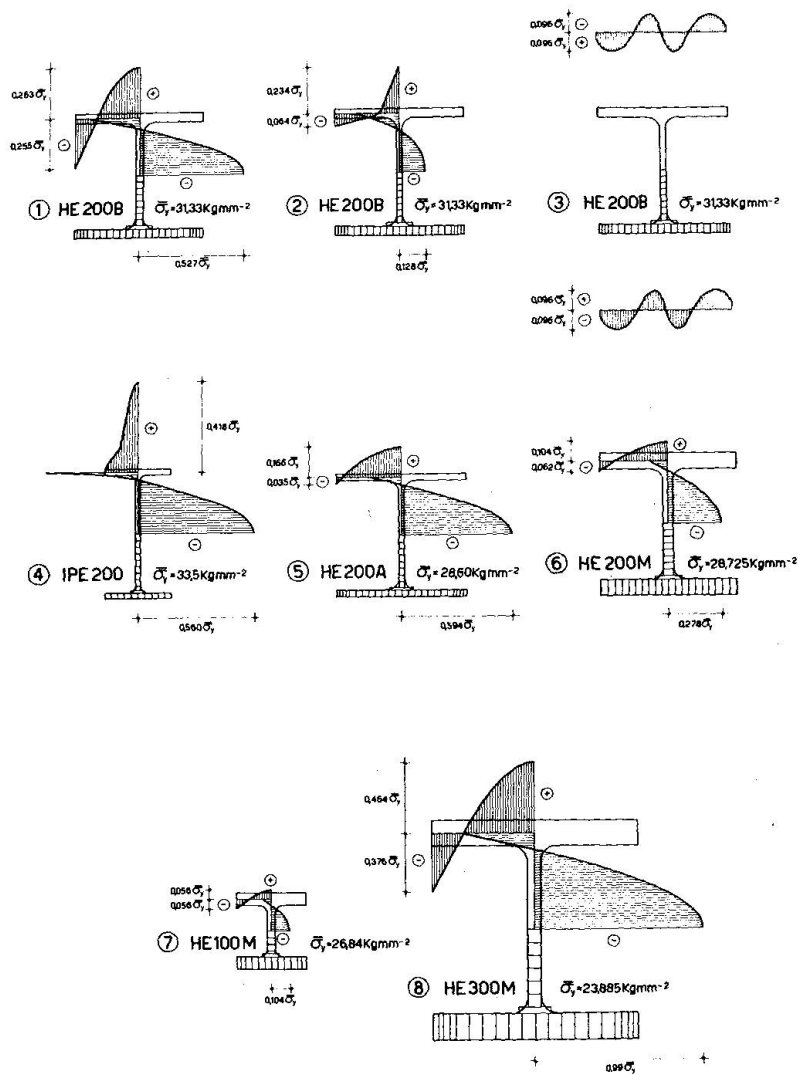


fig. 6

The numerical results have been obtained using the calculation program of the Centro Studi Costruzioni Metalliche (C.S.C.M.) of the Politecnico of Milan (10).

In order to compare the present results with the CECM-ECCS a,b,c curves the residual stresses distributions of fig. 6 have been considered together with the value $f_1 = 1/1000$ as geometrical imperfection, whereas the yield point distribution (fig. 1) has been neglected, using just the average values $\bar{\sigma}_y$ of fig. 6. The obtained dimensionless buckling curves, have been plotted in fig. 7,8,9, referring to the strong (max) and the weak (min) axes.

Fig. 7 shows the comparison among the residual stresses structural pattern (1), the average distribution (2) and the effect of rotorizing (3) in HEB 200 shapes.

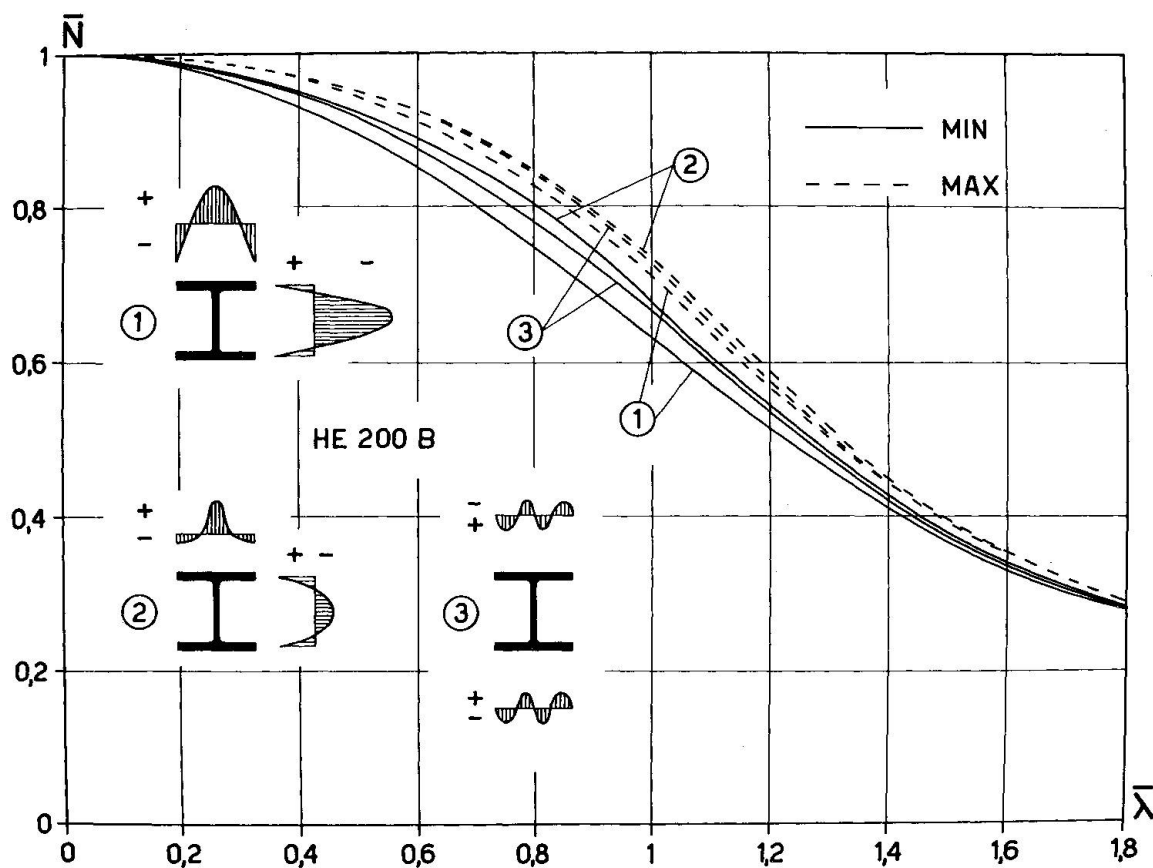


fig. 7

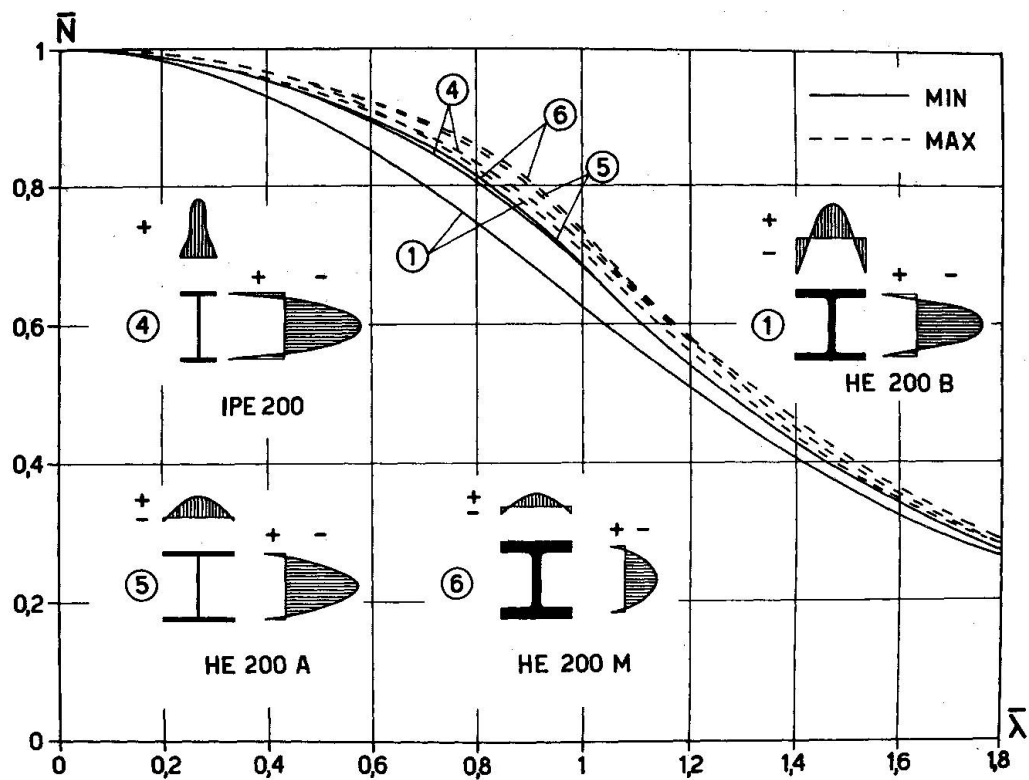


fig. 8

Different distributions of residual stresses on shapes with the same depth (HEA, HEB, HEM, IPE 200) lead to the buckling curves of fig. 8.

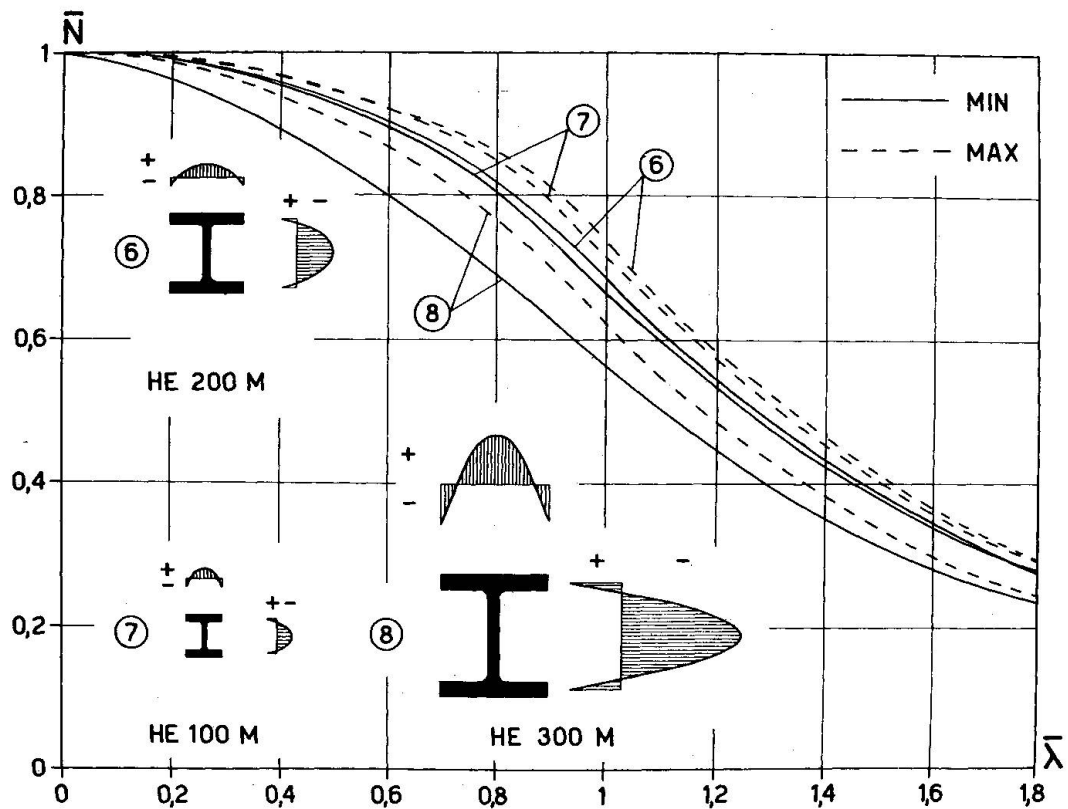


fig. 9

The distribution (1) of residual stresses in the HEB 200 represents always a lower bound for all the curves of fig. 7 and 8. The increase in depth for the same shape (HEM) makes the effect of residual stresses more important (fig. 9).

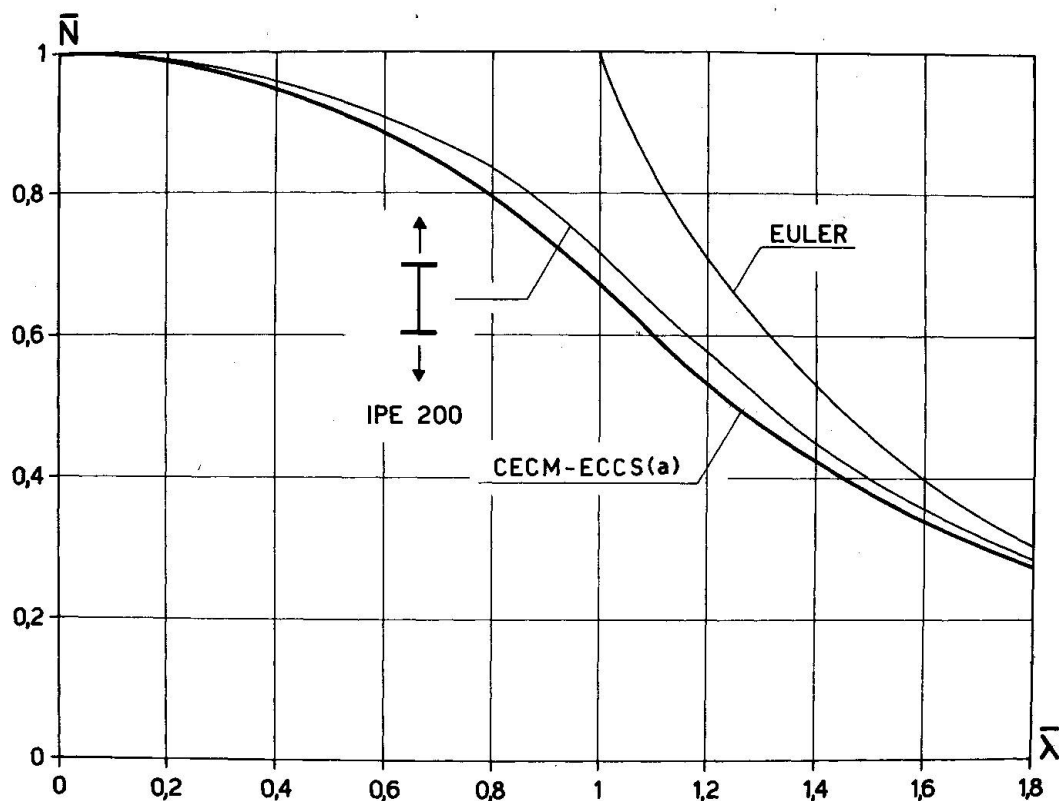


fig. 10

The obtained buckling curves have been plotted together with the CECM-ECCS a,b,c curves (fig. 10, 11, 12). The comparison between each calculated curves and the corresponding standard one shows that the CECM-ECCS curves are always more conservative than the simulation prediction based upon experimental residual stresses data. The difference may be explained, because only thermal residual stresses have been considered in the CECM-ECCS curves, without the favourable effect of rotorizing procedure, commonly used in the hot-rolled shapes manufacturing, which increases column strenght.

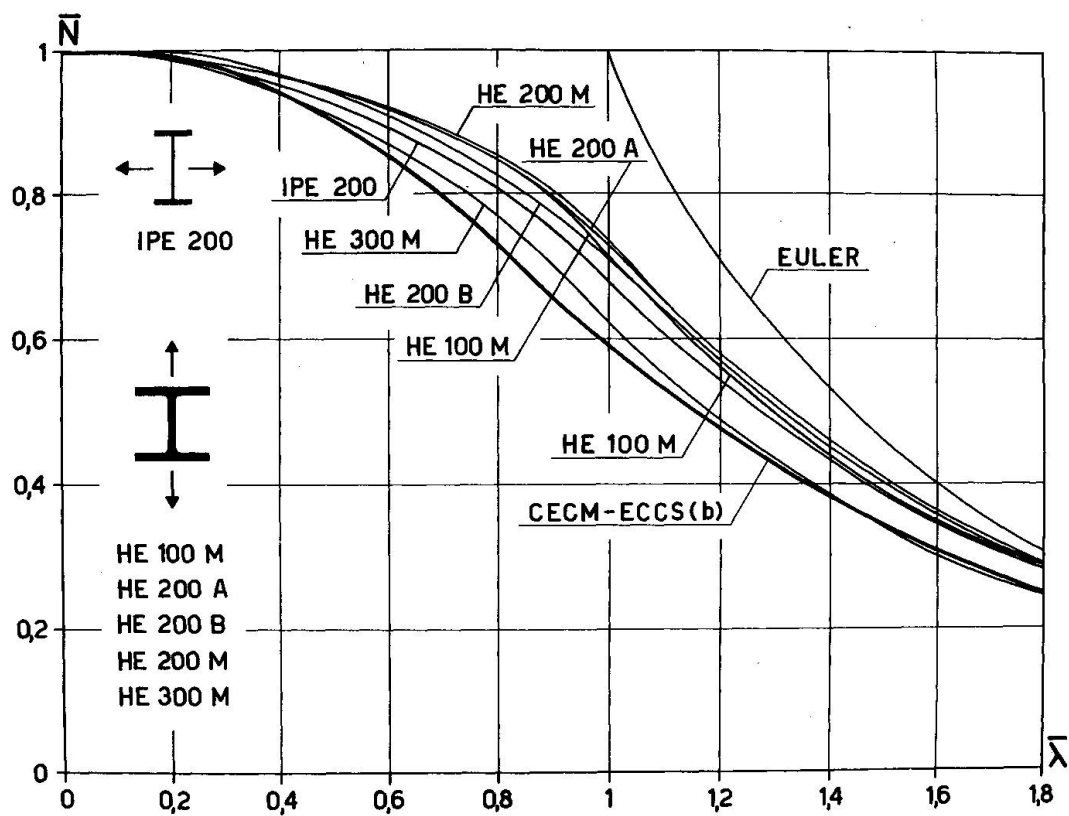


fig. 11

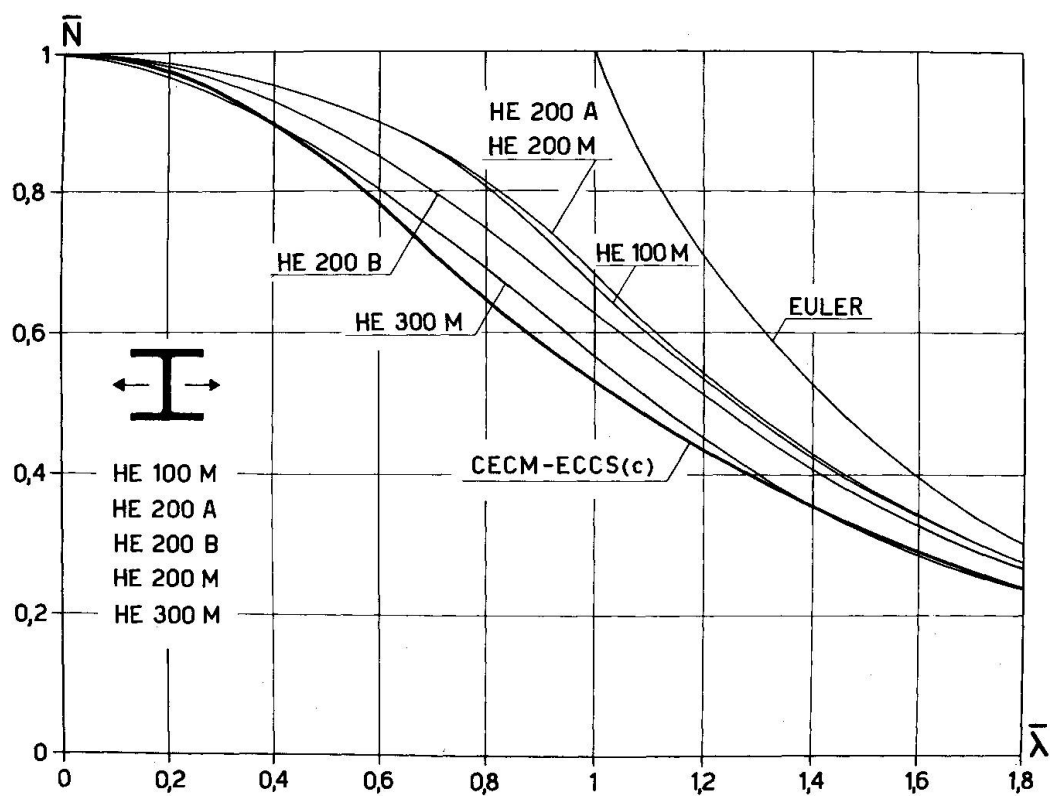


fig. 12

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BEHAVIOUR OF STEEL COLUMNS UNDER FIRE ACTION

Jelle Witteveen and Leen Twilt
Institute TNO for Building
Materials and Building Structures
Delft, The Netherlands

ABSTRACT

In modern deterministic conceptions of structural safety and design fire protection of constructions is no longer an afterthought, but is integrated in the normal statical calculations. A further step can be made by considering fire as a risk together with other risks like overloading and extreme wind loads, treating the problem with a probabilistic approach as is suggested in more advanced theories of structural safety. In view of the above, analytical and experimental studies are undertaken to evaluate the strength of structural members at elevated temperatures.

Probably the largest project in the field of steel construction is the programme of the European Convention for Constructional Steelwork (ECCS). Part of this project is a study on stability at elevated temperatures. In order to stay within the scope of the colloquium the discussion will be limited to the behaviour of individual compressed members.

After an introduction, first the influence of the rate of heating of the member on the bearing capacity is discussed. From small scale model tests it appears that this factor can be neglected for practical cases. This reduces the phenomena to a time independent problem. This implies that columns at elevated temperature can be treated with the theories developed for columns at normal temperatures, provided that the stress-strain relationships for elevated temperatures are known.

1. INTRODUCTION

In order to determine the bearing capacity of columns subjected to fire, the temperature distribution throughout the member as a function of the time must be known first. The ambient temperature in case of a fully developed fire obviously depends on the amount of combustible material (fire load) and some other factors, like the ventilation openings and the shape of the combustible material. All of these are in some way dependent on incidental fluctuations, so that the ambient temperature during a fire is a stochastic quantity. For practical reasons, however, up till now idealized time-temperature relationships are used. Taking into account the heat flow properties of the fire and the heat flow properties of the protected structure, the steel temperature in the relevant parts of the structure can be predicted.

In this paper we will consider the steel temperature as a function of time as known and devote our attention to the load-bearing capacity. Elevated temperatures affect the bearing capacity of the structure in several ways. The most important factors are:

1. The material properties like yield strength or $\sigma_{0.2}$ strength and modulus of elasticity decrease with the temperature.
2. Due to restraint of the thermal expansion stresses are induced, acting in addition to those produced by the applied load.
3. The thermal expansion causes deformations which may produce additional P- Δ effects.
4. When a structure or member is unequally heated, local thermal expansion causes redistribution of forces and moments affecting the load bearing capacity of individual members.

All these factors are included in the ECCS programme. The problem considered in this paper is the buckling behaviour of an individual column subjected to an uniform temperature increase along its length and cross section. Although this represents an idealized case it is the first necessary step to provide insight into more complicated cases.

2. EVALUATION OF THE INFLUENCE OF THE RATE OF HEATING ON THE LOAD-BEARING CAPACITY

In buckling problems at normal temperatures the stress-strain relationship of the material is considered as time independent. Due to creep this may be no longer valid at elevated temperatures more than 250 to 300°C. Creep is the phenomenon of continuous deformation under constant load, and constant temperature (fig. 1). It is clear that creep affects particularly the bearing capacity of compressed members, as the deflections and consequently the bending moments increase as time goes on.

Due to creep a theoretical evaluation of the behaviour of columns at elevated temperatures will become very complicated because time has to be considered as a parameter. In case of fire in general the column will be under constant load while the temperature rises as a function of time. Depending on the isolation of the member the rate of heating can vary. Therefore first experiments were performed to investigate the influence of the rate of heating on the load-bearing capacity, resp. the critical temperature of steel columns at elevated temperatures under fire conditions.

It was decided to study this effect by small scale model tests on beams and columns. Two different approaches were used:

- a. Testpieces with a constant load were heated at different heating rates (fig. 2). Heating rates were chosen of 50°C per minute (approximately corresponding to an unprotected steel member); 10°C per minute (normally protected steel member) and 5°C per minute (highly protected steel member). If the rate of heating would have any influence on the critical temperature^{*}, one would expect this temperature to be systematically higher if the testpiece is heated faster.
- b. Testpieces with a constant load were heated up to a certain temperature level. If no creep appeared at this level (i.e. no time-dependent deformation), temperature was raised to another, higher level and so on (fig. 3). The temperature at which the testpiece starts to creep (what in the end leads to collapse) was defined as the critical (creep) temperature.

It will be clear that both methods are supplementary to each other.

3. EXPERIMENTS ON THE INFLUENCE OF THE RATE OF HEATING

a. Experiments on model beams (Fe E 24, $\sigma_y = 240 \text{ N/mm}^2$)

Rectangular beams (cross-section 6 x 6 mm; span 400 mm) were loaded in a way shown in fig. 4. The applied load P was respectively: $P = 0.19 P_{b200}$, $P = 0.37 P_{b200}$, $P = 0.57 P_{b200}$ and $P = 0.74 P_{b200}$ (P_{b200} is the collapse-load at room temperature, which was determined experimentally).

The influence of the rate of heating was studied in a way described in 3.a. It followed from the tests that the

^{*} The critical temperature is the temperature at which the steel member collapses or a certain deformation criterion is reached.

heating rate does not influence the deformation behaviour in a significant way. This is illustrated in fig. 4 in which, as an example, the temperature is given at which a maximum deflection is reached of 1/30th of the span. Similar figures are obtained for other deformation criteria.

b. Experiments on model-columns (Fe E 24, $\sigma_y = 240 \text{ N/mm}^2$)

Columns with rectangular cross-sections (6 x 12 mm and 9 x 15 mm) were centrically loaded by constant load. The slenderness-ratio was varied ($\lambda = 40, 80, 118, 160$ and 200). For the applied load was chosen: $P = 0.2 P_{b200}$, $P = 0.4 P_{b200}$, $P = 0.6 P_{b200}$ and $P = 0.8 P_{b200}$ (P_{b200} is the collapse load at room temperature, which was determined experimentally and appeared to be in good agreement with the CECM-curve).

The influence of the rate of heating was studied in both ways described in 3.a. and 3.b. As an example, in fig. 5 the behaviour of a column under the conditions according to 3.b. is given. Results of all the tests are given in fig. 6. It appears that the rate of heating does not influence the critical temperature in a significant way. In agreement with this, the critical (creep) temperatures obtained by the experiments according to 3.b. are close too those obtained by experiments according to 3.a. It is noted that the scatter in the observed critical temperatures is rather high, which is also the case with buckling tests at normal temperatures.

4. CONCLUSIONS AND IMPLICATIONS ON THE SIGNIFICANCE OF THE INFLUENCE OF THE RATE OF HEATING

It follows from the test results, that although creep will occur (see fig. 5) the critical temperature of steel members in practical cases will not be affected in a significant way by the

heating rate. This implies that the critical temperature of steel columns under fire conditions can be considered as time independent and consequently not influenced by the "heating history". This conclusion will make possible a theoretical approach of stability-problems which is identical to well-known methods at room temperature.

Condition is that reliable stress-strain relationships at elevated temperatures are available. To find this relationships warm-creep tests were carried out. In fig. 7 is shown in what way the stress-strain relationships can be obtained. The advantage of this method is that the rate of loading, which is an uncertain factor in warm-tensile tests, is eliminated. It is noted that the transformation shown in fig. 7 is only justified for "practical" heating-rates (i.e. between 5 to 50°C per minute and temperatures not over, say 600°C), where the creep behaviour can be considered as incorporated in the stress-strain relationships. In addition also stress-strain relationships were obtained by analyzing the small-scale bending tests. It appeared that the σ - ϵ curves found in this way were in reasonable agreement with those obtained by warm creep tests.

5. PRESENTATION OF EXPERIMENTAL RESULTS

Anticipating on the theoretical approach and full scale tests at the ECCS-Station at Metz, France, experiments on small-scale columns at elevated temperatures were performed. In fig. 8 the results are presented.

Note that it appears that the critical temperature T_{crit} depends on the slenderness-ratio. There is a minimum for T_{crit} at $\lambda = 80$. This phenomenon can be explained as a result of the fact that the stress-strain relationship at normal temperature differs significantly from those at elevated temperatures, whereas the

the stress level in the column is based on the first relationship.

6. STABILITY WORK CURRENTLY UNDERWAY

As already has been stated an analytical approach of columns under fire conditions can be simplified to a time independent problem, and consequently normal stability procedures can be used.

The first step which has already been undertaken is the analytical verification of the experimental results from the model buckling tests (fig. 8). Stress-strain relationships obtained from warm creep tests and bending tests are used as basic data (see chapter 5). In fig. 9 the so derived stress-strain relationships for Fe E 24 ($\sigma_y = 240 \text{ N/mm}^2$) are given. In fig. 10 computed buckling curves are presented together with experimental results. Also the buckling curve based on the Dutch-design code is given. It appears that the present-criterion for the critical temperature of about $300\text{--}400^\circ\text{C}$ is a fairly good estimate. A full presentation of the method used in obtaining stress-strain relationship and buckling curves will be published shortly. The result will lead to buckling curves for elevated temperatures to be used in practice.

Full scale tests at Metz will give additional verification. Thus far the investigations have been limited to the behaviour of individual members. The interaction which occurs between the various members in structures leads to problems which were summarized in chapter 2. In the ECCS programme these problems are studied by theoretical analysis, small scale model tests and full scale tests. Results obtained from this work should lead to the possibility of analyzing the behaviour of building frames in fire.

7. ACKNOWLEDGMENT

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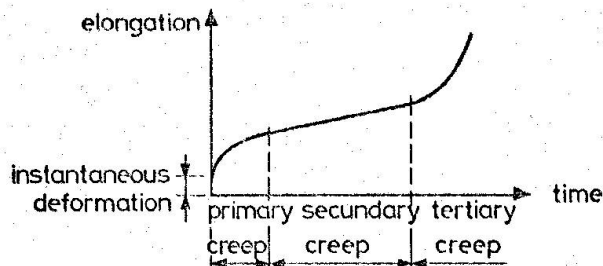


fig. 1: creep behaviour of steel at elevated temp. (schematic)

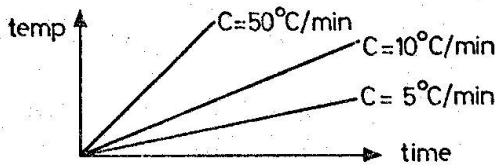


fig. 2: time-temp. relationships during the buckling tests

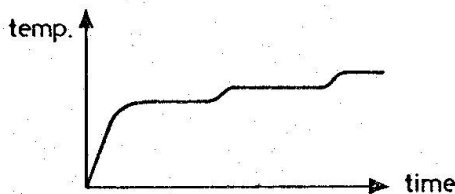
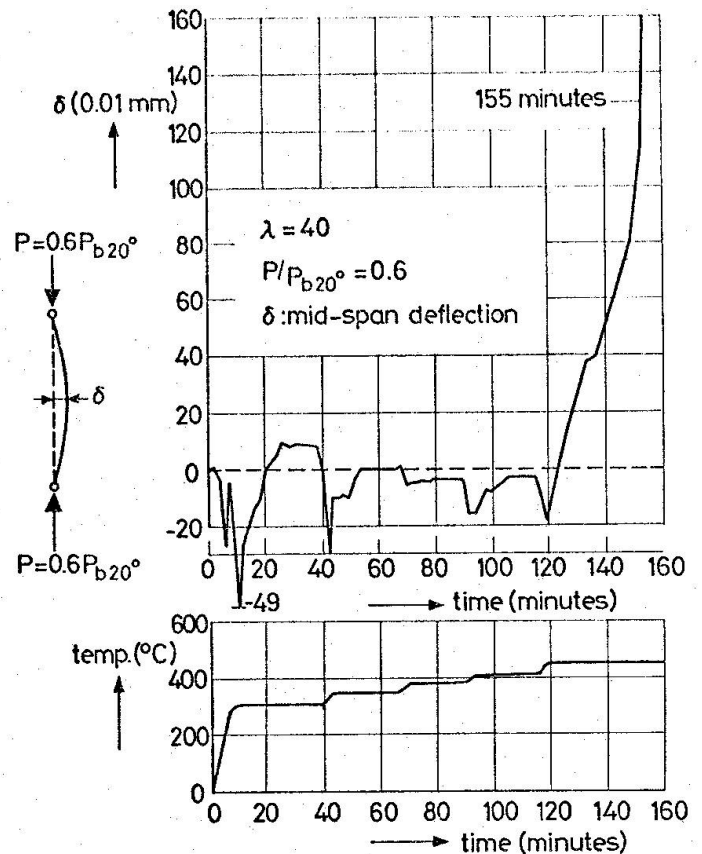
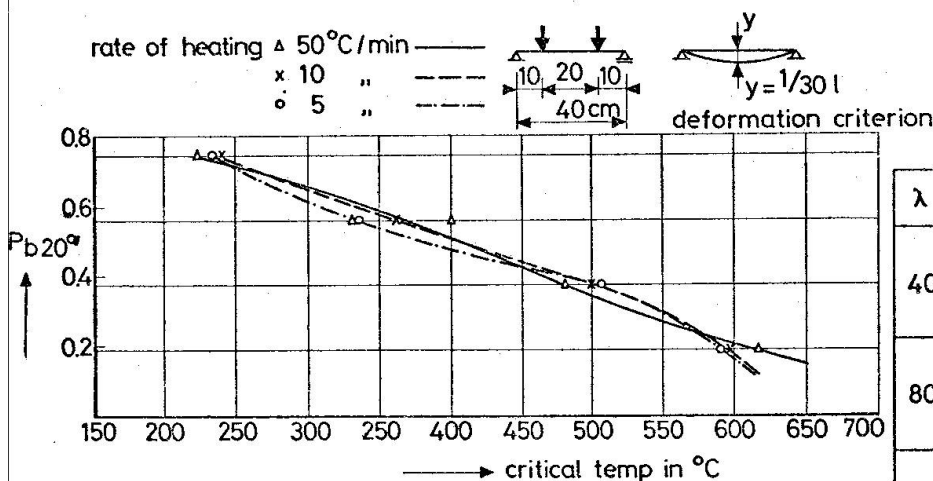


fig. 3: time-temp. relationship during a creep-test



working-method of the creep-buckling-test

fig. 5



temperature at which a critical midspan deflection is reached as function of the load and the rate of heating (FeE 24)

fig. 4

		creep test		buckling-test at different heating-rates C [°C min]		
λ	P/P_{b20°			C = 5	C = 10	C = 50
40	0.4	460°			510°	520°
	0.6	450°			360°	400° 375°
80	0.6	230°		260°	220° 270°	250°
	0.8	210°		205°	230° 180°	220° 175°
118	0.4	490° 520°	510° 525°	480° 430°	495°	
	0.6	210° 235°	320°	255° 285°	295° 320°	
160	0.6	260°		305° 295°	280°	
200	0.4		570°	590° 590°	585°	590° 570°

comparison of T_{cr} [°C] resulting from the creep buckling-tests and from buckling-tests at different heating-rates (FeE 24)

fig. 6

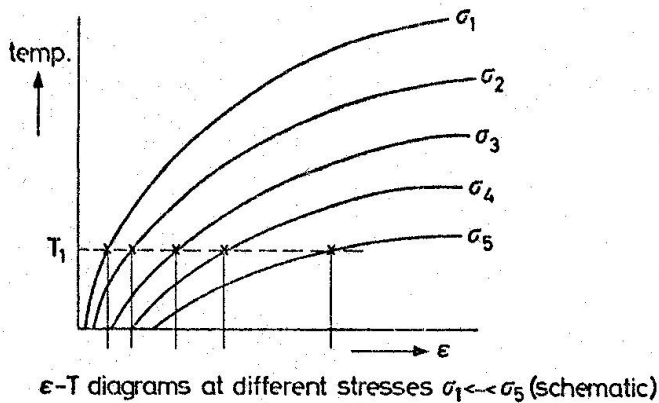


fig.7

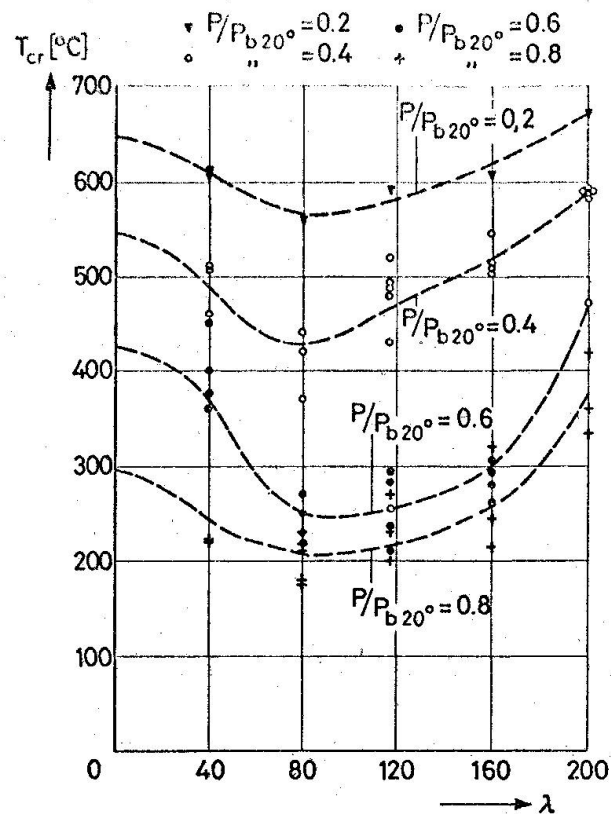


fig.8

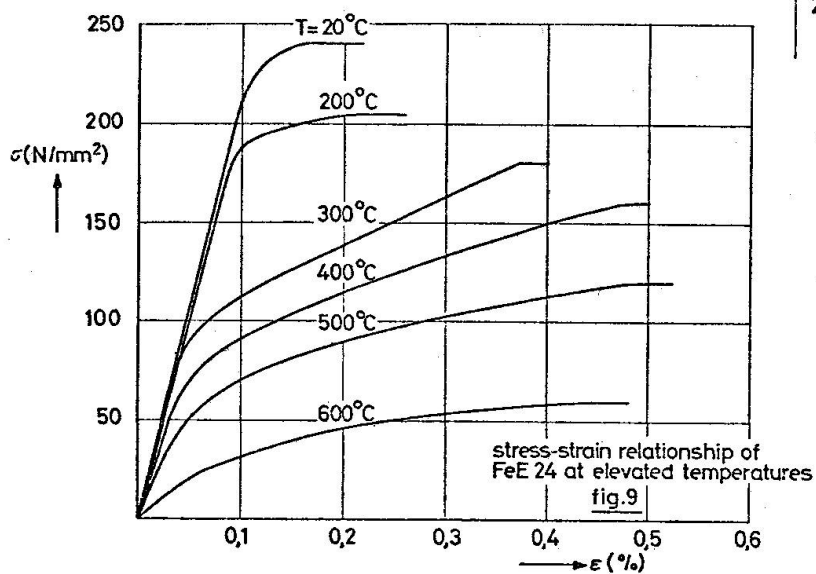


fig.9

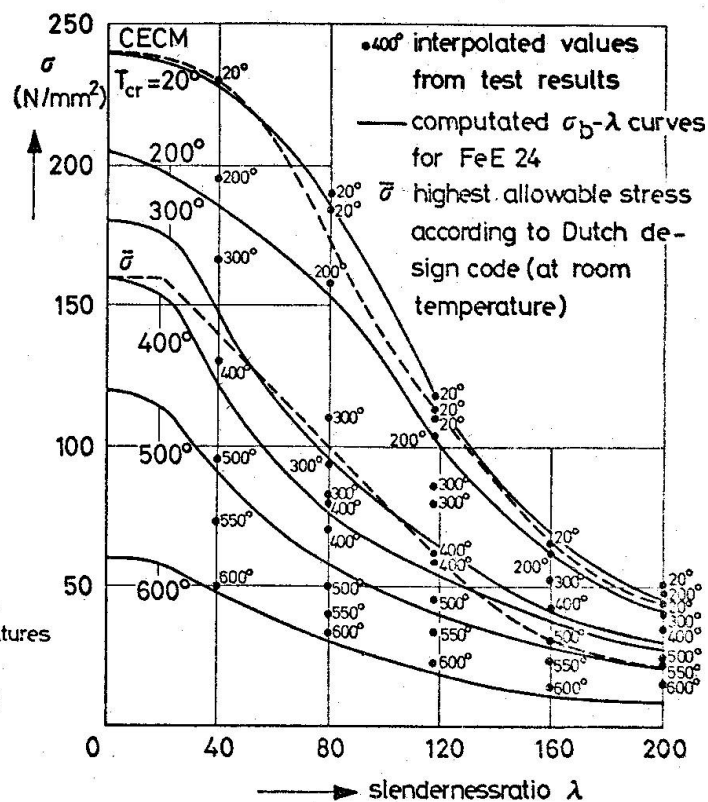


fig.10

ULTIMATE STRENGTH OF ECCENTRICALLY
LOADED COLUMNS

A. Keramati
Assistant Professor of Civil Engineering
University of Science and Industry
Tehran, Iran

E. H. Gaylord
Professor of Civil Engineering, Emeritus
University of Illinois at Urbana-Champaign

A. R. Robinson
Professor of Civil Engineering
University of Illinois at Urbana-Champaign

ABSTRACT

An efficient numerical procedure which enables the ultimate strength of an eccentrically loaded column to be found without developing the load-deflection curve itself is discussed in this paper. Solutions for symmetrical and unsymmetrical uniaxial bending and for the general case of unsymmetrical biaxial bending are considered.

The condition for a maximum value of P for the case of uniaxial bending with equal end eccentricities e is shown to be $\partial e / \partial y_0$, where y_0 is the deflection y at midlength. By constructing an auxiliary curve whose ordinate is $\partial y / \partial y_0$, the problem is reduced to a simultaneous solution of two initial value problems. The equations are solved by numerical integration, proceeding along the z -axis until $\partial y / \partial y_0 = 0$. The corresponding value of z is the half length $L/2$ of the column and $y(L/2)$ is the eccentricity e .

The initial-value problems for the case of uniaxial bending with unequal eccentricities are also developed and the technique for their solution explained. Finally, the extension to the general case of columns with biaxial eccentricities unlike at the two ends is discussed.

Comparisons of results with results of previous investigations are noted.

1. PREVIOUS WORK

Elastic flexural-torsional behavior of beam-columns is discussed by Bleich,¹ Timoshenko and Gere,² and Vlasov.³ In 1935 Jezek⁴ presented an approximate method for determining the ultimate load of a beam-column in uniaxial bending with equal end eccentricities. Galambos and Ketter⁵ and Ketter⁶ developed a more accurate solution, applicable for unequal end eccentricities, in which the effect of residual stresses is considered. Kabaila and Hall⁷ presented a new approach to the solution of the problem for equal eccentricities. Their procedure is extended in this paper to the case of unequal eccentricities, and a technique is developed which results in greater accuracy and a considerable reduction in computational effort.

Investigation of the inelastic behavior of beam-columns under biaxially eccentric load has been undertaken only recently. Birnstiel and Michalos⁸ and Harstead, Birnstiel, and Leu⁹ developed a general procedure for determining the ultimate load with the same eccentricities at each end. The procedure requires considerable computational effort. Sharma and Gaylord¹⁰ gave a simple approximate solution in which the lateral displacements and twist of the cross section are assumed to vary sinusoidally along the axis of the column. The solution of the nonlinear differential equations is simplified by imposing the equilibrium condition only at midlength of the column. Syal and Sharma¹¹ presented a numerical technique for the general case in which eccentricities at one end differ from those at the other. However, it is limited to elastic behavior, so that only the load at first yield is obtained. Residual stresses are taken into account in both (10) and (11) and computed loads are in good agreement with test results.

2. UNIAXIAL BENDING WITH EQUAL END ECCENTRICITIES

2.1 Method of Analysis

The curvature ϕ , the axial load P , and the bending moment M at any cross section of the member are related by

$$\phi = f(P, M) \quad (1)$$

In the elastic range, this equation has the familiar form $\phi = M/EI$. Since displacements are small, ϕ can be taken equal to y'' , the second derivative of displacement with respect to distance along the column:

$$|y''| = f(P, M) \quad (2)$$

With the coordinate axes shown in Fig. 1, y'' is negative and $M = Py$, so that Eq. 2 yields

$$y'' + f(P, Py) = 0 \quad (3)$$

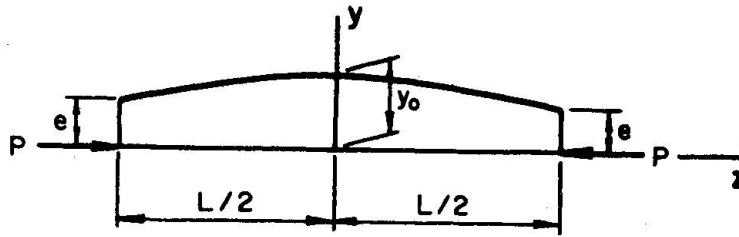


Fig. 1 Eccentrically loaded column

The deflected shape can be found by solving an initial-value problem of Eq. 3. With the following initial values at midlength

$$y(0) = y_0 \quad y'(0) = 0 \quad (4)$$

the solution is obtained in the form

$$y = y(P, y_0) \quad (5)$$

From this equation the end eccentricity e is

$$e = e(P, y_0) \quad (6)$$

Kabaila and Hall⁷ plot on the e - M_0 plane ($M_0 = Py_0$) a family of P-curves for a column of given length (Fig. 2). The peak of each P-curve gives the maximum eccentricity for the corresponding value of P . Thus, the ultimate load is identified by

$$\frac{\partial e}{\partial y_0} = 0 \quad (7)$$

In the analysis presented in this paper the ultimate load is determined without developing the P-curves of Fig. 2. This is done by constructing an auxiliary curve whose ordinate is $\partial y / \partial y_0$ (Fig. 3). To develop this curve, Eq. 3 is differentiated with respect to y_0 . Thus, for any P-curve of Fig. 2

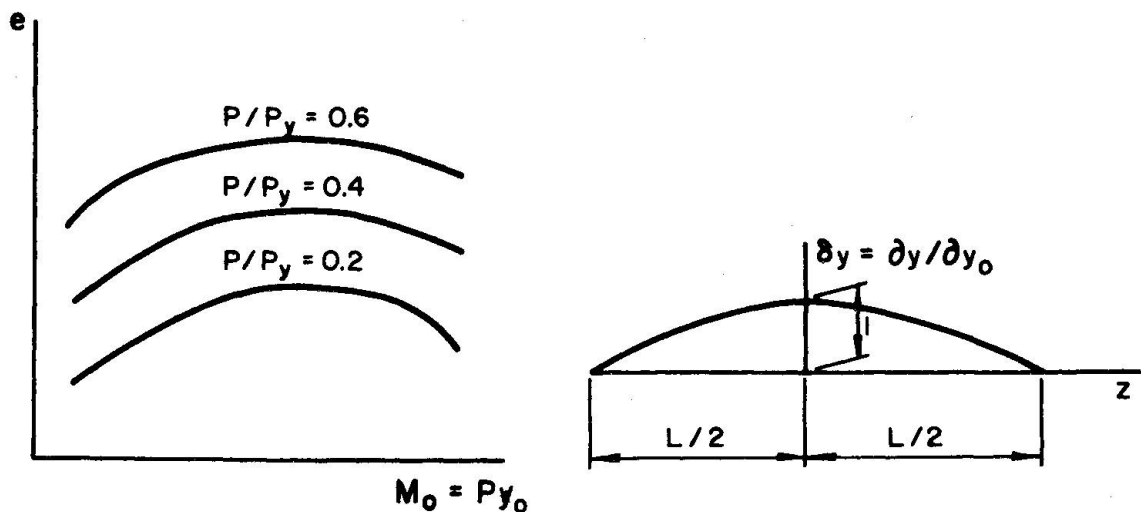


Fig. 2 Plots of P/P_y for column of given length

Fig. 3 Auxiliary curve

$$\frac{\partial y'''}{\partial y_0} + \frac{\partial f(P,M)}{\partial M} P \frac{\partial y}{\partial y_0} = 0 \quad (8)$$

which, with the notation $\delta y = \partial y / \partial y_0$ can be written

$$\delta y''' + f'_M(P,M) P \delta y = 0 \quad (9)$$

The auxiliary curve δy is symmetrical about the δy axis (Fig. 3). The initial values are

$$y(0) = 1 \quad y'(0) = 0 \quad (10)$$

Solutions of Eqs. 3 and 9 with the respective initial values of Eqs. 4 and 10 can be carried out simultaneously by numerical integration, proceeding along the z axis until $\delta y = 0$. This point satisfies the condition expressed by Eq. 7. The corresponding value of z is the half length $L/2$ of the column and $y(L/2)$ is the eccentricity e .

2.2 Evaluation of P , M , and ϕ

The cross section of the column is shown in Fig. 4a. The stress-strain curve of the steel is assumed to have a plateau at the yield stress σ_y and strain hardening is neglected. The member is assumed

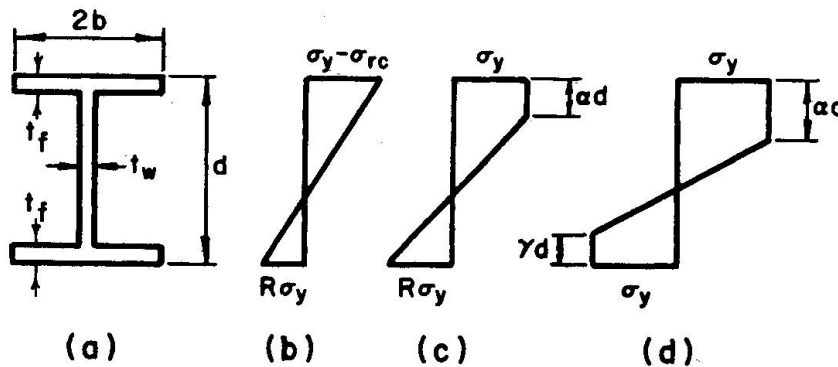


Fig. 4



Fig. 5

to have cooling residual stresses as shown in Fig. 5. This distribution was chosen so that results of the analysis could be compared with those of previous investigations in which it was used (5,10). Ketter, Kaminsky and Beedle¹² developed equations for P , M , and ϕ . Formulas are given for the case of initial yield of the extreme fiber in compression (Fig. 4b), for the compression side partially plastic (Fig. 4c), and for both sides partially plastic (Fig. 4d). For the case shown in Fig. 4d the equations are

$$\frac{P}{\sigma_y} = 2bt_f(R_c + R_t) + t_w(d - 2t_f)R_t + t_w d(\alpha - \gamma) \quad (11a)$$

$$\frac{M}{\sigma_y} = t_f(d - t_f)(2b - t_w) + \frac{t_w d^2}{6} (1 + \alpha + \gamma - 2\alpha^2 + 2\alpha\gamma - 2\gamma^2) \quad (11b)$$

$$\frac{\phi}{\phi_y} = \frac{1}{1 - \alpha - \gamma} \quad (11c)$$

In Eq. 11c $\phi_y = M_y/EI = 2\sigma_y/Ed$. The other symbols are defined in Figs. 4 and 5. Formulas for the other two cases are not repeated here. Figure 6 shows an M- ϕ relationship for a given value of P. Point A corresponds to initial yielding of the extreme fiber in compression, while point B corresponds to the case where yielding is complete through the thickness of the compression flange. For simplification the M- ϕ relation is assumed to be linear between A and B.

Equations 11 can be written in the form

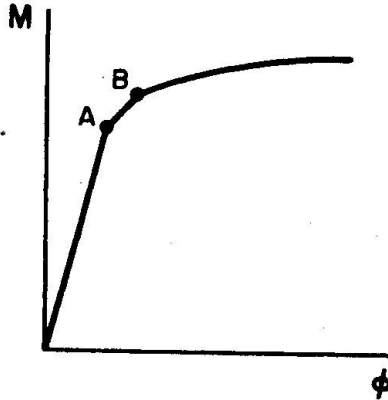


Fig. 6 Moment-rotation curve

$$P = P(\alpha, \gamma) \quad (12a)$$

$$M = M(\alpha, \gamma) \quad (12b)$$

$$\phi = \phi(\alpha, \gamma) \quad (12c)$$

The general forms of the equations for the case shown in Fig. 4c are the same as Eqs. 12 except that R replaces γ . For the case of Fig. 4b, however, the member is elastic and P, M, and ϕ are functions of R only.

2.3 Calculation of y'' and $\delta y''$ for Inelastic Behavior

The numerical integration of Eqs. 3 and 9 requires the evaluation of the curvatures y'' and $\delta y''$ for given values of P and M. Since these cannot be obtained in closed form when the cross section is partially yielded, a numerical procedure which is an extension of the Newton-Raphson method¹³ is used.

Expansion of Eqs. 12 using Taylor's theorem and retaining only the linear terms yields

$$dP = \frac{\partial P}{\partial \alpha} d\alpha + \frac{\partial P}{\partial \gamma} d\gamma \quad (13a)$$

$$dM = \frac{\partial M}{\partial \alpha} d\alpha + \frac{\partial M}{\partial \gamma} d\gamma \quad (13b)$$

$$d\phi = \frac{\partial \phi}{\partial \alpha} d\alpha + \frac{\partial \phi}{\partial \gamma} d\gamma \quad (13c)$$

For a given value of P, $dP = 0$. Therefore, Eq. 13a gives

$$d\alpha = -\frac{\partial P / \partial \gamma}{\partial P / \partial \alpha} d\gamma \quad (14)$$

which upon substitution into Eqs. 13b and 13c gives

$$dM = \left(-\frac{\partial M}{\partial \alpha} \frac{\partial P / \partial \gamma}{\partial P / \partial \alpha} + \frac{\partial M}{\partial \gamma} \right) d\gamma \quad (15a)$$

$$d\phi = \left(-\frac{\partial \phi}{\partial \alpha} \frac{\partial P / \partial \gamma}{\partial P / \partial \alpha} + \frac{\partial \phi}{\partial \gamma} \right) d\gamma \quad (15b)$$

Then, dividing Eq. 15b by Eq. 15a we get

$$\frac{d\phi}{dM} = f'_M(P, M) = \frac{-\frac{\partial \phi}{\partial \alpha} \frac{\partial P / \partial \gamma}{\partial P / \partial \alpha} + \frac{\partial \phi}{\partial \gamma}}{-\frac{\partial M}{\partial \alpha} \frac{\partial P / \partial \gamma}{\partial P / \partial \alpha} + \frac{\partial M}{\partial \gamma}} \quad (16)$$

The following procedure describes the evaluation of y'' and $\delta y''$ for given values of P, M, and δy .

a. Assume $\bar{\alpha}$ and $\bar{\gamma}$ and compute $\bar{P}$ and $\bar{M}$ by Eqs. 12a and 12b.

b. With $dP = P - \bar{P}$ and $dM = M - \bar{M}$, solve Eqs. 13a and 13b to get $d\alpha$ and $d\gamma$. The new values for α and γ are then

$$\alpha = \bar{\alpha} + d\alpha$$

$$\gamma = \bar{\gamma} + d\gamma$$

c. Substitute the values of α and γ from step b into Eqs. 12a and 12b and compare the resulting values of P and M with the given values. If the agreement is not satisfactory, use the new values to start a new cycle. This process is repeated until the desired accuracy is obtained.

d. Substitute the final values of α and γ from step c into Eq. 12c to obtain y'' .

e. Substitute the final values of α and γ from step c into Eq. 16 to obtain $f'_M(P, M)$. Use this and the known value of δy to obtain $\delta y''$ from Eq. 9.

2.4 Numerical Integration of Equations 3 and 9

The procedure for determining the column configuration at any station when its configuration at the preceding station is known is as follows:

Step 1. Having P , y_0 (or M_0), and δy_0 at station 0, the corresponding y_0'' and $\bar{y}_0''$ are found by the procedure outlined in Art. 2.3.

Step 2. Assume y_1'' and $\delta y_1''$ of the next station and use the trapezoidal rule of numerical integration

$$y_1' = y_0' + \frac{h}{2} (y_0'' + \bar{y}_1'') \quad (17a)$$

$$y_1 = y_0 + \frac{h}{2} (y_0' + y_1') \quad (17b)$$

to compute y_1 . Similarly, use

$$\delta y_1' = \delta y_0' + \frac{h}{2} (\delta y_0'' + \delta \bar{y}_1'') \quad (18a)$$

$$\delta y_1 = \delta y_0 + \frac{h}{2} (\delta y_0' + \delta y_1') \quad (18b)$$

to compute δy_1 . In these equations h is the interval between stations.

Step 3. With y_1 from Step 2 compute $M_1 = Py_1$ and through the procedure described in Art. 2.3 determine y_1'' and $\delta y_1''$.

Step 4. If the values of y_1'' and $\delta y_1''$ of Step 3 do not agree with the assumed values of Step 2 start a new cycle with y_1'' and $\delta y_1''$ as new initial values and repeat the procedure. When agreement between the computed values and those of the previous cycle is satisfactory, return to Step 2.

With the method just described, the solutions of Eqs. 2 and 9 are carried out simultaneously until δy becomes zero. As noted before, the corresponding value of z is the half length of the column and the value of y is the end eccentricity e of P . This procedure was programmed for the IBM 360/75 system of the Digital Computer Laboratory of the University of Illinois at Urbana-Champaign. The procedure converges rapidly, and the solution for a given P and M_0 is obtained with a few seconds of computer time. Computed values for the W8 x 31 column of (5) for a number of cases were found to agree within 3 percent of values interpolated from interaction curves in that reference. Further details are given in (14).

It is of interest to note that $y(z)$ and $\delta y(z)$ become zero at the same station if the initial value of y_0 is such that the cross section is elastic at station 0. Of course, this corresponds to a concentrically loaded column and the given P is the Euler load.

3. UNIAXIAL BENDING WITH UNEQUAL END ECCENTRICITIES

3.1 Method of Analysis

The method of analysis for the case of unequal end eccentricities is similar to the case of equal end eccentricities. The procedure starts at the station at which the column has its maximum displacement y_0 measured from the pressure line (Fig. 7a). In general, the auxiliary curve δy is not symmetrical about the y axis and has an initial slope c (Fig. 7b). To obtain the ordinates of this curve it is convenient to consider it as a combination of two curves δy_1 and δy_2 which are determined by the following initial conditions:

$$\delta y_1(0) = 1 \quad \delta y_1'(0) = 0 \quad (19a)$$

$$\delta y_2(0) = 0 \quad \delta y_2'(0) = 1 \quad (19b)$$

The ordinate of the δy curve is then

$$\delta y = \delta y_1 + c \delta y_2 \quad (20)$$

The instability condition $\partial e / \partial y_0 = 0$ of Eq. 7 can be written

$$\delta y(L_i, c) = \delta y_1(L_i) + c \delta y_2(L_i) = 0 \quad i = 1, 2 \quad (21a)$$

which gives

$$c = - \frac{\delta y_1(L_i)}{\delta y_2(L_i)} \quad (21b)$$

where L_i is an end of the column.

The deflected shape $y(z)$ and the values of δy_1 and δy_2 at each station point are found by a straightforward solution of the following initial-value problems:

$$y'' + f(P, M) = 0 \quad (22a)$$

$$y(0) = y_0 \quad y'(0) = 0 \quad (22b)$$

$$\delta y_1'' + f_M'(P, M) P \delta y_1 = 0 \quad (23a)$$

$$\delta y_1(0) = 1 \quad \delta y_1'(0) = 0 \quad (23b)$$

$$\delta y_2'' + f_M'(P, M) P \delta y_2 = 0 \quad (24a)$$

$$\delta y_2(0) = 0 \quad \delta y_2'(0) = 1 \quad (24b)$$

The following steps describe the procedure for given values of P and y_0 :

Step 1. Integrate Eqs. 22a, 23a, and 24a numerically with the prescribed initial values, starting at $z = 0$ and proceeding to the left to any negative value L_1 . The procedure for finding y' and δy at each station point is the same as for the case of equal end eccentricities (Art. 2.3). The corresponding $e_1 = y(L_1)$ is the eccentricity of P at the left end of the column (Fig. 7).

Step 2. Compute c from Eq. 21b.

Step 3. Integrate the same system of equations, with the same initial values, from $z = 0$ to the right until $\delta y(L_2, c) = 0$, using the value of c from Step 2. The corresponding $e_2 = y(L_2)$ is the eccentricity at the right end of the column and $L = -L_1 + L_2$ is the length of the column (Fig. 7), where it is to be remembered that L_1 is negative.

The procedure described above gave results for a number of cases which checked within 3 percent of values according to (15). Further details are given in (14).

4. GENERAL CASE OF BIAXIAL BENDING

The procedure described above was extended to the analysis of columns with biaxial eccentricities unlike at the two ends. The solution is given by

$$\begin{vmatrix} \frac{\partial u}{\partial u_A} & \frac{\partial u}{\partial v_A} & \frac{\partial u}{\partial \beta_A} \\ \frac{\partial v}{\partial u_A} & \frac{\partial v}{\partial v_A} & \frac{\partial v}{\partial \beta_A} \\ \frac{\partial \beta}{\partial u_A} & \frac{\partial \beta}{\partial v_A} & \frac{\partial \beta}{\partial \beta_A} \end{vmatrix} = 0 \quad (25)$$

where u and v are displacements in the x and y directions, respectively, β is the angle of twist, and u_A , v_A , and β_A are first derivatives with respect to z at the end A of the column. Numerical results for a number of cases are reported in (14).

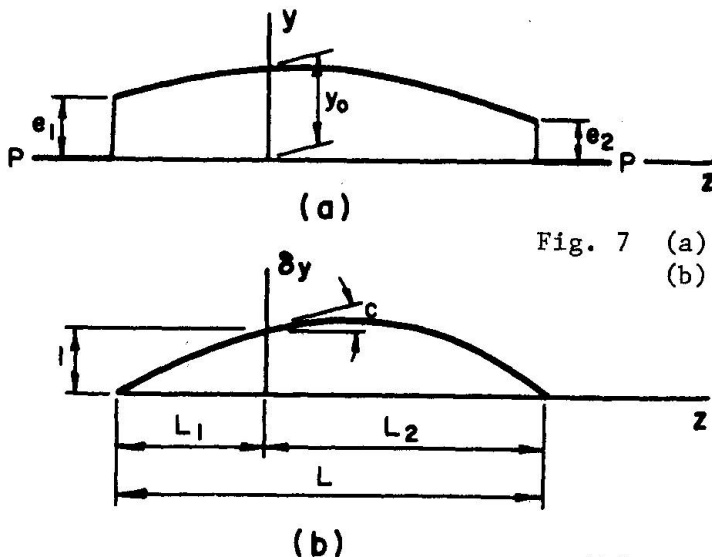


Fig. 7 (a) Column with unequal eccentricities
(b) Auxiliary curve for column in (a)

The concept of an equivalent uniform moment for uniaxial bending with unequal end eccentricities was suggested by Massonnet in 1947, and a formula based on an approximate mathematical investigation was given.¹⁷ Later, other similar formulas were proposed, and the concept was used in codes and specifications. Sharma and Gaylord¹⁰ showed that their interaction curves for biaxially bent columns with eccentricities the same at both ends gave good predictions of the results of four tests¹⁶ on columns with eccentricities unlike at the two ends by using an equivalent uniform moment M_{eq} in each principal plane xz and yz according to

$$M_{eq} = 0.4 M_1 + 0.6 M_2 \geq 0.4 M_2 \quad (26)$$

In this formula, which was suggested by Austin,¹⁸ M_1 and M_2 are positive when the member bends in single curvature and M_1 is the smaller of the two. Furthermore, results for 15 columns with unlike eccentricities analyzed in (14) by Eq. 25 differed from values given by Sharma and Gaylord's interaction curves with the equivalent uniform moment according to Eq. 26 by not more than 5 percent. Thus, it appears that the extended concept of equivalent uniform moment can be relied upon to simplify the analysis of the biaxially bent column with eccentricities unlike at the two ends.

If twist is neglected Eq. 25 reduces to a second-order determinantal equation. Results for 22 columns analyzed by this procedure differed from values given by the interaction curves of (10) by not more than 8.5 percent. This suggests that the analysis of biaxially bent columns can be reduced with good approximation to one of plane bending in the x and y directions. This conclusion was also demonstrated in (10).

5. ACKNOWLEDGEMENTS

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THEORETICAL INVESTIGATIONS OF COLUMNS
UNDER BIAXIAL LOADING

Joachim Lindner
Hünnebeck GmbH
Lintorf/Düsseldorf, Germany

ABSTRACT

Columns in frame structures are often loaded eccentrically with respect to both principal axis. Under load the axis displace and twist themselves. The displacement and the twisting are depending on each other. The result is a space curvature.

The present numerical investigations are based on approximate calculations. In doing so, the moment curvature is improved by an iteration method. In every step of the iteration the plastic behaviour of each part of the column is taken into consideration. The influence of the torsional rigidity is considered as well as that of the residual stresses. It was assumed a parabolic figure of the residual stresses in the flanges and the web of the cross-section. The geometrical imperfections are considered as an eccentricity of the point of application of the compressive forces.

Several numerical values of the geometrical imperfections, of the residual stresses and of the slenderness ratio has been investigated with regard to columns of H-shape. It was possible to put through the extensive calculations only by using a computer. The procedures were programmed for a computer with large storage capacity, type S 4004/45. The various influences are shown in diagrams. The load-carrying capacity of the columns are compared to each other.

1. Introduction

Many extensive papers of the Commission 8 of the European Convention for Constructional Steelwork are in hand concerning compressed hinged columns [1] .

The geometrical imperfection exists in the direction of one principal axis. Other kinds of imperfections are considered, such as residual stresses concerning several types of cross-section.

Columns in frame structures are sometimes loaded eccentrically with respect to both principal axis. Under load the axis displace and twist themselves. The displacement and the twisting are depending on each other. The result is a space curvature.

In the following paper you will see some numerical results. They shall help to decide, if there is a possibility to compute a biaxial loaded column in a similiar way as a column excentrically loaded with respect to one principal axis.

2. Theoretical basis

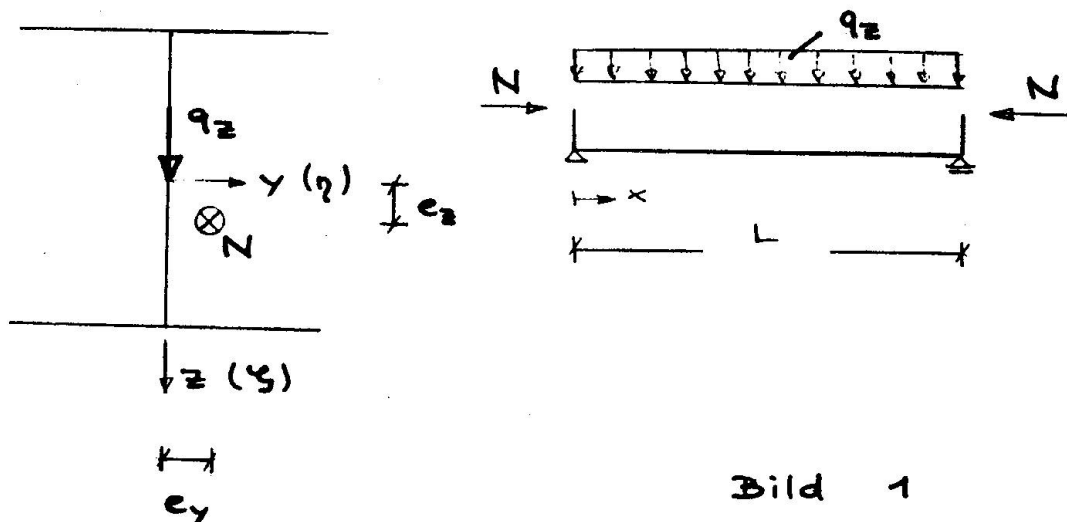
The present investigations are based on approximative calculations. The whole of the mathematical model is given in [2].

In the cases, which are here calculated, the axis of the beam may be straight. But it is also possible that there is an initial imperfection such as a parabolic curve.

The stress-strain relationship of the material is elastic-perfectly plastic. The effects of strain-hardening and shear on the yielding are neglected.

In the given model the potential energy is used in regard to the theory of deformation. The cross-section is bisymmetrical, see equation (1) and figure 1.

$$\pi = \frac{1}{2} \int_{x=0}^L \left\{ GJ_D \vartheta'^2 + EF_{ww} \vartheta''^2 + EF_{zz} \cdot \zeta''^2 + EF_{yy} \eta''^2 + N (\eta'^2 + \zeta'^2 + i_p^2 \cdot \vartheta'^2 + 2e_z \cdot \eta'' \vartheta - 2e_y \zeta'' \vartheta) - 2q_z \zeta \right\} dx \quad (1)$$



To explain the symbols:

GJ_D	torsional rigidity of St. Venant
EF_{ww}	warping rigidity
EF_{zz}	rigidity about Y-axis
EF_{yy}	rigidity about z-axis
i_p^2	$(F_{yy} + F_{zz})/F$
N	axial load
ϑ	twisting
F	area of cross-section

The solution of Eq. (1) is based on assuming polynomial expressions for the displacements η and ξ and the twisting ϑ . We call these polynomial expressions "Hermite'sche Interpolationspolynome", $H_j(x)$, shown in Eq. (2.)

$$\begin{aligned}\vartheta &= \sum_{i=1}^9 a_i \sum_{j=1}^8 H_j(x) \\ \eta &= \sum_{i=1}^9 b_i \sum_{j=1}^8 H_j(x) \\ \xi &= \sum_{i=1}^9 c_i \sum_{j=1}^8 H_j(x)\end{aligned}\quad (2)$$

We will find the bending moments and the twisting moment by the solution of the Eq. (1) by the method of Ritz. After that we will calculate the stresses at any point of the beam as usual. Residual stresses can be added. If we have stresses in some part of the beam, which are greater than the yield stresses, so the corresponding cross-sections have been partly plasticized.

Following that the real curvatures η'' , ξ'' , ϑ'' are defined by a "cross-section-iteration".

The plastification is considered by using the idealized stress-strain relation. By a further iteration we get the real displacements η , ξ , ϑ by numerical integration regarding every station along the member.

From this we find the new bending moments M_y , M_z .

With these improved bending moments the "cross-section-iteration" will be repeated. Both of the iterations are put through as long as there are no more modifications in the displacements.

The ultimate load is reached, as soon as the load-deflection curve do not increase anymore, see fig. 2

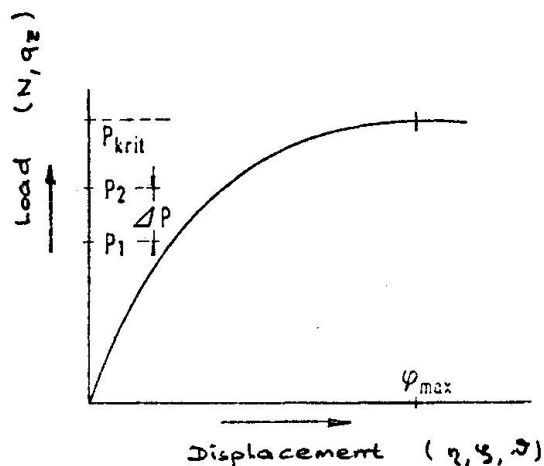


fig. 2

It was possible to put through the extensive calculations only by using a computer. The procedures were programmed for a computer with large storage capacity, type S 4004/45.

3. Numerical results

We will recognize some influences to the ultimate load of biaxial compressed solumns in fig. 3 to 7. All results are presented on a relation between the relative slenderness ratio $\bar{\lambda}$ and the relative axial force $\bar{N}$. Though they are without dimension.

The influence of the type of the residual stresses are shown in fig. 3 and 4. In fig. 3 we assume that the values of the eccentricities e_y and e_z may be one thousandth of the length of bar, ($L/1000$). There are no residual stresses considered in curve no. 1. In curve no. 2 we assume the greatest residual stress of compression by using a value of $0,3 \cdot \sigma_F$. In curve no. 3 we assume a compress stress $0,5 \cdot \sigma_F$. The same curve is also available if the value of compression stress is only $0,3 \cdot \sigma_F$, under the condition that the stress is zero at the point of the connection between the flange and the web.

You can see that in the field of the middle relative slenderness ratio $\bar{\lambda}$ ($\bar{\lambda} \sim 0,8$) the reduction of the ultimate load will reach 20 percent.

In fig. 4 you see two corresponding curves. The eccentricities are assumed as $e_y = L/2000$ and $e_z = L/1000$. In this case too we get a very large reduction in the field of the middle relative slenderness ratio $\bar{\lambda}$ ($\bar{\lambda} \sim 0,8$). We will notice a reduction up to 12 percent.

The effect of various eccentricities has been investigated in fig. no. 5. The curve no. 1 is valid for $e_z = L/1000$, $e_y = 0$, curve no. 2 for $e_z = L/1000$, $e_y = L/2000$. By com-

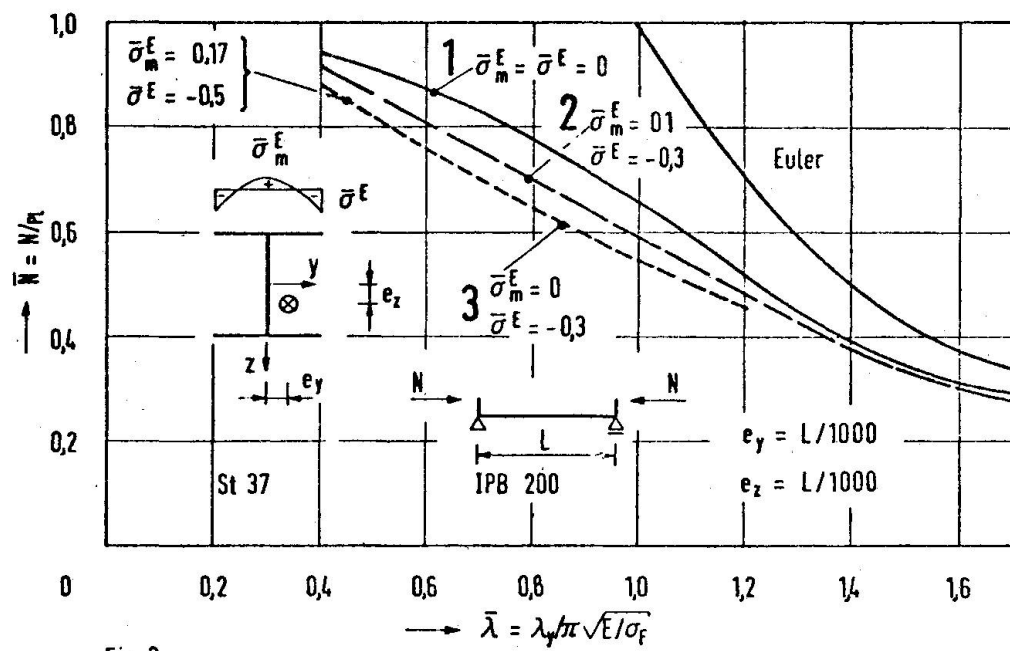


Fig. 3
Bixial loaded column influence of residual stresses
 $e_y = L/1000$ $e_z = L/1000$

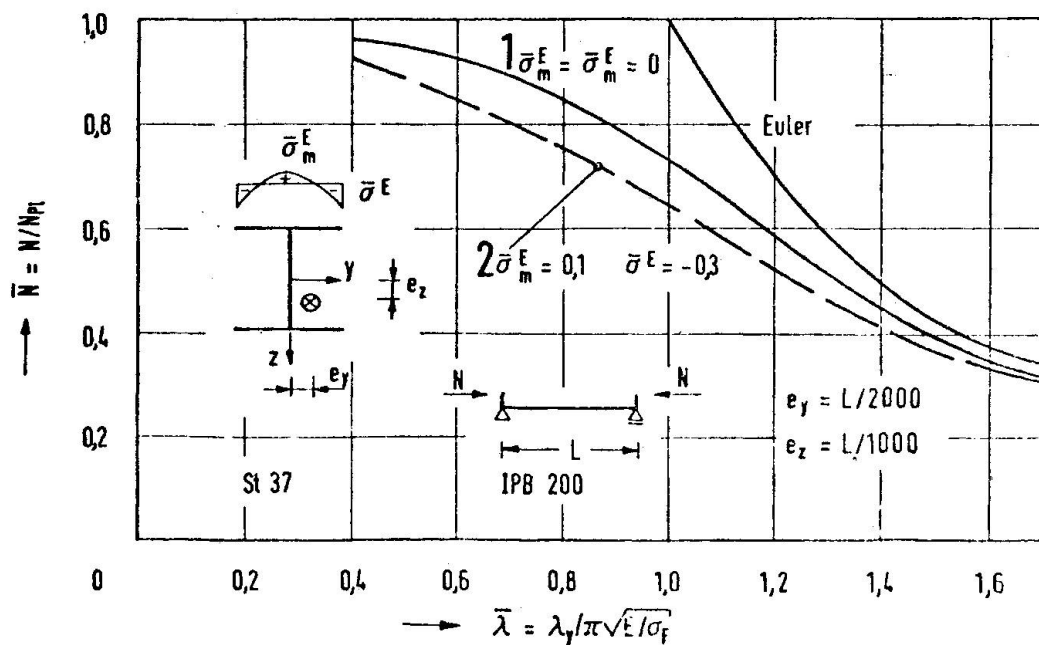


Fig. 4
Bixial loaded column influence of residual stresses
 $e_y = L/2000$ $e_z = L/1000$

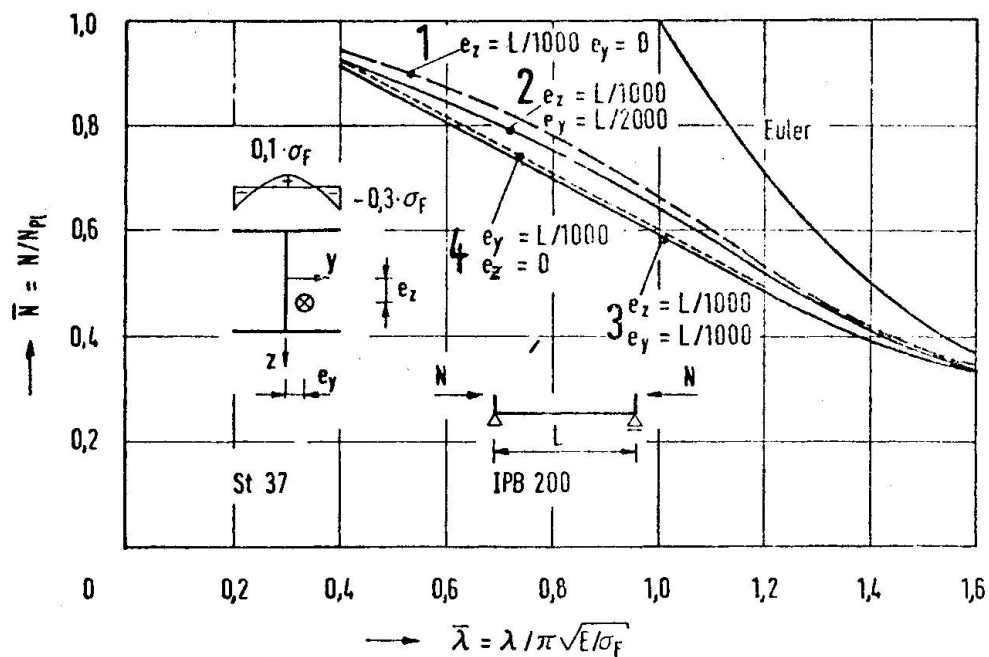


Fig. 5

Bixial loaded column

influence of eccentricities

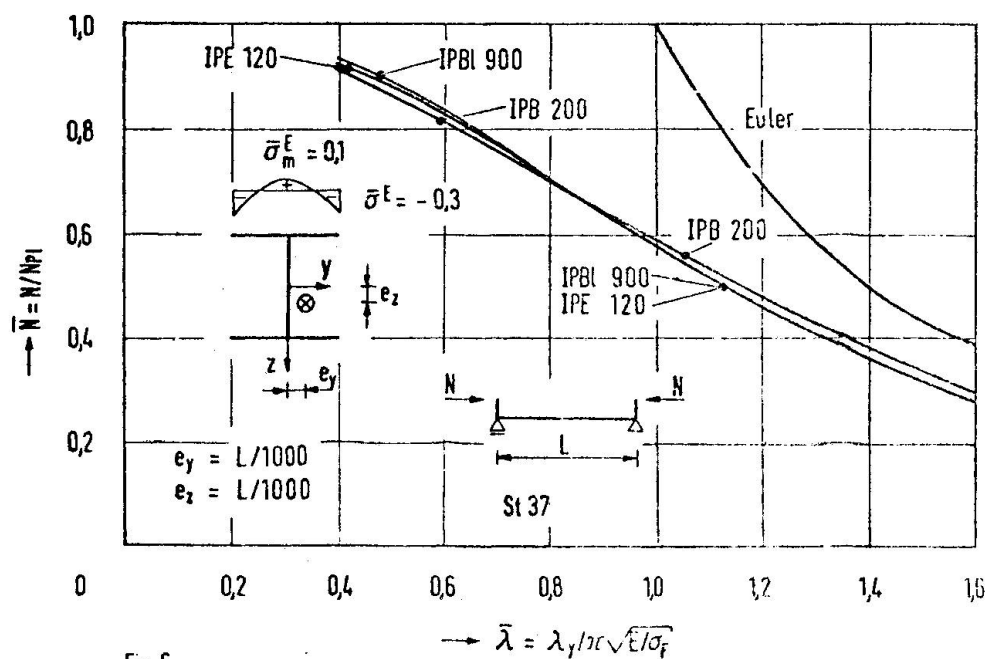


Fig. 6

Bixial loaded column

influence of the type of cross-section

paring these curves we can notice a reduction of nearly 5 percent. If the eccentricity e_y increases to $e_y = L/1000$, an additional reduction is found of nearly 8 percent. (curve no. 3).

If you compare curve no. 3 to curve no. 4 you see that there is only a very small difference.

The influence of three kinds of rolled shapes is shown in fig. no. 6. There are investigated three types of characteristic sections:

- IPE 120 (small flange)
- IPB 200 (wide flange)
- IPB1 900 (high web).

The greatest values of the residual stresses are the same once in each flange. Under this condition the greatest differences amount to 5 percent.

In fig. no. 7 results of calculations are shown, where the column is loaded not only by axial forces N but also by loads q_z .

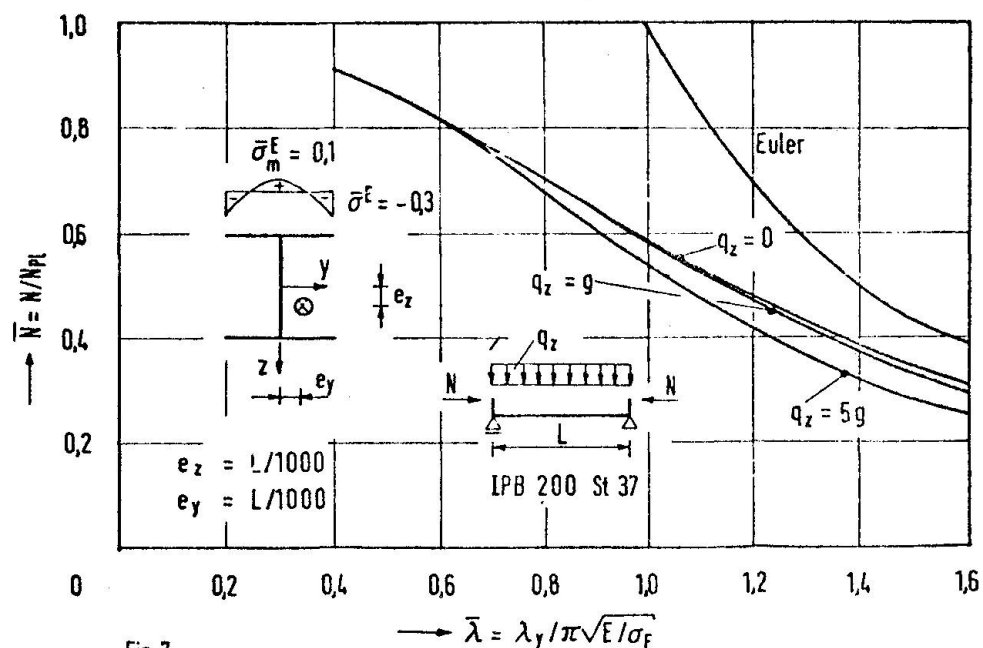


Fig. 7

Bixial loaded column

influence of loads q_z

If these loads are small and if they are as great as the dead weight g , so the influence is unimportant. From $\bar{\lambda} = 0,7$ an important reduction to the ultimate load takes place if the load q_z reaches the value of $5 g$. The reduction amounts to 20%.

4. Conclusion

By the results of part 3 we can see, that the curves of the ultimate load of biaxial loaded columns are similar to those of uniaxial loaded ones. With the assumed eccentricities the absolute values of the ultimate loads however decrease up to 10 percent. The influence of the type of cross-section is very small under the assumed residual stresses, the influence of the loads q_z has to be considered in all cases of great length of member.

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CALCUL ET COMPORTEMENT DES POTEaux ENCASTRES
SOLLICITES EN FLEXION BIAXIALE

Dr. S. Vinnakota
Ecole Polytechnique Fédérale
Lausanne
Suisse

ABSTRACT

Analytical procedures are developed for the determination of the load-deformation characteristics of elastically restrained, inelastic beam-columns under biaxial bending and torsion (Fig. 1).

The equilibrium equations are written with respect to an arbitrary system of axes. The resulting fourth order coupled nonlinear differential equations are solved using the method of finite differences.

The method is programmed for the CDC 6500 Computer at Federal Institute of Technology, Lausanne. This Computer program is utilized for the solution of various numerical examples.

Some of the salient features of the theory are given below :

- i) The member is primastic, originally straight and untwisted.
- ii) The open cross-section considered in the theory is arbitrary in shape. However, the computer program is limited at present to H-shaped sections only.
- iii) Two rotational restraints are provided at each end to simulate the effect of minor and major axis beams on the behaviour of column.
- iv) Two directional restraints are considered at each end to simulate the effect of relative deflection of the ends (Sway or P- Δ effect) on the behaviour of column.
- v) At each end a rotational spring against rotation is also included.
- vi) The bar is subjected to end loads only. At each end, they may consist of : axial load, moments, transverse loads, torsional moment and bi-moment.

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MATERIAL BEHAVIOUR AND STABILITY IN
ALUMINIUM-CONSTRUCTION (E DIN 4113)

Otto Steinhardt
Prof. of Civil Engineering
University (T. H.) Karlsruhe
W.-Germany

ABSTRACT

KEY WORDS:

Alloy-Materials, General Buckling of Struts, Standardisation

Overall buckling by bending and torsional flexural buckling of aluminium alloys can be understood by a set of two formulae, beyond this frame-buckling is cleared up.

Introduction

The up-to-date standard-specifications for compressed members in aluminiumconstruction should reasonably follow the example of steel - construction standardisations. There, the correlations and interrelations between material, section-groups and "imperfections" lead to relativistic general definitions and formulations for "planned-central" (but not yet expressly for "excentrical") loaded struts - as Hermann Beer, his collaborators and others demonstrated in a clear and remarkable manner. The fundamental buckling-curves (fig. 1, overall buckling with excessive bending) offer the collapse-stress σ_S (basic-column-stress) for a given slenderness ratio (probable values, each with a tolerance limit equal 2,3 % of collapse load F_S , found by more than one thousand special tests). -

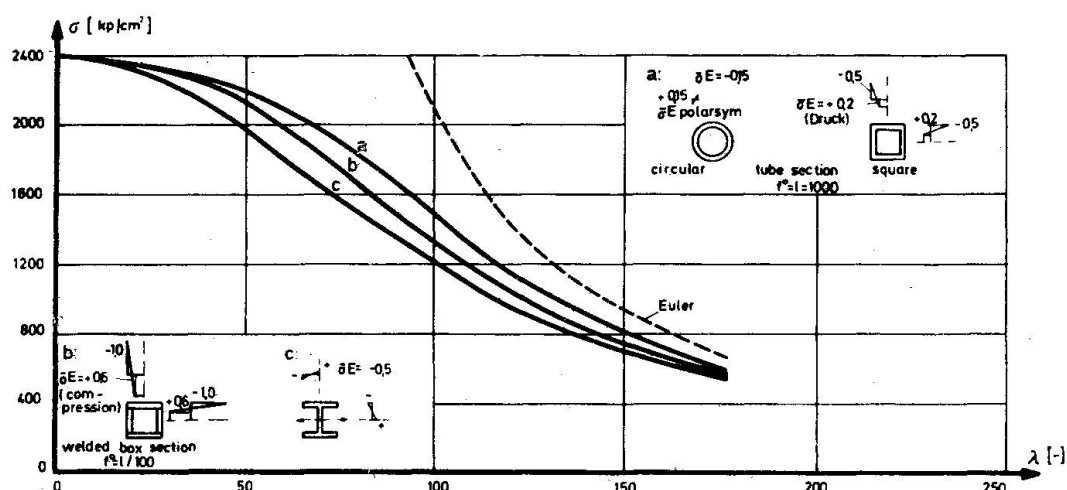


Fig. 1 European Curves of the Basic - Column - Strength
included initial imperfections and residual stresses
(Steel St37)

Here, with regard to aluminiumconstructions (and specially to the E DIN 4113, 1973) one should differentiate (according to six basic alloys), on the one hand between peculiar and disproportionate σ, ϵ -curves (each of them very sufficiently described as a tripartite-polygon); on the other hand there are no residual stresses in not-welded and matrix-pressed sections (fig. 2). -

Additionally, it may be noted that it seems possible to conceive the behaviour of compressed steel-struts, even with residual stresses, by a method which will be explained in the following: For instance, a steel - strut

with only geometrical imperfections according to $u = i(\lambda/320)^2$ (material St. 37, $d_y = 2,4 \text{ Mp/cm}^2$, $n = 3,15$, $\mu = 0,575$) is computable very accurately (for the case fig. 1.8) if the formula of fig. 3 is taken as a basis; hereby the greatest deviation from the Beer-curve may be only 2 to 4 %. -

The Standardisation

In the new E DIN 4113 (in chapter 8.1) the σ - ϵ -(compression) diagram of each alloy will be replaced by a tripartite polygon: At first for flexural buckling it turned out that the exact (computer-)analysis of the elastic-plastic collapse stress σ_s , for single and double symmetrical cross sections over the whole range of slenderness up to $\lambda = 1/i = 250$ produces results not more than 2 or 3 percent higher than given by the formulae (a) and (b) (see figure 3). -

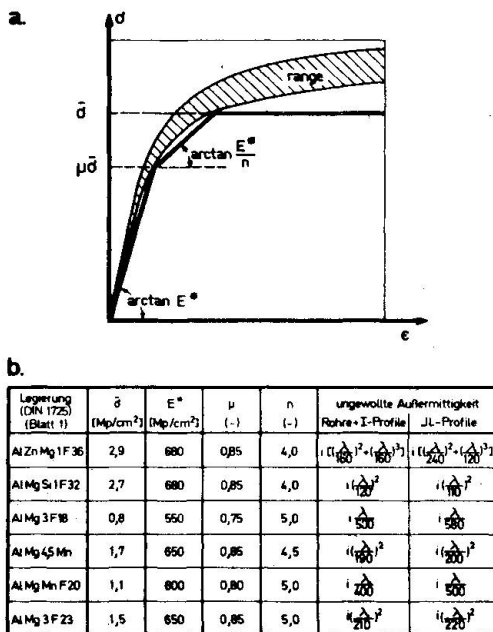


Fig. 2 a. Tripartite σ - ϵ -Polygon (Aluminium)
b. Characteristic alloy values

$$\frac{N_v}{\mu \bar{N}} + \frac{M_v}{(1 - \frac{N_v}{\bar{N}}) \mu M^*} \leq 1 \quad (a)$$

$$\psi \frac{N_v}{\bar{N}} + \frac{M_v}{(1 - \frac{N_v}{\bar{N}}) \cdot M^*} \leq 1 \quad (b)$$

$$\frac{n+1}{2} \quad \frac{N_v > 1 \text{ formula (a)}}{N^* < 1 \text{ formula (b)}} \quad (c)$$

$$\psi = 1 + \frac{n-1}{2} (1 - \mu) \quad \frac{\bar{N}}{N^* \cdot N_v}$$

$$N^* = \text{the smaller value of } \begin{cases} \frac{\pi^2 E^* F}{\lambda^2} & (\text{Euler}) \\ \frac{\pi^2 E^* F}{\lambda^2} & (\text{DIN 4114, R10.13}) \end{cases}$$

$$M^* = \text{the smaller value of } \begin{cases} M_{pl} = Y \bar{\sigma} W_0 (\text{plastic bending moment}) \\ M_{K1} & (\text{Lit. Kolbrunner-Meister}) \end{cases}$$

$$N_v = v \cdot N, \quad M_v = v \cdot [A_m |M_2| + |N_{vorh}| \cdot u]$$

$$A_m = \begin{cases} 0,6 + 0,4 M_2 / M_1 \geq 0,4 & \text{fixed bearing} \\ 0,85 & \text{not fixed bearing} \end{cases}$$

$$v = \text{safety factor}$$

Fig. 3 Formulas (a)(b) and (c) for the Design of Compressed Columns

For the six most important aluminium-alloys, the decisive polygon was shown above in figure 2, (their characteristic dimensions E , $\bar{\sigma}$, n and μ are tabulated). The tripartite polygon is found as the lower limitation of a "bunchy dispersion" of many experimentally determined σ - ϵ diagrams).

The new standard will contain additionally (in chapter 8.2) a series of ω -tables; therefore it will also be possible in future to use (selectively) the so called " ω -procedure". Those ω -tables (according to the DIN 4114, 1952) consider - for the case of the exactly planned-central loaded strut - only partly the unavoidable geometric imperfections $u = i/20 + l/500$; therefore, the greater range of slenderness is determined by an eigen-value analysis and here the buckling factor grows up to $\nu_{Ki} = 2,5$.

The new E DIN 4113 is principally based on the excentrically loaded strut; by the way, it is connected in parallel to the " ω -procedure" by functions $u = f(\lambda)$ (found by attempting) which result in such collapse stresses $\sigma_s = P_u/F$ that $\sigma_s / \nu_s = \sigma_{dzul}$. Now it is essential to state that ν_K is constant over all slenderness ratios. That means that "safety factors" (known from the DIN 4114) ν_{ki} , ν_K and ν_{kr} can be replaced by only one value. -

How far does it seem possible for each case in point to start exclusively from the determined geometric "imperfections" and from the σ - ϵ -polygon? In addition to this the following may be declared up to the present: In cases of flexural bending, there is not only the simple " ω -procedure" to be replaced (for the planned-central strut!) by the formulae given in figure 3, but there result clear σ_s -values also for the excentrically loaded ones; more precisely, there result the relations regulating the forces and moments N and M, according to the formulae (a) to (c). -

The main advantage of the new thinking (not at all an international one, but in view of the German DIN 4114) seems to be given by the fact that one may succeed in integrating also the torsional flexural buckling (besides the "over turning" and the "torsional buckling"). Therefore, the "Karlsruher Formeln" were applied to some limiting cases, partly delivering very clear deviations in comparison with the "theoretical" values of the DIN 4114. On the basis of many calculations and also in borderline cases (this is done in the dissertation of F. Labib, 1972/73), the above mentioned formulae could be estimated as very useful; but additionally a suitable series of tests should verify the real circumstances.

Some aluminium-specimens (fig. 4) were carefully produced (partly with stress relieving by annealing), and the σ - ϵ -compression-diagrams (as well as generally the σ - ϵ tension-diagrams) were registered. Though (see specimen No. 1 to No. 3) a relatively weak aluminium alloy lying outside the new standard, was investigated, it seems to be suited for recording sensitively the aim which should be followed here.

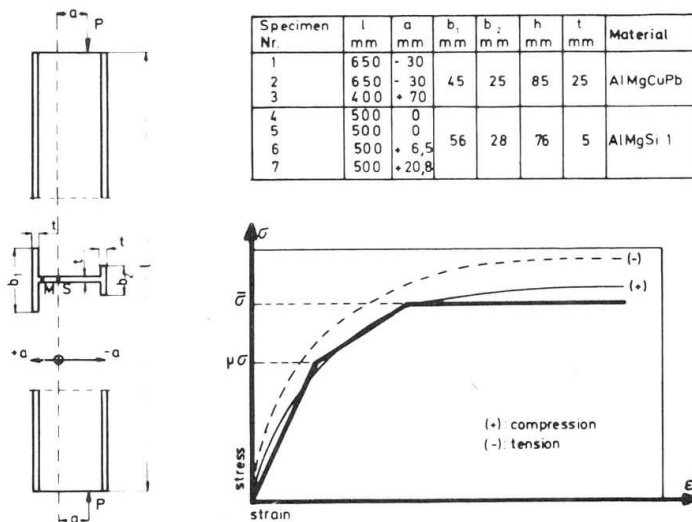


Fig. 4 Investigations on Torsional Flexural Buckling Specimens

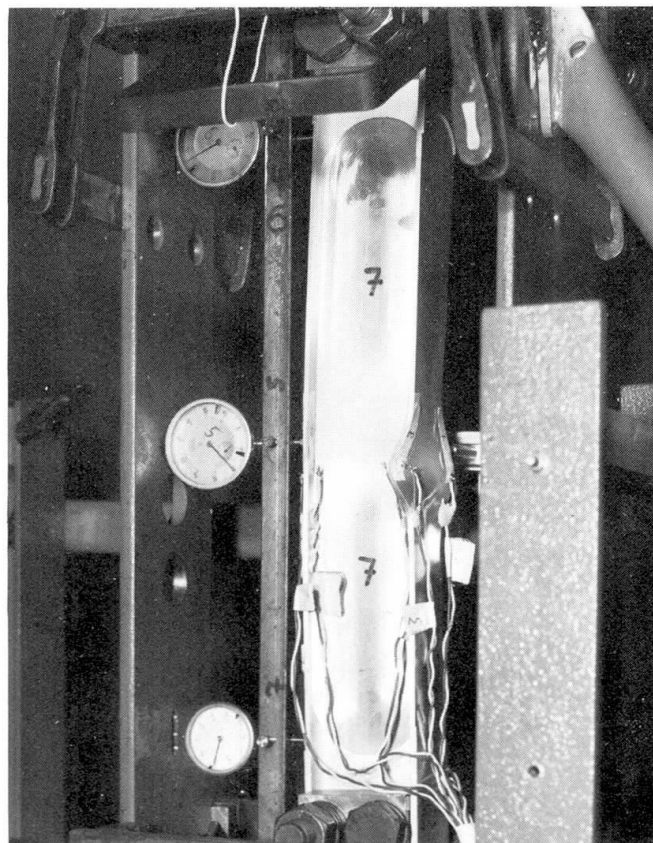


Fig. 5 Setup for Buckling Tests

A photograph ([fig. 5](#)) shows the total mode of the test procedure as well as the normally used measuring instruments.

It was possible to watch permanently not only the strains but also the total and part torsions of the medium cross section of the strut and also the lateral deflections.

The ultimate loads of the tests No. 1 to No. 3 are compiled in figure 6. They are compared with the values appropriate

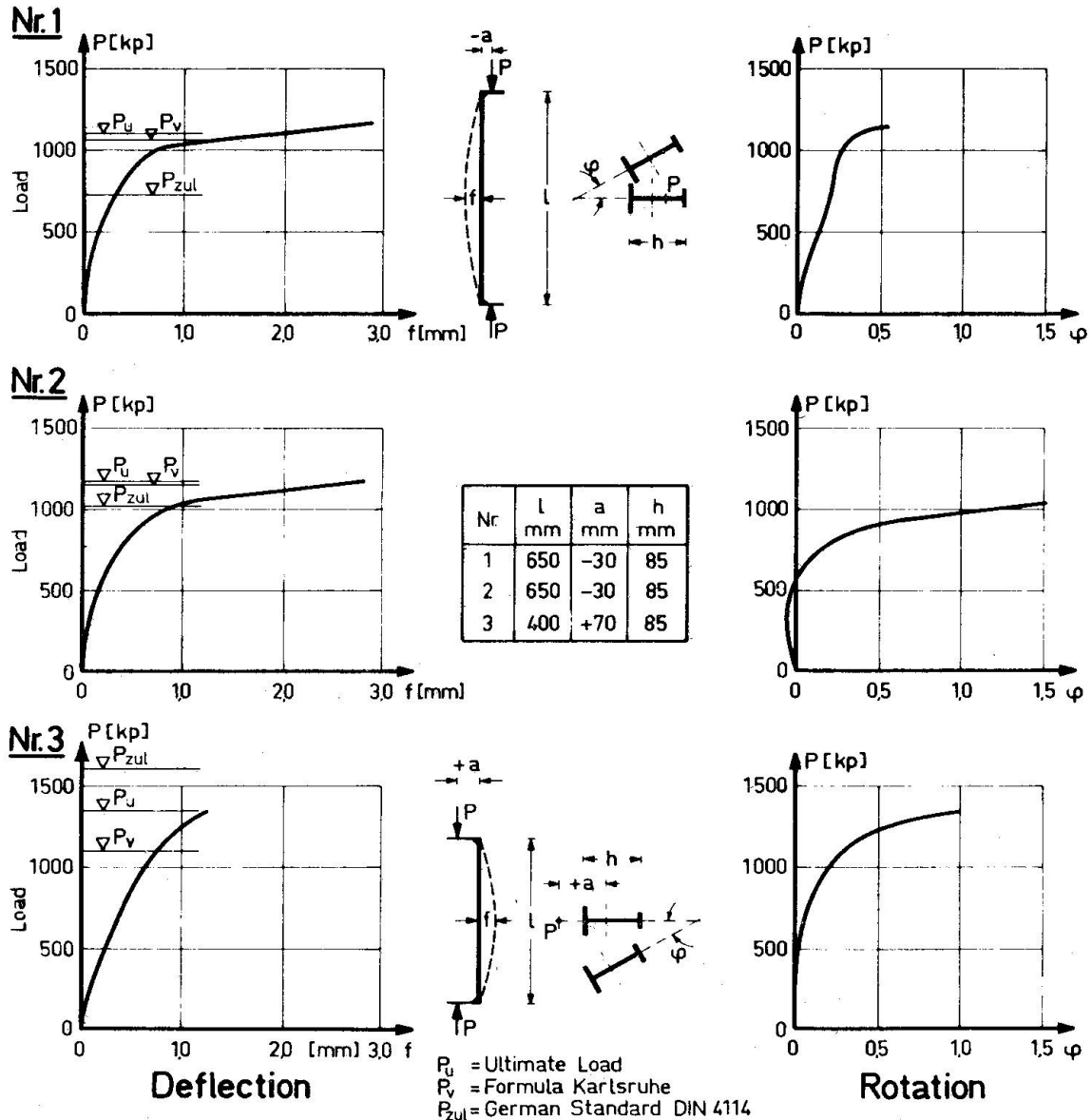


Fig. 6 Load-Deformation-Curves of Aluminium Columns
(Ultimat-Loads and Calculated-Forces)

to the "formulae" (in figure 3) and with the results according to DIN 4114 and they recommend these formulae as most realistic and accurate! At any rate they are very useful (for

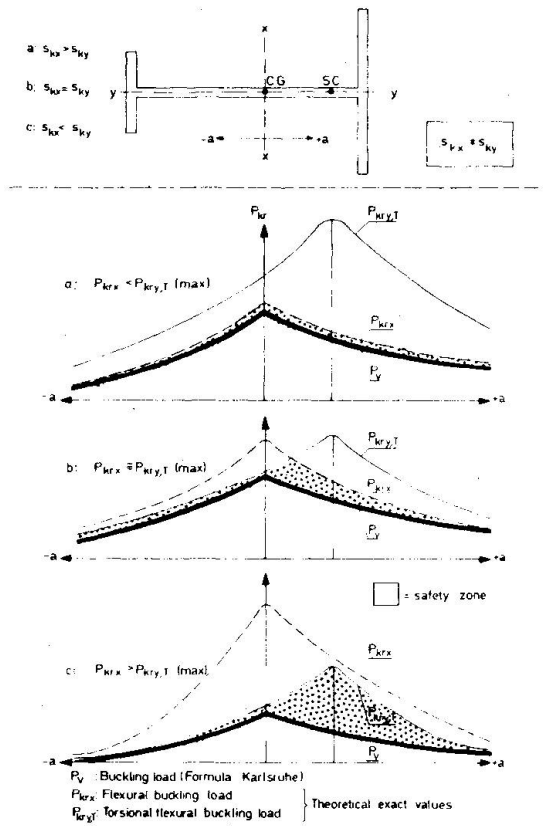


Fig. 7

Qualitative buckling load plots according to Formula Karlsruhe and theoretical values

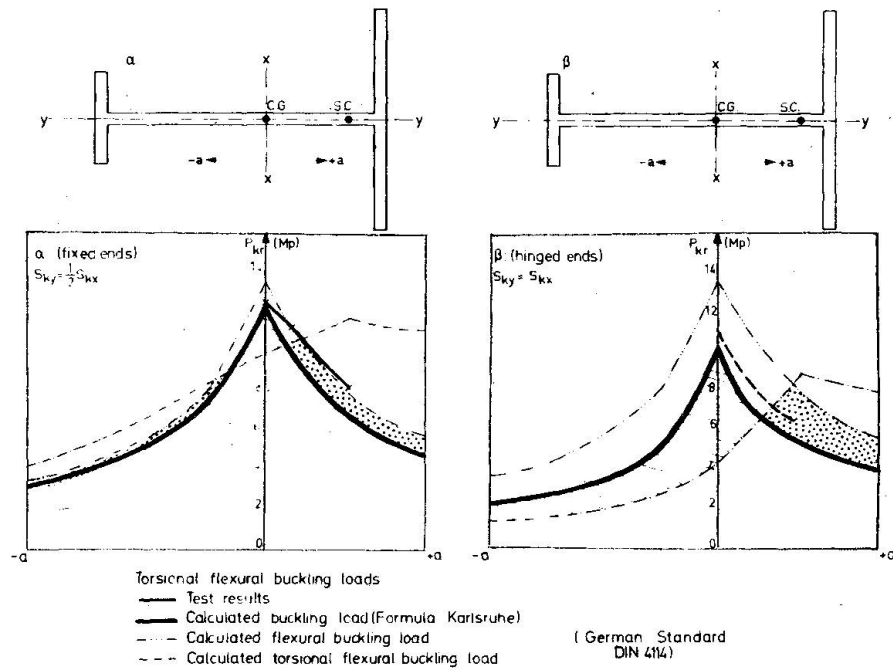


Fig. 8

Comparison of test results with buckling loads due to Formula Karlsruhe and German Standard DIN 4114

horizontally supported or) for braced frames; for unbraced frames, and this holds not only for the "Elastic-Design" but also for the "Plastic-Design" (th.i. german "Traglast-Verfahren"), only the "Nonlinear-Theory" can deliver accurate results. In this connection we maintain that the equilibrium must be considered for the displaced system, not only in the simplified plastic configuration but also in the elastic situation.

Figure 6 shows:

1. The collapse results by way of "torsional flexural buckling"!
2. The deformation curves by flexure and also by torsion, indicate clearly the ultimate load - and not the bifurcation load; but all deformations start directly from initial-values (= geometrical imperfections).
3. The limit-loads can be found to a very good approximation by the formulae (fig. 3); on the other hand roughly estimated values for the "allowable strength" (as in the German standard DIN 4114) are not sufficiently accurate. By the tests No. 1 and No. 2 it was shown that (instead of $\sqrt{K} = 2,5$) the security-factors were only 1.49 and 1.16 respectively; test No. 3 would have been only 0,84, but here simple stress-calculation $N/F \pm M/W < \text{zul } \sigma$ gives protection - but not clearness.

The next series of tests had to clear up the usefulness of the formulae (fig. 3), if the load is piled up within the small area of section between the center of gravity and the shear-center.

The "Classical-Theory" gives us, for this interesting special sphere, the curves of fig. 7; depending on the different effective buckling length in the x- or y-direction, and as a function of the load-point. For practical designing each of the lowest side-branches of these curves should be taken as basis.

The formulae (fig. 3) revale and demonstrate a good and safe adaption, although related to the center of gravity of the cross-section. What's more, the "Classical-Theory" would not be applicable in the elastic-plastic-range.

Left in fig. 8 experimental values can be offered; in this figure, by the way, it is the arithmetical values according to the German-Standard-DIN 4114, which are shown, Not only left but also right one can recognize the weakness of the standardized procedure; on the other hand the formulae of

fig. 3 describe the reality in a satisfactory manner (Probably, on the left side the ultimate loads would have been some what higher, if as a result of entering the elastic-plastic range, buckling of the flanges had not occurred (fig. 9).)

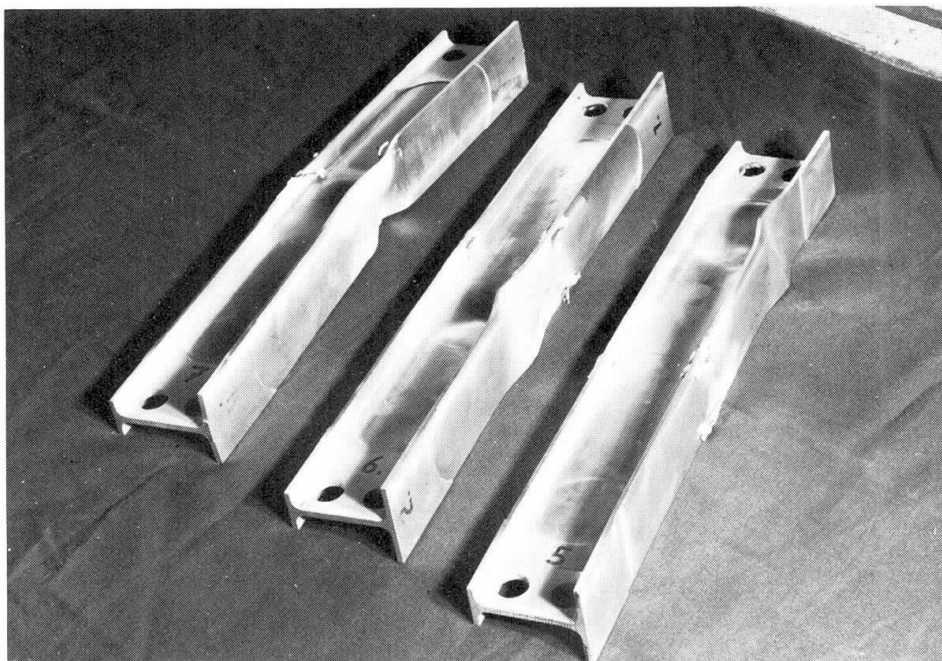


Fig.9 Specimens after Testing

Summary

In regard to the instability of (single) members in a aluminium-structure (i.e. of columns) - and also of members in frames, loaded in a manner which does not produce transverse displacements at the floor levels -, it seems possible to propose a calculating method (see fig. 3) which covers "overall buckling with excessive bending" as well as the "lateral torsional flexural buckling"; this is valid so far as the cross section of the strut has a I-form and is loaded within the web-plane.

For practise design it seems alternatively possible to use tabulated " ω -values" (for planned-central loading) and, as a rough estimate, the "0.9 formula" (DIN 4114) for ex-central loading. -

In both cases knowledge primarily of P_{cr} (that is the bifurcation-load, which is always greater than the stability-limit-load P_S) is an important tool for assessing the strength, also of (braced) systems (see lit. Andelfinger, Der Stahlbau 1964). -

At last the problem of frame-buckling (= general system buckling) with lateral free joints is to be discussed. Here in every case an (elastic) secondary order analysis seems useful (1. Klöppel-Friemann, Der Stahlbau, 1964; 2. H. Rubin, Dissertation, University Karlsruhe, 1971; 3. P. Dubas T.C. 17, Planning and Design of Tall Buildings, S.O.A. Report Nr. 5, 1972). If there are not lateral loads, then initial displacements for the joint levels should be assumed.

In as far as the mentioned (lateral free) frames are only one or two storeys high numerous tabulated values are available for the ideal effective length (S_{ki}); in this connection, the generally known "0.9-formula" (DIN 4114) is indeed being applied, but as regards this important sphere of application investigations are being conducted covering a more realistic and accurate use of the formulae of figure 3.-

RECHERCHES SUR LE COMPORTEMENT AU FLAMBEMENT
DE BARRES EN ALUMINIUM

A. Bernard

F. Frey

J. Janss

Prof. Ch. Massonnet

Faculté des Sciences Appliquées
Université de Liège
Liège, Belgique

RESUME

Ce rapport présente les résultats et conclusions d'une recherche CIDA, effectuée à Liège, préliminaire à l'établissement de courbes européennes de flambement par divergence des profilés en alliages légers. Les principes fondamentaux sont ceux adoptés, dans le domaine de l'acier, par la Commission VIII de la CECM. La partie expérimentale du travail, qui porte sur des profilés extrudés (tube et double T), montre une bonne concordance avec les résultats obtenus à l'aide du programme numérique de simulation mis au point par les auteurs. Par ailleurs, la loi de Ramberg-Osgood apparaît comme parfaitement adaptée pour présenter la loi contraintes-dilatations des différents alliages d'aluminium.

ZUSAMMENFASSUNG

Dieser Bericht bringt die Ergebnisse und Schlüsse einer, an der Universität Lüttich ausgeführten CIDA-Forschungsarbeit, die der Aufstellung von europäischen Knickkurven für Leichtmetallprofile vorausgeht. Die fundamentalen Prinzipien sind diejenigen die die Kommission VIII der EKS im Gebiete des Stahls angenommen hat. Der experimentelle Teil der Arbeit betrifft extrudierte Rohr- und Doppel-T-Profile. Die Übereinstimmung des experimentellen und der mittels des von den Autoren aufgestellten numerischen Simulierungsprogramms auf Computer gerechneten Knickkurven ist mehr als zufriedenstellend. Ausserdem bewährt sich die Ramberg-Osgood Kurve als sehr angemessen zur Darstellung des Spannungs-Dehnungsdiagramms der verschiedenen Aluminiumlegierungen.

ABSTRACT

This report present the results and conclusions of a CIDA research conducted at the University of Liège prior to establishing European buckling curves for aluminium alloy columns. The general principles are those used for steel by the Committee VIII of ECCS. The experimental part of the work, performed on extruded tube and H-shape sections, shows a more than satisfactory concordance with the results computed by the numerical simulation programm written by the authors. The Ramberg-Osgood curve, is shown to fit very well the stress-strain relationship of the different aluminium alloys.

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RECHERCHE SUR LE FLAMBEMENT DES BARRES EN ALUMINIUM

H. Djalaly
Ingénieur au Service Recherches
C.T.I.C.M., Puteaux
France

D. Sfintesco
Directeur des Recherches
C.T.I.C.M., Puteaux
France

ABSTRACT

With reference to a series of tests carried out by P. Arnault and D. Sfintesco on pin-ended axially compressed members in various aluminium alloys, column curves for this type of members have been established by the authors according to a probabilistic method of approach. This method consists in determining statistically the collapse loads with a given probability.

The random parameters concerning the various initial imperfections have a significant influence on the actual behavior of the member in compression. The stochastic analysis of these parameters and of their combined effects require long calculations and extensive use of computer programs, whilst the statistical analysis of test results is relatively simple and easy.

The statistical approach leads to a clear and consistent degree of safety.

Non-dimensional column curves have been thus established for the probable collapse limit, with a constant probability of 2.5 %.

In addition, the probable dispersion is presented as a function of the slenderness ratio, thus allowing to define a variable safety factor in view of obtaining a constant probability.

1. ORIENTATION FONDAMENTALE DE LA RECHERCHE

Cette recherche, dont le but était d'établir une courbe de base du flambement en compression simple, a été réalisée afin de déterminer dans quelles conditions les méthodes de calcul des Règles CM 66 relatives aux constructions en acier pouvaient être appliquées à celles en alliages d'aluminium.

Ce programme d'essais, limité dans son ampleur, n'avait pour objectif que de situer la question. L'exploitation statistique des résultats a permis d'évaluer les paramètres des "courbes Dutheil" de flambement pour les barres en alliages d'aluminium.

La présente étude vise à exposer brièvement l'interprétation des résultats expérimentaux par les moyens statistiques et à présenter les courbes qui en résultent. Nous ne nous attacherons donc pas à décrire les essais et les appareillages.

La méthode probabiliste, employée dans le cadre de cette recherche, consiste à déterminer statistiquement la charge limite d'affaissement pour une probabilité intégrale donnée moyennant un nombre d'essais jugé satisfaisant, effectués sur des barres industrielles qui ne sont ni géométriquement, ni matériellement parfaites. On sait, en effet, que les imperfections initiales (défauts de rectitude et de centrage, tolérances dimensionnelles de la section, contraintes rémanentes, etc...) de nature aléatoire changent sensiblement le comportement des barres prévu par la théorie de l'instabilité élastique ou de l'instabilité élasto-plastique.

Les facteurs aléatoires des imperfections initiales intervenant dans le comportement de la barre comprimée sont à l'origine de divergences importantes. L'étude stochastique des combinaisons de tous ces facteurs aléatoires permet de déterminer théoriquement l'intensité des divergences entre le comportement réel de la barre industrielle et celui d'une barre idéalement parfaite.

Bien entendu, l'étude stochastique du comportement de la barre, compte tenu de ces variables aléatoires, serait non seulement longue et fastidieuse mais aussi fort complexe. C'est pourquoi, ce type de problème requiert, de préférence, une étude expérimentale.

L'étude statistique des résultats d'essais permet aussi de clarifier le problème de la sécurité et d'en donner des bases cohérentes.

Lorsqu'il s'agit de flambement simple, pour un élanement donné et un certain nombre d'essais effectués pour cet élanement, on peut définir des contraintes probables d'affaissement et des contraintes conventionnelles d'affaissement correspondant à une probabilité intégrale de 2,5 %.

Comme on le sait, la détermination de cette probabilité ne peut se faire que si l'on connaît la loi de répartition de la population totale, ici la répartition des contraintes d'affaissement auxquelles on se réfère.

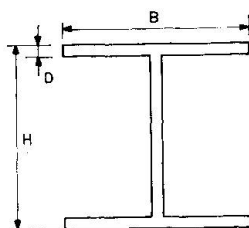
Une éprouvette de flambement en aluminium est caractérisée, avant même sa mise en charge, par un certain nombre de paramètres qui font que, dès que cette éprouvette est sollicitée, la charge d'affaissement est influencée par un nombre de variables aléatoires. Pour que dans ces conditions nous puissions calculer la charge d'affaissement probable, nous devons envisager une loi de répartition telle que celle de Laplace-Gauss, mais encore faut-il s'assurer de la justesse de cette hypothèse.

Pour y parvenir, deux méthodes dites tests d'hypothèses ont été appliquées aux résultats d'essais, ce sont : la méthode de la droite de Henry et la méthode par les coefficients γ_1 et γ_2 de Karl Pearson, qui montrent respectivement si la distribution est symétrique¹ et si elle correspond à une loi normale ou non.

2. CLASSES D'ALLIAGES DES BARRES

Les essais ont porté sur des barres industrielles fabriquées dans les conditions normales et plus précisément sur le profilé unique H de 63 x 63 x 4 mm défini par la norme A 65-162, dont les tolérances dimensionnelles sont rassemblées dans le tableau ci-dessous.

	<i>H</i> mm	<i>B</i> mm	<i>D</i> mm	Section mm ²
Dimensions	63	63	4	737
Tolérances	± 0,95	± 0,95	± 0,24	—



Le tableau I donne les compositions chimiques et les caractéristiques mécaniques des alliages d'aluminium utilisés dans cette recherche.

Les lots livrés provenaient d'une même fusion pour l'A-G5, l'A-SG et l'AS-GM et de trois fusions pour l'AU4G.

TABLEAU I. — Valeurs nominales des compositions chimiques et des caractéristiques mécaniques

		Fe	Si	Cu	Mn	Zn	Mg	Cr	Ti	masse spéci- fique g/cm ³	limite élastique à 0,2 %		limite de rupture		État
											ga- rantie	moyen- ne	ga- rantie	moyen- ne	
A-G5	min.	—	—	—	0,2	—	4,5	—	—	2,63	12	15	27	32	Brut de fabrication
	max.	0,5	0,4	0,10	1,0	0,2	5,5	0,4	0,2	—	—	—	—	—	
A-U4G	min.	—	0,3	3,5	0,3	—	0,4	—	—	2,80	25	28	39	42	Trempé mûri
	max.	0,7	0,8	4,7	0,8	0,25	1,0	0,1	0,2	—	—	—	—	—	
A-Z5G		—	—	—	—	4,8	1,2	0,2	—	2,80	—	30	—	37	Trempé revenu
A-SG	min.	—	0,8	—	—	—	0,7	—	—	2,70	—	26	—	32	Trempé revenu dur
	max.	0,5	1,5	0,10	0,2	0,25	1,3	—	0,2	—					
AS-GM	min.	—	0,6	—	0,1	—	0,6	—	—	—					
	max.	0,5	1,6	0,10	1,0	0,25	1,4	0,3	0,2	—					

3. EXPLOITATION STATISTIQUE DES RESULTATS

La loi de Laplace-Gauss constitue une famille de lois de probabilité dont la fonction de densité est :

$$f(x) = \frac{1}{s\sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{x_i - m}{s} \right)^2 \right] \quad (1)$$

et dont la fonction de distribution est :

$$F(X) = \frac{1}{s\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2} \left(\frac{x_i - m}{s} \right)^2} dx \quad (2)$$

où m , valeur moyenne ou moyenne arithmétique, est égal à $\frac{\sum n_i x_i}{N}$ et s , qui est l'écart-type, est égal à :

$$s = \sqrt{\frac{\sum n_i (x_i - m)^2}{N - 1}}$$

Avec un changement de variable $t = \frac{x_i - m}{s}$, on peut écrire :

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-t^2/2} dt \quad (3)$$

La loi normale est une distribution symétrique. Pour appliquer la loi de distribution gaussienne, il faut que la répartition expérimentale ou le raccordement opéré soit légitime.

4. METHODE DE LA DROITE DE HENRY

La méthode de la droite de Henry est une méthode graphique ; elle indique à simple vue si la répartition expérimentale est une distribution légitime de Gauss, et montre aussi si certaines données ne s'éloignent pas trop de la droite passant par les autres points d'essais, permettant ainsi d'écarter des données aberrantes qui seraient conservées dans un calcul trop systématique.

Considérant la formule (2), on peut à toute valeur x_i de X , faisant partie de la série de mesures, faire correspondre la valeur de la fréquence expérimentale égale au rapport du nombre de mesures inférieures ou égales à x_i au nombre total de mesures. Puis, grâce à une table relative à la relation (3) on trouve la valeur t , en portant les valeurs x et t sur des axes de coordonnées cartésiennes, on constate que tous ces points d'une distribution normale sont en ligne droite, que cette ligne droite coupe l'axe des abscisses au point m et que la pente de cette droite est égale à $\frac{1}{s}$. Si la répartition de la série de mesures est une distribution légitime de Gauss, les points de mesures se trouvent au voisinage ou sur la droite de Henry. Le graphique fait ainsi connaître les deux paramètres m et s respectivement moyenne et écart-type. La pratique courante est de graduer l'axe des ordonnées non pas suivant les valeurs de t , mais directement en fréquence cumulée. Cette graduation n'est évidemment pas linéaire, aussi pour faciliter encore la mise en oeuvre de la méthode, Dumas et Maheu préconisent-ils, de faire correspondre à chaque mesure de rang n , la fréquence cumulée $\frac{n-0,5}{N}$, c'est-à-dire :

$$F_n(t) \simeq \frac{n-0,5}{N} \quad (4)$$

Il existe des tables donnant la valeur de t en fonction de N et n . On peut, dans ces conditions, travailler directement sur papier millimétré au lieu de papier logarithmique. Sur un graphique représentant une loi normale, l'aire comprise entre les valeurs $m - s$ et $m + s$, c'est-à-dire $t = 1$, est égale à 68,3 % de l'aire totale située sous la courbe en cloche.

4.1. Méthode des moments

Afin de mieux vérifier si la répartition des mesures est symétrique et si elle est pointue par rapport à la distribution normale, on exécute les calculs suivants :

1. Calcul de la valeur moyenne :

$$m = \frac{\sum n_i x_i}{N}$$

2. Calcul des écarts :

$$\delta_i = x_i - m$$

3. Calcul des valeurs :

$$S_2 = \sum_{i=1}^N \delta_i^2, \quad S_3 = \sum_{i=1}^N \delta_i^3, \quad S_4 = \sum_{i=1}^N \delta_i^4$$

4. $K_2 = \frac{S_2}{N-1}$ d'où l'écart-type $s = \sqrt{K_2}$

$$K_3 = \frac{N}{(N-1)(N-2)} S_3$$

$$K_4 = \frac{N}{(N-1)(N-2)(N-3)} \left[(N+1)S_4 - \frac{3(N-1)}{N} S_2^2 \right]$$

$$5. \quad G = K_3/K_2^{3/2}, \quad G_2 = K_4/K_2^2$$

$$6. \quad s'_1 = \sqrt{\text{var.}(G_1)} = \sqrt{6 N(N-1)/(N-2)(N+1)(N+3)}$$

$$s'_2 = \sqrt{\text{var.}(G_2)} = \sqrt{\frac{24 N(N-1)^2}{(N-2)(N-3)(N+3)(N+5)}}$$

7. Les rapports γ_1 et γ_2 sont tels que :

$$\gamma_1 = \frac{G_1}{s'_1}, \quad \gamma_2 = \frac{G_2}{s'_2}$$

pour une distribution parfaitement normale, on a $\gamma_1 = \gamma_2 = 0$.

Pour toute distribution symétrique, on aurait encore $\gamma_1 = 0$. γ_1 indique combien et comment, par sa valeur et son signe, la distribution n'est pas symétrique. γ_2 indique si la distribution est plus ou moins pointue par rapport à la distribution normale.

Lorsque γ_1 et γ_2 ont des valeurs assez grandes, on ne peut pas accepter l'hypothèse de la normalité de la distribution de la série de mesures. Ainsi pour γ_1 et γ_2 supérieurs à 3, l'hypothèse doit être réfutée.

5. INTERVALLE DE CONFIANCE POUR LA MOYENNE D'UNE POPULATION NORMALE DE REFERENCE

5.1. Estimation de la moyenne quand la variance σ^2 n'est pas connue

Comme la variance réelle n'est pas connue, nous devons déterminer l'intervalle de Confiance de la moyenne pour une probabilité α donnée par le procédé suivant.

Nous savons que la distribution des moyennes d'échantillons de taille N , prélevés non exhaustivement dans une population normale (m, σ) est elle-même normale et telle que :

$$E(m') = m \text{ (moyenne de la population)}$$

Pour tourner la difficulté créée par cette estimation, les statisticiens se sont efforcés de trouver une valeur caractéristique de l'échantillon dont la loi de distribution (contrairement à celle de m'), ne fasse pas intervenir σ^2 . Cela a conduit Student

à préconiser la variable $z = \frac{m' - m}{\sigma}$. Par la suite, R.A. Fischer a préconisé la variable :

$$t = \frac{m' - m}{\frac{\sigma'}{\sqrt{N-1}}} = \frac{(m' - m)\sqrt{N}}{\sigma_1} \quad (5)$$

Cette variable a pour loi de probabilité la loi de Student-Fischer représentée par l'équation ci-après :

$$f(t) = \frac{1}{\sqrt{N-1} B\left(\frac{N-1}{2}, \frac{1}{2}\right) \left(1 + \frac{t^2}{N-1}\right)^{N/2}} \quad (6)$$

$B(a, b)$ étant la fonction bêta qui a pour expression :

$$B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} = \int_0^1 u^{(a-1)} (1-u)^{(b-1)} du$$

avec $a > 0$ et $b > 0$.

Il s'agit d'une loi symétrique, comme la loi normale mais un peu plus aplatie. Le t_α qui en résulte, pour un seuil de probabilité $(1 - \alpha)$ donné, est donc légèrement supérieur à celui qui fournit la loi normale.

Par référence à la loi de Student-Fischer, l'intervalle de probabilité $(1 - \alpha)$ de $\frac{m' - m}{\frac{\sigma_1}{\sqrt{N}}}$ s'écrit en effet :

$$-t_\alpha \leq \frac{m' - m}{\frac{\sigma_1}{\sqrt{N}}} \leq t_\alpha \quad (7)$$

ou encore :

$$m - \frac{\sigma_1}{\sqrt{N}} t_\alpha \leq m' \leq m + \frac{\sigma_1}{\sqrt{N}} t_\alpha, \quad \sigma_1 = s \quad (8)$$

où t_α est déterminé pour la probabilité de $\frac{\alpha}{2}$ ou $1 - \frac{\alpha}{2}$ dans les tables correspondantes avec $\nu = N - 1$ degrés de liberté.

6. LIMITE DE CONFIANCE DES VARIANCES

Comme la valeur de la moyenne réelle est inconnue, l'intervalle de confiance des variances doit être défini pour ce cas.

Etant donné une série de N mesures extraite d'une population normale $N(m, \sigma)$, la somme quadratique est égale à :

$$X^2 = \sum_i \frac{(x_i - m')^2}{\sigma^2} = \frac{N\sigma'^2}{\sigma^2} \quad (9)$$

et suit une loi de X^2 à $\nu = N - 1$ degrés de liberté, deux quantités $X^2\alpha_1$ et $X^2\alpha_2$ telles que :

- probabilité (X^2 expérimental $> X^2\alpha_1$) = α_1

- probabilité (X^2 expérimental $> X^2\alpha_2$) = α_2

α_1 et α_2 étant les seuils de probabilité. On a alors :

$$X^2\alpha_1 \leq \frac{N\sigma'^2}{\sigma^2} \leq X^2\alpha_2 \quad (10)$$

on passe à l'intervalle de confiance :

$$\frac{(N-1) \sigma_1^2}{X^2 \alpha_2} \leq \sigma^2 \leq \frac{(N-1) \sigma_1^2}{X^2 \alpha_1}$$

ou encore

$$\sqrt{\frac{\sum_i (x_i - m)^2}{X^2 \alpha_1}} < \sigma < \sqrt{\frac{\sum_i (x_i - m)^2}{X^2 \alpha_2}} \quad (11)$$

Nous avons effectué les calculs en prenant $\alpha_1 = \frac{\alpha}{2} = 0,025$ et $\alpha_2 = 1 - \frac{\alpha}{2} = 0,975$, c'est-à-dire qu'il y a 95 % de chances pour que les variances vraies se trouvent dans les intervalles de confiance. Ces valeurs sont rassemblées dans les tableaux 2, 3, 4, 5 et 6. Dans chacun de ces tableaux, on trouve la moyenne, les limites de confiance de la moyenne et de l'écart-type, les coefficients de Pearson, la valeur des variables Student-Fischer, les contraintes limites d'affaîssement compte tenu de 2,5 % de probabilité intégrale et finalement la plus petite et le plus grande valeur observées dans les séries de mesures. Un examen rapide de ces tableaux nous montre que la plupart des séries de mesures correspondant aux différents élancements ont une distribution normale légitime, sauf pour les élancements 120 et 150 de tous les alliages utilisés.

Il est bon de remarquer que les répartitions de la contrainte d'affaîssement des barres en A-SG aux élancements 85, 100, 120, 150 ne sont pas du tout des répartitions normales légitimes.

TABLEAU II. — Résultats des calculs statistiques pour des barres à profil H en A - U 4 G

Élancements	λ	10	50	70	85	100	120	150
Nombre d'essais	N	19	17	17	16	12	5	9
Valeurs moyennes	$\bar{x} = m$	31,884	22,980	14,658	9,951	7,587	5,244	3,447
Limites de confiance de la moyenne	m'	30,880	22,183	14,327	9,628	7,478	5,122	3,171
	m''	32,888	23,777	14,989	10,274	7,696	5,366	3,723
Écarts-types	S	2,084	1,550	0,644	0,607	0,171	0,098	0,359
Limite de confiance des écarts-types	S'	1,577	1,155	0,491	0,502	0,122	0,058	0,242
	S''	3,082	2,358	1,030	1,203	0,290	0,282	0,687
Coefficients de Pearson	γ_1	- 0,076	- 1,113	1,046	- 2,279	0,509	- 0,560	3,481
	γ_2	- 0,169	- 0,036	0,583	0,627	1,050	51,024	4,901
Variable de Stu. Fischer	t_α	2,337	2,400	2,400	2,437	2,655	4,275	4,275
$m - t_\alpha \cdot S$		27,014	19,260	13,112	8,471	7,133	4,825	2,382
$A_{1-\alpha}$		1,390	1,421	1,421	1,437	1,551	2,372	1,711
$S'_{in} = A_{1-\alpha} \cdot S$		2,897	2,202	0,915	0,872	0,265	0,232	0,614
La plus petite valeur expérimentale	x_e	27,4	19,78	13,84	8,50	7,33	5,13	3,18
La plus grande valeur expérimentale	x_u	35,9	25,49	15,95	10,50	7,89	5,35	4,36
Dispersion en %	δ	19,3	20,5	14,0	24,1	7,6	10,7	20

TABLEAU III. — Résultats des calculs statistiques pour des barres à profil H en A - Z 5 G

Élancements	λ	10	50	70	85	100	120	150
Nombre d'essais	N	24	10	10	10	9	5	5
Valeurs moyennes	$\bar{x} = m$	32,771	24,204	14,051	9,908	7,233	4,860	3,134
Limites de confiance de la moyenne	m'	32,108	23,384	13,582	9,702	7,082	4,750	3,040
	m''	33,434	25,025	14,519	10,114	7,384	4,970	3,228
Écart-types	S	1,571	1,146	0,655	0,288	0,196	0,089	0,076
Limite de confiance des écart-types	S'	1,220	0,789	0,450	0,198	0,132	0,053	0,0450
	S''	2,203	2,092	1,196	0,525	0,375	0,255	0,218
Coefficients de Pearson	γ_1	- 1,607	0,163	0,941	0,865	- 0,645	0,114	0,884
	γ_2	1,037	- 0,355	0,831	- 0,067	0,237	61,65	31,35
Variable de Stu. Fischer	t_α	2,225	2,839	2,839	2,839	2,967	4,275	4,275
$m - t_\alpha \cdot S$		29,275	20,950	12,191	9,090	6,651	4,479	2,809
$A_{1-\alpha}$		1,338	1,645	1,645	1,645	1,711	2,372	2,372
$S'_{in} = A_{1-\alpha} \cdot S$		2,102	1,885	1,077	0,474	0,335	0,211	0,180
La plus petite valeur expérimentale	x_e	28,6	22,4	13,05	9,5	6,95	4,76	3,07
La plus grande valeur expérimentale	x_u	35,4	26,00	15,40	10,40	7,53	4,98	3,24
Dispersion en %	δ	13,4	17,2	17,0	10,6	10,3	10,5	13,8

TABLEAU IV. — Résultats des calculs statistiques pour des barres à profil H en A S G

Élancements	λ	10	50	70	85	100	120	150
Nombre d'essais	N	26	10	10	10	10	5	5
Valeurs moyennes	$\bar{x} = m$	29,381	21,775	14,677	9,547	6,687	4,760	2,986
Limites de confiance de la moyenne	m'	29,103	20,840	14,364	9,423	6,586	4,586	2,959
	m''	29,659	22,710	14,990	9,671	6,788	4,934	3,013
Écart-types	S	0,688	1,307	0,438	0,173	0,141	0,140	0,022
Limite de confiance des écart-types	S'	0,539	0,899	0,302	0,119	0,036	0,034	0,010
	S''	0,95	2,386	0,799	0,316	0,257	0,402	0,062
Coefficients de Pearson	γ_1	- 0,982	- 0,235	0,366	- 1,575	1,374	2,037	—
	γ_2	- 0,465	- 0,695	0,408	6,441	4,021	2,684	—
Variable de Stu. Fischer	t_α	2,193	2,839	2,839	2,839	2,839	4,275	4,275
$m - t_\alpha \cdot S$		27,872	18,064	13,433	9,056	6,287	4,161	2,892
$A_{1-\alpha}$		1,308	1,645	1,645	1,645	1,645	2,372	2,372
$S'_{in} = A_{1-\alpha} \cdot S$		0,900	2,15	0,720	0,284	0,232	0,332	0,052
La plus petite valeur expérimentale	x_e	28,00	19,65	14,00	9,15	6,50	4,64	2,95
La plus grande valeur expérimentale	x_u	30,60	23,70	15,50	9,82	7,00	5,00	3,02
Dispersion en %	δ	6,5	21,9	10,9	7,8	7,7	16,8	—

TABLEAU V. — Résultats des calculs statistiques pour des barres à profil H en A - S G M

Élancements	λ	10	50	70	85			
Nombre d'essais	N	9	10	10	9			
Valeurs moyennes	$\bar{x} = m$	32,111	24,586	15,186	10,037			
Limites de confiance de la moyenne	m'	31,879	23,363	14,347	9,683			
	m''	32,879	25,809	16,025	10,390			
Écart-types	S	0,302	1,710	1,173	0,460			
Limite de confiance des écart-types	S'	0,204	1,176	0,807	0,311			
	S''	0,578	3,122	2,141	0,881			
Coefficients de Pearson	γ_1	- 0,043	0,815	0,951	1,707			
	γ_2	- 0,089	- 0,904	0,323	1,727			
Variable de Stu. Fischer	t_α	2,967	2,839	2,839	2,967			
$m - t_\alpha \cdot S$		31,215	19,731	11,856	8,672			
$A_{1-\alpha}$		0,517	2,813	1,929	0,787			
$S'_{tn} = A_{1-\alpha} \cdot S$		0,517	2,813	1,929	0,787			
La plus petite valeur expérimentale	x_e	31,6	22,4	13,7	9,55			
La plus grande valeur expérimentale	x_u	32,6	27,1	17,54	10,97			
Dispersion en %	δ	1,8	12,7	14	8,8			

TABLEAU VI. — Résultats des calculs statistiques pour des barres en A G 5

Élancements	λ	10	50	70	85	100	120	150
Nombre d'essais	N	27	8	8	8	8	5	5
Valeurs moyennes	$\bar{x} = m$	12,00	9,580	7,810	7,040	6,229	4,850	3,070
Limites de confiance de la moyenne	m'	11,871	9,396	7,504	6,747	6,038	4,803	2,983
	m''	12,129	9,764	8,116	7,333	6,420	4,897	3,156
Écart-types	S	0,326	0,220	0,366	0,351	0,229	0,038	0,089
Limites de confiance des écart-types	S'	0,257	0,145	0,242	0,231	0,151	0,022	0,053
	S''	0,447	0,448	0,745	0,714	0,466	0,108	0,255
Coefficients de Pearson	γ_1	- 1,239	0,189	- 1,318	- 0,211	- 0,183	—	- 0,947
	γ_2	- 0,704	0,234	1,013	- 0,424	0,388	—	30,35
Variable de Stu. Fischer	t_α	2,178	3,136	3,136	3,136	3,136	4,275	4,275
$m - t_\alpha \cdot S$		11,29	8,89	6,66	5,94	5,51	4,69	2,69
$A_{1-\alpha}$		1,300	1,797	1,797	1,797	1,797	2,373	2,372
$S' = A_{1-\alpha} \cdot S$		0,424	0,393	0,658	0,631	0,411	0,091	0,211
La plus petite valeur expérimentale	x_e	11,3	9,29	7,09	6,51	5,9	4,81	2,94
La plus grande valeur expérimentale	x_u	12,5	9,90	8,27	7,54	6,56	4,90	3,16
Dispersion en %	δ	7,5	10,2	19,1	20,3	15	12	12

TABLEAU VII. — Comparaison sommaire des moyennes de contraintes en compression et en traction

		A-Z5G	A-U4G	A-SG	A-G5
Compression	N_A	48	44	51	23
	S_A	1,739	1,687	0,665	0,906
	m_A	32,642	36,176	29,374	15,987
Traction	N_B	39	38	41	27
	S_B	1,743	1,187	0,944	0,326
	m_B	31,47	32,384	28,956	12,000
S_p		1,741	1,477	0,801	0,659
Valeur appr. de v_f		83	78	71	27
$t'_{0,975}$		1,989	1,991	1,996	2,052
$m_A - m_B$		1,171	3,792	0,418	3,987
u^*		0,439	0,635	0,348	0,408

* $u = t_1 - \alpha S_p \sqrt{\frac{1}{N_A} + \frac{1}{N_B}}$ ou $u = t'_1 - \alpha \sqrt{\frac{S_A^2}{N_A} + \frac{S_B^2}{N_B}}$ Pour tous les alliages $m_A - m_B > u$.

TABLEAU VIII. — Résultats des calculs statistiques à l'éclatement nul sur l'A - G 5

		Compression			Traction		
Contraintes limites de		$\sigma_{0,02}$	$\sigma_{0,2}$	$\sigma_{rup.}$	$\sigma_{0,02}$	$\sigma_{0,2}$	$\sigma_{rup.}$
Nombre d'essais	N	27	27	28	23	23	23
Valeur minimale	x_u	7,30	11,30	18,50	12,00	14,50	30,90
Valeur maximale	x_e	8,80	12,50	21,10	16,00	17,90	33,90
Valeur moyenne	m	7,963	12,00	19,964	13,930	15,987	31,783
Écart-type	S	0,373	0,326	0,612	0,866	0,906	0,718
	γ_1	0,103	- 1,239	- 0,314	0,397	2,142	3,598
	γ_2	0,344	- 0,704	0,029	1,110	1,756	3,613
	t_α	2,612	2,612	2,595	2,697	2,697	2,697
	$m - t_\alpha S$	6,990	11,149	18,376	11,000	13,543	29,847

TABLEAU IX. — Résultats des calculs statistiques à l'éclatement nul sur l'A - S G

		Compression			Traction		
Contraintes limites de		$\sigma_{0,02}$	$\sigma_{0,2}$	$\sigma_{rup.}$	$\sigma_{0,02}$	$\sigma_{0,2}$	$\sigma_{rup.}$
Nombre d'essais	N	51	51	51	42	41	40
Valeur minimale	x_u	23,50	28,0	28,4	26,5	27,3	30,9
Valeur maximale	x_e	27,4	30,6	30,8	30,5	32,9	45,8
Valeur moyenne	m	25,47	29,37	29,68	28,10	28,96	32,26
Écart-type	S	0,924	0,665	0,623	0,878	0,944	2,293
	γ_1	- 1,217	- 1,401	- 0,912	1,575	4,312	14,654
	γ_2	0,292	- 0,810	- 0,924	0,910	9,145	45,334
	t_α	2,379	2,379	2,379	2,438	2,445	—
	$m - t_\alpha S$	23,27	27,79	28,20	25,95	26,65	—

TABLEAU X. — 1^{re} et 2^e livraison (A - U 4 G)

		Compression			Traction		
Contraintes limites de		$\sigma_{0,02}$	$\sigma_{0,2}$	$\sigma_{rup.}$	$\sigma_{0,02}$	$\sigma_{0,2}$	$\sigma_{rup.}$
Nombre d'essais	N	34	34	35	28	28	28
Valeur minimale	x_u	22,0	27,0	35,3	30,1	31,7	42,5
Valeur maximale	x_e	20,7	35,9	38,9	38,0	39,6	53,0
Valeur moyenne	m	26,74	33,02	37,15	35,64	37,17	50,61
Écart-type	S	2,142	1,745	0,992	2,249	2,393	2,572
	γ_1	- 1,496	- 3,902*	- 0,932	- 2,339	- 1,853	- 4,243*
	γ_2	- 0,268	4,664*	- 0,831	0,255	0,259	3,592*
	t_α	2,513	2,513	2,501	2,595	2,595	2,595
	$m - t_\alpha S$	21,36	28,64	34,66	29,80	30,96	43,94*

TABLEAU XI. — 3^e livraison (A - U 4 G)

		Compression			Traction		
Contraintes limites de		$\sigma_{0,02}$	$\sigma_{0,2}$	$\sigma_{rup.}$	$\sigma_{0,02}$	$\sigma_{0,2}$	$\sigma_{rup.}$
Nombre d'essais	N	10	10	10	10	10	10
Valeur minimale	x_u	15,0	27,4	36,1	27,5	30,3	43,5
Valeur maximale	x_e	23,7	31,2	38,1	32,9	35,3	47,0
Valeur moyenne	m	19,08	30,22	36,99	29,68	33,39	45,43
Écart-type	S	2,738	1,094	0,671	1,674	1,645	1,091
	γ_1	- 0,036	- 3,107*	0,398	1,606	- 1,247	- 0,927
	γ_2	- 0,593	- 4,206*	- 0,821	0,530	- 0,045	0,054
	t_α	3,532	3,532	3,532	3,532	3,532	3,532
	$m - t_\alpha S$	9,67	26,36*	34,62	23,77	27,58	41,58

TABLEAU XII. — Résultats des calculs statistiques à l'éclatement nul sur l'A - Z5G

		Compression			Traction		
Contraintes limites de		$\sigma_{0,02}$	$\sigma_{0,2}$	$\sigma_{rup.}$	$\sigma_{0,02}$	$\sigma_{0,2}$	$\sigma_{rup.}$
Nombre d'essais	N	48	48	48	39	39	39
Valeur minimale	x_u	25,00	26,6	29,2	22,7	25,8	32,7
Valeur maximale	x_e	32,8	35,4	35,8	31,1	34,2	40,1
Valeur moyenne	m	28,746	32,642	33,548	27,931	31,471	36,302
Écart-type	S	1,628	1,739	1,419	1,973	1,743	1,991
	γ_1	0,123	- 3,740	- 3,154	- 2,590	- 4,501	- 2,642
	γ_2	0,502	3,799	2,202	1,144	5,279	2,937
	t_α	2,385	2,385	2,385	2,472	2,472	2,472
	$m - t_\alpha S$	24,863	28,495*	30,164*	23,05	27,16*	31,38

(*) La distribution de la série de mesures n'est pas symétrique.

7. INTERPRETATION DES RESULTATS PAR LA METHODE DUTHEIL

Si l'on soumet une barre prismatique pratiquement droite à un effort de compression centré constamment croissant, appliqué à ses extrémités articulées, on constate qu'elle se courbe dès le début du chargement (à condition d'utiliser des appareils de mesure suffisamment sensibles). La relation empirique de la flèche au milieu de la barre soumise à l'effort de compression simple, aussi bien centré que possible, est définie d'après Dutheil par :

$$f = a \frac{\sigma_k}{\sigma_k - (1+b)\sigma} (1+b)\sigma \frac{I}{vN_k} = \frac{a(1+b)\sigma}{\sigma_k - (1+b)\sigma} \frac{I}{Av} \quad (12)$$

où :

$$\sigma_k = \frac{\pi^2 E}{\lambda^2} \quad \text{et} \quad N_k = A\sigma_k$$

a et b étant les paramètres empiriques (déterminés à partir des résultats statistiques de la charge d'affaissement probable).

Cette flèche provoque un moment fléchissant qui a pour valeur :

$$M = Nf = A\sigma f = \frac{a(1+b)\sigma^2}{\sigma_k - (1+b)\sigma} \frac{I}{v} \quad (13)$$

La contrainte maximale a donc pour expression :

$$\sigma_{max} = \frac{N}{A} + \frac{Mv}{I} = \sigma + \frac{a(1+b)\sigma^2}{\sigma_k - (1+b)\sigma} = \sigma \left[\frac{\sigma_k - (1+b)(1-a)\sigma}{\sigma_k - (1+b)\sigma} \right] \quad (14)$$

La valeur de la contrainte limite d'affaissement σ_s pour laquelle $\sigma_{max} = \sigma_e$ satisfait alors à :

$$\sigma_s \left[\frac{\sigma_k - (1+b)(1-a)\sigma_s}{\sigma_k - (1+b)\sigma_s} \right] = \sigma_e \quad (15)$$

d'où :

$$\sigma_s^2 (1+b)(1-a) - \sigma_s [\sigma_k + \sigma_e (1+b)] + \sigma_e \sigma_k = 0 \quad (16)$$

ou encore :

$$\sigma_s = \frac{\sigma_k + (1+b)\sigma_e}{2(1+b)(1-a)} - \sqrt{\left[\frac{\sigma_k + (1+b)\sigma_e}{2(1+b)(1-a)} \right]^2 - \frac{\sigma_e \sigma_k}{(1+b)(1-a)}} \quad (17)$$

Ainsi nous avons introduit trois paramètres a , b , et σ_e qui se déterminent à l'aide des résultats expérimentaux.

7.1. Détermination de a , b et σ_e

Si l'on considère trois valeurs de l'élancement λ_1 , λ_2 et λ_3 auxquelles correspondent les contraintes critiques d'Euler σ_{k1} , σ_{k2} , σ_{k3} et trois valeurs expérimentales σ_{s1} , σ_{s2} , σ_{s3} de la contrainte d'affaissement, on peut écrire en vertu de l'équation (16) le système de trois équations suivant :

$$\left. \begin{aligned} \sigma_{s1}^2 (1+b)(1-a) - \sigma_{s1} (1+b)\sigma_e + \sigma_{k1}\sigma_e &= \sigma_{s1}\sigma_{k1} \\ \sigma_{s2}^2 (1+b)(1-a) - \sigma_{s2} (1+b)\sigma_e + \sigma_{k2}\sigma_e &= \sigma_{s2}\sigma_{k2} \\ \sigma_{s3}^2 (1+b)(1-a) - \sigma_{s3} (1+b)\sigma_e + \sigma_{k3}\sigma_e &= \sigma_{s3}\sigma_{k3} \end{aligned} \right\} \quad (18)$$

qui permet de calculer les trois inconnues a , b , et σ_e .

Après avoir résolu le système de trois équations (18), on obtient :

$$\sigma_e = \frac{\sigma_{s1}(\sigma_{k2} - \sigma_{k1}) + \sigma_{s2}(\sigma_{k3} - \sigma_{k2}) + \sigma_{s3}(\sigma_{k1} - \sigma_{k3})}{\sigma_{s1}\left(\frac{\sigma_{k2}}{\sigma_{s2}} - \frac{\sigma_{k3}}{\sigma_{s3}}\right) + \sigma_{s2}\left(\frac{\sigma_{k3}}{\sigma_{s3}} - \frac{\sigma_{k1}}{\sigma_{s1}}\right) + \sigma_{s3}\left(\frac{\sigma_{k1}}{\sigma_{s1}} - \frac{\sigma_{k2}}{\sigma_{s2}}\right)} \quad (19)$$

$$(1+b) = \frac{\sigma_{s1}\left[\sigma_{k2}\left(\frac{\sigma_k}{\sigma_s}\right)_3 - \sigma_{k3}\left(\frac{\sigma_k}{\sigma_s}\right)_2\right] + \sigma_{s2}\left[\sigma_{k3}\left(\frac{\sigma_k}{\sigma_s}\right)_1 - \sigma_{k1}\left(\frac{\sigma_k}{\sigma_s}\right)_3\right] + \sigma_{s3}\left[\sigma_{k1}\left(\frac{\sigma_k}{\sigma_s}\right)_2 - \sigma_{k2}\left(\frac{\sigma_k}{\sigma_s}\right)_1\right]}{\sigma_{s1}(\sigma_{k2} - \sigma_{k3}) + \sigma_{s2}(\sigma_{k3} - \sigma_{k1}) + \sigma_{s3}(\sigma_{k1} - \sigma_{k2})} \quad (20)$$

$$(1+b)(1-a) = \frac{\sigma_{k1}\left(\frac{\sigma_{k2}}{\sigma_{s2}} - \frac{\sigma_{k3}}{\sigma_{s3}}\right) + \sigma_{k2}\left(\frac{\sigma_{k3}}{\sigma_{s3}} - \frac{\sigma_{k1}}{\sigma_{s1}}\right) + \sigma_{k3}\left(\frac{\sigma_{k1}}{\sigma_{s1}} - \frac{\sigma_{k2}}{\sigma_{s2}}\right)}{\sigma_{s1}\left(\frac{\sigma_{k2}}{\sigma_{s2}} - \frac{\sigma_{k3}}{\sigma_{s3}}\right) + \sigma_{s2}\left(\frac{\sigma_{k3}}{\sigma_{s3}} - \frac{\sigma_{k1}}{\sigma_{s1}}\right) + \sigma_{s3}\left(\frac{\sigma_{k1}}{\sigma_{s1}} - \frac{\sigma_{k2}}{\sigma_{s2}}\right)} \quad (21)$$

Dans le tableau ci-dessous nous avons rassemblé les valeurs de a , b , et σ_e pour chaque alliage d'aluminium dans cet article,

TABLEAU

	A-U4G	A-SG	AS-GM	A-Z5G	A-G5
a	0,0954	0,2605	0,2091	0,1174	0,8885
b	0,2346	0,0370	0,252	0,0378	- 0,1545
σ_e	27,12	28,11	31,53	29,15	11,24
E	7400	7000	7000	7200	7000

8. CONCLUSIONS

8.1. Essais sur matériaux

A l'examen des tableaux 8, 9, 10, 11 et 12 on constate que :

- l'alliage A-U4G présente, pour les différentes fusions, des écarts considérables aussi bien en traction qu'en compression ;
- l'alliage A-SG ne manifeste pas d'anomalie appréciable, mais qu'il possède, contrairement à l'alliage précédent, une limite d'élasticité de compression légèrement supérieure à celle de traction.

8.2. Essais de flambement et comparaison avec la courbe Dutheil

Les coefficients de Pearson, insérés dans les tableaux 2, 3, 4, 5 et 6, nous permettent de relever quelques élancements pour lesquels la série de mesures de contrainte d'affaissement n'a pas de répartition normale. Dans le souci de faciliter la détermination de la contrainte probable d'affaissement, on a utilisé une même relation, c'est-à-dire $X = m - t_{\alpha} s_{\alpha}$ où t_{α} varie avec le nombre de mesures à chaque élancement.

Comme le montrent les figures 1 à 5, la courbe théorique de chaque alliage ne dépasse presque en aucun cas la limite des contraintes d'affaissement ; ces courbes peuvent donc servir de base au calcul de la résistance au flambement simple des barres en alliage d'aluminium à traitement thermique, compte tenu du facteur de sécurité de la barre.

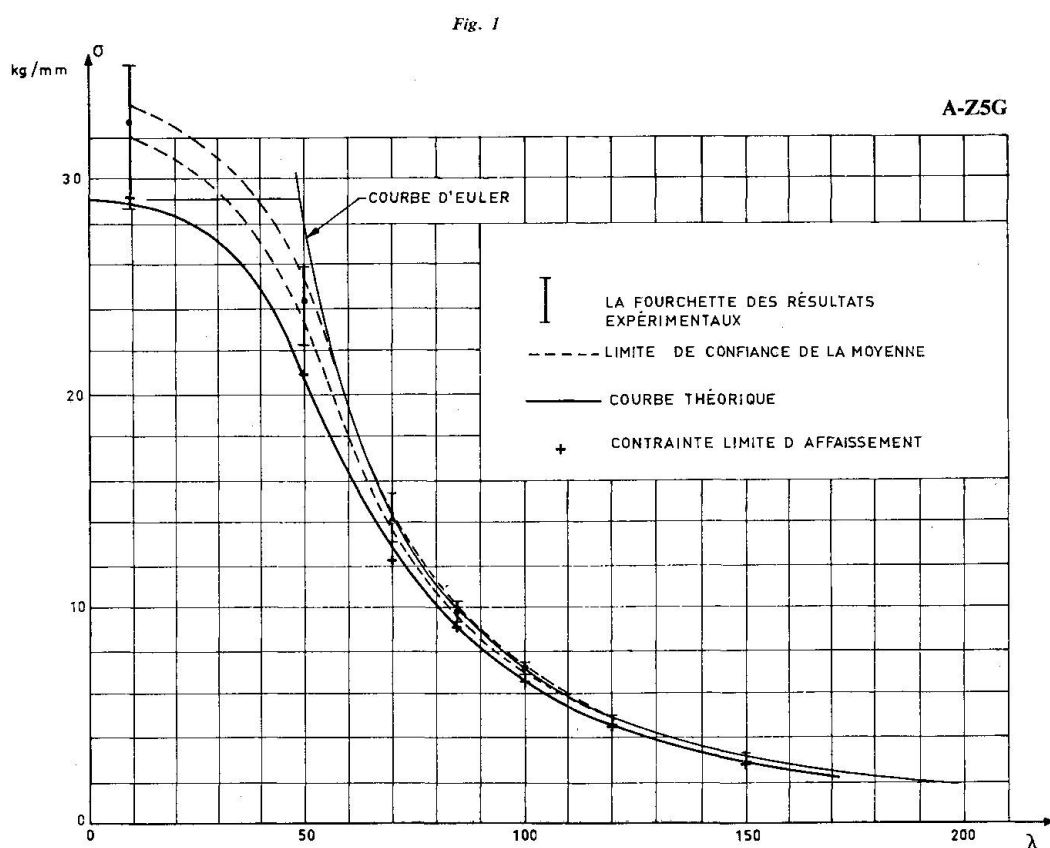
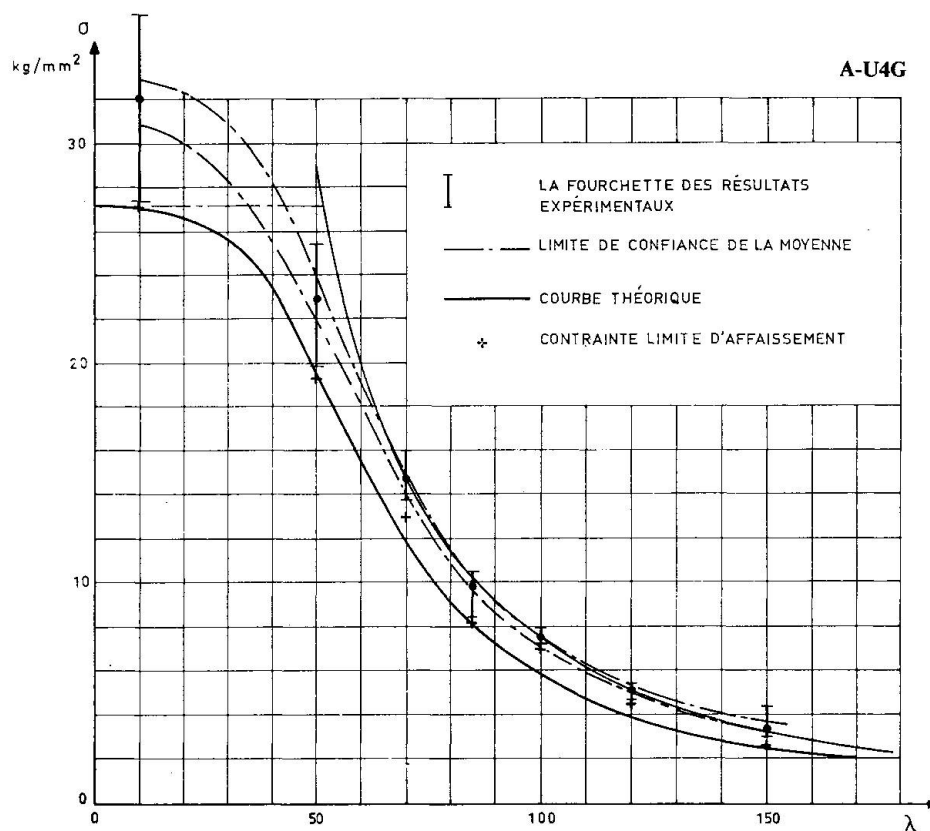
Sur la figure 6, on voit que les courbes non-dimensionnelles de flambement pour les alliages à traitement thermique, plus précisément A-U4G, A-SG, A-SGM et A-Z5G, sont proches les unes des autres.

Afin de simplifier le calcul, H. Djalaly a été amené à proposer une courbe moyenne (représentée par des points sur la figure 6) ayant pour expression :

$$\bar{\sigma} = \frac{0,53993}{\bar{\lambda}^2} + 0,59041 - \sqrt{\left[\frac{0,53993}{\bar{\lambda}^2} + 0,59041\right]^2 - \frac{1,07987}{\bar{\lambda}^2}}$$

$\bar{\sigma}$ et $\bar{\lambda}$ étant les valeurs non-dimensionnelles de la contrainte et de l'élancement.

Il est intéressant de remarquer que la loi de comportement de l'alliage A-G5 (alliage sans traitement thermique) présente un écrouissage important, par conséquent la courbe de flambement eulérienne ne peut répondre à la résistance ultime des barres en A-G5 pour des barres relativement élancées. Cela est parfaitement visible sur la figure 5. Bien entendu, en ce qui concerne les barres élancées la courbe eulérienne est valable pour les matériaux dont le phénomène d'écrouissage n'est pas considérable, notamment pour le cas des alliages d'aluminium à traitement thermique.



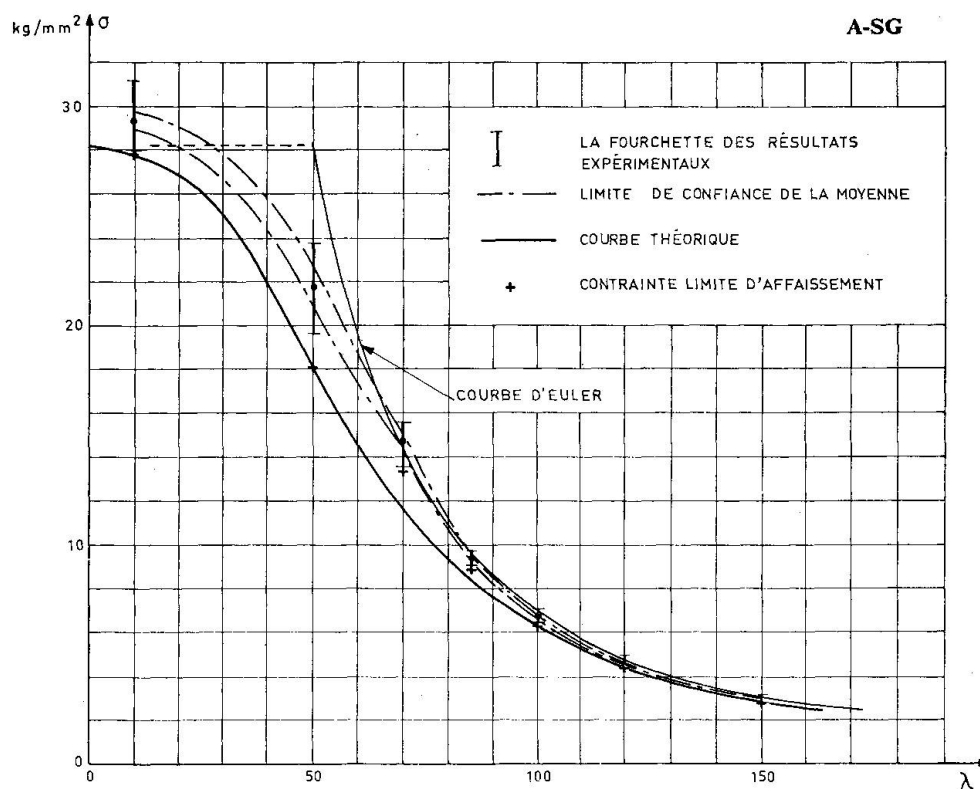


Fig. 3

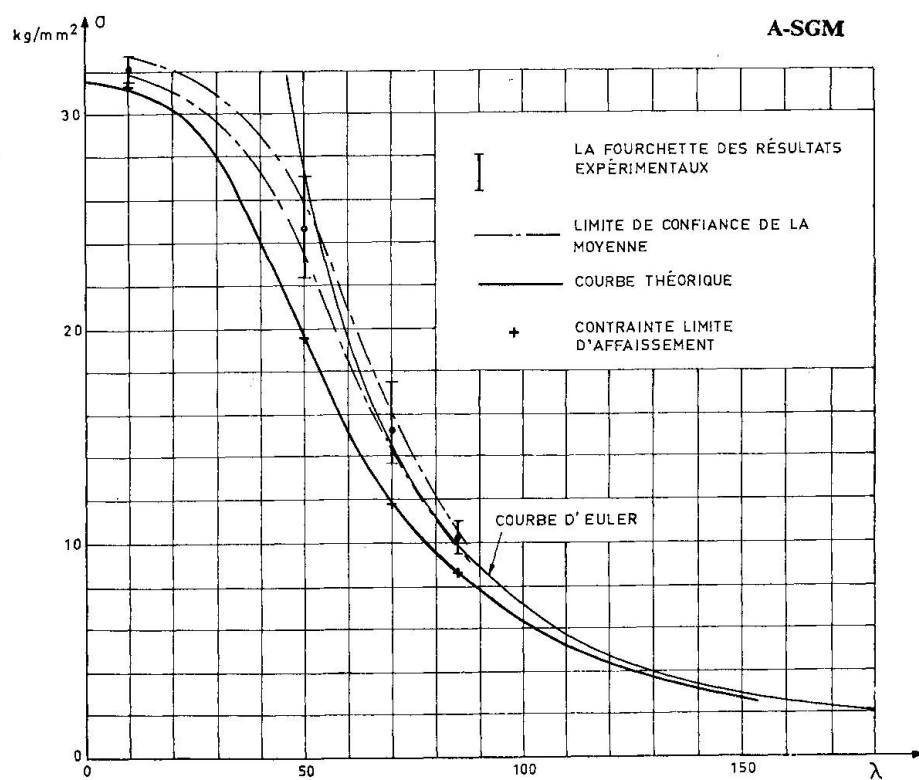


Fig. 4

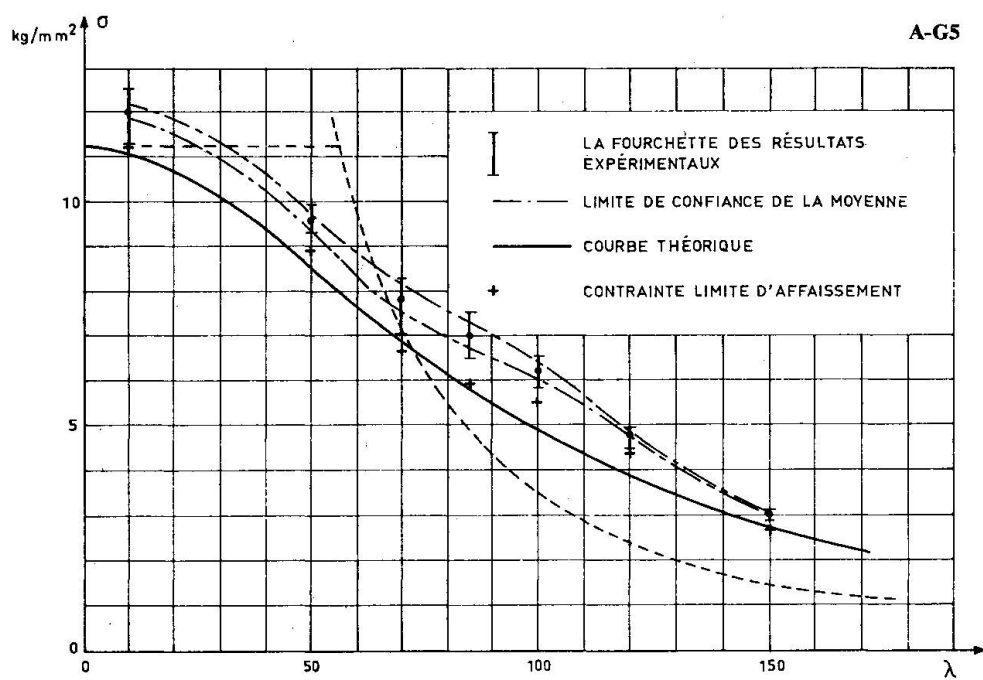


Fig. 5

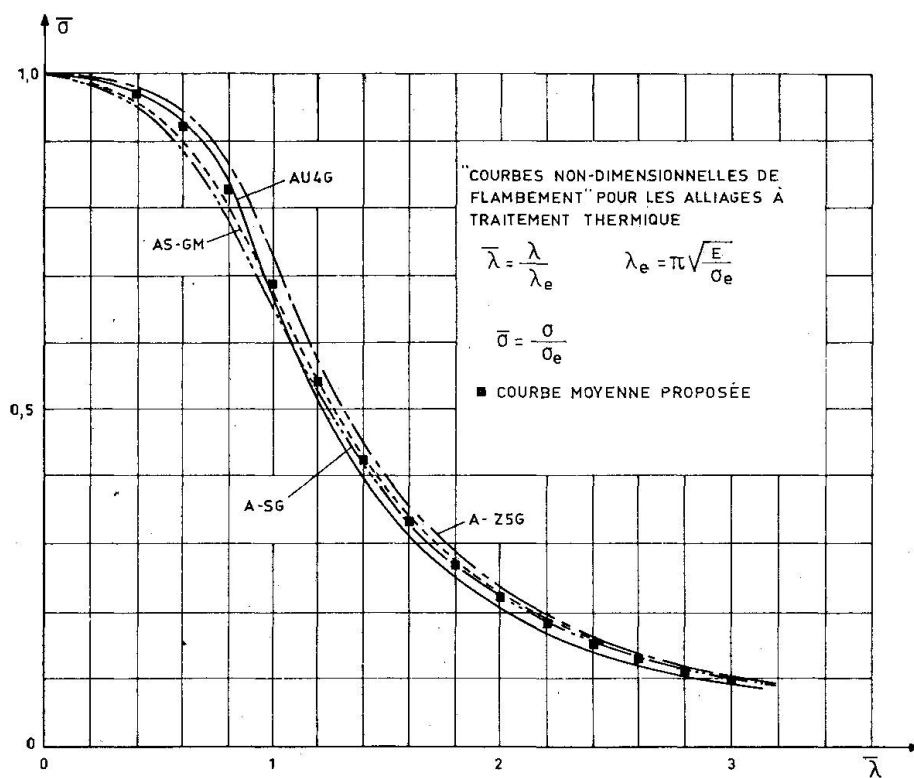


Fig. 6

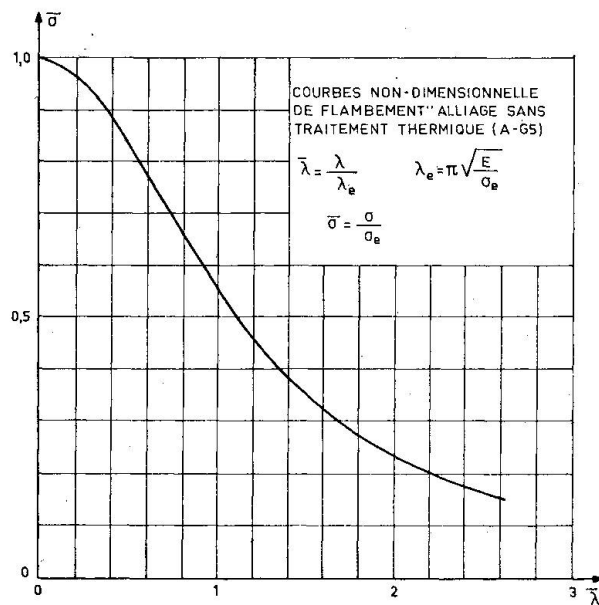


Fig. 7

La partie expérimentale de cette recherche a été effectuée avec l'appui de l'Aluminium Français, au Laboratoire du C.E.B.T.P. par MM. Dauphin et Texier.

M. P. Arnault, Ingénieur au Service Recherches du C.T.I.C.M., était chargé de cette recherche et a effectué la première exploitation des résultats, sous la direction de D. Sfintesco, à l'usage des règles de conception et de calcul des charpentes en alliages d'aluminium.

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