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**Autor:** De Donato, O. / Franchi, A. / Gioda, G.

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## Finite Element Elastoplastic Analysis of Underground Openings by Quadratic Programming

*Analyse élastoplastique avec éléments finis d'ouvertures souterraines  
par programmation quadratique*

*Elastoplastische Berechnung von Tunneln durch ein finite Elemente,  
quadratische programmierung Verfahren*

O. De DONATO, A. FRANCHI, G. GIODA

Technical University (Politecnico), Milan, Italy

### Summary

A finite element approach to the analysis of soils and underground openings is presented allowing for material non linearities under the assumption of elastic plastic "associated" constitutive laws. The procedure is based on the piecewise-linearization of the yield condition and on the subdivision of the loading history in finite loading steps with modification of the constitutive laws from step to step in order to take into account the irreversibility of plastic deformations and "local unloading" phenomena. For a generic step the elastoplastic analysis is then formulated as a quadratic programming problem whose solution is obtained by a gradient technique suitably modified in order to handle large size problems.

As an application, the stress, strain (plastic and total) and displacement distribution around an underground opening is solved for various loading situations, under the assumption of the extended Mohr-Coulomb yield condition.

### Resumé

Une étude utilisant les éléments finis est présentée, en vue de l'analyse de sols et d'ouvertures souterraines, qui permet de tenir compte des non-linéarités du matériau sous l'hypothèse de lois constitutives élasto-plastiques "associées". La méthode est basée sur la linéarisation de la condition de plasticité et sur la subdivision de l'histoire de charge en échelons de charge finis avec modification des lois constitutives pour tenir compte de l'irréversibilité des déformations plastiques et du phénomène de "décharge locale".

A chaque échelon de charge l'analyse élastoplastique est alors formulée comme un problème de programmation quadratique où la solution est obtenue grâce à une technique au gradient modifiée de manière appropriée pour permettre

de traiter des problèmes avec grand nombre de variables.

Come application, la distribution des contraintes des déformations (plastiques et totales) et des déplacements au voisinage d'une cavité souterraine est résolue pour différent états de charge et en respectant la conditions de plasticité généralisée de Mohr-Coulomb.

### Zusammenfassung

Es wird ein finite Elemente-Verfahren für Tunnel vorgestellt, mit der Voraussetzung dass die Materialeigenschaft nichtlinear, elasto-plastisch ist und das Normalitätsgesetz gehorcht wird.

Die Methode beruht auf einer stück-weisen Linearisierung der Fließbedingungen und in einer Unterteilung der Belastungsverfahren in endlichen Schritten. Dabei werden die Materialgesetze von Schritt zu Schritt unter Berücksichtigung des nicht umkehrbaren Verhaltens von plastischen Verformungen und lokalen Entladung angepasst.

Für eine allgemeingültige Betrachtung wird dann die elasto-plastische Analyse als ein quadratisches Programmierungsproblem dargestellt, dessen Lösung durch eine Gradiententechnik in einer geeigneter Weise modifiziert um so Probleme höhere Abmessungen lösen zu können, erhalten ist.

Als Anwendungsbeispiel wird die Verteilung von Spannung, Verformung (plastischer Verformung und Gesamtverformung) und Verschiebung um ein Tunnel für verschiedene Belastungsfälle betrachtet, wobei eine verallgemeinerte Mohr-Coulomb Fließbedingung angenommen wird.

### 1.- Introduction

It has been recognized for many years that there are many phenomena in soil mechanics which cannot be adequately described by the elasticity theory.

Permanent deformations, path-dependency, non linear and dilatative behavior, work softening are some of them. Among the numerous constitutive models so far proposed (see the overview [1]) the elastoplastic model combined with the incremental theory of plasticity is one of the most frequently and generally accepted, although the important question regarding the appropriateness of plasticity theory [2] and the validity of constitutive laws to predict stress, strain and displacement fields in soil and rock media is still open. In fact, the adoption of yield criteria dependent on the hydrostatic component of stresses (such as Mohr-Coulomb, or the extended Mohr-Coulomb criterion proposed by Drucker and Prager [3]) combined with the normality principle, imply [3] [4] a plastic dilatation which, when unconstrained plastic flow (i.e. elastic strains are negligible with respect the plastic ones), increases exponentially with the size of the plastic zone [2]; this may require the use of non associated flow rules for unrestricted plastic flow [5][6]. However practical engineering problems are often of contained plastic deformations (i.e. elastic and plastic strain components of the same order) with a total volumetric expansion which may be positive negative or zero, and then the traditional plasticity theory may provide a reasonable estimate in a statistical sense of stress, strain and displacement fields [2].

With the above constitutive model many practical problems have been solved.

The numerical solution was mainly derived with an extensive use of the finite element method by adopting any of the well known non linear analysis formulations already developed for structural and continuum mechanics (e.g. the incremental method, initial stress or initial strain method, ecc.). A comprehensive review of the finite element analyses carried out with elastic

plastic models has been presented by Desai et al. [1].

However only a few contributions have taken into account path-dependency behavior in soil mechanics. It is one of the more significant aspects related either to the non-holonomic (irreversible) character of constitutive laws or to the history of loading and unloading due to sequential construction. Among the above non linear finite element procedures, the incremental method appears, and was in fact frequently used, the more suitable. However, many circumstances can make its application cumbersome in passing from rates to finite (though small) increments (as required in practice to follow the evolution of the system under a given loading history). Some of them are: accumulative violations of the yield conditions and of equilibrium, or, when piecewise linear yield surfaces are considered, the exceedingly high number of steps which may be required for structures with many elements and many yielding modes [7][8].

It would be more advantageous to make a preliminary subdivision of the loading history into finite steps independent in number from the structure discretization model, according to the path-dependency of the behavior for constitutive laws and sequential construction.

This is achieved by a consistent approach previously proposed in [8] for the analysis of elastic-plastic structures subjected to general non proportional loading paths, and here applied to soil mechanics. The procedure involves piecewise linearization of the yield surface and holonomic (i.e. reversible) finite flow laws (i.e. in the spirit of "deformation" plasticity theory) [9] but the irreversible (path-dependent) nature of plastic flow is fully allowed for by a modification of the constitutive law in passing from one finite step to another. Then, the analysis problem for each loading step is reduced into a quadratic programming problem whose solution is obtained by a modified gradient technique suitably devised [7] for handling large size finite element problems.

The method is applied to the stress, strain and displacement analysis around an underground opening, under the final and some intermediate particular loading situations which arise during the construction of the opening.

Notation: underlined symbols denote matrices and column vectors, 0 is a matrix or vector whose entries are zero. A tilde  $\sim$  means transpose, a dot time derivative. A vector inequality applies to each pair of corresponding components.

## 2.- Constitutive laws and analytical description

2.1.- Constitutive model and assumptions. Consider the problem of stress, strain and displacement field determination around an underground opening under given dead and live loads taking into account the non linear and path-dependent nature of the materials (i.e. soil and concrete tunnel covering). The usual assumption of an elastic-perfectly plastic constitutive model for the soil is here made, according an already widely accepted opinion in the literature [2] of its capability to describe soil behavior beyond the elastic limit, without postulating any particular mechanism of permanent deformation (such as recrystallization, dislocation movements, intergranular slip, or microfracturing).

Reference will be made in what follows to the extended Mohr-Coulomb yield condition (proposed by Drucker and Prager [3][4]), which is defined by the equation:

$$\phi = \alpha J_1 + J_2^{\frac{1}{2}} = k \quad (1)$$



where  $\alpha$  and  $k$  are positive constants and  $J_1$  is the first stress invariant and  $J_2$  is the second invariant of the stress deviation. In the principal stress space for  $\alpha=0$  the surface (1) defines the right circular cylinder of the Mises criterion, while for  $\alpha>0$  the surface is a right circular cone with its axis equally inclined to the coordinate axis and with apex in the tension octant.

A relevant consequence of the normality rule is the presence of both the effects of the volumetric and deviatoric stress components, i.e. the plastic strain rate  $\dot{\epsilon}_{ij}^p$  is given by:

$$\dot{\epsilon}_{ij}^p = \dot{\lambda} \frac{\partial \phi}{\partial \sigma_{ij}} = \dot{\lambda} [ \alpha \delta_{ij} + s_{ij}/2J_2^{1/2} ] \quad (2)$$

where  $\dot{\lambda}$  is a non negative factor of proportionality (plastic multiplier),  $\delta_{ij}$  is the Kronecker delta and  $s_{ij}$  the stress deviation.

From (2) follows:

$$\dot{\epsilon}_1^p + \dot{\epsilon}_2^p + \dot{\epsilon}_3^p = 3\alpha\dot{\lambda} \quad (3)$$

i.e. for  $\alpha \neq 0$  plastic deformation must be accompanied by an increase in volume (dilatancy property).

In the case of plain strain, as in the problem in hand, it was shown by Drucker and Prager [3] that, if  $c$  represents cohesion and  $\phi$  the friction angle, and assuming

$$3\alpha^2 = \frac{\sin^2 \phi}{3 + \sin^2 \phi} ; \quad k^2 = \frac{3c^2 \cos^2 \phi}{3 + \sin^2 \phi} \quad (4)$$

(1) reduces to the Mohr-Coulomb criterion, i.e. in the  $\sigma_1, \sigma_2$  stress plane, to the curve:

$$\sigma_1 - \sigma_2 = \pm [ 2c \cos \phi + (\sigma_1 + \sigma_2) \sin \phi ] \quad (5)$$

and, in the  $\sigma_x, \sigma_y, \tau_{xy}$  stress space, to the surface:

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = [ 2c \cos \phi + (\sigma_x + \sigma_y) \sin \phi ]^2 \quad (6)$$

The surface (6) is an elliptic cone (fig. 1a) with its axis lying in the  $\sigma_x, \sigma_y$  plane and equally inclined to the axis  $\sigma_x, \sigma_y$  and apex  $V \equiv [ -c/\tan \phi; -c/\tan \phi; 0 ]$ . In what follows the coefficients  $c, \phi$  and the elastic properties will be considered time independent (viscous effects ignored).†

The Mohr-Coulomb yield criterion (6) with small tensile strength ( $c \neq 0$ ) and elastic-perfectly plastic behavior are lastly assumed also for the concrete covering of the tunnel. Both these assumptions, previously made in [10] gave good results, according to the experimental evidence, in recent works [10] [11] dealing with the bearing capacity of concrete blocks (or rock) even if, as discussed by Chen and Drucker [10], the perfectly plastic idealization of concrete is a questionable point so far as its application to limit analysis theorems is concerned (limited ductility, brittleness and falling stress-strain curve after maximum strength are in fact usually observed). In particular in [10] and [11] the tensile strength assumption, adopted by modifying the Mohr-Coulomb condition with a small tension cut-off, was found to play an essential role in the ability to carry loads. Viscous effects will be ignored.

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† Cohesionless soils ( $c=0$ ) are a typical example where viscous effects generally can be ignored.

2.2.- Analytical description. The present description refers to a piecewise linearization of the yield surface (6) in  $r$  planes, see e.g. fig. 1b, according to the desired degree of approximation. Let  $\underline{\varepsilon} = \{ \varepsilon_x, \varepsilon_y, \gamma_{xy} \}$  and  $\underline{\sigma} = \{ \sigma_x, \sigma_y, \tau_{xy} \}$  be the strain and stress vectors relevant to a generic point; denoting with  $\underline{N} = [ \underline{N}_1, \dots, \underline{N}_j, \dots, \underline{N}_r ]$  the  $(3 \times r)$ -matrix of the outward vectors  $\underline{N}_j$  normal to each of the  $r$  planes, the incremental constitutive laws for the ideal plasticity case can be written in the form [8][9]:

$$\underline{\varepsilon} = \underline{\varepsilon}^e + \underline{\varepsilon}^p = \underline{S}^{-1} \underline{\sigma} + \underline{N} \underline{\lambda} \quad (7)$$

$$\underline{\phi} = \underline{\tilde{N}} \underline{\sigma} - \underline{k} ; \quad \underline{\phi} \leq \underline{0} \quad (8)$$

$$\underline{\lambda} \geq \underline{0} , \quad \underline{\tilde{\phi}} \underline{\lambda} = 0 , \quad \underline{\gamma} \underline{\lambda} = 0 \quad (9)$$

where the  $r$ -vectors  $\underline{\phi}$ ,  $\underline{\lambda}$ ,  $\underline{k}$  collect plastic potentials, plastic multiplier rates, and distances from the origin relevant to each of the  $r$  planes and  $\underline{S}$  is the  $(3 \times 3)$ -plain strain elasticity matrix.

Equations (8) define the polyhedral yield surface (in this case a pyramid) approximate to the original yield condition; eqs. (9) represent the flow rules; the first together with (7) expresses the outward normality rule of the plastic strain rate vector, the second selects as potentially activable those yield planes containing the stress point, the third give as mutually exclusive the events of plastic flow ( $\underline{\lambda}_j > 0$ ) and local unloading ( $\underline{\phi}_j < 0$ ). The path-dependent (non-holonomic) relations (7) to (9) allow the strain history to be followed whatever stress path is given; however, for any stress path which rules out loading-unloading sequence, the above incremental laws can be substituted by the corresponding holonomic (reversible) laws in finite terms [8], i.e. by the eqs. (7), (8) together with the new relations:

$$\underline{\lambda} \geq \underline{0} , \quad \underline{\tilde{\phi}} \underline{\lambda} = 0 \quad (10)$$

Referring now to the structure, it may be said that typical loading paths that exclude loading-unloading sequences (or that if present are immaterial), generally are those for which the loads increase all together proportionally (i.e. characterized by a family of loads governed by one parameter monotonically increasing). Referring in such a case to the stress path  $0 \rightarrow \underline{\sigma}$  the above finite constitutive laws fully apply and give the same description of the non holonomic (irreversible) actual law. It may be seen furthermore [8] that the above finite constitutive laws apply even when subsequent loading stages with different families of one parameter loads are considered, i.e. in cases in which important loading-unloading phenomena from stage to stage easily appear. This is possible only by a suitable adjustment of the constitutive laws passing from one stage to the next so as to allow unloading of stress points from activated planes in the previous stage. Considering in fact a new finite proportional path which brings the stresses from the level  $\underline{\sigma}$  to  $\underline{\sigma} + \Delta \underline{\sigma}$  (i.e. ruling out yielding-unloading sequences within this path, but not unloading with respect to yield planes activated in the previous path  $0 \rightarrow \underline{\sigma}$ ) the relation between  $\Delta \underline{\varepsilon}$  and  $\Delta \underline{\varepsilon}^p$  may be derived [8] in the following form:

$$\Delta \underline{\varepsilon} = \underline{S}^{-1} \Delta \underline{\sigma} + \underline{N} \Delta \underline{\lambda} \quad (11)$$

$$\Delta \underline{\phi} = \underline{\tilde{N}} \Delta \underline{\sigma} ; \quad \Delta \underline{\phi} \leq -\underline{\phi} \quad (12)$$

$$\Delta \underline{\lambda} \geq \underline{0} ; \quad \Delta \underline{\tilde{\phi}} \Delta \underline{\lambda} = -\underline{\tilde{\phi}} \Delta \underline{\lambda} \quad (13)$$

where  $\phi$  denotes the plastic potential at the level  $\sigma$ . Eqs. (11) to (13) are fully equivalent to the corresponding (7), (8), (10) of the preceding path, the only difference being the presence of the total potential  $\phi$  at the end of the preceding stress path. In conclusion any non proportional loading history can be studied by making use of the above finite laws, provided that a preliminary subdivision in finite, individually proportional, loading stages is made. This procedure will be adopted for the problem in hand.

### 3.- Mathematical programming formulation of the analysis problem

3.1.- Loading stage subdivision. The more significant loading conditions of the ground around the opening can be summarised as follows: A) initial situation before the opening, in which the dead and live loads  $X$ ,  $q$  are the only external actions to be considered; B) ground consolidation by injections and then modification by  $\Delta X$  of the dead loads around the opening in order to improve the material properties (cohesion and friction angle) before the excavation; C) opening, i.e. in loading terms, the vanishing of the boundary forces  $F$  present, in the original situation A), around the contour line of the tunnel; D) centring of the excavation in order to reduce any additional ground movement after situation B); E) concrete casting along the contour line of the opening and application of the corresponding additional dead load. In view of the subdivision of the loading history into finite individually proportional loading stages, some of the above loading conditions can be condensed in only one. In fact, although their concomitance or succession in time greatly depends on the technology adopted, the rapidity by which in practice some opening phases usually follow one another must also be considered. This is the case in conditions C), D), E) which frequently can be considered as simultaneous and then as a unique one-parameter loading stage. On the other hand the slight (compared to the pre existing) additional dead loads  $\Delta X$  of condition B) do not generally involve substantial modifications of the initial state of stress and strain or local unloading phenomena; then it can be considered as a part of the preceding or of the following loading condition. Therefore the loading history, until the completion of the opening, can be split into the following two subsequent stages:

- I) a first loading stage defined by dead and live loads before making the opening (state A);
- II) a second loading stage, defined by the sum of the remaining load conditions, i.e. B), C), D), E), which leads directly, with a finite increment, to the final values of the loads.

Then the problem of an underground opening transforms into that of the analysis of the first loading stage and of the response due to the additional loads of the second. The stress, strains distributions will be given by the sum of the increments relevant to each loading stage; the displacements involved by the opening will be given by the increment from the end of the first stage to the final situation.

3.2.- First loading stage. Consider a discretization of the problem in  $m$  triangular constant strain elements and  $n$  nodes. Denote with  $\underline{\sigma}_I$  and  $\underline{\epsilon}_I$  the  $(3 \times n)$ -vectors of the stresses and strains in all the elements at the end of the application of the  $(2 \times n)$ -dead and live load vectors  $X$  and  $q$ ; then the equilibrium and compatibility equations can be expressed in the form:

$$\underline{\sigma}_I = \underline{\sigma}_I^e(q) + \underline{\sigma}_I^e(X) + \underline{\sigma}_I^S = \underline{\sigma}_I^e(q, X) + \underline{\sigma}_I^S \quad (14)$$

$$\underline{\varepsilon}_I = \underline{\varepsilon}_I^{e(q)} + \underline{\varepsilon}_I^{e(X)} + \underline{\varepsilon}_I^{ep} + \underline{\varepsilon}_I^p = \underline{\varepsilon}_I^{e(q,X)} + \underline{\varepsilon}_I^{ep} + \underline{\varepsilon}_I^p \quad (15)$$

where: suffix e means elastic response of the system to the loads indicated in brackets and index I means " at the end of the first loading stage";  $\underline{\sigma}_I^S$  is a selfequilibrated stress distribution corresponding in elastic range to the unknown plastic strains  $\underline{\varepsilon}_I^p$ , i.e.:

$$\underline{\sigma}_I^S = \underline{Z} \underline{\varepsilon}_I^p \quad (16)$$

$\underline{Z}$  being the  $(3m \times 3m)$ -symmetric, negative semidefinite matrix which transforms plastic strains into elastic stresses;  $\underline{\varepsilon}_I^{ep}$  is the elastic strain field compatible with  $\underline{\varepsilon}_I^p$  and corresponding in elastic range to  $\underline{\sigma}_I^S$ . On the other hand denoting with  $\underline{\phi}$ ,  $\underline{\lambda}$ ,  $\underline{k}$  the  $(r, X, m)$ -vectors that collect all the corresponding element r-vectors  $\underline{\phi}_I^r$ ,  $\underline{\lambda}_I^r$ ,  $\underline{k}_I^r$ , and with  $\underline{N} = \text{diag } |\underline{N}^i|$  ( $i=1, \dots, m$ ), then the constitutive stress-strain laws (7) (8) (10) become, now for the whole system:

$$\underline{\varepsilon}_I^p = \underline{N} \underline{\lambda}_I \quad (17)$$

$$\underline{\phi}_I = \underline{N} \underline{\sigma}_I - \underline{k} \quad ; \quad \underline{\phi}_I \leq \underline{0} \quad (18)$$

$$\underline{\lambda}_I \geq \underline{0} \quad ; \quad \underline{\phi}_I \underline{\lambda}_I = \underline{0} \quad (19)$$

Eqs. (14) to (19) is the complete set of the equations governing the elastoplastic analysis problem of the first loading stage. By substitution of (14) into (18) taking into account (16), eqs. (18) (19) transforms into a set of relationships which can be recognised, through the Kuhn-Tucker theorem of mathematical programming, as fully equivalent to the following quadratic programming problem [8] [9]:

$$(QP1) \quad \max \{ \frac{1}{2} \underline{\lambda}_I^T \underline{N} \underline{Z} \underline{N} \underline{\lambda}_I + \underline{\lambda}_I^T (\underline{N} \underline{\sigma}_I^{e(X,q)} - \underline{k}) \mid \underline{\lambda}_I \geq \underline{0} \} \quad (20)$$

in the unknown vector  $\underline{\lambda}_I$ . The semidefiniteness in sign of the  $\underline{Z}$  matrix cannot guarantee in general the existence and uniqueness of the solution vector for any given loads  $X$  and  $q$ ; lack of existence would imply a load distribution greater than the carrying capacity of the structural system. From the vector solution  $\underline{\lambda}_I$  (see sect. 4 about the adopted optimization algorithm) any other information on the distribution of stresses and strains can be easily obtained by eqs. (14) to (17). The interaction forces  $F$  across the contour line of the opening can be obtained by overlapping and suitably arranging the element nodal forces ●

$$\underline{F}^i = \int_{V^i} \underline{B}^i \underline{\sigma}_I^i dv^i \quad (21)$$

where  $i$  refers to the boundary elements around the opening in the final situation. The nodal displacements will be finally given by:

$$\underline{u}_I = \underline{u}^e + \underline{u}^s \quad (22)$$

where  $\underline{u}^e$  and  $\underline{u}^s$  correspond to the elastic  $\underline{\varepsilon}_I^{e(X,q)}$  and plastic  $\underline{\varepsilon}_I^{ep} + \underline{\varepsilon}_I^p$  strain fields respectively.

(22) is the displacement field from which the incremental displacement field due to the opening has to be evaluated.

3.3.- Second loading stage. The finite increments  $\Delta \underline{\sigma}$ ,  $\Delta \underline{\varepsilon}$ ,  $\Delta \underline{\lambda}$  etc. correspon-

ding to the incremental loads of the second stage are here determined. Denoting with index II the quantities at the end of the second loading stage, stresses, strains etc. at the completion of opening will be given by:

$$\begin{aligned}\underline{\sigma}_{II} &= \underline{\sigma}_I + \Delta \underline{\sigma} \\ \underline{\varepsilon}_{II} &= \underline{\varepsilon}_I + \Delta \underline{\varepsilon} \\ \underline{\lambda}_{II} &= \underline{\lambda}_I + \Delta \underline{\lambda}\end{aligned}\quad (23)$$

Again regular progression (i.e. absence of loading-unloading sequence within the stage) is assumed, but local unloading with respect to yield planes activated in the first loading stage are fully allowed for. This means that constitutive laws of the type of (11) to (13) hold and eqs. (17) to (19) now modify into the new relations:

$$\Delta \underline{\varepsilon}^p = \underline{N} \Delta \underline{\lambda} \quad (24)$$

$$\Delta \underline{\phi} = \underline{\tilde{N}} \Delta \underline{\sigma} \quad ; \quad (\underline{\phi}_I + \Delta \underline{\phi}) \leq 0 \quad (25)$$

$$\Delta \underline{\lambda} \geq 0 \quad ; \quad (\underline{\phi}_I + \Delta \underline{\phi}) \Delta \underline{\lambda} = 0 \quad (26)$$

where  $\underline{\phi}_I$  represents the plastic potential vector at the end of the first stage; the equilibrium and compatibility equations now read:

$$\Delta \underline{\sigma} = \Delta \underline{\sigma}^{e(\Delta \underline{X}, -\underline{F})} + \Delta \underline{\sigma}^s \quad (27)$$

$$\Delta \underline{\varepsilon} = \Delta \underline{\varepsilon}^{e(\Delta \underline{X}, -\underline{F})} + \Delta \underline{\varepsilon}^{ep} + \Delta \underline{\varepsilon}^p \quad (28)$$

with the same meaning of the symbols as in sect. 3.2 ( $\Delta \underline{X}$  are the increments of dead loads in the consolidated region and where concrete is cast).

Again by algebraic manipulations and using the matrix  $\underline{Z}$  defined by (16), the set of governing equations (24) to (28) of the problem can be seen fully equivalent to the following new quadratic programming problem [8]:

$$(QP2) \quad \max \{ \frac{1}{2} \Delta \underline{\lambda}^T \underline{\tilde{N}} \underline{Z} \underline{N} \Delta \underline{\lambda} + \Delta \underline{\lambda}^T (\underline{\tilde{N}} \Delta \underline{\sigma}^{e(\Delta \underline{X}, -\underline{F})} + \underline{\phi}_I) / \Delta \underline{\lambda} \geq 0 \} \quad (29)$$

in the unknown vector  $\Delta \underline{\lambda}$ . The same remarks as in sect. 3.2 on the existence and uniqueness of the solution and on the derivation of all other information on the corresponding stress and strain fields, still hold, taking into account also eqs. (23). However the nodal displacements  $\underline{u}$  due to the opening will be:

$$\underline{u} = \Delta \underline{u}^e + \Delta \underline{u}^s \quad (30)$$

since now  $\Delta \underline{u}^e$  and  $\Delta \underline{u}^s$  correspond to the elastic  $\Delta \underline{\varepsilon}^{e(\Delta \underline{X}, -\underline{F})}$  and plastic  $\Delta \underline{\varepsilon}^{ep} + \Delta \underline{\varepsilon}^p$  strain fields respectively.

#### 4.- Solution algorithm and computational remarks

The essential remarks on which the adopted algorithm to solve the quadratic programming problems QP1 or QP2 is based, can be summarised in the following points:

a) the non vanishing components of the unknown vector  $\underline{\lambda}$  correspond to the planes of the linearized yield condition which will be activated at the solution.

This means that the real size of QP1 or QP2 problem is in general very much reduced with respect to the total number ( $r \times m$ ) of components of the vector  $\underline{\lambda}$  as not all finite elements are in general in the plastic range at the solution. It is not known "a priori" what particular  $\hat{\underline{\lambda}}$  subvector of  $\underline{\lambda}$  will be involved in the problem because generally it is difficult to forecast exactly the finite elements which will be plastic and, even worse, for each of them which yield planes will be activated.

A procedure [7] which resolves a preliminary "reduced problem" on the basis of a prevision of the possibly activable yield planes, and then checks the non positivity of all plastic potentials is easy to conceive, and has in fact been used. If some plastic potential is positive, a new reduced problem larger in size than the preceding one and including the violated yielding modes must be considered, and so on until no plastic potential is positive; in that case the solution is achieved. A simple choice of the subset  $\hat{\underline{\lambda}}_1$  for the first reduced problem may be suggested by the indices of the violated yielding modes in the linear elastic solution;

b) in calculating, handling and storing the matrix  $\hat{\underline{N}} \underline{Z} \underline{N}$  according to any classical quadratic programming algorithms some computational difficulties arise due to its size ( $r \times m$ ,  $r \times m$ ). It is possible however to overcome all these drawbacks when some gradient optimization technique is adopted by virtue of a particular mechanical meaning that was found attributable to the gradient of the objective functions QP1 or QP2. Referring, e.g., to the first stage analysis problem, in fact, the gradient, apart from constant terms with respect to  $\underline{\lambda}$ , becomes (by virtue of (16)):

$$\underline{g}(\underline{\lambda}) = \hat{\underline{N}} \underline{Z} \underline{N} \underline{\lambda} = \hat{\underline{N}} \underline{\sigma}^s(\underline{\lambda}) \quad (31)$$

which is a suitable projection of the elastic selfequilibrated stresses due to the imposition of the plastic strain vector  $\underline{\epsilon}^p$  corresponding to  $\underline{\lambda}$ . In other words the gradient may be evaluated by an elastic calculation under given dislocations without using the computationally cumbersome matrix  $\hat{\underline{N}} \underline{Z} \underline{N}$ .

Starting from this interpretation any optimization climbing technique of the gradient family can be applied, noting that every iteration (i.e. transition from an approximate solution  $\underline{\lambda}^k$  to a better  $\underline{\lambda}^{k+1}$ ) will require an elastic analysis of the problem under imposed inelastic strains  $\underline{\epsilon}^k = \underline{N} \underline{\lambda}^k$ , as the transition is based on the gradient vector at  $\underline{\lambda}^k$ ;

c) among the gradient techniques, the conjugate gradient method is one of the most commonly used in quadratic programming because of its well known capacity to achieve the solution in a finite number of iterations (always less or equal the number of the problem variables). However this property is strictly related to the absence of constraints to the objective function. When constraints are present this is generally not true, except for the case in which the solution point is interior to the feasible domain of constraints. The modified conjugate method proposed in [7] for the present special case of variables only restricted in sign, transforms the constrained conjugate gradient method into a sequence of unconstrained optimizations in suitable subspace of the variables  $\underline{\lambda}_i$ , thus ensuring solution in a finite number of step. More details can be found in [7].

So an idea of the computing effort required in order to achieve the elastic plastic solution by the above procedure appears to be given by the size of the elastic problem instead of the number of variables  $\underline{\lambda}_i$ , as the plastic solution is obtained by a sequence of linear elastic problem analyses.

However the number of iterations was shown [7] to be quite sensitive to



the number of the non vanishing components in the solution vector, i.e. in mechanical terms to the extent of the plastic zone. For this reason no useful extension can be expected of the above procedure to limit analysis problems.

### 5.- Numerical example

The geometry and final loading state of the plain strain underground opening problem considered is given in fig. 2. In order to take into account the fact that the elastic properties depend on the stress state, the soil was subdivided into six layers with Young modulus increasing from the first to the last. Fig. 3 and 4 give the meshes and the elastic and plastic properties assumed for the two sequential loading stages considered in setc. 3. The regions VII and VIII of fig. 4 correspond to the injected zone and to the concrete covering respectively.

The Mohr-Coulomb condition (6) was piecewise linearized with a six lateral plane pyramid.

Three different analyses were carried out. Fig. 5a) gives the plastic zone found by the first (two stages) analysis at the completion of the opening.

From the second two analyses (still two stages each), fig. 5b) and 5c) give, for different values of  $c$  and  $\phi$  in the injected region, the corresponding extension which arises if, in the second loading stage, no concrete casting or centring is considered after the opening. Then fig. 5b) and 5c) represent transient situations which may arise in practice, depending on the technology adopted.

Surface displacements, ground stress fields and concrete covering stresses are plotted in fig. 6, 7, 8, 9 with reference to the first analysis problem considered (fig. 5a). In fig. 7 the surface displacements of the second two analyses of fig. 5b) and 5c) are superimposed (lines b and c respectively).

The dashed line of fig. 8 corresponds to the third analysis case of fig. 5c); the numerical values of the diagrams of fig. 9 refer to 1 m of depth.

The QP1 and QP2 problems were solved by the modified conjugate gradient technique of ref. [7] using the reduced problem procedure mentioned in sect. 4. The CPU time per iteration was 6.5, 8.5, 8.7 sec., for each of the three analysis problems considered, on a Univac 1106 computer. The number of iterations was 13, 50, 85 respectively.

Without using reductions in size, the first analysis (which was in 2394 variables) was carried out with 10 iterations in 20 sec. of CPU time per iteration.

### Conclusions

A quadratic programming approach has been presented for the analysis of soils with the assumption of elastic perfectly plastic behavior and of the extended Mohr-Coulomb yield condition.

Material and sequential construction path-dependency is fully allowed for a suitable modification of the piecewise linearized constitutive laws which is able to take into account unloading phenomena in passing from one to another of each of the finite loading stages in which the loading path is "a priori" subdivided. The solution is obtained by a gradient technique modified in order to allow for handling large size finite element problems.



### Captions of the figures

- Fig. 1- a) Mohr-Coulomb yield condition for plain strain problems in the  $\sigma_x$ ,  $\sigma_y$ ,  $\tau_{xy}$  stress space under normality assumption, b) piecewise linearization.
- Fig. 2- Geometrical and loading conditions of the opening.
- Fig. 3- Mesh and material properties in the first loading stage.
- Fig. 4- Mesh and material properties in the second loading stage.
- Fig. 5- Extension of the plastic zone for some typical loading conditions: a) at the completion of the opening, b) at the end of the second loading stage without concrete casting and centreing ( $c=2 \text{ kg/cm}^2$ ,  $\phi=40^\circ$ ), c) as in b) with  $c=1 \text{ kg/cm}^2$  and  $\phi=33^\circ$ .
- Fig. 6- Surface and tunnel displacements and principal stresses relevant to the analysis problem of fig. 5a.
- Fig. 7- Comparison of the surface displacements in the three loading conditions of fig. 5a, 5b, 5c.
- Fig. 8-  $\sigma_x$  and  $\sigma_y$  distribution along the axis of symmetry: the continuous line corresponds to the loading condition of fig. 5a, and the dashed line to the condition of fig. 5c.
- Fig. 9- Distribution around the concrete covering of the tunnel of the: a) nodal forces, b) hydrostatic pressure, c) shear stress, d) bending moment, e) axial force, f) shear force.

### Légende des figures

- Fig. 1- a) Critere de plasticité de Mohr-Coulomb dans l'espace des contraintes  $\sigma_x$ ,  $\sigma_y$ ,  $\tau_{xy}$  pour les problèmes de déformation plane suivant la condition de normalité, b) linearisation du critere de plasticité.
- Fig. 2- Conditions géométriques et de charge du tunnel.
- Fig. 3- Maille et propriétés du matériau pour le premier état de charge.
- Fig. 4- Maille et propriétés du matériau pour le second état de charge.
- Fig. 5- Extension de la zone plastifiée pour quelques typiques conditions de charge: a) en fin de réalisation du tunnel, b) en fin du second état de charge sans bétonnage et cintrage ( $c=2 \text{ kg/cm}^2$ ,  $\phi=40^\circ$ ), c) comme an b) avec  $c=1 \text{ kg/cm}^2$ ;  $\phi=33^\circ$ .
- Fig. 6- Déplacements en surface et du tunnel et contraintes principales en rapport au problème de la fig. 5a.

Fig. 7- Comparaison des déplacements en surface pour les trois conditions de charges des fig. 5a, 5b, 5c.

Fig. 8- Distribution de  $\sigma_x$  et de  $\sigma_y$  le long de l'axe de symétrie: la ligne en trait continu correspond à la condition de charge a (voir fig. 5) et la ligne en trait discontinu à la condition c.

Fig. 9- Distribution le long de l'anneau en béton du tunnel: a) des forces nodales, b) de la pression hydrostatique, c) de la contrainte tangentielle, d) du moment fléchissant, e) de la force axiale, f) de l'effort tranchant.

#### Untertitel der Abbildungen

Abb. 1- Mohr-Coulomb Fliessbedingung im  $\sigma_x$ ,  $\sigma_y$ ,  $\tau_{xy}$  Raum für Probleme der ebenen Verformung, b) stück-weise Linearisierung.

Abb. 2- Geometrische Charakteristiken und Belastungsbedingungen.

Abb. 3- Geometrische Masche und Materialcharakteristiken für die erste Belastungs Bedingung.

Abb. 4- Geometrische Masche und Materialcharakteristiken für die zweite Belastungs Stufe.

Abb. 5- Ausdehnung des plastischen Bereichs für die folgenden Belastungsbedingungen: a) nach der Follendung des Tunnelbaues, b) nach der zweiten Belastungsbedingung ohne Bedeckung mit Beton und ohne Bogenkrümmung ( $c=2 \text{ kg/cm}^2$  ;  $\phi=40^\circ$ ), c) wie in b) mit  $c=1 \text{ kg/cm}^2$  ;  $\phi=33^\circ$ .

Abb. 6- Verschiebungen der Bodenoberfläche und des Tunnels: Haupt-Spannungen in Beziehung zur Belastungsbedingung von Abb. 5a.

Abb. 7- Vergleich der Oberflächlichverschiebungen in den drei Belastungsbedingungen von Abb. 5a, 5b, 5c.

Abb. 8- Verteilung der Spannungen  $\sigma_x$ ,  $\sigma_y$  längs den Symmetrie-Axis; ununterbrochene Linie entspricht zur Belastungsbedingung von Abb. 5a und die unterbrochen Linie zur Bedingung von Abb. 5c.

Abb. 9- Verteilung längs der Tunnelbedeckung von: a) Knotenkräften, b) hydrostatischem Druck, c) Schubspannung, d) Biegemoment, e) axialer Kraft, f) Schubkraft.

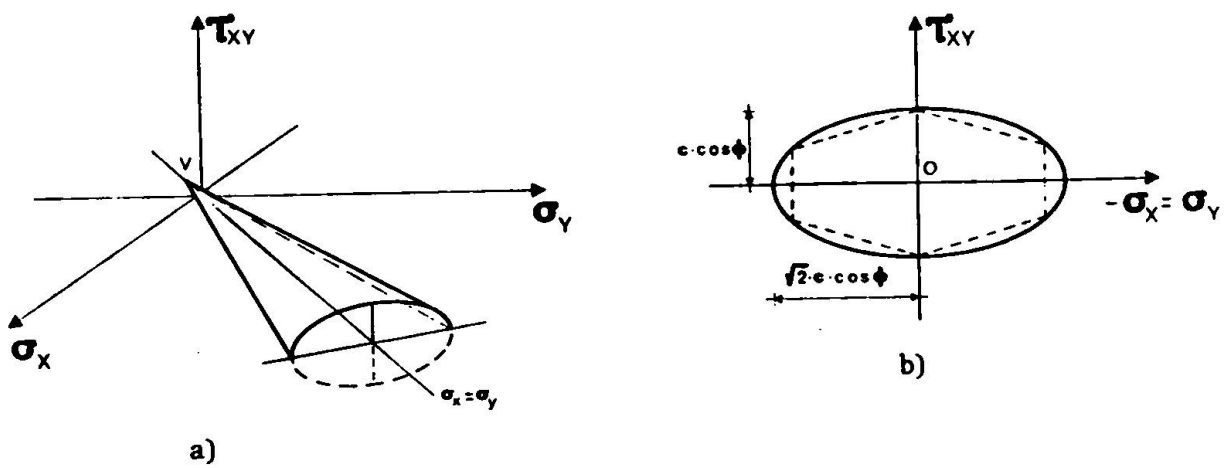


Fig. 1.

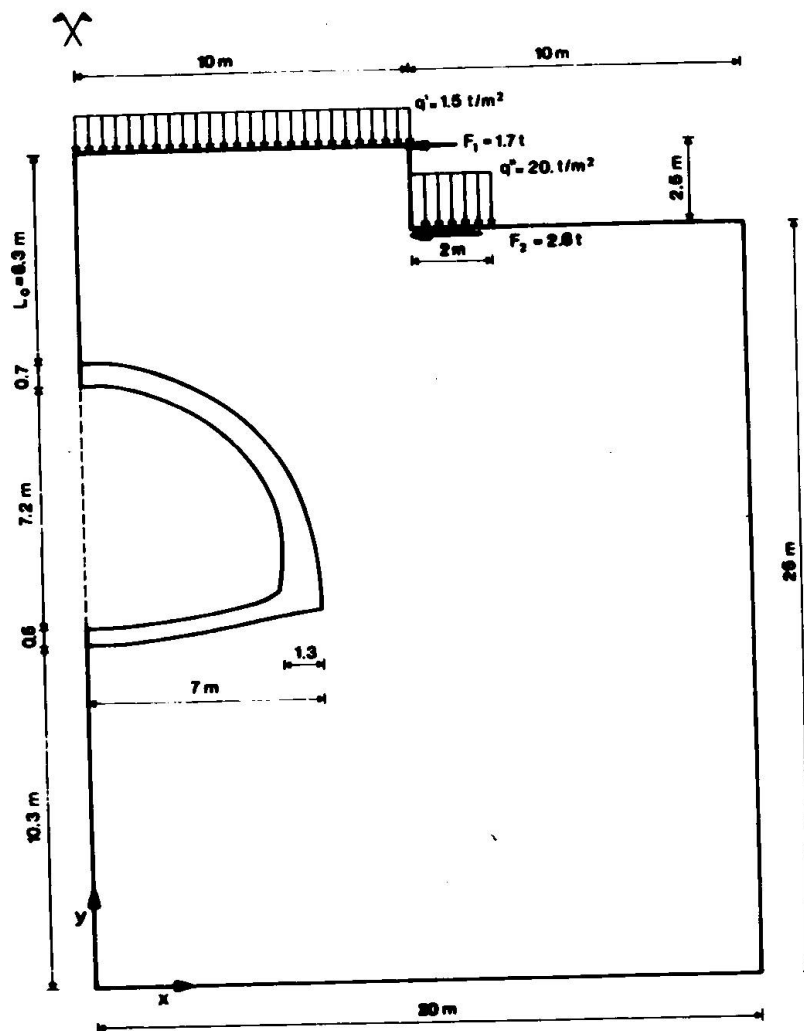
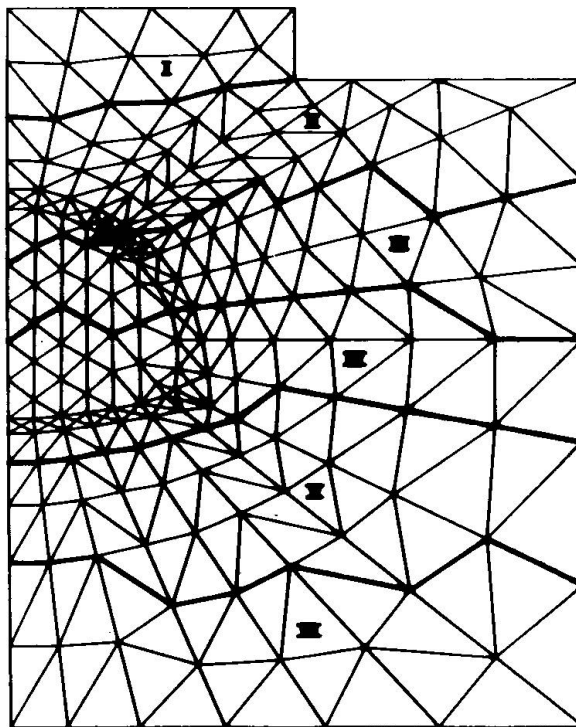


Fig. 2.

# MESH and MATERIAL PROPERTIES

(1<sup>st</sup> loading stage)



## MESH

No. Nodes = 219  
No. Elements = 399

## ELASTIC PROP.

$E_1 = 500 \text{ Kg/cm}^2$   
 $E_2 = 1450 \text{ Kg/cm}^2$   
 $\rho = 1900 \text{ Kg/m}^3$   
 $\nu = 0,35$

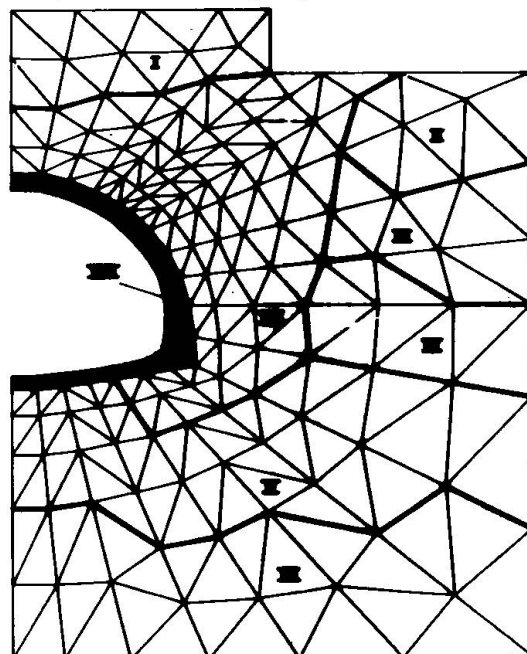
## PLASTIC PROP.

Mohr-Coulomb (6 planes)  
 $\sigma = 0 \text{ Kg/cm}^2$   
 $\phi = 33^\circ$

Fig. 3.

# MESH and MATERIAL PROPERTIES

(2<sup>nd</sup> loading stage)



## MESH

No. Nodes = 191  
No. Elements = 329

## ELASTIC PROP.

$E_1 \dots E_6$  as in 1<sup>st</sup> loading stage  
 $E_7 = 1000 \text{ Kg/cm}^2$   
 $E_8 = 250,000 \text{ Kg/cm}^2$   
 $\rho_1 \dots \rho_6$  as in 1<sup>st</sup> l. s.  
 $\rho_7 = 2150 \text{ Kg/m}^3$   
 $\rho_8 = 2500 \text{ Kg/m}^3$   
 $\nu_1 \dots \nu_6$  as in 1<sup>st</sup> l. s.  
 $\nu_7 = 0,30$   
 $\nu_8 = 0,15$

## PLASTIC PROP.

Mohr-Coulomb (6 planes)  
 $\sigma_1 \dots \sigma_6$  as in 1<sup>st</sup> l. s.  
 $\sigma_7 = 2 \text{ Kg/cm}^2$   
 $\sigma_8 = 52,5 \text{ Kg/cm}^2$   
 $\phi_1 \dots \phi_6$  as in 1<sup>st</sup> l. s.  
 $\phi_7 = 40^\circ$   
 $\phi_8 = 51^\circ$

Fig. 4.

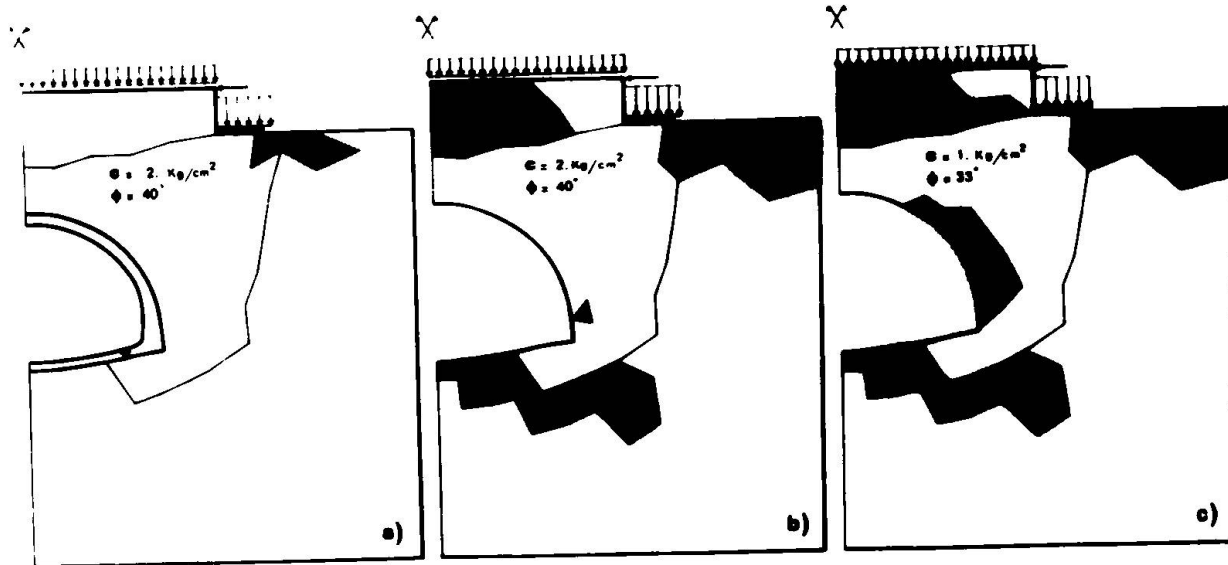
PLASTIC ZONES

Fig. 5.

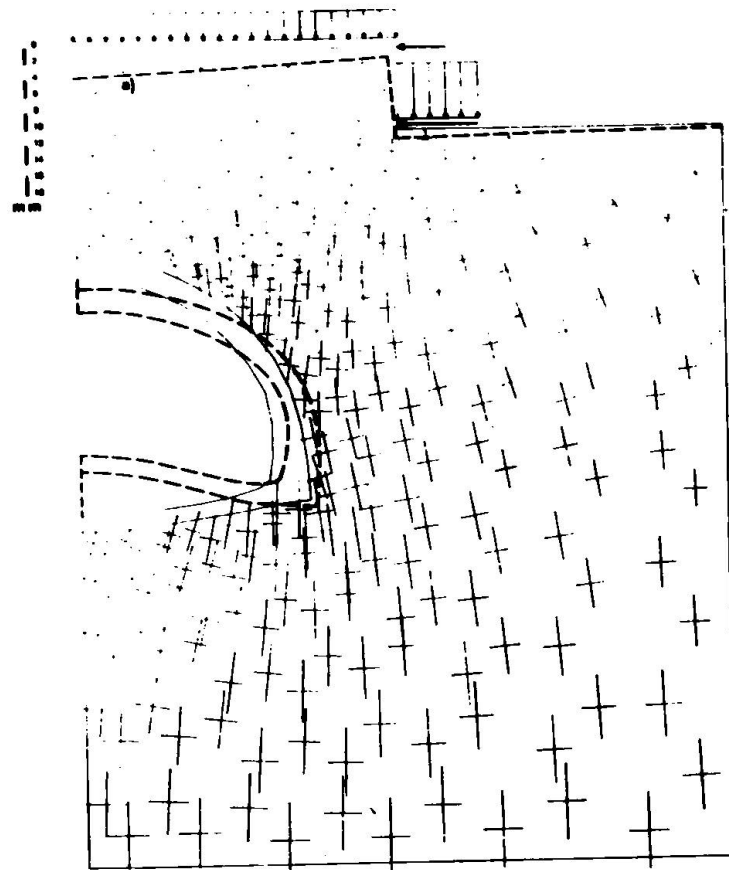


Fig. 6.

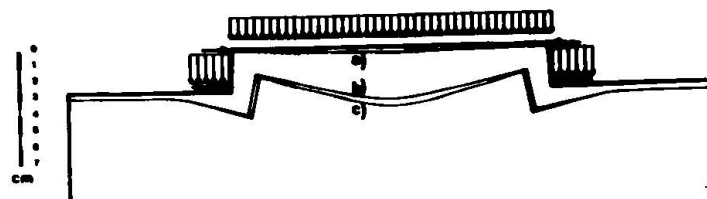


Fig. 7.

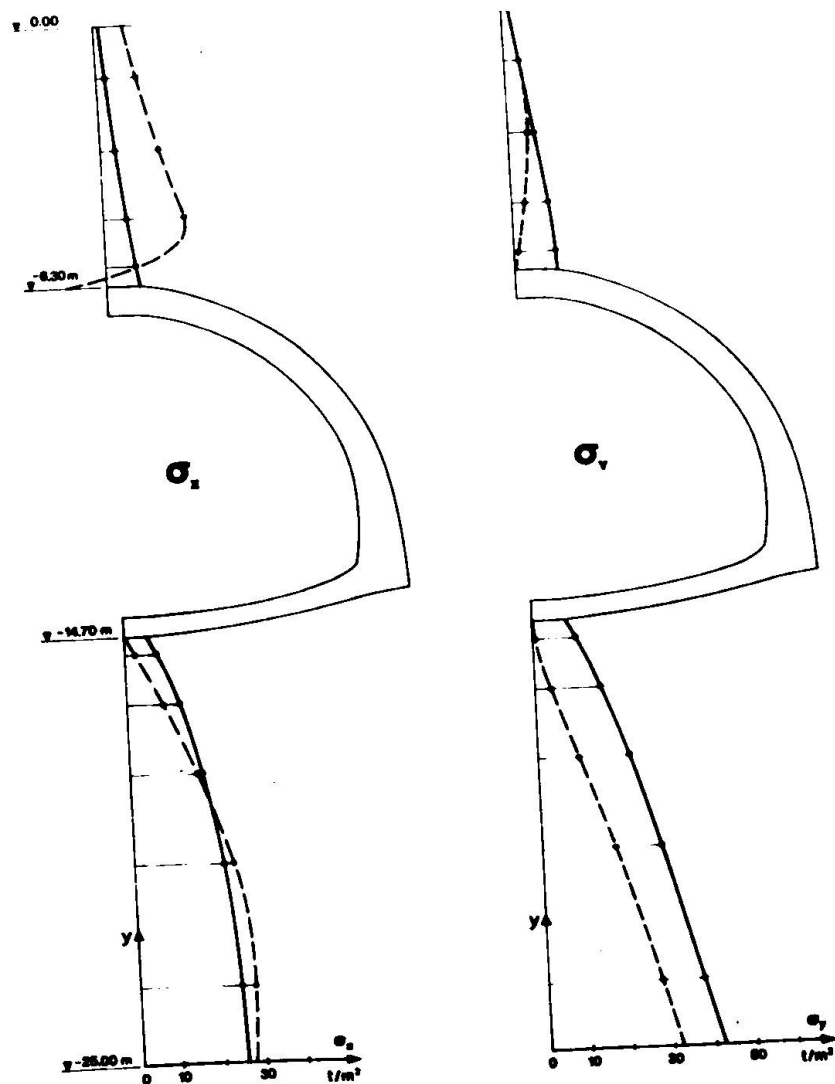


Fig. 8.

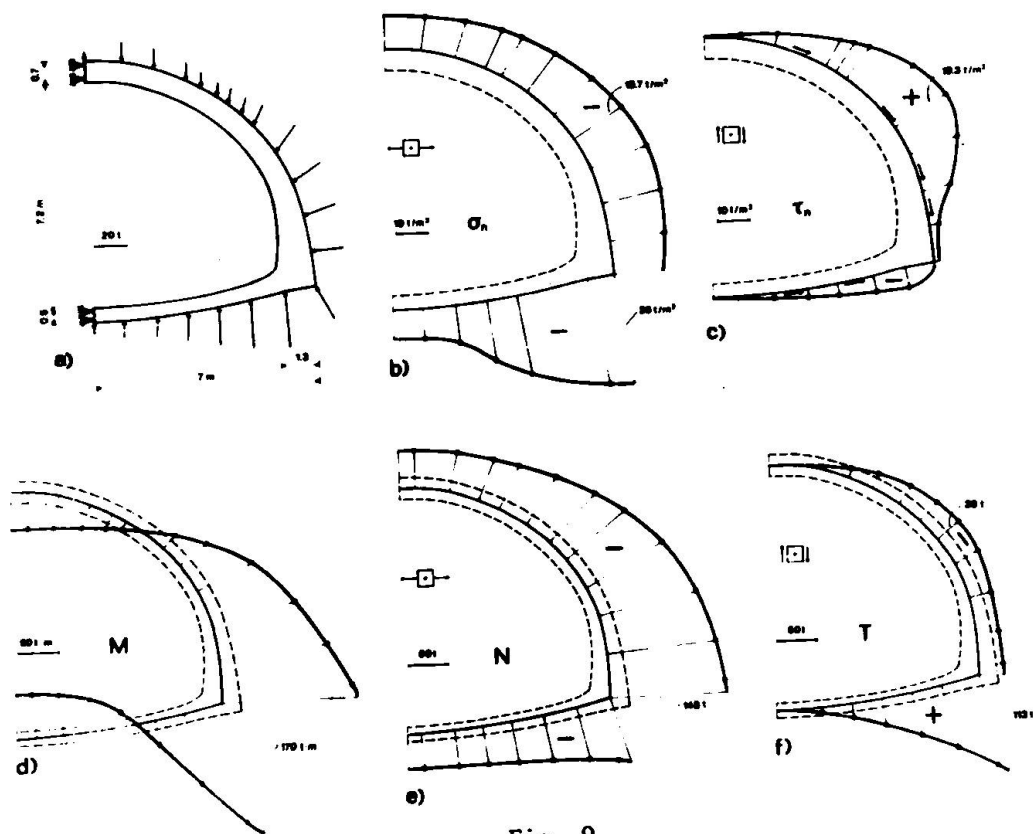


Fig. 9.

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