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Inelastic Analysis of Reinforced Concrete Beam Columns

Calcul non-élastique de poutres-colonnes en béton armé

Unelastische Berechnung von Balken-Stützen-Systemen aus Stahlbeton

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INTRODUCTION

This paper presents the results of a pilot study on the application of an incremental, tangent stiffness finite element analysis technique to the solution of beam-column problems. Results of numerical investigations on reinforced concrete and steel beam-columns subjected to concentrated midspan lateral loads were compared to results obtained using the Column-Curvature-Curve (CCC) method (Ref.1) and, in some cases, the Column-Deflection-Curve (CDC) method (Ref.5) via interaction diagrams. The results have been presented in Ref.4. Only those examples dealing with reinforced concrete beam-columns will be presented here. In each case it will be assumed that the complete axial load is applied first, then the lateral load is applied.

BASIC MODEL

Consistent with the finite element method, the beam-column will be assumed to be divided into elements along its length as shown in Fig.1. A frame or beam type finite element will be used and bending will be assumed to occur about only one axis. Inelastic biaxial bending could also be analyzed by an extended version of the same element. There will be three degrees of freedom at each node point used to define the elements. They are the axial displacement, U , the lateral displacement, V , and the bending rotation, θ . These displacements will occur along an arbitrary reference axis shown as the X axis in Fig.1.

The elements are subdivided into layers which are also indicated in Fig.1. Each layer is assumed to be in uniaxial tension or compression with the strain in each layer analytically related to the strain at the reference axis by the assumption that the cross-section is a plane before and after bending. Each layer may have its own stress-strain relation. Ascending portions of stress-strain curves are idealized using a Ramberg-Osgood formulation for either steel or concrete (Ref.3). Unloading legs of stress-strain curves are modeled as straight line segments.

The displacements within the elements are described by the polynomials below.

$$U = \alpha_1 + \alpha_2 X \quad (1)$$

$$V = \alpha_3 + \alpha_4 X + \alpha_5 X^2 + \alpha_6 X^3 \quad (2)$$

$$\theta = -dV/dx \quad (3)$$

The generalized stresses are chosen as the normal force and moment at the reference axis. The corresponding strains are the axial strain and curvature. Utilizing the assumption of plane sections it is possible to define the elasticity matrix as:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} \bar{A} & \bar{S} \\ \bar{S} & \bar{I} \end{bmatrix} \begin{Bmatrix} dU/dx \\ -d^2V/dx^2 \end{Bmatrix} \quad (4)$$

in which:

$$\bar{A} = \sum_{i=1}^J E_i A_i, \quad \bar{S} = \sum_{i=1}^J E_i A_i z_i, \quad \bar{I} = \sum_{i=1}^J E_i A_i z_i^2 \quad (5)$$

The tangent stiffness matrix given below was developed following the well established procedures of the finite element method, e.g. Ref. 7.

$$\begin{bmatrix} K_E \end{bmatrix} = \frac{1}{\ell^3} \begin{bmatrix} \bar{A}\ell^2 & & & & \\ 0 & 12\bar{I} & & & \text{symm} \\ \bar{S}\ell^2 & -6\bar{I}\ell & 4\bar{I}\ell^2 & & \\ -\bar{A}\ell^2 & 0 & -\bar{S}\ell^2 & \bar{A}\ell^2 & \\ 0 & -12\bar{I} & 6\bar{I}\ell & 0 & 12\bar{I} \\ -\bar{S}\ell^2 & -6\bar{I}\ell & 2\bar{I}\ell^2 & \bar{S}\ell^2 & 6\bar{I}\ell & 4\bar{I}\ell^2 \end{bmatrix} \quad (6)$$

The arbitrary reference axis mentioned earlier has been used in the development of the stiffness matrix above. This facilitates consideration of the change in the position of the neutral axis as nonlinear action proceeds. The equilibrium equations are applied in incremental form so as to treat a nonlinear problem as a series of piecewise linear problems.

The $P-\Delta$ effect caused by the deflection of the beam-column can be included by using the geometric stiffness matrix. The particular form used here was found in Ref. 6. The bending displacements are related to the axial force by Eq. 7 in which the axial force P is positive if it causes tension.

$$\begin{bmatrix} K_G \end{bmatrix} = \frac{P}{30\ell} \begin{bmatrix} 0 & & & & & \\ 0 & 36 & & & & \text{symm} \\ 0 & -3\ell & 4\ell^2 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -36 & 3\ell & 0 & 36 & \\ 0 & -3\ell & -\ell^2 & 0 & 3\ell & 4\ell^2 \end{bmatrix} \quad (7)$$

Combining Eqs. 6 and 7 results in the equilibrium equations for the displaced beam-column element.

$$\{F\} = \left[\begin{bmatrix} K_G \end{bmatrix} + \begin{bmatrix} K_E \end{bmatrix} \right] \{\delta\} \quad (8)$$

The total stiffness matrix of each element can then be assembled to form the global equilibrium equations. After application of the boundary conditions these equations can be solved for each increment of load.

A more complete discussion of the basic model is contained in Ref. 3. Comparisons of analytic and experimental load-deflection curves for reinforced and prestressed concrete beams are also presented as verification of the model.

ITERATION SCHEME

The iteration procedure for a given lateral load increment is started by solving the global equilibrium equations for the increments of displacement. Strain increments are computed from the displacement increments. Using the

latest level of stress available new tangent moduli are computed for each layer, the tangent stiffness matrix, Eq.6, is regenerated and added to the geometric stiffness matrix. The equilibrium equations are solved again. If the new increments of displacement are within a relative tolerance of the previous set, convergence is said to have occurred. If convergence is not attained in several trials the load increment is reduced and the process is repeated. If no convergence is attained after a number of reductions in load the process is stopped. If convergence is attained in relatively few trials the load increment to be applied for the next load step is increased.

Once convergence has been attained for the load step, consideration is given to cracking and crushing if appropriate. The first phase in this step is a pre-scanning process in which all the layers are checked to see if they have exceeded the allowable tensile or compressive stress tolerances by an excessive amount. If this occurs the basic load step is reduced and the problem of finding a converged displacement increment for the basic load step is repeated.

Once it has been determined that no stress criteria are exceeded by more than their tolerances any alteration in stiffness required by the cracking or crushing of a layer is made. The downward legs of the analytic stress-strain curves are used to convert strain increments into "fictitious stresses" which are used to unload layers which have been found to exceed cracking or crushing criteria. The "fictitious stresses" are used to compute nodal "fictitious forces" which hold the rest of the beam-column in equilibrium. This process produces a globally adequate but not locally exact redistribution of stresses. The global equilibrium problem corresponding to that set of "fictitious forces" is solved until convergence is attained. The layers are then rechecked to see if subsequent cracking or crushing has occurred. If so the cracking-crushing analysis is repeated. Execution of a given beam-column analysis is terminated in one of two ways: 1) the total stiffness matrix given by Eq.8 ceases to be positive-definite, or, 2) the process of cracking or crushing results in an inability to find a total solution for a given load step. The first mode of termination was most common in the examples presented here.

Alterations in the stiffness matrix arising from plastic flow like phenomena in reinforcing steels or in steel beam-columns are automatically accounted for by employing the appropriate Ramberg-Osgood curve.

NUMERICAL RESULTS

The rectangular, doubly reinforced section used here is shown in Fig.1. Beam-columns using the same cross-section have been analyzed using the CCC method (Ref.2). The section is 356mm deep, 305mm wide and has equal compressive and tensile areas of 21.68cm^2 each. The compressive strength of 17.58 MN/m^2 used in the CCC analysis was also used here. The yield strength of the steel was 310.27 MN/m^2 . The 26 beam-columns which will be discussed herein are enumerated in the table below.

ANALYTIC REINFORCED CONCRETE BEAM COLUMNS

$e = 0.0$	L/t	10	10	10	10	10	20	20	20	20	20	30	30	30	30
	P/P _o	0.0	0.2	0.4	0.6	0.8	0.0	0.2	0.4	0.6	0.8	0.0	0.2	0.4	0.6
$e = 0.3t$	L/t	10	10	10	10	20	20	20	20	30	30	30	30	30	30
	P/P _o	0.0	0.1	0.2	0.3	0.0	0.1	0.2	0.3	0.0	0.1	0.2	0.3	0.0	0.1

The interaction curves produced by both the finite element and CCC methods for a concentric axial load are shown in Fig.2. It can be seen that the results of both analyses agree quite well for the curves with L/t=30 and L/t=20. The agreement with the CCC results for L/t=10 is not as good but is still within about 5% of the same Q/Q_o value for a given value of P/P_o.

The load-deflection curves produced using the current work for the case with $L/t=20$ is shown in Fig.3. The load-deflection curves do not appear to form a systematic pattern as was obtained when the same analysis was applied to a steel wide-flange beam-column (Ref.4). There are several reasons for this: 1) as seen in Fig.2, there are some values of P/P_o for which a reinforced concrete beam-column loaded in this manner can support a larger lateral load than is possible when $P/P_o = 0$. This was not true for the steel beam-columns, and 2) the effect of cracking is evident in these load-deflection curves as a relatively early change in slope. The amount of change is dependent on the extent of cracking along and through the beam-column.

Figure 4 is a comparison of interaction diagrams for an eccentrically loaded reinforced concrete beam-column. Good agreement with the corresponding CCC results is again noted. The finite element results do not extend as far along these interaction curves because of a limitation in the current iterative procedure. For the higher values of P/P_o in both the eccentric and concentric cases the axial load alone caused enough nonlinear behavior to result in a failure to converge to the first displacement increment. This is because the axial load is applied in one load step in the current algorithm while the subsequent lateral load is applied in small steps. The algorithm could be modified so as to apply the axial load in several steps. For the concentric load case it was relatively easy to circumvent this problem by using an initial stress field which satisfied equilibrium and strain compatibility.

Figure 5 is a set of load-deflection curves for the eccentric load case with $L/t=20$. The almost horizontal offset at the beginning of each curve represents the effect of the application of the total axial load and the first increment of lateral load.

This analysis technique has also been applied to steel wide-flange beam columns and the results were compared to those obtained using both the CCC and CDC methods. The corresponding interaction curves have been reported in Ref.4 and show even better agreement than indicated herein.

CONCLUSIONS

It can be concluded from this study that this incremental iterative analysis technique using a simple layered beam-type finite element can provide solutions to inelastic beam-column problems. While there is already a large body of information in this area, this method does have several advantages which may prove useful in future beam-column studies: 1) a wide range of loadings can be handled. There is no intrinsic difference between one concentrated, several concentrated, uniform, symmetric or unsymmetric loads. 2) boundary conditions can also be handled easily. There is no change in the formulation for different boundary conditions. 3) There is no need for an a-priori moment-thrust-curvature curve. 4) There is nothing conceptually prohibitive about changing the order of loading or using simultaneous (but proportional) axial and lateral loads. 5) Previous work on prestressed concrete beams using basically the same simple model would indicate that prestressed concrete beam-columns could also be treated by this technique.

NOMENCLATURE

\bar{A}	= Generalized area
E	= Initial modulus of elasticity
\bar{I}	= Generalized moment of inertia
J	= Number of layers in an element
L	= Beam-column length
M	= Bending moment
N	= Normal force
P	= Axial load
Q	= Lateral load
\bar{S}	= Generalized statical moment
e	= Eccentricity of axial load

ℓ = Element length
 U = Axial displacement
 V = Lateral displacement
 θ = Bending rotation
 A_c = Concrete area
 A_s = Steel area
 P_o = $f''_{c,c} A_c + f_y A_s$
 Q_o = Ultimate load of a concrete beam with no axial load
 f''_c = $0.85f'_c$
 f_y = Yield stress
 $[K_E]$ = Stiffness matrix
 $[K_G]$ = Geometric stiffness matrix
 $\{\alpha\}$ = Constants in displacement polynomials

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SUMMARY

An efficient model has been developed which predicts the flexural load-deformation behaviour and stress history of inelastic reinforced concrete and steel beam-columns. The beam-column is discretized into an assemblage of layered beam type finite elements and is analyzed using an incremental, iterative, tangent stiffness approach, good agreement between interaction curves developed by this method and the column-curvature-curve method is demonstrated herein.

RESUME

On a développé un modèle efficace permettant de prévoir le comportement flexion-déformation et l'évolution des sollicitations de poutres-colonnes non élastiques en béton armé ou en acier. La poutre-colonne est traitée comme un assemblage d'éléments finis en forme de lamelles. Le calcul se base sur une approche progressive par itérations du module de rigidité tangentiel; on montre également une bonne concordance entre les courbes d'interaction obtenues par cette méthode et la méthode des courbes de courbure de colonne.

ZUSAMMENFASSUNG

Ein leistungsfähiges Modell wurde entwickelt, mit welchem die Vorhersage des Verformungsverhaltens und des Beanspruchungsverlaufs in unelastischen Balken - Stützen - Systemen aus Stahlbeton bzw. Stahl möglich ist. Das Balken - Stützen - System wird dabei aufgelöst in schichtförmige Finite Elemente und untersucht mittels eines stufenweisen iterativen Vorgehens, bei welchem die jeweilige Steifigkeit Verwendung findet. Die gute Übereinstimmung der mittels der verliegenden Methode ermittelten Interaktions-Diagramme mit denjenigen aus einer Stützen - Krümmungs - Methode hergeleiteten wird aufgezeigt.

FIGURES

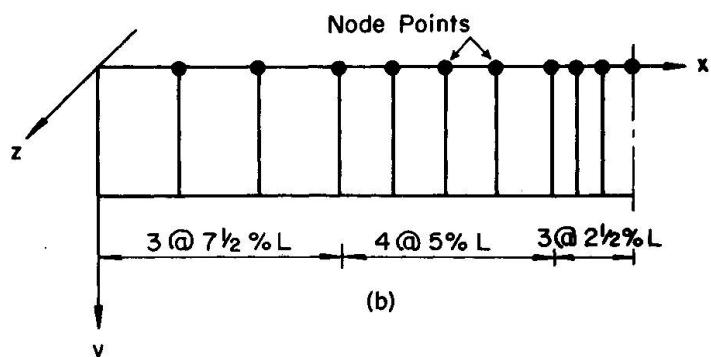
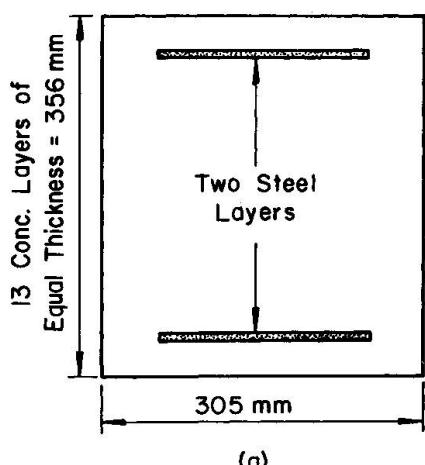


Fig.1 Discretization Models

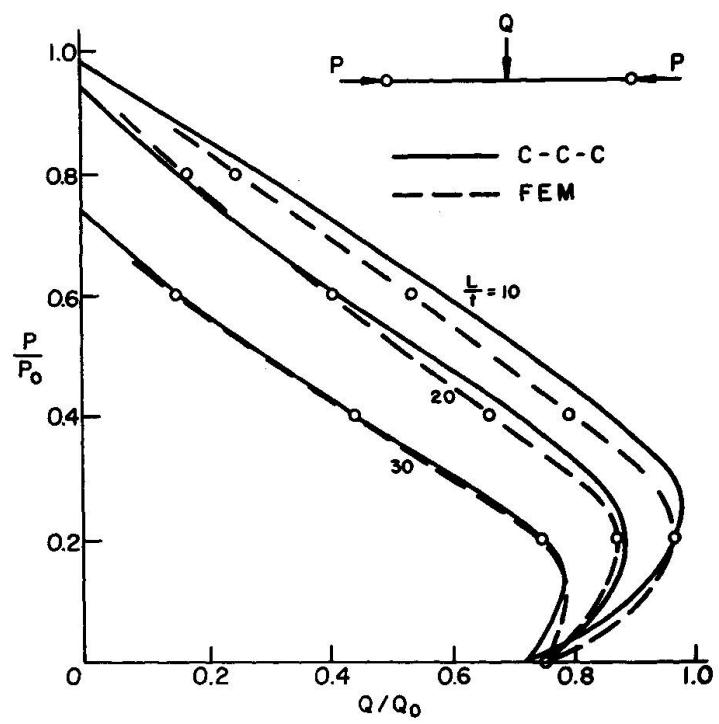


Fig. 2 Interaction Curves, $e/t=0$

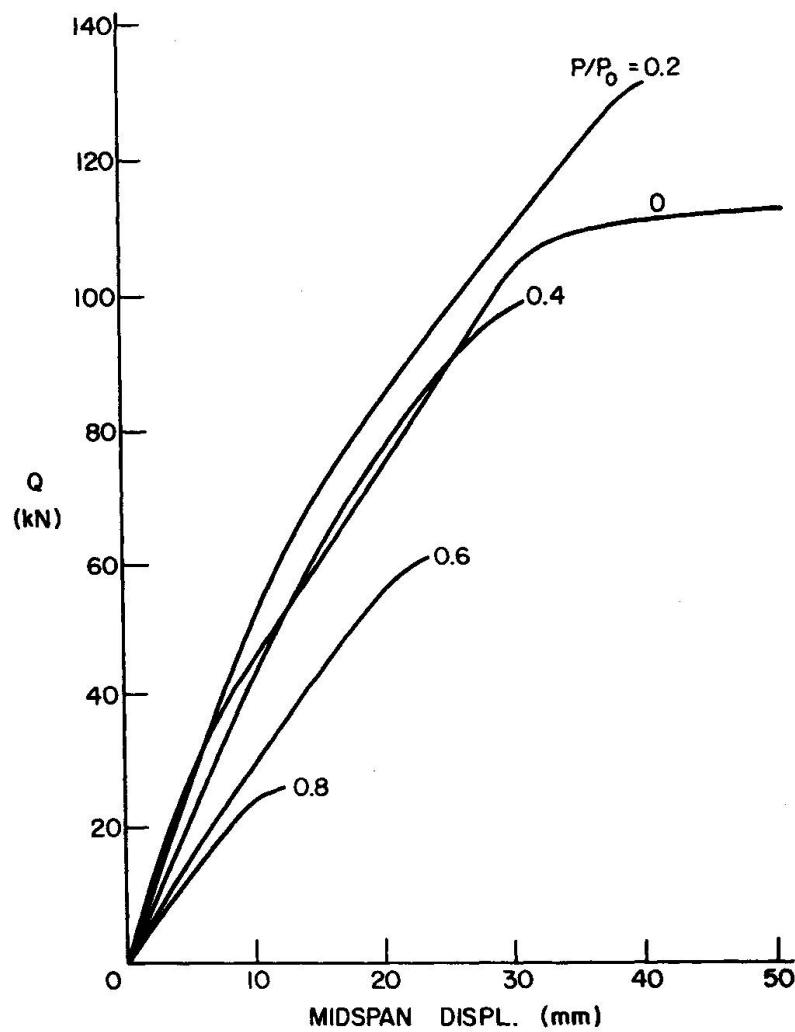


Fig. 3 Load-Deflection Curves, $e/t=0$

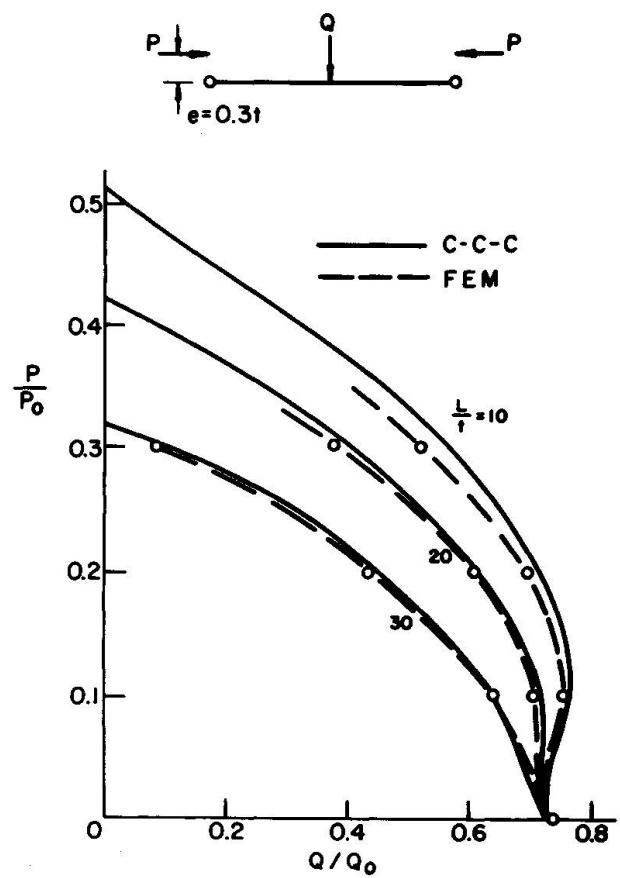


Fig.4 Interaction Curves, $e/t=0.3$

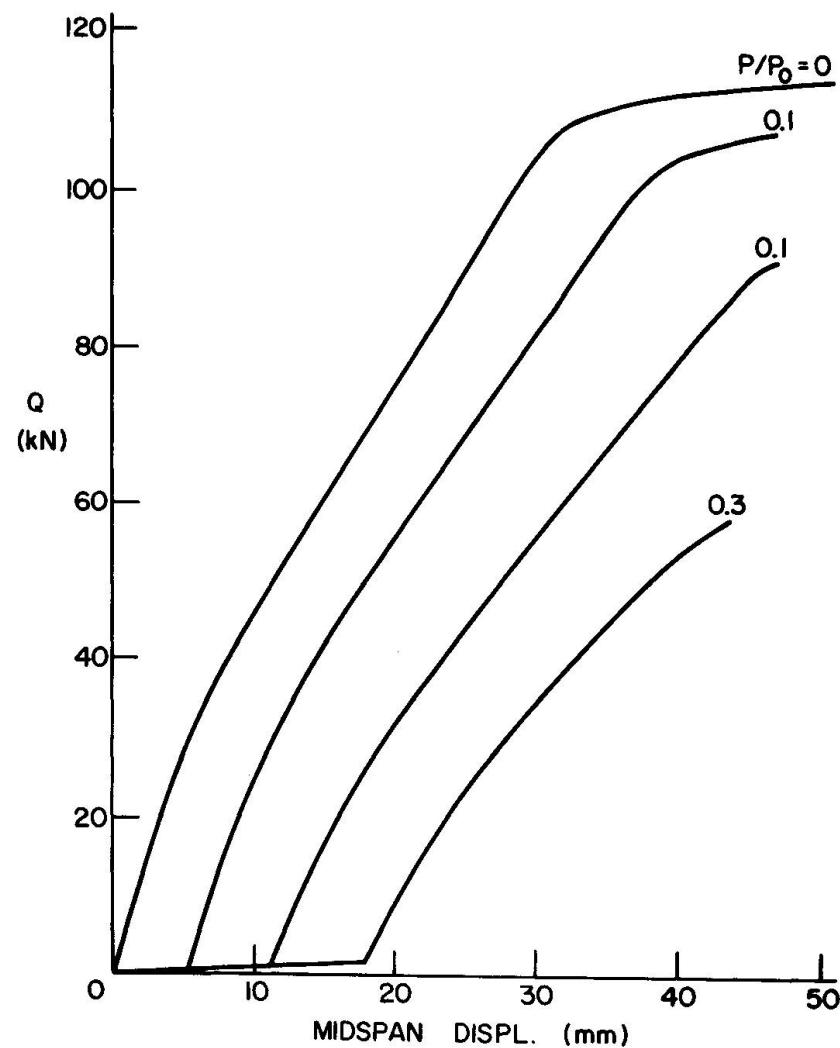


Fig.5 Load-Deflection Curves, $e/t=0.3$