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A Simplified Model for Nonlinearly Viscoelastic Columns

Un modèle simplifié pour le calcul visco-élastique non-linéaire des colonnes

Ein vereinfachtes Modell für nichtlineare viskoelastische Stützen

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1. INTRODUCTION

Adequate theories exist for the analysis of deformation and failure of linearly viscoelastic columns. J. N. Distéfano, /1/, has studied the problem in a series of papers of great generality, considering arbitrary end conditions, lateral loads and initial imperfections, and the most general expression for linear creep.

However, both instantaneous and time dependent deformations of concrete are nonlinear, specially at high stresses. In fact, the behaviour of concrete ranges from almost linear, bounded creep at low stresses to highly nonlinear, unbounded creep at stresses near the compressive strength.

The effect of nonlinear behaviour on creep buckling is analyzed in this paper. In Section 2 a nonlinear rheological model apt to describe the behaviour of concrete for the whole range of stresses is introduced.

In Section 3, the creep buckling problem is studied for the above mentioned rheological model, using a simplified model for the column.

In seccion 4, the model is refined by considering additional effects present in real situations, as the influence of axial thrust on the bending rigidity and the different behaviour of concrete in loading and unloading processes /2/.

For the simpler situations, analytical solutions to the differential equations are used. In the general case, a step by step numerical analysis is necessary. The effect of ageing of concrete may be easily taken into account by taking age dependent coefficients.

2. RHEOLOGICAL MODEL

The proposed model, shown in Fig.1 is similar to the well known standard solid. Its particular feature is the nonlinear stress-strain relation assumed for the spring elements 1 and 2. We denote with $\varepsilon_1(t)$ the strain due to the deformation of spring element 1 and with $\varepsilon_2(t)$ the strain corresponding to spring 2 and dashpot. The total strain is $\varepsilon(t)=\varepsilon_1(t)+\varepsilon_2(t)$, where t denotes the time.

For spring 1 we assume the stress-strain relation

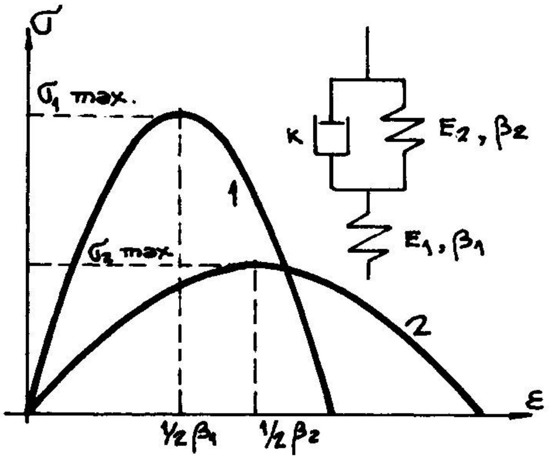

FIG. 1: RHEOLOGICAL MODEL

$$\sigma = E_1\varepsilon_1(1-\beta_1\varepsilon_1) \quad ; \quad 0 \leqslant \varepsilon_1 \leqslant \frac{1}{\beta_1} \tag{1}$$

Here $\sigma(t)$ is the stress and E_1, β_1 are material constants. This equation defines a maximum stress $\sigma_1^{max}=\frac{E_1}{4\beta_1}$ and a corresponding deformation $1/2\beta_1$. Increasing deformations from $1/2\beta_1$ to $1/\beta_1$ are possible for decreasing stresses. No physical meaning is attached to deformations larger than $\frac{1}{\beta_1}$.

The nonlinear Kelvin element constituted by the spring 2 and the dashpot is responsible for the time dependent behaviour. The spring 2 and the dashpot are defined by the relationships

$$\begin{aligned} \sigma' &= E_2\varepsilon_2(1-\beta_2\varepsilon_2) \quad ; \quad 0 \leqslant \varepsilon_2 \leqslant \frac{1}{\beta_2} \\ \sigma'' &= K\dot{\varepsilon}_2 \end{aligned} \tag{2}$$

where E , β and K are material constants and the dot indicates differentiation with respect to time. Being $\sigma=\sigma'+\sigma''$ we obtain the equation for the nonlinear Kelvin element

$$\dot{\varepsilon}_2 - \frac{\beta_2 E_2}{K}\varepsilon_2^2 + \frac{E_2}{K}\varepsilon_2 = \frac{\sigma}{K} \tag{3}$$

which is a Riccati's first order nonlinear differential equation.

We shall consider now the case of a constant stress $\sigma(t)=\sigma_0$ applied at time t=0 and mantained thereafter. Inversion of Eq.(1) provides the expression

$$\varepsilon_1 = \frac{1}{2\beta_1}\left(1-\sqrt{1-\frac{\sigma_0}{\sigma_1^{max}}}\right) \quad ; \quad 0 \leqslant \varepsilon_1 \leqslant \frac{1}{2\beta_1} \tag{4}$$

for the instantaneous deformation. In order to determine the value of the delayed deformations, we must replace the value σ_0 into (3) and solve it (for more details see Ref./3/). Adding instantaneous and time dependent deformations, the final expressions are

$$\varepsilon(t) = \frac{1}{2\beta_1}\left(1-\sqrt{1-\frac{4\beta_1\sigma_0}{E_1}}\right) + \frac{1}{2\beta_2}\left(1-\sqrt{1-\gamma}\;\frac{-1-\sqrt{1-\gamma}+(-1+\sqrt{1-\gamma})\exp\left(-\frac{E_2}{K}\sqrt{1-\gamma}\;t\right)}{-1-\sqrt{1-\gamma}-(-1+\sqrt{1-\gamma})\exp\left(-\frac{E_2}{K}\sqrt{1-\gamma}\;t\right)}\right) \tag{5}$$

$$\varepsilon(t) = \frac{1}{2\beta_1}\left(1-\sqrt{1-\frac{4\beta_1\sigma_0}{E_1}}\right) + \frac{1}{2\beta_2}\left(\frac{t}{\frac{2K}{E_2}+t}\right) \tag{6}$$

$$\varepsilon(t) = \frac{1}{2\beta_1}\left(1-\sqrt{1-\frac{4\beta_1\sigma_0}{E_1}}\right) + \frac{1}{2\beta_2}\left(\gamma\;\frac{1+\sqrt{\gamma-1}\;\sin\frac{E_2}{K}\sqrt{\gamma-1}\;t - \cos\frac{E_2}{K}\sqrt{\gamma-1}\;t}{\gamma+2\sqrt{\gamma-1}\;\sin\frac{E_2}{K}\sqrt{\gamma-1}\;t+(\gamma-2)\cos\frac{E_2}{K}\sqrt{\gamma-1}\;t}\right) \tag{7}$$

for $\gamma<1$, $\gamma=1$ and $\gamma>1$ respectively, being $\gamma = 4\beta_2\sigma_0/E_2 = \sigma_0/\sigma_2^{max}$ (8)

The behaviour of the model under constant stress is indicated in Fig.2. We

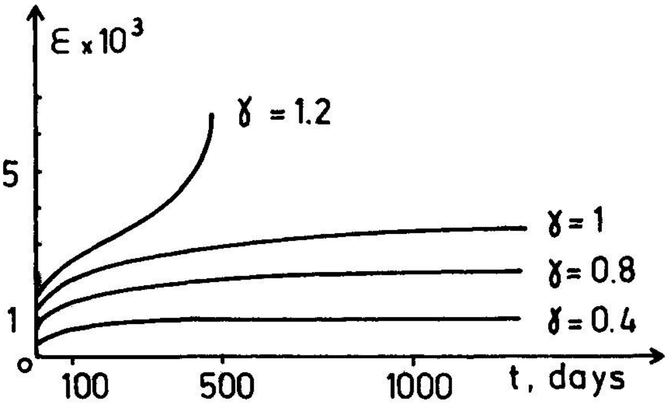

FIG.2:CREEP CURVES FOR THE RHEOLOGICAL MODEL

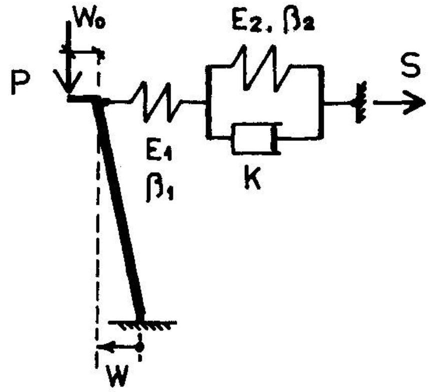

FIG.3:COLUMN MODEL

may see that for $\gamma>1$,i.e. for $\sigma_0 > \sigma_2^{max}$ a condition similar to failure is reached after a finite time. As σ_1^{max} is the strength under instantaneous loading, σ_2^{max} may be interpreted as the strength under sustained load (static fatigue). In Ref./3/ a comparison of this model behaviour with the experimental results of Rüsch /4/ is given.

3. SIMPLIFIED COLUMN MODEL

Let us consider now the system in Fig.3. In this structure the deflexion of the hinged bar due to the action of the force P is prevented by a viscoelastic element which in turn reacts with a force S. This simple model contains many of the more interesting features of considerably more complex systems. For small, quasi-static deflexions, equilibrium provides the relation

$$S = P\,\frac{w + w_0}{L} \tag{9}$$

We are interested in the behaviour of this column model in the presence of nonlinear creep. For the sake of clarity, we shall study first some simpler situations.

<u>3.1. Linear spring</u> ($\beta_1=0; E_2=\infty$): The force in the spring is $S=E_1 w$; from this and (9) we obtain

$$E_1\,w = \frac{P(w+w_0)}{L} \qquad \text{or} \qquad w = \frac{w_0}{\frac{E_1 L}{P}-1} \tag{10}$$

When $P \to LE_1$ we have $w \to \infty$;$P_E = LE_1$ may be considered the buckling load for this case.

<u>3.2. Linear Kelvin material</u> ($\beta_2=0; E_1=\infty$): The force in the viscoelastic element is $S=E_2 w+K\dot{w}$; the differential equation for equilibrium is

$$\dot{w} + w\left(\frac{E_2}{K} - \frac{P}{LK}\right) - \frac{Pw_0}{LK} = 0 \tag{11}$$

It may be seen that the solution of (11) shall be bounded for $t\to\infty$ whenever the the coefficient of w is possitive; thus, the creep buckling load in this case is $P_k = LE_2$, while the instantaneous load is infinite.

<u>3.3. Linear standard material</u> ($\beta_1=\beta_2=0$): Proceeding in a similar way, we find

in this cae $P_E = LE_2$; $P_K = LE_1E_2/(E_1+E_2)$.

Both 3.2 and 3.3 are particular cases of the general linear viscoelastic problem as studied by Distéfano/1/. The creep buckling load is given (as it should) by the reduced modulus load. Physically, this may be interpreted saying that, in order to obtain the load stable for $t\to\infty$, only the spring constants are significant,as the action of the dashpot vanishes for $t\to\infty$ (*).

3.4. Nonlinear spring ($E_2=\infty$): The force in the spring is now $S=E_1w(1-\beta_1w)$; from this and (9) we have

$$E_1 w(1-\beta_1 w) = \frac{P(w+w_0)}{L} \tag{12}$$

Solving for w we find that for each pair of values (P,w_0) there exist two equilibrium points, defined by

$$w = -\frac{1}{2\beta_1}\left(\frac{P}{LE_1}-1\right) \pm \frac{1}{2}\sqrt{\frac{1}{\beta_1^2}\left(\frac{P}{LE_1}-1\right)^2 - \frac{Pw_0}{LE_1\beta_1}} \tag{13}$$

and represented by points A and B in Fig.4. It is easy to see that A corresponds to stable equilibrium and B to unstable equilibrium. The maximum load that allows stable equilibrium is obviously that corresponding to point C and may be obtained making the square root equal to zero. We obtain

$$P_E = LE_1\left\{(2w_0\beta_1+1) - 2w_0\beta_1\sqrt{1+\frac{1}{w_0\beta_1}}\right\}$$
$$w_{cr} = w_0\left(\sqrt{1+\frac{1}{w_0\beta_1}} - 1\right) \tag{14}$$

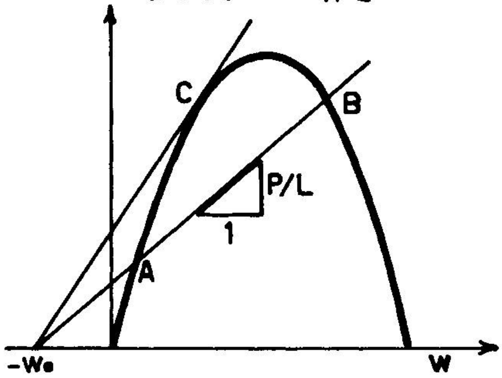

FIG.4: NONLINEAR SPRING

Also from Fig.4 we may see that, in order to reach the critical condition, one may increase the load P to P_E or increase the initial excentricity w_0. Thus, a column whose material is characterized by a nonlinear stress-strain relation for instantaneous loading has a critical deflexion at which the applied load is the critical load. This fact is specially important in the treatment of creep buckling problems.

3.5. Nonlinear Kelvin material ($E_1=\infty$):In this case we have $S=E_2w(1-\beta_2w)+K\dot{w}$; From this and (9) we obtain

$$\dot{w} - \frac{E_2\beta_2}{K}w^2 + \frac{E_2}{K}\left(1-\frac{P}{LE_2}\right)w - \frac{Pw_0}{LK} = 0 \tag{15}$$

Eq.(15) is formally identical with (3). Following the same procedure we obtain the corresponding solution for w(t), namely

(*) This is of course only true for materials with bounded deformations, i.e. solids.

$$w(t)=\frac{1}{2\beta_2}\left(1-\frac{P}{LE_2}\right)\left\{1-\sqrt{1-\Gamma}\;\frac{1+\sqrt{1-\Gamma}+(1-\sqrt{1-\Gamma})\exp\left(-\frac{E_2}{K}\left(1-\frac{P}{LE_2}\right)\sqrt{1-\Gamma}\,t\right)}{1+\sqrt{1-\Gamma}-(1-\sqrt{1-\Gamma})\exp\left(-\frac{E_2}{K}\left(1-\frac{P}{LE_2}\right)\sqrt{1-\Gamma}\,t\right)}\right\} \tag{16}$$

$$w(t)=\frac{1}{2\beta_2}\left(1-\frac{P}{LE_2}\right)\left\{\frac{t}{\frac{2K}{E_2\left(1-\frac{P}{LE_2}\right)}+t}\right\} \tag{17}$$

$$w(t)=\frac{1}{2\beta_2}\left(1-\frac{P}{LE_2}\right)\left\{\Gamma\;\frac{1+\sqrt{\Gamma-1}\sin\frac{E_2}{K}\sqrt{\Gamma-1}\,t-\cos\frac{E_2}{K}\sqrt{\Gamma-1}\,t}{\Gamma+2\sqrt{\Gamma-1}\sin\frac{E_2}{K}\sqrt{\Gamma-1}\,t+(\Gamma-2)\cos\frac{E_2}{K}\sqrt{\Gamma-1}\,t}\right\} \tag{18}$$

for $\Gamma<1$, $\Gamma=1$ and $\Gamma>1$ respectively, being

$$\Gamma=\frac{\frac{4P}{LE_2}w_0\beta}{\left(1-\frac{P}{LE_2}\right)^2} \tag{19}$$

Thus, the critical load is given by

$$P_E=LE_2\left(1+2\beta w_0-\sqrt{4\beta w_0(\beta w_0+1)}\right) \tag{20}$$

Comparing this with $(14)_1$ we see that, as for the linear case, the load for infinite stability corresponds to the instantaneous critical load of a similar column where the Kelvin body has been replaced by the spring.

<u>3.6. Nonlinear standard material:</u> In this case, a closed solution has not been found. The problem has been solved numerically, using a step by step procedure.

The time interval of interest is divided into small (*) time intervals Δt. Then, at a time t, the force $S(t)$ satisfies the equation

$$S(t)=\frac{P}{L}(w_0+w)=\frac{P}{L}(w_0+w_D+w_E)=\frac{P}{L}\left(w_0+w_D(t)+\frac{1-\sqrt{1-\frac{4\beta_1 S(t)}{E_1}}}{2\beta_1}\right) \tag{21}$$

where w_D is the delayed deflexion for time t (of course, $w_D=0$ for $t=0$), and w_E is the elastic deflexion. In the following time interval $(t,t+\Delta t)$ we consider the spring 1 frozen, while the Kelvin element deforms under the action of force $S(t)$ assumed constant during the interval. The corresponding creep deformation is

$$\Delta w_D=\frac{S(t)-E_2w_D(t)\left(1-\beta_2w_D(t)\right)}{K}\Delta t \tag{22}$$

and the delayed deformation now amounts to $w_D(t+\Delta t)=w_D(t)+\Delta w_D$. Then, $w_D(t+\Delta t)$ is replaced into (21) and the process continues in the same fashion. The outlined procedure is very easily programmed for a digital computer.

The analysis of the results may be better understood looking at Fig.5. Line OA represents the t=0 isochronous curve for the material and corresponds to the $\sigma-\varepsilon$ relation for spring 1 in Fig.1. Line OFR corresponds to the $t=\infty$ isochronous curve and represents the behaviour of springs 1 and 2 in series.

(*) Small when compared with the characteristic retardation time of the model.

Different loads, applied with excentricity w_0 are represented by straight lines beggining at N. We observe that:

1)The instantaneous buckling load P_E (line NA) depends only on the characteristics of spring 1 and the initial excentricity w_0.

2)The creep buckling load P_K (line NF) depends on the characteristics of springs 1 and 2 and the initial excentricity w_0. More precisely, it may be evaluated as the instantaneous buckling load of an ideal elastic nonlinear column with a spring equivalent to springs 1 and 2 in series.

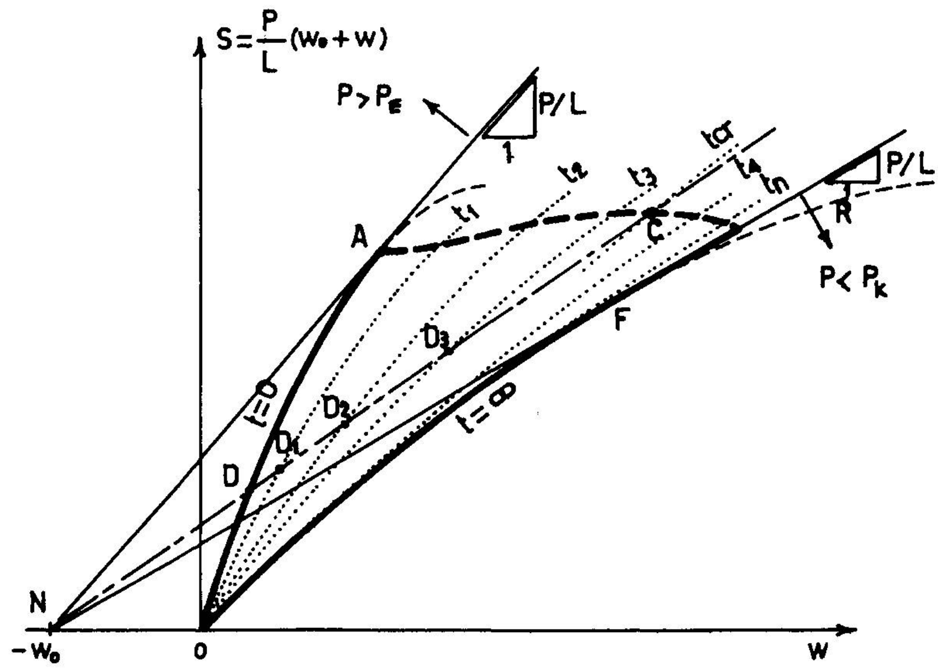

FIG.5: NONLINEAR STANDARD MATERIAL

This behaviour resembles in some ways the case of linear viscoelastic columns (see Section 3.3) where a reduced modulus exists. Of course, being the present problem nonlinear, a unique reduced modulus does not exists, and the load P_K depends on the initial excentricity.

3)A column loaded with $P_\phi < P_K$ reaches a limit deflexion with a velocity that depends on the value of K.

4)If a load $P_E > P_\phi > P_K$ (represented by NC) is applied, an instantaneous equilibrium position D is reached at t=0. As time goes on,successive equilibrium positions $D_1, D_2, \ldots$are reached.At some time t_{cr} (which depends on the value of K) the delayed deflexion w_D reaches a value for which P_ϕ is critical for spring 1. At this time, the column fails suddently. Failure points lay along a frontier indicated by AC.

The pattern of behaviour described closely resembles that observed experimentally /5,6/.

4. A REFINED MODEL

The model studied in Section 3 does not take into account:

a)the effect of axial thrust P on the bending rigidity

b) the different behaviour of concrete in loading and unloading processes. In order to take account of this influences, a refined model may be used, as indicated in Fig.6. Accordingly, the stress-strain relation for the springs in the rheological model has been generalized, as indicated in Fig.7

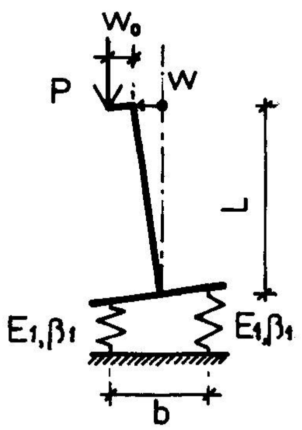

FIG.6: REFINED COLUMN MODEL

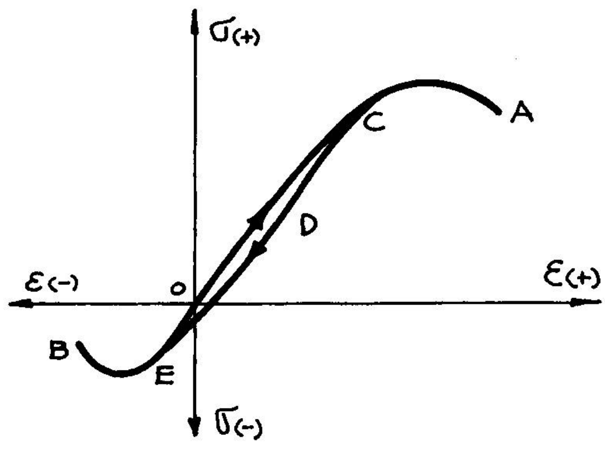

FIG.7:STRESS-STRAIN RELATION

Stresses and strains are now taken with their corresponding signs (compression:positive; tension:negative). Then, the base curve OA in compression is given by Eq.(1) with E_1 and $\beta_{1C} > 0$; the base curve OB in tension is given by the same Eq.(1) with E_1 and $\beta_{1T} < 0$. During unloading, the material behaves along line CDE with origin in C and E_1 , $\beta_1^* = \frac{\beta_{1C}\,\beta_{1T}}{\beta_{1C} - \beta_{1T}} < 0$. This curve contacts the base curve OB at point E smoothly (both curves have a common tangent at point).

By putting together the equilibrium and compatibility equations for the column model, and the constitutive relations of the springs, a system of equations is obtained that allows the study of the stability of the model. Comparing the critical loads obtained using this model (P_{cr2}) and the model in Section 3 (P_{cr1}), the effect of axial deformation may be evaluated.

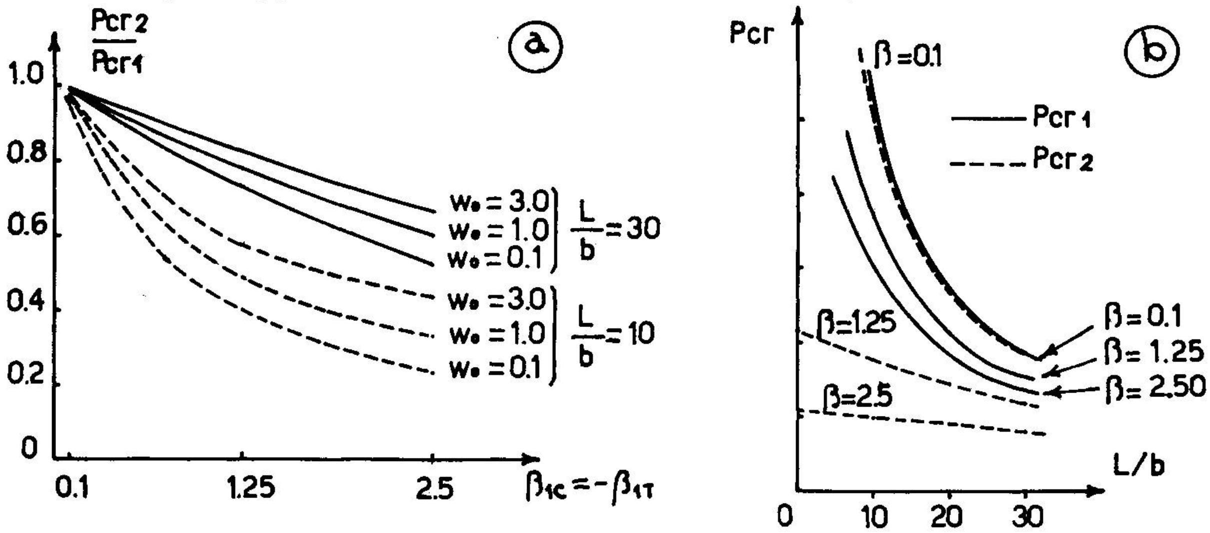

FIG.8: COMPARISON BETWEEN COLUMN MODELS

In Fig.8-a we observe that the influence of axial deformation increases with the nonlinearity coefficient β_{1C}. The influence of the initial excentricity w_0

and the slenderness ratio L/b is also shown. In Fig.8-b we may see how the critical loads for both models depend on β_{1c}, w_o and L/b.

ACKNOWLEDGEMENT

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SUMMARY

The effect of nonlinear behaviour on creep buckling is analized, using a nonlinear rheological element to express material properties and simplified models for the column.

RESUME

On analyse l'influence du comportement non-linéaire sur le flambement dû au fluage en utilisant un élément non-linéaire pour exprimer les caractéristiques du matériau, ainsi que des modèles simplifiés pur la colonne.

ZUSAMMENFASSUNG

Unter Verwendung eines nichtlinearen rheologischen Elementes für die Beschreibung der Materialeigenschaften und eines vereinfachten Modelles für die Stütze wird die Wirkung des nichtlinearen Verhaltens auf das Kriechknicken untersucht.