

**Zeitschrift:** IABSE reports of the working commissions = Rapports des commissions de travail AIPC = IVBH Berichte der Arbeitskommissionen  
**Band:** 13 (1973)  
**Artikel:** Plastic shakedown of structures with stochastic local strengths  
**Autor:** Augusti, Giuliano / Baratta, Alessandro  
**DOI:** <https://doi.org/10.5169/seals-13781>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 12.12.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## **Plastic Shakedown of Structures with Stochastic Local Strengths**

Ruine des structures en domaine plastique avec résistance locale empirique

Plastisches Versagen von Tragwerken mit stochastischen lokalen Festigkeiten

**Giuliano AUGUSTI**  
Istituto di Ingegneria Civile  
Università di Firenze  
Italia

**Alessandro BARATTA**  
Istituto di Scienza delle Costruzioni  
Università di Napoli  
Italia

### **1. INTRODUCTION**

A procedure for evaluating bounds on the plastic collapse probability of redundant, ductile structures with stochastic local strengths has been presented elsewhere [1] and brought to the direct knowledge of I.A.B.S.E. members by a contribution to the Free Discussion at the Amsterdam Congress [2]. The original form of the procedure allowed the determination of bounds to the reliability of structures subjected to load systems measured by a single parameter with known probability density law; these bounds could be made closer by successive approximations.

Later, the procedure has been extended to multi-parameter loading [3], and shown to be just a particular case of a more general formulation applicable to all structural problems whose solution is given by a maximum or minimum condition [4].

In this and a companion paper [5], the procedure is further extended and applied to the shakedown/incremental collapse problem of plastic structures. In fact, it is well known in classical limit analysis (see e.g. Refs. 6, 7) that a ductile structure can be led to gradual collapse ("incremental collapse") by repeated loads of smaller intensity than the corresponding non-repeated loads, even if the lowering of the yield limit with plastic cycles of stress is (unsafely) neglected. This effect, which is quantitatively significant in a number of cases of practical engineering importance, appears not to have been examined before for structures made up of elements with random strengths; also Professor Ferry-Borges' Introductory Report for this Symposium [8] considers explicitly effects of the "fatigue" type only.

The theorems which allow the introduction of this effect into probabilistic limit analysis are presented in detail in the quoted Ref. [5], and are briefly summarized below. This paper is more specifically devoted to a numerical example.

## 2. BOUNDS ON THE PROBABILITY OF SHAKEDOWN

For simplicity's sake, only structures with one significant strength component (bending moment  $M$ ) will be considered.

Let  $m_e \max$  and  $m_e \min$  be the elastic diagrams of maximum and minimum bending moment  $M_e \max$  and, respectively,  $M_e \min$ , calculated allowing all the admissible loadings to act in turn on the structure. If  $m'_0$  and  $m''_0$  are the diagrams of positive and negative limit moments  $M'_0$  and, respectively,  $M''_0$ , the static theorem states that shakedown takes place if at least one diagram  $m^*$  of self-equilibrated moments exists such that the inequality:

$$M''_0 \leq M_e \max + M^* \leq M'_0; \quad M''_0 \leq M_e \min + M^* \leq M'_0 \quad (2.1)$$

holds in every section of the structure.

On the contrary, by the kinematic theorem, shakedown cannot take place if a rigid-plastic mechanism exists such that:

$$\sum_i M_i^e \max \Delta \phi_i + \sum_j M_j^e \min \Delta \phi_j > \sum_i M'_{0i} \Delta \phi_i + \sum_j M''_{0j} \Delta \phi_j \quad (2.2)$$

$i$  and  $j$  being the dummy indices which distinguish sections undergoing positive and, respectively, negative plastic rotations. Therefore, if  $m'_0$  and  $m''_0$  are random functions and if a number of self-equilibrated diagrams  $m^*_1, m^*_2, \dots, m^*_n$  is considered, denoting by  $E_1$  the event that (2.1) holds at least for one  $m^*_i$ , and by  $S_d$  the actual probability of shakedown, one gets:

$$S_{d\psi} = \text{Prob} \{E_1\} \leq S_d \quad (2.3)$$

Similarly, consider a number of mechanisms  $k_1, \dots, k_r$  and denote by  $E_2$  the event that (2.2) does not hold for any  $k_i$ . Then:

$$S_{d\gamma} = \text{Prob} \{E_2\} \geq S_d \quad (2.4)$$

From (2.3) and (2.4), bounds on the probability of incremental collapse  $P_d = 1 - S_d$  follow obviously:

$$P_{d\psi} = 1 - S_{d\psi} \geq P_d \geq 1 - S_{d\gamma} = P_{d\gamma} \quad (2.5)$$

The two events (2.1) and (2.2) are mutually exclusive and exhaustive. Therefore, bounds (2.3) and (2.4) are sharp, in the sense that the difference  $S_{d\gamma} - S_{d\psi}$  approaches zero when all self-equilibrated diagrams and all possible mechanisms are considered for the definition of events  $E_1$  and  $E_2$ .

## 3. NUMERICAL EXAMPLE

Consider the simple frame in Fig. 1, already studied in [1,2,3], of nominally constant section throughout, subjected to loading dependent on two parameters,  $W_1$  and  $W_2$ .

Assume that only the eleven critical sections shown in Fig. 1 are significant, and that in each critical section  $|M''_{0i}| = M'_{0i} = \bar{M}_{0i}$ . Assume further that the  $\bar{M}_{0i}$  ( $i = 1, \dots, 11$ ) constitute a set of independent random variables distributed according to the Gauss probability law  $P_{0i}(\bar{M}_{0i})$  with mean value  $\bar{M}_{0i} = \bar{M}_0 = 10$  tm and standard deviation  $\sigma_{0i} = \sigma_0 = 1$  tm in every section. The set of admissible loading is defined by the following inequalities:

$$0 \leq W_1 \leq W_{1m}; \quad 0 \leq W_2 \leq W_{2m} \quad (3.1)$$

where  $W_{1m}$  and  $W_{2m}$  are the "extreme load-parameters" on which the probability of incremental collapse depends. The results of the analysis are presented

mainly as level curves, respectively  $C_{\psi P}$  and  $C_{\gamma P}$ , of the bounding incremental collapse probabilities  $P_{d\psi}$  and  $P_{d\gamma}$ , drawn in the plane of the load-parameters  $W_{1m}$ ,  $W_{2m}$ .

To obtain the curves  $P_{d\psi}(W_{1m}, W_{2m}) = P$  (curves  $C_{\psi P}$ ), consider three diagrams  $\bar{m}_1^*$ ,  $\bar{m}_2^*$ ,  $\bar{m}_3^*$  which, on the expected (or average) structure [1,2,3], can be associated with the three failure mechanisms shown in Fig. 2. Fig. 3 shows such diagrams for a particular value of  $\alpha = W_{2m}/W_{1m}$ . Each of these diagrams is also associated with extreme loads which lead the expected structure to incremental collapse according to one of the quoted mechanisms. Therefore, if the actual loading  $(W_{1m}, \alpha W_{1m})$  has to be multiplied by a factor  $\gamma_{\alpha i}$  in order to cause collapse of the expected structure according to mechanism  $k_i$ , three self-equilibrated diagrams are associated to each pair  $(W_{1m}, \alpha W_{1m})$  by setting

$$m_i^* = \frac{\bar{m}_i^*}{\gamma_{\alpha i}} \quad (i = 1, 2, 3) \quad (3.2)$$

Now, put:

$$M_{ij} = \max \{ |M_{i \max}^e + M_i^*|, |M_{i \min}^e + M_i^*| \} \quad (3.3)$$

$$\left. \begin{aligned} P_r &= \prod_j [1 - P_{oj}(M_{rj})] \\ P_{rs} &= \prod_j [1 - P_{oj}(\max \{M_{rj}, M_{sj}\})] \\ P_{123} &= \prod_j [1 - P_{oj}(\max \{M_{1j}, M_{2j}, M_{3j}\})] \end{aligned} \right\} \begin{aligned} j &= 1, \dots, 11 \\ r, s &= 1, 2, 3 \\ s &> r \end{aligned} \quad (3.4)$$

From (3.3) (3.4) it is possible to obtain the expression for  $P_{d\psi}$ :

$$1 - P_{d\psi}(W_{1m}, W_{2m}) = (1 - P_1) + (1 - P_2) + (1 - P_3) - P_{12} - P_{23} - P_{13} + P_{123} \quad (3.5)$$

Function  $P_{d\psi}(W_{1m}, \alpha W_{1m})$  is plotted in Fig. 4 in a semi-logarithmic diagram versus the load-parameter  $W_{1m}$ , for a number of values of the ratio  $\alpha$ , and for  $P_{d\psi} \leq 0.1$ . Intersecting these curves by horizontal lines it is possible to obtain points of the level curves  $C_{\psi P}$ . These curves are shown in Fig. 5, for  $P = 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 0.5$ .

As for level curves  $P_{d\gamma}(W_{1m}, W_{2m}) = P$  (curves  $C_{\gamma P}$ ), it is necessary to associate to the three collapse mechanisms so far considered also the ones that lead to localized fracture the expected structure, paying attention to the critical sections 1,4,6,8,11. Furthermore, it is convenient, in order to keep the calculations simple, to consider separately each failure mechanism, and to calculate only the individual failure probabilities. By this simplification, it turns out that curves  $C_{\gamma P}$  are composed by a number of straight lines. In order to illustrate the procedure, consider, for instance, mechanism  $k_2$  (Fig. 2). The equation of virtual work for this mechanism is:

$$3 W_{2m} + 2.1816 W_{1m} = \hat{M}_{o1} + 2 \hat{M}_{o6} + 2 \hat{M}_{o8} + \hat{M}_{o11} \quad (3.6)$$

The second member of (3.6) is a Gaussian random variable  $\hat{D}_a$  with mean value  $\bar{D}_a = 6 \bar{M}_o$  and variance  $\sigma_D^2 = 10\sigma_o^2$ . In order to keep the failure probability lesser than  $P$ , it is necessary that the first member of (3.6) remains lesser than  $\bar{D}_a + Z_P \cdot \sigma_D$ ,  $Z_P$  being the value of the standard normal variable  $Z$  which is not exceeded with probability  $P$ . For  $P = 0.1$ , the equation of a side of the polygonal  $C_{\gamma P}$  is given by:

$$3 W_{2m} + 2.1816 W_{1m} = 6 \bar{M}_o - 1.28 \sqrt{10} \sigma_o \quad (3.7)$$

Similarly, the equation for the alternate plasticity mechanism relevant, for instance, to section 1 is:

$$M_{lmax}^e - M_{lmin}^e = 0.1818 W_{lm} + 0.8864 W_{2m} = 2 \bar{M}_{o1} \quad (3.8)$$

For  $P = 0.1$ , the equation of another side of the polygonal  $C_{YP}$  would be:

$$0.0909 W_{lm} + 0.4432 W_{2m} = \bar{M}_o - 1.28 \sigma_o \quad (3.9)$$

Note now that the same equation (3.9) is obtained when section 11 is considered as critical, and rupture occurs if eq. (3.8) holds in section 1 or in section 11. Therefore, if  $Q$  is the probability that eq. (3.8) holds in only one section, the compound probability will be

$$2Q - Q^2 = P \quad (3.10)$$

if

$$Q = 1 - \sqrt{1 - P} \quad (3.11)$$

Replacing  $Z_P$  by  $Z_Q$ , it is possible to obtain directly the overall probability of rupture relevant to both mechanisms: for  $P = 0.1$ , eq. (3.11) yields  $Q = 0.05$ ,  $Z_Q = 1.65$ , and the equation (3.9) becomes:

$$0.0909 W_{lm} + 0.4432 W_{2m} = \bar{M}_o - 1.65 \sigma_o \quad (3.12)$$

Curves  $C_{YP}$  are shown in Fig. 6 for  $P = 10^{-5}$ ,  $10^{-4}$ ,  $10^{-3}$ ,  $10^{-2}$ ,  $10^{-1}$ , 0.5. It is soon noted that only the first three mechanisms yield significant bounds on the probability of incremental collapse. This is clearly due to the definition (3.1) assigned to the set of admissible loading; however, even under these conditions, Fig. 7 shows a contraction of the shakedown domains with respect to the corresponding domains relevant to monotonic loading.

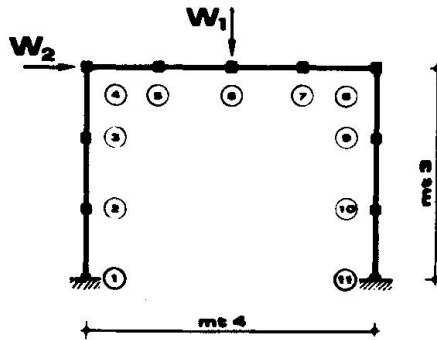


Fig. 1: Example Frame

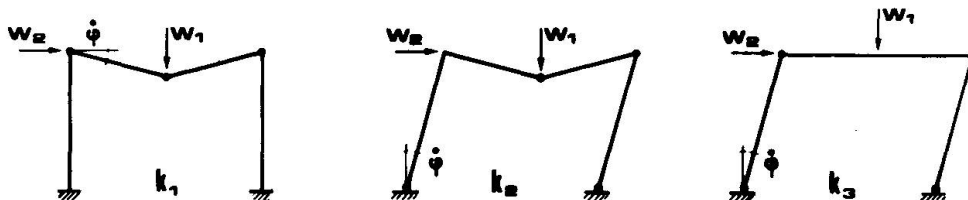


Fig. 2: Basic mechanisms

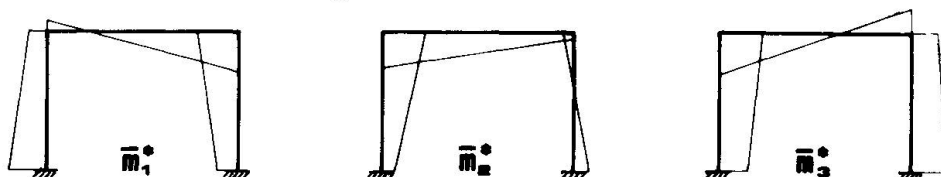


Fig. 3: Self-equilibrated diagrams of bending moment

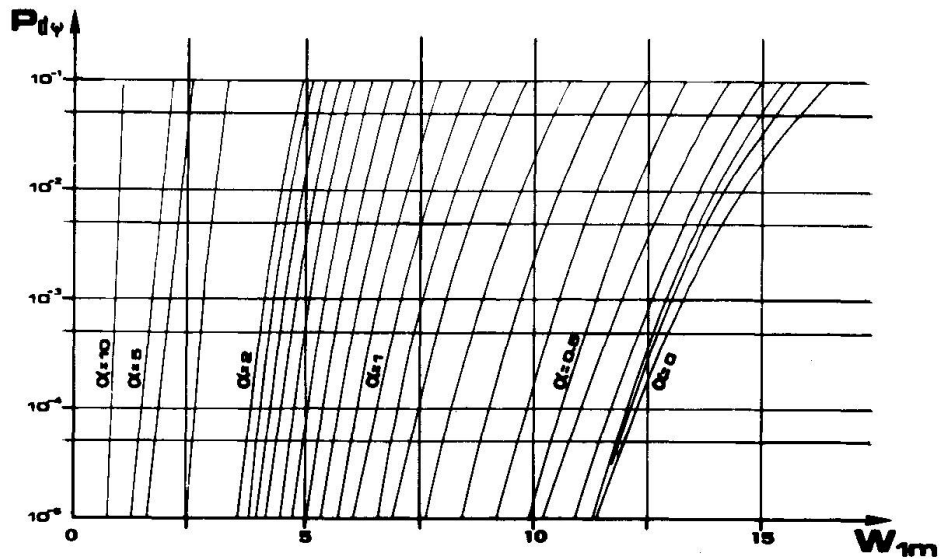


Fig. 4: Upper bound probability  $P_{d\psi}$  vs. load parameter  $W_{1m}$  ( $\alpha = W_{2m}/W_{1m}$ )

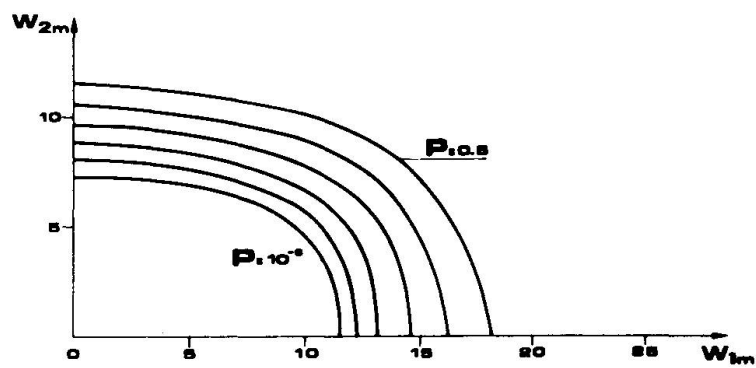


Fig. 5: Level curves  $C_{\psi P}$  of the upper bound probability  $P_{d\psi}$

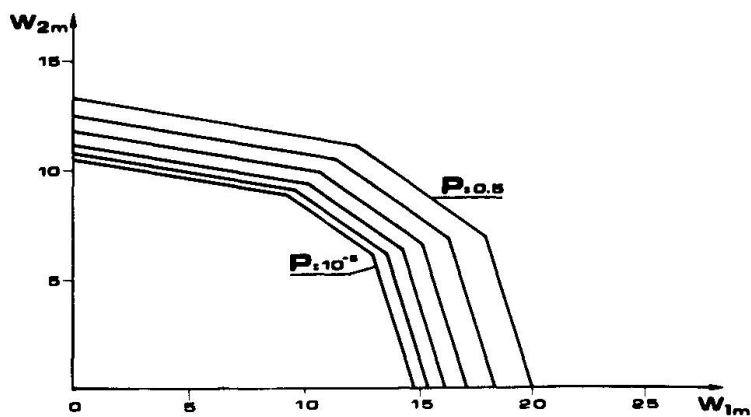


Fig. 6: Level curves  $C_{\gamma P}$  of the lower bound probability  $P_{d\gamma}$

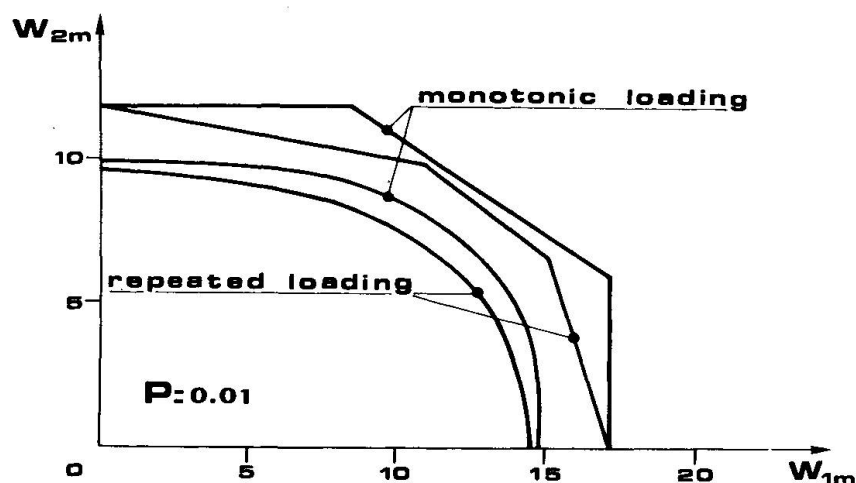


Fig. 7: Comparison between monotonic and repeated loading

#### REFERENCES:

- [1] G. AUGUSTI, A. BARATTA: Journ.Struct.Mech., Vol.1, No.1, 1972.
- [2] G. AUGUSTI: Fin. Rept., 9th Congr. I.A.B.S.E., Amsterdam, 1972.
- [3] G. AUGUSTI, A. BARATTA: in Foundations of Plasticity, Nordhoff, 1972.
- [4] A. BARATTA: (in preparation).
- [5] G. AUGUSTI, A. BARATTA: 2nd Int.Conf.Struct.Mech., Berlin, Sept. 1973.
- [6] J. HEYMAN: Plastic Design of Frames, Vol.2, Cambr. Univ. Press, 1971.
- [7] V. FRANCIOSI et al.: Journ.Struct.Div. A.S.C.E., Vol.90, No.ST1, 1964.
- [8] J. FERRY-BORGES: Introd. Rept. I.A.B.S.E. Symp., Lisbon, 1973.

#### ACKNOWLEDGEMENT:

This research has been supported by the Italian National Research Council (C.N.R.).

#### SUMMARY

A method for bounding the shakedown probability of an elastic-perfectly plastic structure with stochastic local yield condition, subjected to repeated cycles of loading is presented. The procedure is then illustrated by a numerical example with reference to a framed structure.

#### RESUME

On présente dans ce travail une méthode pour estimer la probabilité de ruine d'une structure élastique parfaitement plastique avec condition de fluage local stochastique, soumise à des cycles de charge répétés. Le procédé est ensuite illustré par un exemple numérique traitant un portique.

#### ZUSAMMENFASSUNG

In dieser Arbeit wird eine Methode zur Begrenzung der Shakedown-Wahrscheinlichkeit eines elastisch-idealplastischen Tragwerkes mit stochastischen lokalen Fließbedingungen unter wiederholten Belastungszyklen vorgelegt. Das Vorgehen wird sodann anhand eines numerischen Beispiels, bezogen auf ein Rahmentragwerk, veranschaulicht.