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# A Design Method using Weighted Fractiles as Design Values

Une méthode de dimensionnement utilisant des valeurs pondérées

Eine Bemessungsmethode unter Benutzung gewichteter Fraktile

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## INTRODUCTION

The theoretical basis of the design method by which weighted fractiles are used as design values was presented in [4] and it is thus unnecessary to study its derivation here in detail. The main extension by the present contribution compared with the former is the application of the method to load-effects which are variable in time. This study is included in the latter part of the paper.

There are anyhow good reasons for presenting the main principles of this design method generally first.

## DESCRIPTION OF THE METHOD

We study a general case of a design problem in which the condition for failure can be described by the corresponding inequality:

$$g(X_1,...,X_n) \leq 0 \qquad ... (1)$$

where $g(\cdot)$ = the corresponding limit state function, obtainable from the handbooks of statics; $X_1,...,X_n$ = the various quantities such as properties of materials, dimensions, loads or load-effects. All these quantities, invariable in time are random variables and we assume that their distributions are given in the standards.

As a design criterion we write

$$p_f = P(g(X_1,...,X_n) \leq \,) \leq p_{fa} \qquad ... (2)$$

where $p_f$ = failure probability, which simply means that the failure probability should be $p_{fa}$ at most. The $p_{fa}$ - values should be given in standards and depend on the type of structure.

As will be shown later, the method can also be applied to a case when the distributions are unknown, and we define the level of reliability not by $p_{fa}$, but by ß, defined by

$$m_Z - \beta \cdot \sigma_Z = 0 \qquad ... (3)$$

where

$$Z = g(X_1,...,X_n) \qquad ... (4)$$

and $m_Z$, $\sigma_Z$ are the mean value and standard deviation of Z.

With distributions differing from the normal distribution, this simplification leads to considerable errors, in the aimed reliability.

We write the design equation in a deterministic form

$$g(x_1^*,\ldots,x_n^*) = 0 \qquad \ldots (5)$$

with such design values that (2) is valid. As was shown in [4] there are several combinations of values $x_1^*,\ldots,x_n^*$ which satisfy the equation (5). We choose the following values for use in (5):

$$x_i^* = m_i - \beta_i \cdot \alpha_i \cdot \sigma_i \qquad \ldots (6)$$

where $m_i$ and $\sigma_i$ are the mean and st.d. of $X_i$ and the parameters $\alpha_i$ and $\beta_i$ are defined as follows:

The parameters $\beta_i$ describe the desired level of reliability, according to the form of the corresponding distribution:

$$F_i(m_i - \beta_i^- \cdot \sigma_i) = p_{fa} \qquad \ldots (7a)$$

$$1-F_i(m_i + \beta_i^+ \cdot \sigma_i) = p_{fa} \qquad \ldots (7b)$$

The parameters $\alpha_i$ describe the significance of the corresponding quantity and are defined as follows:

$$\alpha_i = a_i \;/\; \left(\sum_{j=1}^{n} a_j^2\right)^{1/2} \qquad \ldots (8)$$

where

$$a_j = \beta_j \cdot \sigma_j \cdot \partial g/\partial x_j \qquad \ldots (9)$$

From (9) it can be noted that the significance of the different quantities is thus proportional to the distance of the $p_{fa}$ - fractile from the mean and to the derivative $\partial g/\partial x_j$. In the design, the $m_i$ - values are chosen so that using the design values defined by (6), (5) is valid. Choice of the design values in this way is the main principle of the design method presented.

In cases in which the distributions of the different quantities are unknown the method can easily be applied distribution-free using the following relations:

$$x_i^* = m_i - \beta \cdot \alpha_i \cdot \sigma_i \qquad \ldots (10)$$

where $\alpha_i$ is defined by (8) but

$$a_i = \frac{\partial g}{\partial x_i} \cdot \sigma_i \qquad \ldots (11)$$

In this case the level of the reliability is not given by $p_{fa}$ but by $\beta$, in the way shown in (3).

## CHARACTERISTICS OF THE METHOD

Before studying the application of the method in cases which also include variable quantities, it is necessary to study some special features of this method.

Firstly, using (5) and (6) as design equations, it is not necessary to know the values of $\alpha_i$ exactly, while

$$\left.\frac{\partial p_f}{\partial a_i}\right|_{\vec{x} = \vec{x}^*} = 0 \qquad \text{by } i = 1,\dots,n \qquad \dots (12)$$

and therefore small variations of $\partial g/\partial x_i$, $\beta_i$ and $\sigma_i$ in definition of the parameters $\alpha_i$ do not influence the results.

It is only significant to note that always

$$\sum_{i=1}^{n} \alpha_i^2 = 1 \qquad \dots (13)$$

With (8) and (13), the α-values can be estimated so that they correspond approximately to the significance of the various quantities. A more exact method is division of function g(•) into subfunctions.

If we assume that $Y_i$ i = 1,...,m are the subfunctions of g(•), according to (6), we obtain

$$y_i^* = m_i - \beta_i \cdot \alpha_i \cdot \sigma_i \qquad \dots (14)$$

This is, however, only a $p_{fai}$ - fractile of the quantity $Y_i$ and we can treat the quantities $X_{ij}$ included in it in a way analogous to the way $Y_i$ was treated. We then obtain

$$x_{ij}^* = m_{ij} - \beta_{j/i} \cdot \alpha_{j/i} \cdot \sigma_{ij} \qquad \dots (15)$$

where

$$F_{ij}(m_{ij} \overset{-}{-} \beta_{j/i} \cdot \sigma_{ij}) = p_{fai} \qquad \dots (16a)$$

$$1 - F_{ij}(m_{ij} \overset{+}{+} \beta_{j/i} \cdot \sigma_{ij}) = p_{fai} \qquad \dots (16b)$$

$$\alpha_{j/i} = a_{j/i} / \left(\sum_{k=1}^{n} a_{k/i}^2\right)^{1/2} \qquad \dots (17)$$

where

$$a_{k/i} = \frac{\partial y_i}{\partial x_{ik}} \cdot \beta_{k/i} \cdot \sigma_{ik} \qquad \dots (18)$$

Should $Y_i$ and $X_{ij}$ have approximately the same distribution, we have

$$\beta_{j/i} = \beta_i \cdot \alpha_i = \beta_{ij} \cdot \alpha_i \qquad \dots (19)$$

where $\beta_{ij}$ is defined as in (7). In this way we obtain the design values

$$x_{ij}^* = m_{ij} - \beta_{ij} \cdot \alpha_i \cdot \alpha_{j/i} \cdot \sigma_{ij} \qquad \dots (20)$$

where $\beta_{ij}$ is defined as in (7), $\alpha_{j/i}$ as in (17) and $\alpha_i$ by (8) where

$$a_j = \frac{\partial g}{\partial y_i} \cdot \beta_j \cdot \sigma_j \qquad \text{... (21)}$$

It should be noted that the definition of $\alpha_i$ - values is approximate but because of (12) the small errors are not significant.

APPLICATION TO VARIABLE LOADS

All the quantities in the design criterion (1) are assumed to have distributions which remain invariable with time. We will now study a case in which some of these quantities are variable in time.

We first make a limitation concerning the type of these quantities, the variable load-effects. We assume that they all have the same dimension i.e. they are all e.g. moments or normal forces. Secondly, we assume that all the variable quantities have, to use the terminology of Ferry Borges [3] , the same duration of elementary interval and the same number of independent repetitions. These are both considerable simplifications of the general case treated earlier in [1] .

We assume now that the number of independent repetitions, r, is given, as well as the types of the momentary distributions of the variable loads as fundamental information in standards. To be on the safe side, it seems better to assume that the number r is rather small and to define the momentary distribution as the distribution of the maximum-value in the corresponding, arbitrary elementary interval.

The definition of the design values will now be made stepwise, so that the invariable quantities are first combined with the extreme-value of the combination of the different, momentary variable load -effects.

The momentary distribution of the combination can be trivially defined from the distributions of the various, usually additional quantities. After that we only need define approximately the mean and standard deviation of the distribution of the extreme-value.

We combine this approximate distribution with the distributions of the invariable quantities, and obtain with (8) the α-parameters for the different invariable quantities and for the extreme-value of the combination of the variable quantities.

The design values of the invariable quantities can then be defined with (6). We further study the approximate design value of the extreme value of the sum of the variable quantities:

$$y^* = m_y - \beta_y \cdot \alpha_y \cdot \sigma_y \qquad \text{... (21)}$$

In (21) all the parameters are approximate, but using (7) we know that

$$1-F_y(m_y+\beta_y \cdot \sigma_y) = p_{fa} \qquad \text{... (22)}$$

According to the form of the distribution we may now solve a new probability:

$$1-F_y(m_y+\beta_y\cdot\alpha_y\cdot\sigma_y) = p_{fay} > p_{fa} \qquad \text{... (23)}$$

It should be noted that in many cases the relation $p_{fay}$: $p_{fa}$ is independent of the values $m_y$ and $\sigma_y$, e.g. with the extreme type 1 distribution. Because of (12) and (13) the slight errors in $\alpha$ are not significant. For these reasons we can note that in spite of the approximate way of defining $\alpha_y$, we have a rather reliable value, $p_{fay}$ as a basis for the determination of the design values of the variable quantities.

The second step is to study the momentary combination of the different variable loads. The $p_{fay}$ - fractile with the extreme distribution corresponds to

$$1-(1-p_{fay})^{1/r} \approx p_{fay}/r \text{ - fractile} \qquad \text{... (24)}$$

with the momentary distribution. We denote this with

$$p_{fay}/r = p_{fam} \qquad \text{... (25)}$$

The design values of the different variable quantities may now be defined as earlier:

$$x_i^* = m_i-\beta_i\cdot\alpha_i\cdot\sigma_i \qquad \text{... (26)}$$

where $m_i$ and $\sigma_i$ are the mean value and the standard deviation of the different momentary distributions. The other parameters are defined as follows:

$$1-F(m_i+\beta_i\cdot\sigma_i) = p_{fam} \qquad \text{... (27)}$$

$$\alpha_i = a_i/(\sum_{j=1}^{m} a_j^2)^{1/2} \qquad \text{... (28)}$$

$$\text{where } a_j = \frac{\partial y}{\partial x_j}\cdot\beta_j\cdot\sigma_j \qquad \text{... (29)}$$

in which y simply means the sum $x_1+,\dots,+x_m$, and m is the number of the variable loads.

EXAMPLE

We now study a simple function $g(\cdot)$ with resistance R, invariable load-effect $S_g$, and two variable load-effects $S_{p1}$ and $S_{p2}$:

$$g(\cdot) = R-S_g-S_{p1}-S_{p2} \leq 0 \qquad \text{... (30)}$$

We take as fundamental information, which is supposed to be given in the standards, the following values:

$$p_{fa} = 10^{-6} \quad ; \quad r = 100$$

Further, we assume that the variation-coefficients and the types of distributions are given in table 1. The dependence $p_{fa}/\beta$ is given in fig. 1.

| | | Type of distr. | $m_i$ | $V_i$ |
|---|---|---|---|---|
| R | $X_1$ | Log.normal | 6,0 | 0,10 |
| $S_g$ | $X_2$ | Normal | 1,0 | 0,05 |
| $S_{p1}$ | $X_3$ | Extreme I | 1,0 | 0,15 |
| $S_{p2}$ | $X_4$ | Extreme I | 0,5 | 0,10 |
| | | | | |

Table 1

| | $m_i$ | $\beta_i$ | $\sigma_i$ | $\beta_i \cdot \sigma_i$ | $\alpha_i$ | $x_i^*$ |
|---|---|---|---|---|---|---|
| $X_1$ | 5,9 | 3,8 | 0,59 | 2,24 | 0,81 | 4,08 |
| $X_2$ | 1,0 | 4,7 | 0,05 | 0,24 | 0,08 | 1,02 |
| $X_3+X_4$ | 2,1 | 10,4 | 0.16 | 1,67 | 0,59 | |
| $X_3$ | 1,0 | 9,7 | 0,15 | 1,46 | 0,95 | 2,39 |
| $X_4$ | 0,5 | 9,7 | 0,05 | 0,49 | 0,32 | 0,66 |

Table 2 $\Phi = 2{,}36$

| | $m_i$ | $\beta_i$ | $\sigma_i$ | $\beta_i \cdot \sigma_i$ | $\alpha_i$ | $x_i^*$ |
|---|---|---|---|---|---|---|
| $X_1$ | 5,0 | 3,8 | 0,50 | 1,90 | 0,75 | 3,57 |
| $X_2$ | 1,0 | 4,7 | 0,05 | 0,24 | 0,10 | 1,02 |
| $X_3$ | 1,0 | 10,4 | 0,15 | 1,56 | 0,62 | 1,97 |
| $X_4$ | 0,5 | 10,4 | 0,05 | 0,52 | 0,21 | 0,61 |

Table 3 $\Phi = 2{,}00$

| | $m_i$ | $\beta_i$ | $\sigma_i$ | $\beta_i \cdot \sigma_i$ | $\alpha_i$ | $x_i^*$ |
|---|---|---|---|---|---|---|
| $X_1$ | 6,0 | 3,8 | 0,60 | 2,28 | 0,82 | 4,13 |
| $X_2$ | 1,0 | 4,7 | 0,05 | 0,24 | 0,08 | 1,02 |
| $X_3$ | 1,52 | 10,4 | 0,15 | 1,56 | 0,54 | 2,36 |
| $X_4$ | 0,67 | 10,4 | 0,05 | 0,52 | 0,18 | 0,76 |

Table 4 $\Phi = 2{,}40$

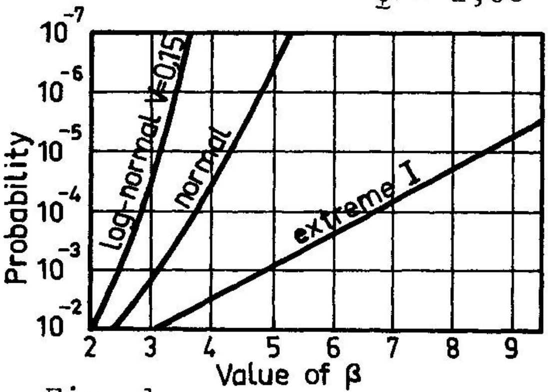


Fig. 1.

We first obtain as mean and standard deviation of $(S_{p1}+S_{p2})_e$ :

$$m = 2{,}1 \quad ; \quad \sigma = 0{,}16$$

where index e indicates the extreme. According to (25) and (27) we obtain

$$p_{fam} = 4 \cdot 10^{-6} \quad ; \quad \beta_3 = \beta_4 = 9{,}7$$

and the results with (26) are given in table 2.

After this, the same example is calculated in two different ways. Firstly, by studying only the combination of the momentary load-effects. This is a kind of lower-bound solution which approaches the "exact" solution with small values or r. The results of this calculation are given in table 3. Secondly, we study the combination of the distributions of the extreme values, which contrarily to the former case is an upper bound solution and is approached by the "exact" solution in cases in which the significance of the variable loads is concentrated in one of them. In our example, it can be seen that the significance of quantity $X_3$ is superior in relation to $X_4$ and therefore the results in table 2 and table 4 are rather close to each other.

The differences between the cases may be described by the fictive central safety-factors $\Phi$ corresponding to the mean values obtained in the different cases.

## CONCLUSIONS

The present design method implies standardized distributions for the most usual quantities needed in the design of structures. This will anyhow be a requirement in future irrespective of the type of design method chosen. Statistical research on the types of distributions of the various quantities is therefore necessary and useful.

The method has several applications, some of which are more exact but complicated and the others which are simplier. These will be presented on a larger scale in another paper to be published later.

LITERATURE

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SUMMARY

A statistical design method is presented which is characterized by using weighted fractiles as design values. This paper gives special attention to the application in which quantities invariable in time are combined with quantities variable in time, primarly load-effects. An example is also presented to illustrate the method.

RESUME

On présente une méthode statistique de dimensionnement utilisant des valeurs pondérées en-dessous d'un certain seuil de la courbe de fréquence (courbe de Gauss) comme grandeurs de dimensionnement. Ce travail traite tout spécialement le cas de grandeurs invariables dans le temps combinées avec des grandeurs variables, en premier lieu les influences de la charge. Un exemple numérique est présenté pour illustrer la méthode.

ZUSAMMENFASSUNG

Es wird eine statistische Bemessungsmethode gezeigt, die durch gewichtete Fraktilen als Bemessungsgrössen charakterisiert ist. Die Abhandlung beachtet speziell jene Anwendung, bei der die zeitunabhängigen Grössen mit zeitabhängigen Grössen kombiniert werden, hauptsächlich bei Belastungseffekten. Ein Beispiel veranschaulicht die Methode.

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