V

Reliability Under Uncertain Parameters, Stochastic Loads and Resistances

Notion de sécurité, en dépit de paramètres incertains, de charges et de résistances stochastiques

Zuverlässigkeit unter Berücksichtigung unsicherer Parameter, zufälliger Lasten und Festigkeit

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I Description of the Problem

The feasibility of developing the mathematical tools to investigate the reliability of structural systems under probabilistic and stochastic loading is studied. The following cases are considered: 1) The reliability of systems when the probability density function of the resistance to loading is fully known, but the mean of the load distribution is a random variable governed by a probability distribution. 2) Reliability of systems subjected to narrowband, Gaussian loading when the resistance is given by a known density function. 3) The reliability of systems with constant (with respect to time), but probabilistic loads, where the resistance distribution is dependent on the load level and time.

II Reliability of Systems with One Distribution Partially Known, the Other Fully Known

Often, it is not possible to assume that the parameters of the resistance or load distribution are fully known. In general, sampling is done in batches, i.e., a set of parameter values is obtained for each sample batch. Large differences between the various means, for example, make point estimates inaccurate. Hence, it is necessary to consider the parameters as random variables with their own density functions. In this paper only the mean will be considered as a random variable; all other parameters are assumed to be known.

Consider the case where the mean of the load is a random variable and all other parameters are known. Let

$f_r(r_o) \sim$ p.d.f. of the resistance

$f_\ell(\ell_o) \sim$ p.d.f. of the load with unknown mean

$f_{\mu_1}(\mu_{1o}) \sim$ p.d.f. of the mean of the load

$f_{r,1}(r_o,1_o) \sim$ joint p.d.f. of load and resistance

Then,

$$\rho/\mu_1 = P(r > \ell/\mu_1)$$

$$\rho/\mu_1 = \int_0^\infty dr_o \int_0^r dl_o \cdot f_{r,1}(r_o, l_o) = g(\mu_1) \qquad (1)$$

where ρ is the reliability. The above defines the relationship between ρ and μ_1. It is now possible to use the change of variable procedure, yielding

$$f_\rho(\rho_o) = \frac{1}{\left|\frac{\partial \rho}{\partial \mu_1}\right|} f_{\mu_1}[g^{-1}(\mu_1)] \qquad (2)$$

This result will now be applied to the particular case where all distributions are Gaussian and r,l are independent, i.e.,

$$f_r(r_o) \sim N(\mu_r, \sigma_r^2)$$

$$f_{\ell/\mu_1}(\ell_o/\mu_1) \sim N(\mu_1, \sigma_1^2)$$

$$f_{\mu_1}(\mu_{1_o}) \sim N(\mu_{\mu_1}, \sigma_{\mu_1}^2)$$

$$f_x(x/\mu_1) = f_{r,1/\mu_1}(r_o, l_o/\mu_1) \sim N(\mu_r - \mu_1, \sigma_r^2 + \sigma_1^2)$$

Using eqn. 1 and 2, it can be shown that the distribution of reliability is given by,

$$f_\rho(\rho_o) = \frac{\sqrt{\sigma_r^2+\sigma_1^2}}{\sigma_{\mu_1}} \exp\left[[\mathrm{erf}^{-1}(2\rho-1)]^2 - \frac{(\mu_r - \sqrt{\sigma_r^2+\sigma_1^2}\{\mathrm{erf}^{-1}(2\rho-1)\} - \mu_{\mu_1})^2}{\sigma_{\mu_1}^2}\right] \qquad (3)$$

Applying equation 3 to a given problem is complicated, since it requires obtaining the inverse of an error function. It is simpler to use a numerical approach. The method recommended is perfectly general and can easily be applied to distributions other than normal. The same procedure was programmed for the computer for the case where

$$f_r(r_o) \sim N(4, \tfrac{1}{\sqrt{2}}) \ , \quad f_{1/\mu_1}(\mu_1) \sim N(\mu_1, \tfrac{1}{\sqrt{2}}) \ , \quad f_{\mu_1}(\mu_1) \sim N(1.5, \tfrac{1}{12})$$

Figure 1 shows probability distribution function for ρ. It can be seen that $f_\rho(\rho_o)$ is skewed to the left. Furthermore, the density function approaches zero asymptotically on both sides (see Figure 1).

The sensitivity of the distribution was tested by running the computer program for various values of the parameters: 1) In Figure 2 the distribution sensitivity for μ_{μ_1} is shown. Note that, as μ_{μ_1} becomes smaller, the mean of the reliability distribution approaches 1.0 and the spread of $f_\rho(\rho_o)$ decreases. Thus, one can conclude that the standard deviation of $f_\rho(\rho_o)$ is quite sensitive to the mean of the μ_1 probability density function. 2) Figure 3 shows the distribution sensitivity for σ_{μ_1}. The mean of $f_\rho(\rho_o)$ is not strongly influenced by small changes in σ_{μ_1}. As expected, the spread of $f_\rho(\rho_o)$ decreases as σ_{μ_1} gets smaller. 3) The distribution sensitivity for σ_1 is shown in Figure 4. It can be seen that σ_1 influences both the mean of $f_\rho(\rho_o)$, μ_ρ, and the variance σ_ρ^2. 4) Similarly, influence of σ_r and μ_r on $f_\rho(\rho_o)$ was investigated. The effect of σ_r on $f_\rho(\rho_o)$ is similar to the effect of σ_1. A smaller value of μ_r is associated with a smaller μ_ρ and a greater spread of $f_\rho(\rho_o)$.

III <u>Reliability of a Stochastically Loaded System with Probabilistic Resistance</u>

Since the resistance is generally not known accurately, it should be considered as a random variable. Let the loading be described in the following form. $L(t) = L_o + \ell(t)$ where L_o = constant and $\ell(t)$ = stochastic loading. Let $\ell(t)$ be narrowband, stationary, Gaussian loading with zero mean and known variance. The distribution of peaks above level α is Rayleigh (ref. 2). By definition of conditional probability

$$P\{\text{reliable}\} = P\{\text{reliable load is above } \alpha\} \cdot P\{\text{load is above } \alpha\} \quad 4$$

where α is any load level or

$$P\{r > \ell\} = P\{r > \ell/\ell > \alpha\}P\{\ell > \alpha\} \quad 5$$

Let

$$\rho_o = P\{r > \ell/\ell = \alpha\} \quad 6$$

which will be called $g(\alpha)$. Note that ρ_o is a limiting value of ρ, since ℓ cannot be exactly α. Whenever ℓ is larger than α, ρ will be smaller than ρ_o. Using equations 4, 5 and 6 and Rayleigh distribution for peaks, the probability distribution for reliability is given by the following equation (see ref. 1).

$$f_\rho(\rho_o) = \frac{d}{d\rho_o}\left[\rho_o \cdot \exp\frac{\{L_o - g^{-1}(\rho_o)\}}{2\sigma_y^2}\right] = \exp\left\{\frac{L_o - g^{-1}(\rho_o)}{2\sigma_y^2}\right\}\left[1 - \frac{\rho_o}{2\sigma_y^2}\frac{d\{g^{-1}(p_o)\}}{d\rho_o}\right] \quad 7$$

Since finding $\dfrac{d\{g^{-1}(\rho_o)\}}{d\rho_o}$ is complicated for most probability distributions a numerical approach is presented here as an example. Consider an example with the following parameters

$$L_o = 0 \;, \quad \sigma_y = \frac{1}{\sqrt{2}} \;, \quad f_r(r_o) \sim N(4,1)$$

The probability density function, $f_\rho(\rho_o)$, is shown in Figure 5. Note that $f_\rho(\rho_o)$ approaches a limit of $\rho_o = 0.999968$ asymptotically. Since the Rayleigh distribution is only defined for positive values of α, the reliability associated with the smallest α must be the maximum ρ_o, i.e., for the numerical values given above $(\rho_o)_{max} = 0.999968$. The sensitivity of the distribution was tested by running a computer program for various values of the parameters: 1) It was observed that $f_\rho(\rho_o)$ is hardly influenced by small changes of σ_y. 2) Figure 6 shows the distribution sensitivity for L_o. For increasing mean load level, L_o, the reliability distribution shifts to the left. Furthermore, the maximum ρ_o becomes smaller, e.g., for $L_o = 0.0$, $(\rho_o)_{max} = 0.999968$; $L_o = 0.5$, $(\rho_o)_{max} \doteq 0.999730$; $L_o = 1.0$, $(\rho_o)_{max} = 0.998462$. 3) The distribution sensitivity for σ_r is shown in Figure 7. It seems that the reliability distribution is quite strongly influenced by changes in σ_r. Furthermore, $(\rho_o)_{max}$ changes with σ_r, e.g., for $\sigma_r = 0.75$, $(\rho_o)_{max} = 0.99999988$; $\sigma_r = 1.00$, $(\rho_o)_{max} = 0.999968$;

$\sigma_r = 1.25$, $(\rho_o)_{max} = 0.999233$. 4) The influence of μ_r on $f_\rho(\rho_o)$ is shown in Figure 8. As expected $(\rho_o)_{max}$ is affected by μ_r, e.g., $\mu_r = 3.50$, $(\rho_o)_{max} = 0.999730$; $\mu_r = 4.00$, $(\rho_o)_{max} = 0.999968$.

IV Reliability of Systems with Stochastic, Load Dependent Resistance

Structural systems subjected to loads will generally experience a deterioration of strength with time due to such physical phenomena as creep, metal fatigue, etc. The rate of loss of strength often depends on the load level, e.g., a reinforced concrete beam will usually experience a greater rate of creep if it is subjected to a larger load than if it is subjected to a smaller load. Consider a model whose behavior is: a) Loads are random, but not time varying. b) The only parameter of the resistance distribution, which is load and time dependent, is the mean, $\mu_r(r\ t,\ell)$. c) A family of functions exists completely defining the change of the mean μ_r with respect to time and load. Condition (c) stated above is shown in Figure 9. Given a load ℓ_1 and time t_1, it is possible to find the mean of the resistance which in turn defines the probability density function of the resistance. Let $f_\ell(\ell_o) \sim$ p.d.f of the load, $\mu_r = g(\ell,t)$, $f_{r/\mu_r}(r_o/\mu_r) \sim$ p.d.f of the resistance given the mean. Consider a particular time, t_o. The random variable μ_r at time, t_o is only a function of the random variable ℓ. It is possible to find f_{μ_r} by using the change of variable procedure. $f_{\mu_r}(\mu_r) = \dfrac{1}{\left|\dfrac{\partial g(\mu_r)}{\partial \ell}\right|} f_\ell\{g^{-1}(\ell)\}$

Following the same procedure discussed in previous section one obtains

$$f_\rho(\rho_o/t) = \frac{1}{\left|\dfrac{\partial \rho}{\partial \mu_r}\right|} f_{\mu_r}\{g^{-1}(\mu_r)\} \tag{8}$$

where $\rho/\mu_r = g(\mu_r) = \int_o^\infty dr_o \int_o^r d\ell_o f_{r,\ell}(r_o,\ell_o)$.

This result will now be applied to the particular case where both the resistance and load distributions are normal and the mean is related to the load and time by a linear function, i.e.,

$$\text{1)}\ f_\ell(\ell_o) \sim N(\mu_\ell,\sigma_\ell) \quad \text{2)}\ \mu_r = -(m\ell + b)t + c = g(\ell,t) \qquad \text{3)}\ f_{r/\mu_r}(r_o/\mu_r) \sim N(\mu_r,\sigma_r) \tag{9}$$

Then, $\rho = P(r>\ell/\mu_r) = P(r-\ell>0/\mu_r) = \dfrac{1}{\sqrt{2\pi}\sqrt{\sigma_r^2+\sigma_\ell^2}} \int_o^\infty \exp\left[-\dfrac{\{x-(\mu_r-\mu_\ell)\}^2}{2\sqrt{\sigma_r^2+\sigma_\ell^2}}\right] dx$

Following similar procedure as mentioned in previous sections, it can be shown that

$$f_{\rho/t}(\rho_o/t) = \frac{\sqrt{\sigma_r^2+\sigma_\ell^2}}{mt \cdot \sigma_\ell} \cdot \exp\left[\{\text{erf}^{-1}(2\rho-1)\}^2 - \frac{1}{2(\sigma_\ell mt)^2} \{\mu_\ell\sqrt{\sigma_r^2+\sigma_\ell^2}[\text{erf}^{-1}(2\rho-1)] - c + t[b-m\mu_\ell]\}^2\right] \tag{10}$$

This equation cannot be used directly, since it involves the inverse of the error function. Hence, it is necessary to develop a numerical method. Consider the following numerical example

$$f_{\ell}(\ell_o) \sim N(5.1) \ , \quad \mu_r = -0.5 \cdot t \cdot \ell + 10.0 \ , \quad f_{r/\mu_r}(r_o/\mu_r) \sim N(\mu_r,1) \quad (11)$$

The results of the computation are shown in Figure 10. At T = 0 the reliability distribution is asymptotic to $\rho = 1$ with half of the probability mass concentrated between $\rho = 0.99999921$ and $\rho = 1.0$. At T = 2 the reliability function is symmetric about $\rho = 0.5$ and asymptotic to both zero and one. At T = 4 there is a probability of 0.5 that ρ will lie between 0.0 and 0.000005413. Thus, it can be seen that the probability mass shifts from a reliability close to 1.0 to a reliability close to 0.0 with the passage of time. Since ρ is a function of time, $f_\rho(\rho_o)$ is also time dependent. In Figure 11 $f_\rho(\rho_o)$ vs. time is shown. Prior to t = 1 most of the probability mass is concentrated between $\rho = 0.99$ and $\rho = 1.00$. Close to time period 1 the probability mass moves through the point $\rho = 0.99$. During the subsequent time periods the probability of ρ being 0.99 decreases. The sensitivity of the distributions was tested by running a computer program for various values of the parameters and three different time periods (t = 0.0, t = 1.5, t = 3.0; circled numbers on subsequent graphs indicate the corresponding reliability distribution). 1) In Figure 12 the distribution sensitivity for μ_ℓ is shown. As μ_ℓ becomes larger, the probability mass shifts from a high to a low reliability more rapidly. Furthermore, for larger values of μ_ℓ the asymptote $\rho = 1.0$ is approached faster. 2) Figure 13 shows the distribution sensitivity for σ_ℓ. Except for higher peaks corresponding to lower values of σ_ℓ (Figure 13) small changes of this parameter do not influence $f_\rho(\rho_o)$ significantly. 3) The distribution sensitivity for σ_r is shown in Figure 14. Similarly to σ_ℓ, σ_r hardly influences the reliability distribution. Note that in this case the peaks are higher for larger values of σ_r. 4) The influence of the slope, m, on the reliability distribution is shown in Figure 15. Small changes in m cause fairly large changes in $f_\rho(\rho_o)$. As the slope decreases, the shifting of probability mass from a high reliability to a low reliability takes place more rapidly, i.e. a system has a greater probability of survival over time if the slope is large. 5) The influence of the intercept, c, shown in Figure 16 is quite pronounced. For large values of c the shifting of probability mass takes place at a later time period.

V Conclusions

As was shown, the derivations of the reliability distributions is fairly simple for the case where only one parameter is a random variable. However, even in this case it is not possible to find a usable, analytical solution for the distributions chosen. Further work needs to be done in trying to develop simple closed form solutions — possibly using approximations or investigating various distributions. It would be interesting to see this idea expanded to the case where not only the mean, but also the variance is random.

References:

1. Reliability under Uncertain Parameters, Stochastic Loads and Resistances, K. Steven Ott, Engineer's Thesis, Stanford 1969.
2. Random Vibrations, Crandall and Mark, Academic Press, 1963.
3. Probability, Random Variables, and Stochastic Processes, A. Papoulis, McGraw-Hill, New York, 1965.

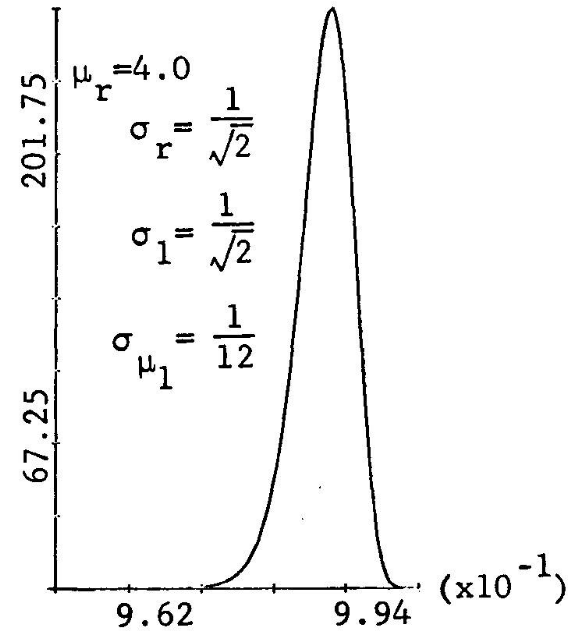

FIG. 1

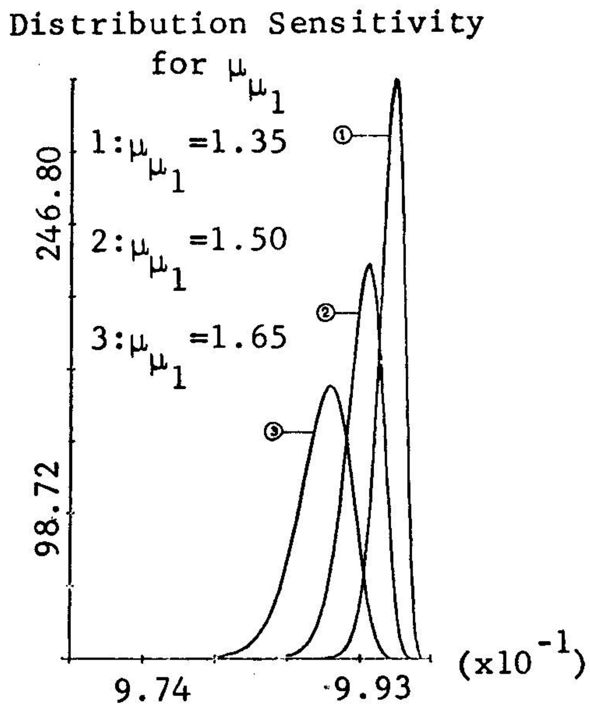

FIG. 2

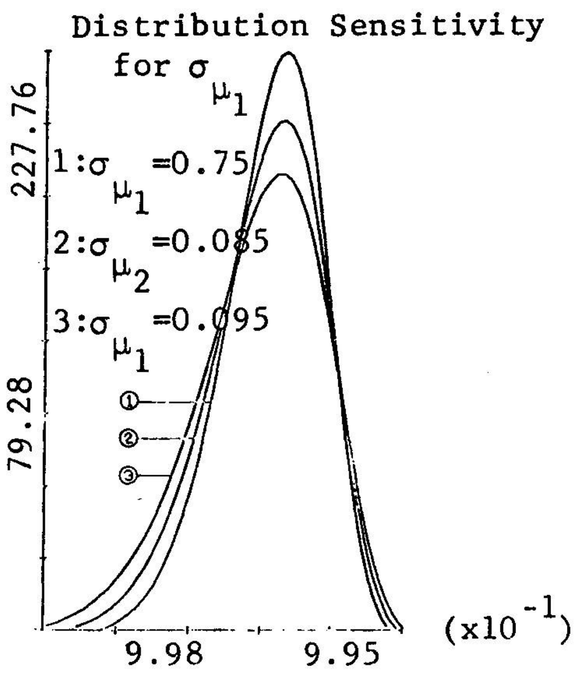

FIG. 3

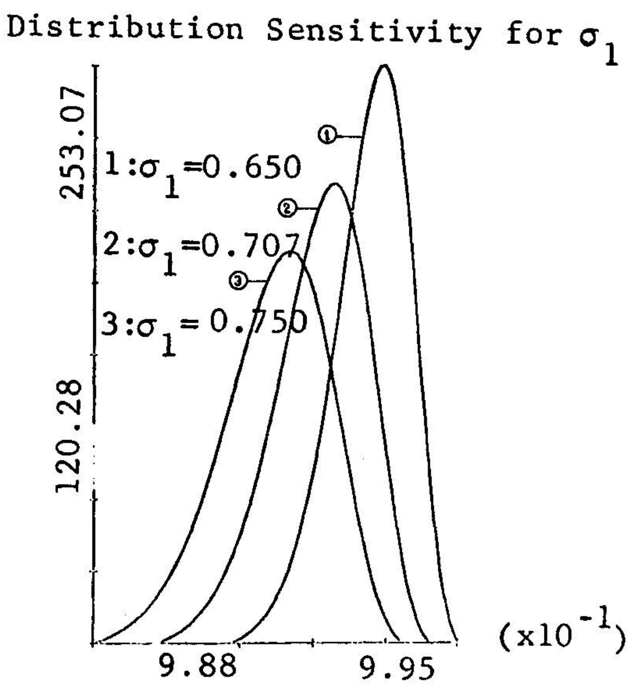

FIG. 4

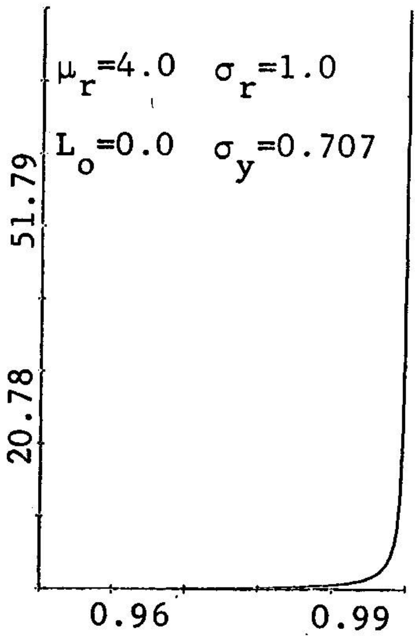

FIG. 5

Distribution Sensitivity for σ_y

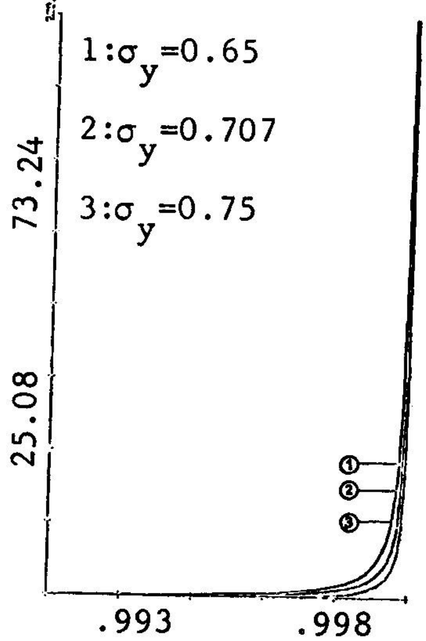

FIG. 6

Distribution Sensitivity for L_o

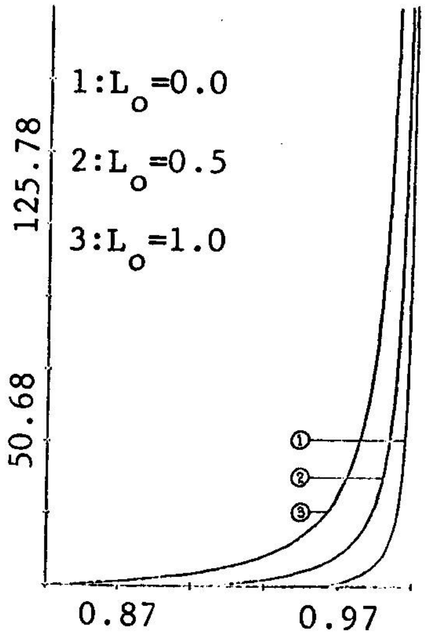

FIG. 7

Distribution Sensitivity for σ_r

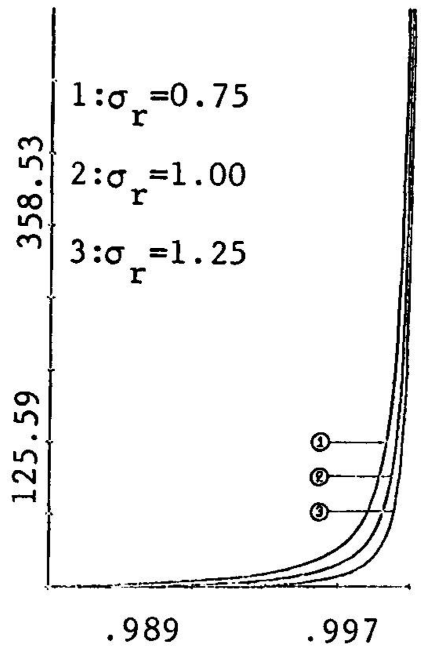

FIG. 8

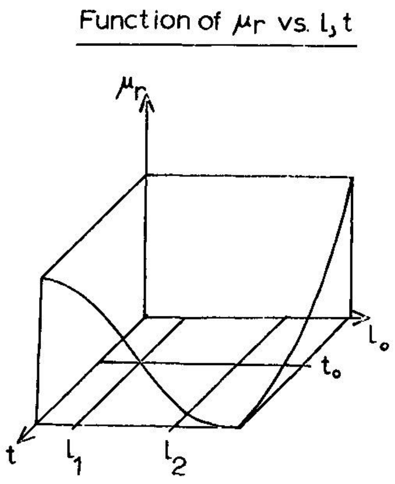

Figure 9

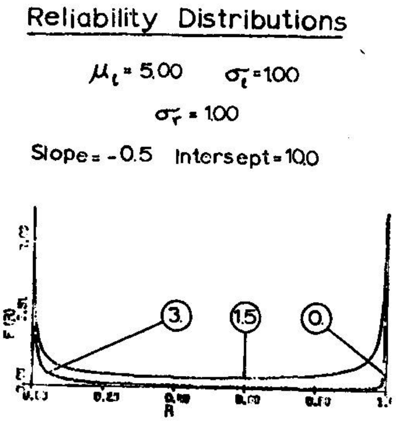

Figure 10

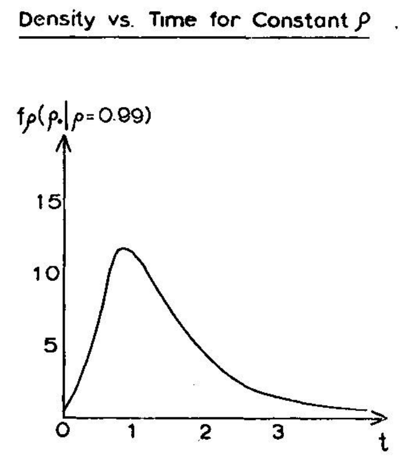

Figure 11

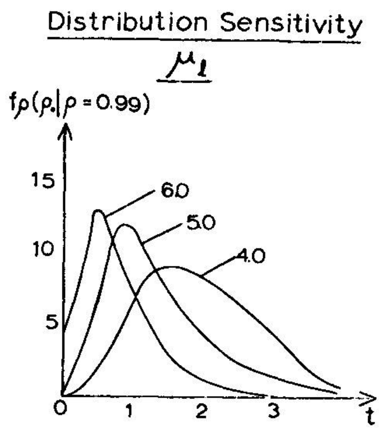

Figure 12

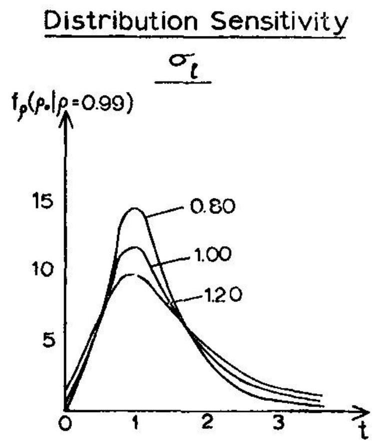

Figure 13

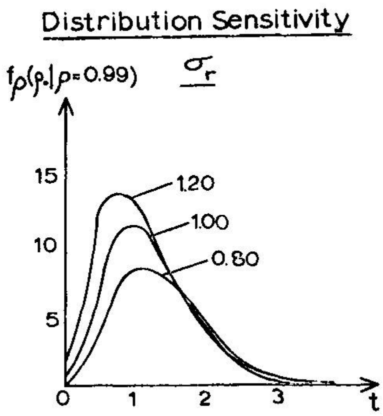

Figure 14

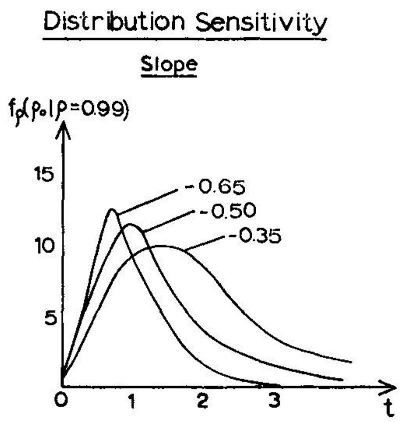

Figure 15

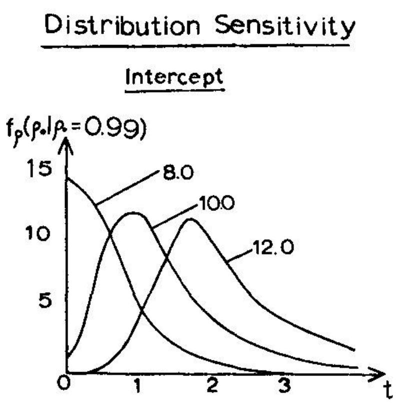

Figure 16