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## **Deformation of Concrete Structures Theoretical Basis for the Calculation**

**Déformation des structures en béton  
Base théorique pour le calcul**

**Formänderungen von Stahlbetontragwerken  
Theoretische Grundlagen für die Berechnung**

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### **SUMMARY**

This document presents the theoretical basis for a calculation of deformation consistent with the 1978 CEB/FIP Model Code. It has been elaborated by the authors within Commission V of CEB in view of a "Cracking and Deformation" manual and was approved by the General Assembly of CEB in Budapest in June 1980. It deals with the calculation of the curvatures and deformations resulting from bending moments with possible axial load. Cracking, creep and shrinkage are taken into account. A further publication will present practical applications and a comparison with laboratory test results.

### **RÉSUMÉ**

Ce document présente la base théorique pour un calcul des déformations conforme au Code-Modèle CEB/FIP 1978. Il a été élaboré par les auteurs au sein de la Commission V du CEB en vue de la rédaction d'un manuel «Fissuration et Déformation» et a été approuvé par l'Assemblée générale du CEB à Budapest en juin 1980. Il traite le calcul des courbures et des déformations par suite de moments de flexion avec un effort normal éventuel, tout en tenant compte de la fissuration, du fluage et du retrait. Une publication ultérieure présentera des applications pratiques et une comparaison avec des résultats d'essais en laboratoire.

### **ZUSAMMENFASSUNG**

Im vorliegenden Bericht werden die theoretischen Grundlagen für die Berechnung von Formänderungen gemäss CEB/FIP-Mustervorschrift dargestellt. Diese Grundlagen wurden von den Verfassern im Rahmen der Kommission V des CEB im Hinblick auf ein Handbuch «Rissbildung und Verformung» erarbeitet und von der Generalversammlung des CEB in Budapest im Juni 1980 bestätigt. Sie behandeln die Berechnung von Krümmungen und Verformungen, welche aufgrund von Biegemomenten eventuell mit einer Normalkraft zusammen entstehen. Der Rissebildung sowie dem Kriechen und Schwinden wird dabei Rechnung getragen. Eine später erscheinende Publikation wird Anwendungsbeispiele und einen Vergleich mit Resultaten aus Laborversuchen beinhalten.



## INTRODUCTION

The 1978 CEB-FIP Model Code gives in chapter 16 the principles for the calculation of deformations.

In order to enable the practising engineer to use these principles, it is important to supply him with as clear and simple tools as the subject allows. The object is to introduce a physical model of very general validity by means of which the entire cracking problem can be reduced to the simple state I and state II<sub>0</sub> cases. The distribution ratio  $c$  between these states is given by formula [15.5] of the Model Code. With the physical model proposed, the calculation of curvatures remains the same whatever the value of  $c$ . Any improvement to formula [15.5] in future years will not change in any way the principle of the method.

The considerations apply to linear systems subjected to bending; indications with regard to expanding to plane systems are also given. The influence of shear and torsion will be dealt with separately.

## 1. BASIC ASSUMPTIONS

### 1.1 Mean strain of the reinforcement

Let  $\Delta\ell$  be the total elongation of a reinforced concrete tie member of length  $\ell$ , subjected to a constant axial tensile force  $N = \sigma_{s2} \cdot A_s$ .

The mean strain of the reinforcement is equal to (fig. 1.1):

$$\varepsilon_{sm} = \frac{\Delta\ell}{\ell} = \varepsilon_{s2} - \Delta\varepsilon_s \quad (1.1)$$

where  $\Delta\varepsilon_s$ , representing the contribution of the concrete in tension between the cracks, follows a hyperbolic law above  $\sigma_{sr}$ , which has been confirmed experimentally:

$$\Delta\varepsilon_s = \Delta\varepsilon_{smax} \cdot (\sigma_{sr}/\sigma_{s2}) \quad (1.2)$$

By replacing  $\Delta\varepsilon_s$  in (1.1) one obtains successively:

$$\begin{aligned} \varepsilon_{sm} &= \varepsilon_{s2} - \Delta\varepsilon_{smax} \cdot (\sigma_{sr}/\sigma_{s2}) \\ &= \varepsilon_{s2} - (\varepsilon_{sr} - \varepsilon_{cr}) \cdot (\sigma_{sr}/\sigma_{s2}) \\ &= \varepsilon_{s2} \cdot (1 - (\sigma_{sr}/(\varepsilon_{s2} \cdot \varepsilon_{s2})) \cdot (\sigma_{sr}/\sigma_{s2})) + \varepsilon_{s1} \cdot (\sigma_{sr}/\sigma_{s2})^2 \\ &= \varepsilon_{s2} \cdot (1 - (\sigma_{sr}/\sigma_{s2})^2) + \varepsilon_{s1} \cdot (\sigma_{sr}/\sigma_{s2})^2 \end{aligned}$$

wherefrom

$$\varepsilon_{sm} = (1-c) \cdot \varepsilon_{s1} + c \cdot \varepsilon_{s2} \quad (1.3)$$

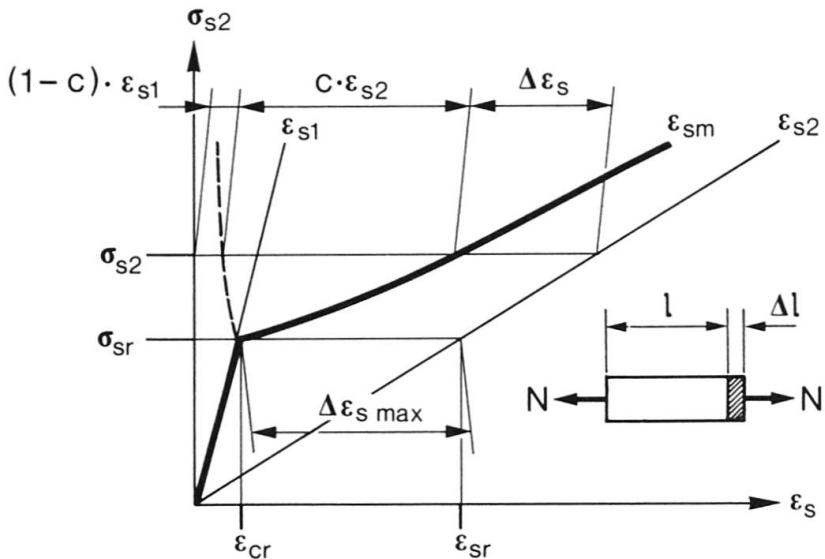


Fig. 1.1 Stress-strain curve for the reinforcing steel

Notation :

$$c = 1 - \frac{\sigma_{sr}^2}{\sigma_{s2}^2} \quad - \text{distribution ratio; } \quad (1.4)$$

$\sigma_{s2}$  - stress in the reinforcement in the cracked section, under the combination of actions under consideration;

$\sigma_{sr}$  - stress in the reinforcement calculated on the assumption of a cracked section, where the maximum tensile stress in the concrete (uncracked section) is taken equal to (clause 16.2.1 of the Model Code):

$f_{ctk \ 0,05}$  : in order to prevent damage

$f_{ctm}$  : in order to calculate the camber;

$\epsilon_{s1}$  - strain in the reinforcement situated in the embedment zone in state I, i.e. taking into consideration the uncracked section;

$\epsilon_{s2}$  - strain in the reinforcement situated in the embedment zone in cracked state II, i.e. neglecting the contribution of the concrete in tension between the cracks;

$\epsilon_{sr}$  - strain in the reinforcement  $\epsilon_{s2}$ , corresponding to  $\sigma_{sr}$ ;

$\epsilon_{cr}$  - strain in the concrete at the level of the reinforcement, corresponding to  $\sigma_{sr}$ ;

$A_s$  - area of the reinforcement.



The formulation (1.3) of the mean strain  $\varepsilon_{sm}$  in the reinforcement enables us to calculate the deformations of an element subjected to tension; the same expression is proposed for the calculation of deformations due to bending.

*Note: In order to calculate the crack widths, one only needs the increase of the mean strain of the steel with respect to that of the adjacent concrete ( $c \cdot \varepsilon_{s2}$ ), denoted by  $\varepsilon_{sm}$  in eq. [15.5] of the Model Code.*

In order to take into account the bond properties of the bars and the influence of the duration of the application and the repetition of loads, the Model Code proposes in clause 15.2.3 to introduce two coefficients  $\beta_1$  and  $\beta_2$  which correct the expression giving  $c$ :

$$c = 1 - \beta_1 \cdot \beta_2 \cdot \left( \frac{\sigma_{sr}}{\sigma_{s2}} \right)^2 \quad (1.5)$$

with

$\beta_1 = \frac{1}{2,5 \cdot K_1}$  coefficient which characterizes the bond properties of the bars:

$$K_1 = 0,4 \text{ for high bond bars,}$$

$$K_1 = 0,8 \text{ for plain bars;}$$

$\beta_2$  - coefficient representing the influence of the duration of the application and repetition of the loads:

$$\beta_2 = 1,0 \text{ at first loading,}$$

$\beta_2 = 0,5$  for loads applied in a sustained manner or for a large number of load cycles.

## 1.2 Calculation models

From the expression (1.3):

$$\varepsilon_{sm} = \frac{\Delta\ell}{\ell} = (1-c) \cdot \varepsilon_{s1} + c \cdot \varepsilon_{s2}$$

calculation models are established, which are equally valid for elements subjected to pure tension or to bending.

Elements subjected to pure tension ( $N = \text{const.}$  along the length  $\ell$ )

The real element can be represented by means of a model composed of two parts (fig. 1.2):

- one acting in state I (uncracked sections),
- the other in state  $II_0$  (naked state II, cracked sections: only the concrete in compression and the reinforcement are considered).

The respective lengths  $\ell_1$  and  $\ell_2$  are to be defined:

As

$$\varepsilon_{sm} = \frac{\Delta\ell}{\ell} = \frac{\varepsilon_{s1} \cdot \ell_1}{\ell} + \frac{\varepsilon_{s2} \cdot \ell_2}{\ell}$$

the comparison with expression (1.3) gives:

$$\ell_1 = (1-c) \cdot \ell \quad \ell_2 = c \cdot \ell$$

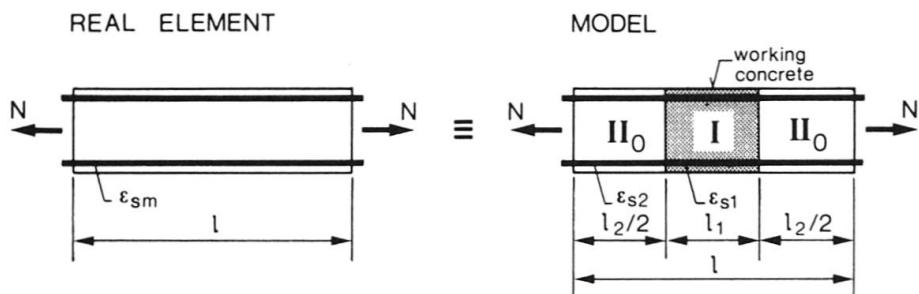


Fig. 1.2 Calculation model - pure tension

*Note: It is also possible to consider a model composed of two elements of length  $\ell$ , one of which acting in state I, is subjected to  $(1-c)$ -times the load effects and the other, acting in state  $II_0$ , subjected to  $c$ -times the load effects (fig. 1.2a):*

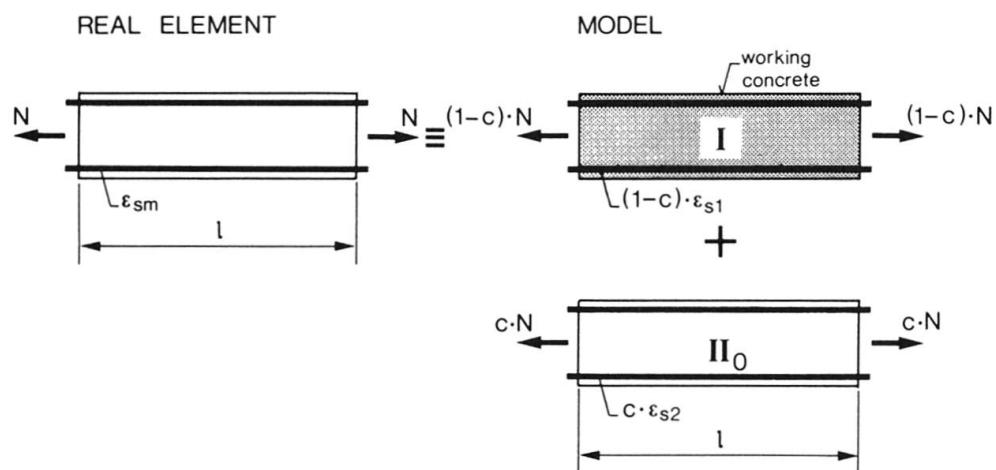


Fig. 1.2a Calculation model - pure tension

#### Elements subjected to pure bending ( $M = \text{const.}$ along a length $s$ )

The real element of length  $s$ , along which the bending moment can be assumed constant, is once more replaced by a model composed of two parts (fig. 1.3):

- one acting in state I,
- one acting in state  $II_0$ .



From the mean strain in the reinforcement in tension

$$\varepsilon_{sm} = \frac{\Delta s_s}{s} = \frac{\varepsilon_{s1} \cdot s_1}{s} + \frac{\varepsilon_{s2} \cdot s_2}{s} = (1-c) \cdot \varepsilon_{s1} + c \cdot \varepsilon_{s2}$$

one obtains lengths  $s_1$  (state I) and  $s_2$  (state  $II_0$ ):

$$s_1 = (1-c) \cdot s \quad s_2 = c \cdot s$$

Therefore, the mean strain in the concrete in the upper fibre is:

$$\varepsilon_{cm} = \frac{\Delta s_c}{s} = \frac{\varepsilon_{c1} \cdot s_1}{s} + \frac{\varepsilon_{c2} \cdot s_2}{s} = (1-c) \cdot \varepsilon_{c1} + c \cdot \varepsilon_{c2}$$

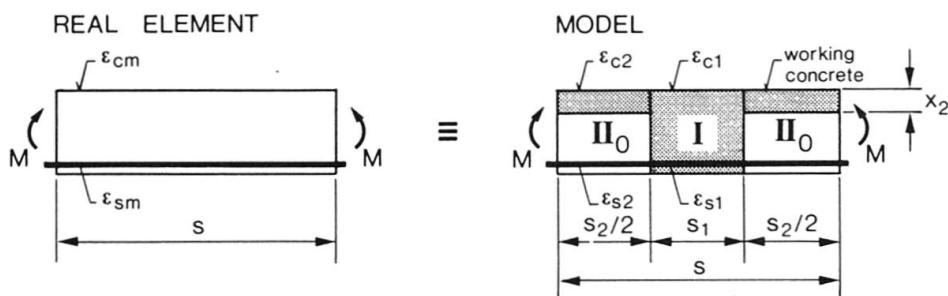


Fig. 1.3 Calculation model - pure bending

Note: One can once again consider a model composed of two elements of length  $\ell$ , one of which, acting in state I, is subjected to  $(1-c)$ -times the load effects and the other, acting in state  $II_0$ , is subjected to  $c$ -times the load effects (fig. 1.3a).

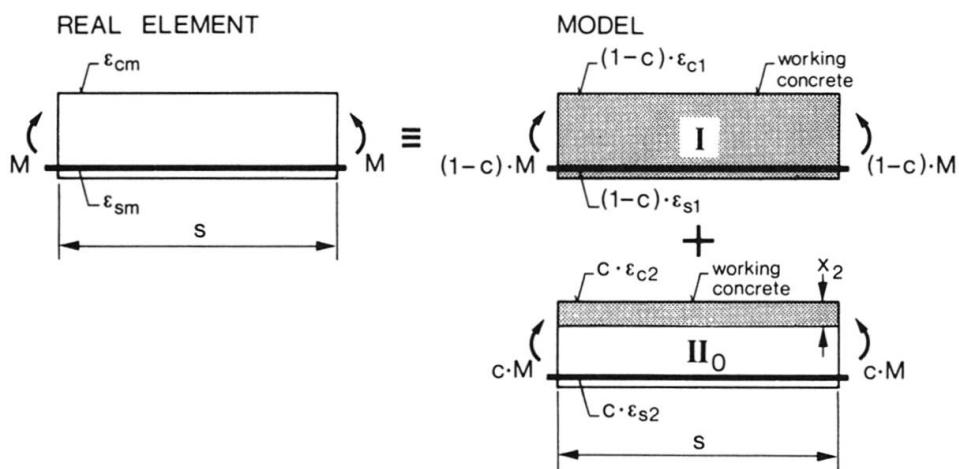


Fig. 1.3a Calculation model - pure bending

Elements subjected to combined bending (M and N = const. along a length s)

The same calculation models can be retained (fig. 1.4). One should however remark that neutral axis depends on the load effects ( $x = f(M/N)$ ); the calculation, especially in state  $II_0$ , is therefore more complicated.

In a further article, we will deal in more detail with the problem of combined bending as well as the possible simplifications for practical applications.

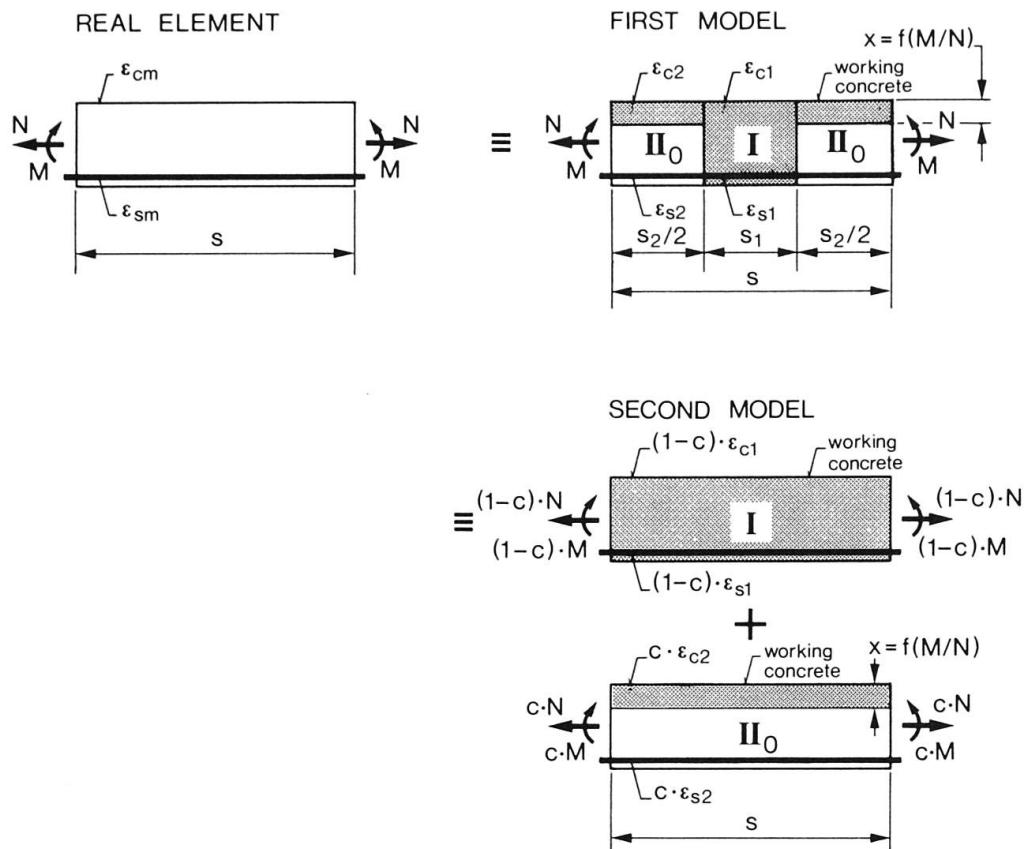


Fig. 1.4 General calculation model



## 2. CURVATURES

At every point of the structure, the curvature (see fig. 2.1) is given by:

$$\kappa = -\frac{1}{r} = \frac{M}{EI} = \frac{\varepsilon_s - \varepsilon_c}{d}$$

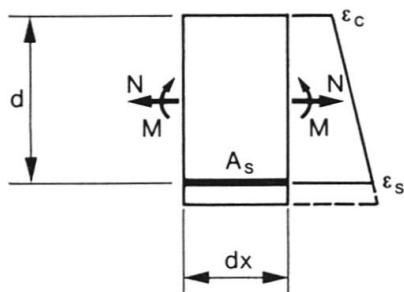


Fig. 2.1

The calculation of the curvatures in states I and II<sub>0</sub> is made from the basic curvature

$$\kappa_c = \frac{M}{EI_c} \quad (2.1)$$

where  $EI_c$  is the bending stiffness corresponding to the section of concrete alone. The basic curvature  $\kappa_c$  is modified by the correction coefficients  $k$  (see § 2.2) in order to take into account the reinforcement and creep; the effect of shrinkage is considered separately.

The calculation of the mean curvatures is carried out by means of the appropriate models (fig. 1.3, 1.4).

### 2.1 Curvatures in the different states

Curvature in state I:  $\kappa_1 = \kappa_{10} + \kappa_{1\phi} + \kappa_{1s}$

The curvature  $\kappa_1$  in state I represents the lower limit of the mean curvature  $\kappa_m$ .

Calculated from the basic curvature  $\kappa_c$  by introducing the effect of the reinforcement (correction coefficient  $k_A$ ), it can be expressed at the time  $t = 0$  by:

$$\kappa_{10} = k_{A1} \cdot \kappa_c \quad (2.2a)$$

At the time  $t$ , taking into account only the permanent loads (g), the increase of curvature due to creep (creep coefficient  $\phi$ , correction coefficient  $k_\phi$ ) can be expressed by:

$$\kappa_{1\phi} = (k_{A1} \cdot k_{\phi 1} \cdot \phi) \cdot \kappa_{cg} \quad (2.2b)$$

The curvature due to shrinkage (strain in the concrete due to shrinkage  $\varepsilon_{cs}$ , correction coefficient  $k_s$ ) is given by:

$$\kappa_{1s} = k_{s1} \cdot \frac{|\varepsilon_{cs}|}{d} \quad (2.2c)$$

$$\text{Curvature in state } II_o: \quad \kappa_2 = \kappa_{20} + \kappa_{2\varphi} + \kappa_{2s}$$

The curvature  $\kappa_2$  in state  $II_o$  represents the upper limit of the mean curvature  $\kappa_m$ .

Similarly to what was done in state I, one can write:

$$\kappa_{20} = k_{A2} \cdot \kappa_c \quad - \quad \text{for } t = 0 \quad (2.3a)$$

$$\kappa_{2\varphi} = (k_{A2} \cdot k_{\varphi 2} \cdot \varphi) \cdot \kappa_{cg} \quad - \quad \text{effect of creep} \quad (2.3b)$$

$$\kappa_{2s} = k_{s2} \cdot \left| \frac{\varepsilon_{cs}}{d} \right| \quad - \quad \text{effect of shrinkage} \quad (2.3c)$$

Mean curvature  $\kappa_m$ :

The mean curvature  $\kappa_m$  is given in a general manner (clause 16.2.2 of the Model Code) by

$$\kappa_m = \frac{M}{EI_m} = \frac{\varepsilon_{sm} - \varepsilon_{cm}}{d} \quad (2.4)$$

Referring to the model presented previously, one obtains (see fig. 2.2):

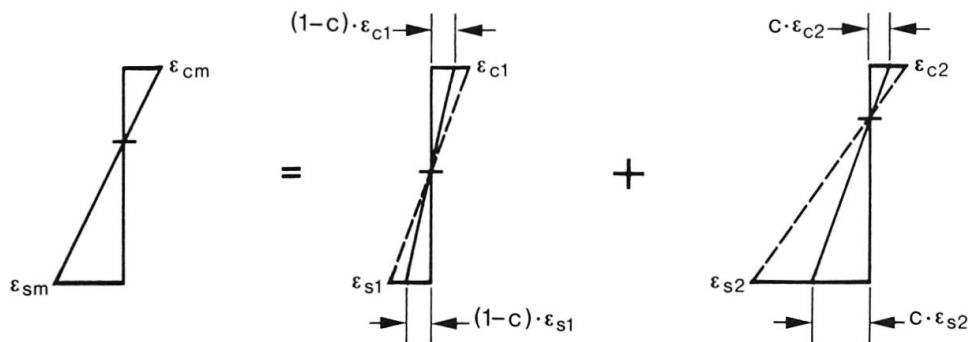


Fig. 2.2 Mean curvature

$$\varepsilon_{sm} = (1-c) \cdot \varepsilon_{s1} + c \cdot \varepsilon_{s2} \quad (2.5a)$$

$$\varepsilon_{cm} = (1-c) \cdot \varepsilon_{c1} + c \cdot \varepsilon_{c2} \quad (2.5b)$$

and the mean curvature (eq. 2.4) is expressed by:

$$\kappa_m = (1-c) \cdot \kappa_1 + c \cdot \kappa_2 \quad (2.6)$$



Figure 2.3 shows an example of evolution of a curvature through the different states, at times  $t = 0$  and  $t$ .

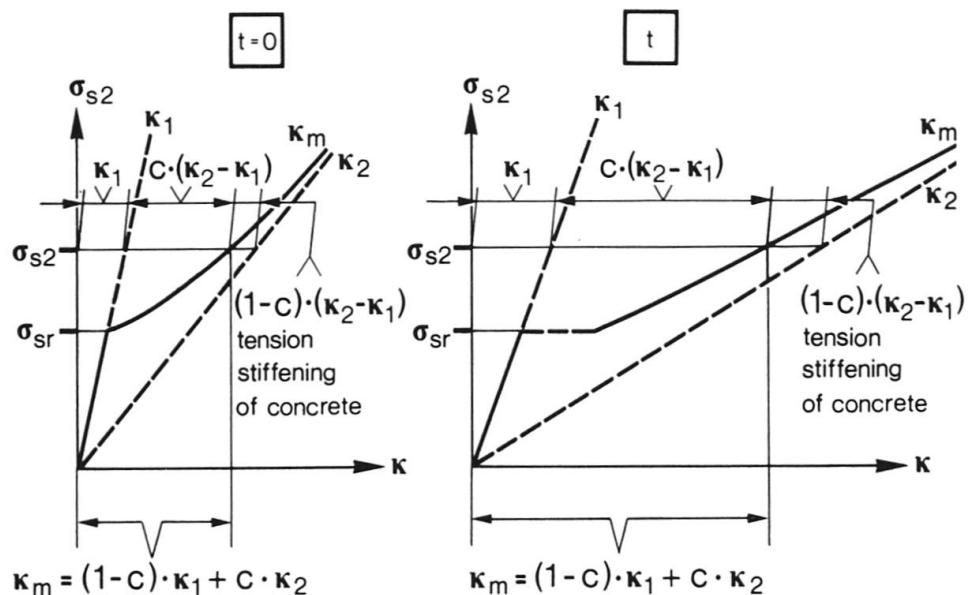


Fig. 2.3 Curvatures

## 2.2 Indications on the establishment of the k coefficients

The establishment of the k coefficients is shown for a section with one axis of symmetry (fig. 2.4).

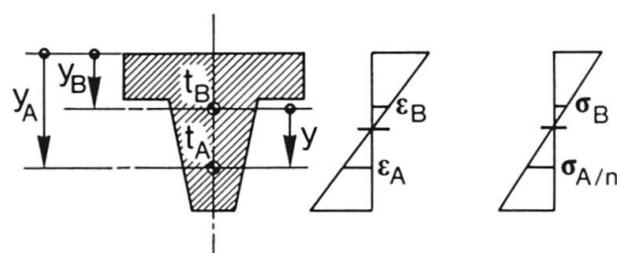


Fig. 2.4 Section

The following notation is used:

$t_B$  - centre of gravity of the active section of concrete (in state I: the entire section of concrete; in state II: the section of concrete in compression);

$t_A$  - centre of gravity of the entire reinforcement;

$d$  - effective height;

$x$  - position of the neutral axis ( $x_1$  in state I and  $x_2$  in state II);

$y$  - oriented distance between the centres of gravity of the active section of concrete ( $t_B$ ) and the entire reinforcement ( $t_A$ );

$A_B$  - active section of concrete;

$A_A$  - entire section of reinforcement;

$I_c$  - moment of inertia of the section of concrete with respect to its centre of gravity;

$I_x$  - moment of inertia of the idealized active section (taking into account the reinforcement) with respect to the neutral axis;

$I_B$  - moment of inertia of the active section of concrete with respect to its centre of gravity;

$I_A$  - moment of inertia of the entire reinforcement with respect to its centre of gravity;

$I_{Bx}$  - moment of inertia of the active section of concrete with respect to the neutral axis;

$I_{Ax}$  - moment of inertia of the entire reinforcement with respect to the neutral axis;

$\sigma_A, \sigma_{AO}, \sigma_{Ap}$  - stresses at the centre of gravity of the entire reinforcement in general, at times  $t = 0$  and  $t$  (taking creep into account) respectively;

$\sigma_B, \sigma_{BO}, \sigma_{Bp}$  - stresses at the centre of gravity of the active section of concrete;

$\epsilon_A, \epsilon_{AO}, \epsilon_{Ap}$  - strains at the centre of gravity of the entire reinforcement;

$\epsilon_B, \epsilon_{BO}, \epsilon_{Bp}$  - strains at the centre of gravity of the active section of concrete.



Coefficient of correction  $k_A$ :

Let us express the curvature at the time  $t = 0$  ( $\kappa_0$ ,  $\kappa_{10}$  for state I,  $\kappa_{20}$  for state  $II_{0'}$ ) by means of the basic curvature  $\kappa_c$ ; this relation being linear for a given section

$$\kappa_0 = k_A \cdot \kappa_c$$

one obtains for  $k_A$ :

$$k_A = \frac{\kappa_0}{\kappa_c} = \frac{M/(EI_x)}{M/(EI_c)} = \frac{EI_c}{EI_x} = \frac{E_c I_c}{E_c I_{Bx} + E_s I_{Ax}} = \frac{I_c}{I_{Bx} + n I_{Ax}}$$

Note: The neutral axis, in state I or  $II_{0'}$ , is determined by

$$\frac{x}{d} = \frac{\frac{y_B}{d} + \frac{nA_A}{A_B} \cdot \frac{y_A}{d}}{1 + \frac{nA_A}{A_B}}$$

Coefficient of correction  $k_\varphi$ :

Let us express by means of the curvature at the time  $t = 0$  due to the permanent loads

$$\kappa_0 = k_A \cdot \kappa_{cg}$$

the increase of curvature due to creep ( $\kappa_\varphi$ ,  $\kappa_{1\varphi}$  for state I,  $\kappa_{2\varphi}$  for state  $II_{0'}$ ):

$$\kappa_\varphi = \varphi \cdot k_\varphi \cdot \kappa_0$$

Coefficient  $k_\varphi$  can be developed as follows:

$$\begin{aligned} k_\varphi &= \frac{1}{\varphi} \cdot \frac{\kappa_\varphi}{\kappa_0} \\ &= \frac{1}{\varphi} \cdot \frac{\frac{1}{y} \cdot (\varepsilon_{B\varphi} - \varepsilon_{A\varphi})}{\frac{1}{y} \cdot (\varepsilon_{B0} - \varepsilon_{A0})} = \frac{1}{\varphi} \cdot \frac{\frac{\sigma_{B0}}{E_c} \cdot \varphi + \frac{\sigma_{B\varphi}}{E_c} \cdot (1 + \chi \cdot \varphi) - \frac{\sigma_{A\varphi}}{E_s}}{\frac{\sigma_{B0}}{E_c} - \frac{\sigma_{A0}}{E_s}} \end{aligned}$$

Note:  $\chi$  - aging coefficient - see Model Code - appendix e.

By introducing the condition of equivalence between the internal forces ( $M, N, \dots$ ) and the stresses ( $\sigma, \tau$ ) in the section

$$\sigma_B \cdot A_B + \sigma_A \cdot A_A = 0$$

one finds:

$$k_{\varphi} = \frac{\frac{E_s A_A}{E_c A_B} + \frac{1}{\varphi} \cdot \frac{\sigma_{A\varphi}}{\sigma_{AO}} \cdot (1 + \frac{E_s A_A}{E_c A_B} \cdot (1 + \chi \cdot \varphi))}{1 + \frac{E_s A_A}{E_c A_B}}$$

The only unknown is the ratio  $\sigma_{A\varphi}/\sigma_{AO}$  which can be determined by means of the conditions of compatibility (between the concrete and steel) and of equivalence; one obtains:

$$\frac{\sigma_{A\varphi}}{\sigma_{AO}} = \varphi \cdot \frac{1 - \frac{E_s A_A}{E_c A_B} \cdot \frac{E_s I_A}{E_c I_B} \cdot (1 + \chi \cdot \varphi)}{D}$$

and therefore for coefficient  $k_{\varphi}$ :

$$k_{\varphi} = \frac{1}{1 + \frac{nA_A}{A_B}} \cdot \left[ \frac{nA_A}{A_B} + \frac{1 - \frac{nA_A}{A_B} \cdot \frac{nI_A}{I_B} (1 + \chi \cdot \varphi)}{D} \cdot (1 + \frac{nA_A}{A_B} \cdot (1 + \chi \cdot \varphi)) \right]$$

with

$$D = 1 + \frac{nA_A}{A_B} \cdot (1 + \frac{A_B}{I_B} \cdot y^2) \cdot (1 + \chi \cdot \varphi) + \\ + \frac{nI_A}{I_B} (1 + \chi \cdot \varphi) \cdot (1 + \frac{nA_A}{A_B} \cdot (1 + \chi \cdot \varphi))$$

Coefficient of correction  $k_s$  :

Let us express the curvature due to shrinkage  $\kappa_s$  ( $\kappa_{s1}$  for state I,  $\kappa_{s2}$  for state II<sub>0</sub>):



$$\kappa_s = \frac{1}{y} \cdot (\varepsilon_{Bs} - \varepsilon_{As}) = \frac{1}{y} \cdot \left( \frac{\sigma_{Bs}}{E_c} \cdot (1 + \chi \cdot \varphi) + \varepsilon_{cs} - \frac{\sigma_{As}}{E_s} \right)$$

By introducing the condition of equivalence one finds:

$$\kappa_s = \frac{\varepsilon_{cs}}{y} \cdot \left( 1 - \frac{\sigma_{As}}{E_s \cdot \varepsilon_{cs}} \cdot \left( 1 + \frac{E_s I_A}{E_c I_B} \cdot (1 + \chi \cdot \varphi) \right) \right)$$

The only unknown, the ratio  $\sigma_{As}/(E_s \cdot \varepsilon_{cs})$ , can be determined by means of the conditions of compatibility and of equivalence; one obtains:

$$\frac{\sigma_{As}}{E_s \cdot \varepsilon_{cs}} = \frac{1 + \frac{E_s I_A}{E_c I_B} \cdot (1 + \chi \cdot \varphi)}{D}$$

and therefore for the curvature  $\kappa_s$ :

$$\kappa_s = \frac{\varepsilon_{cs}}{d} \cdot \frac{\frac{nA_A}{I_B} \cdot y \cdot d \cdot (1 + \chi \cdot \varphi)}{D}$$

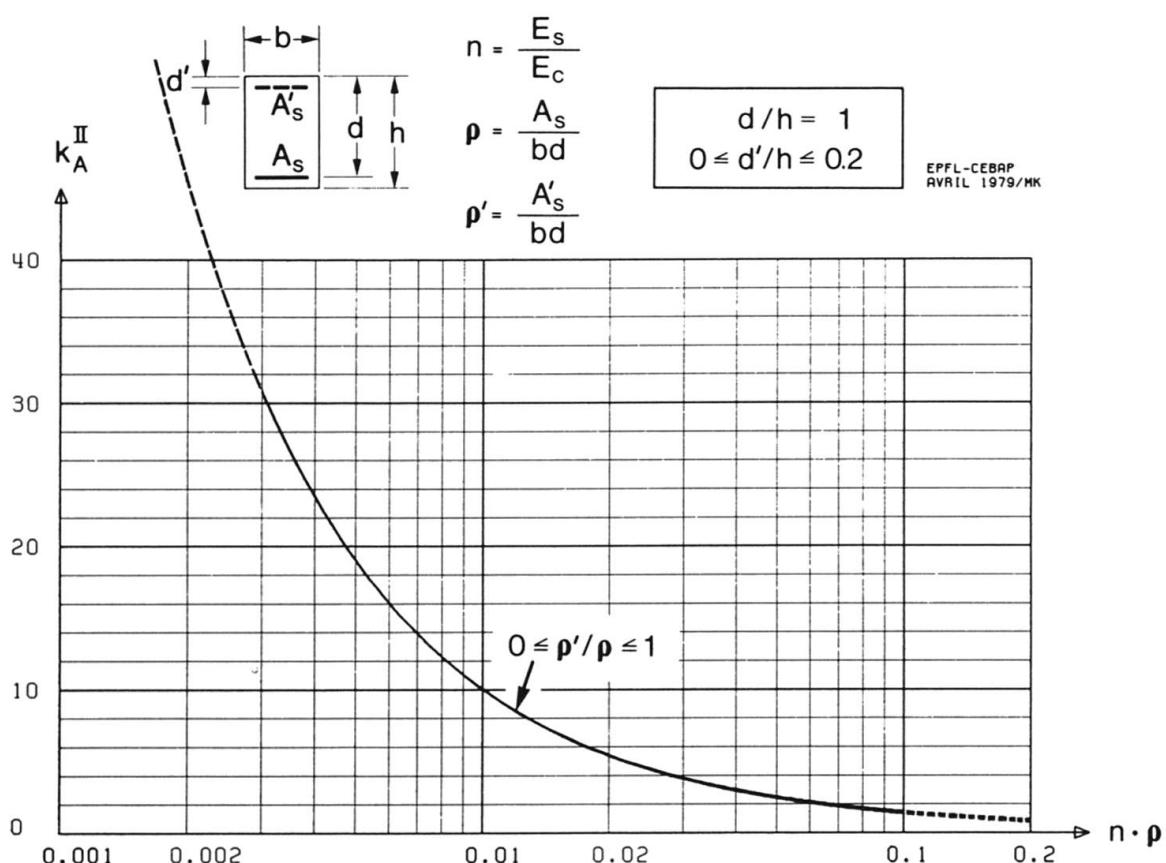
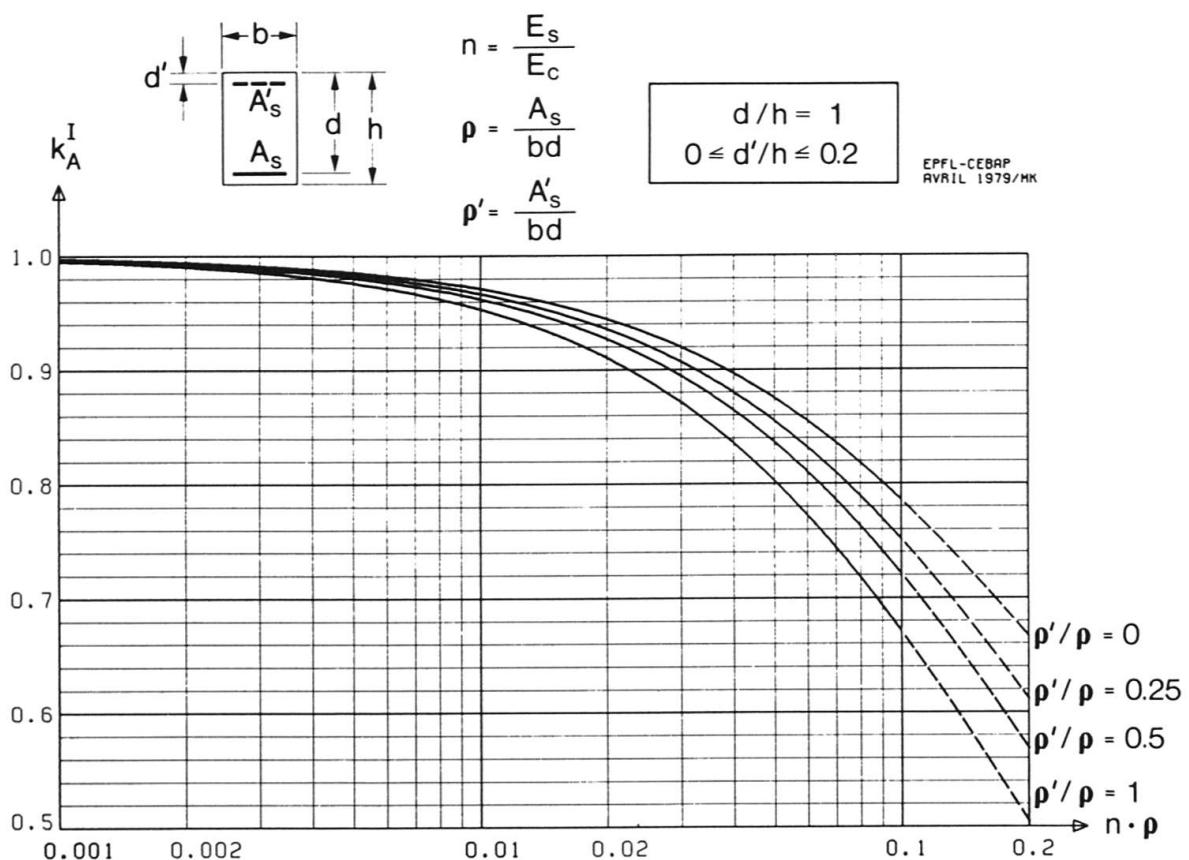
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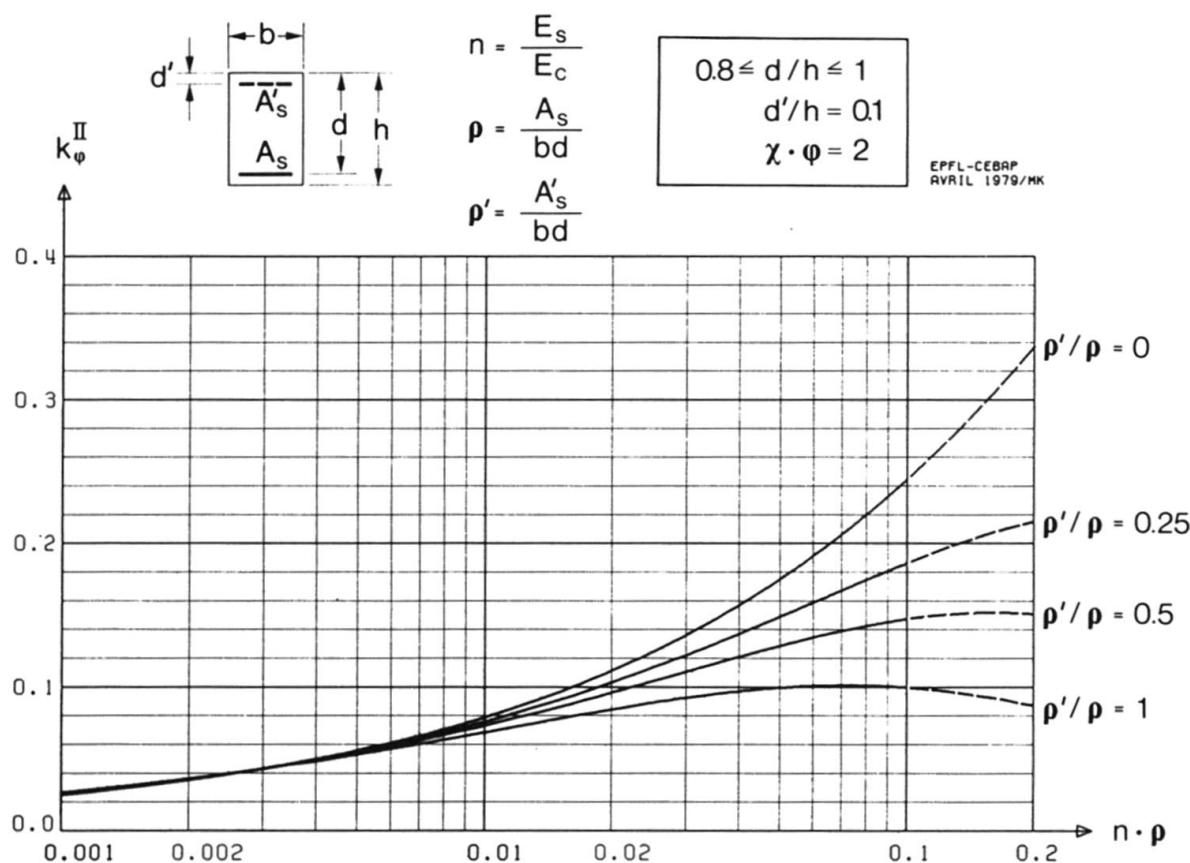
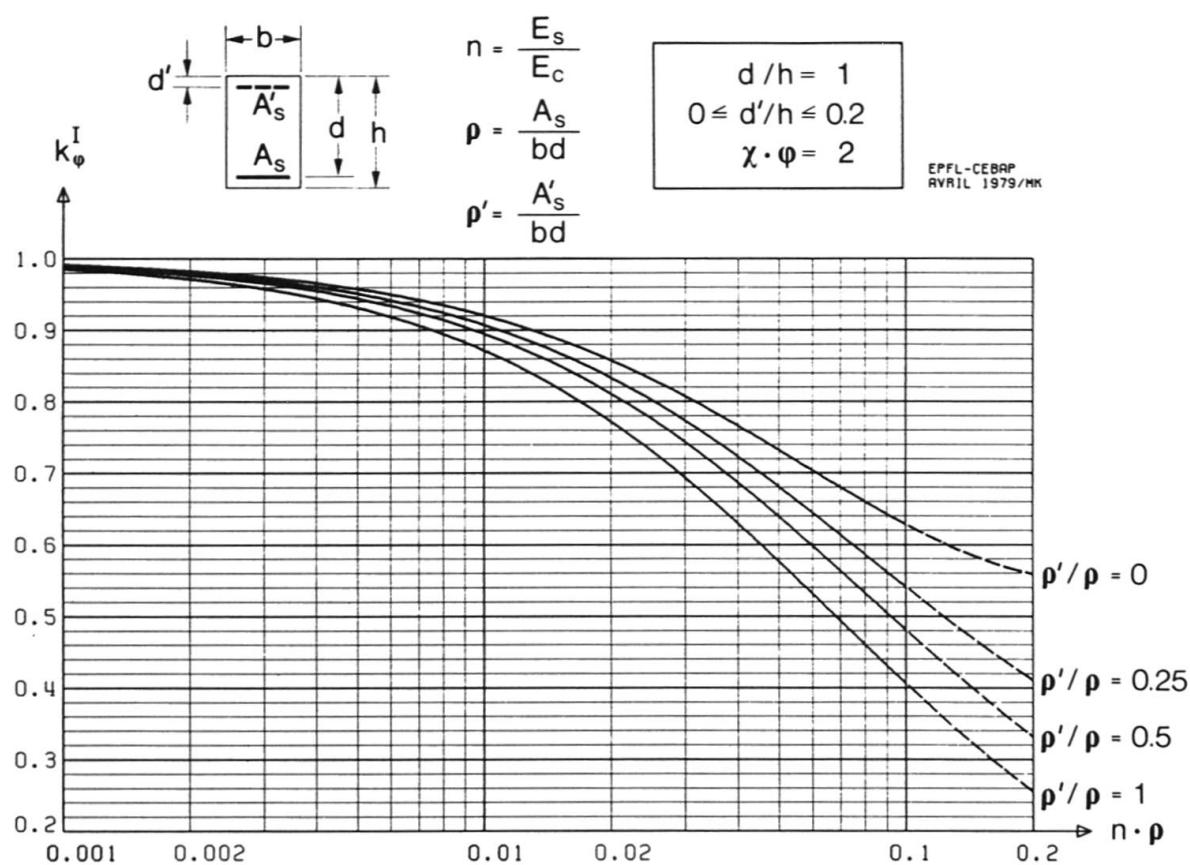
$$\kappa_s = \frac{\varepsilon_{cs}}{d} \cdot k_s$$

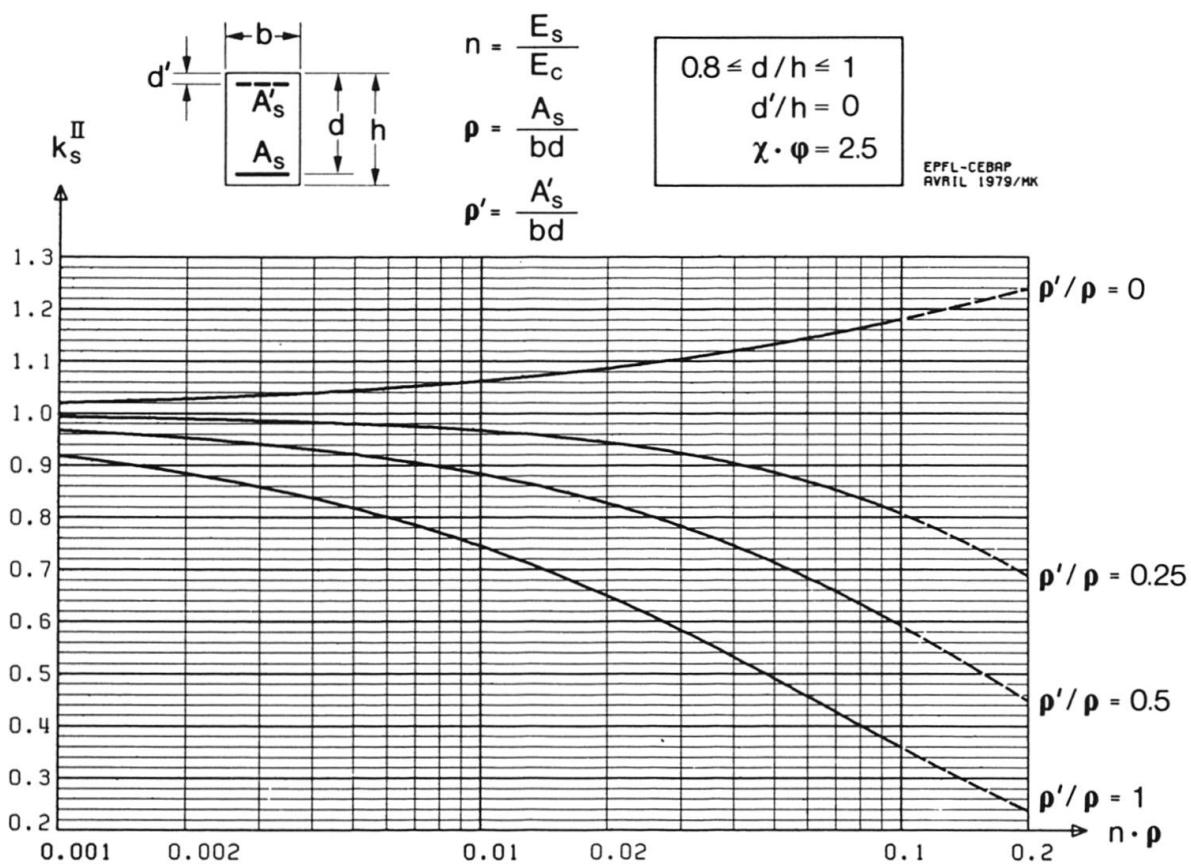
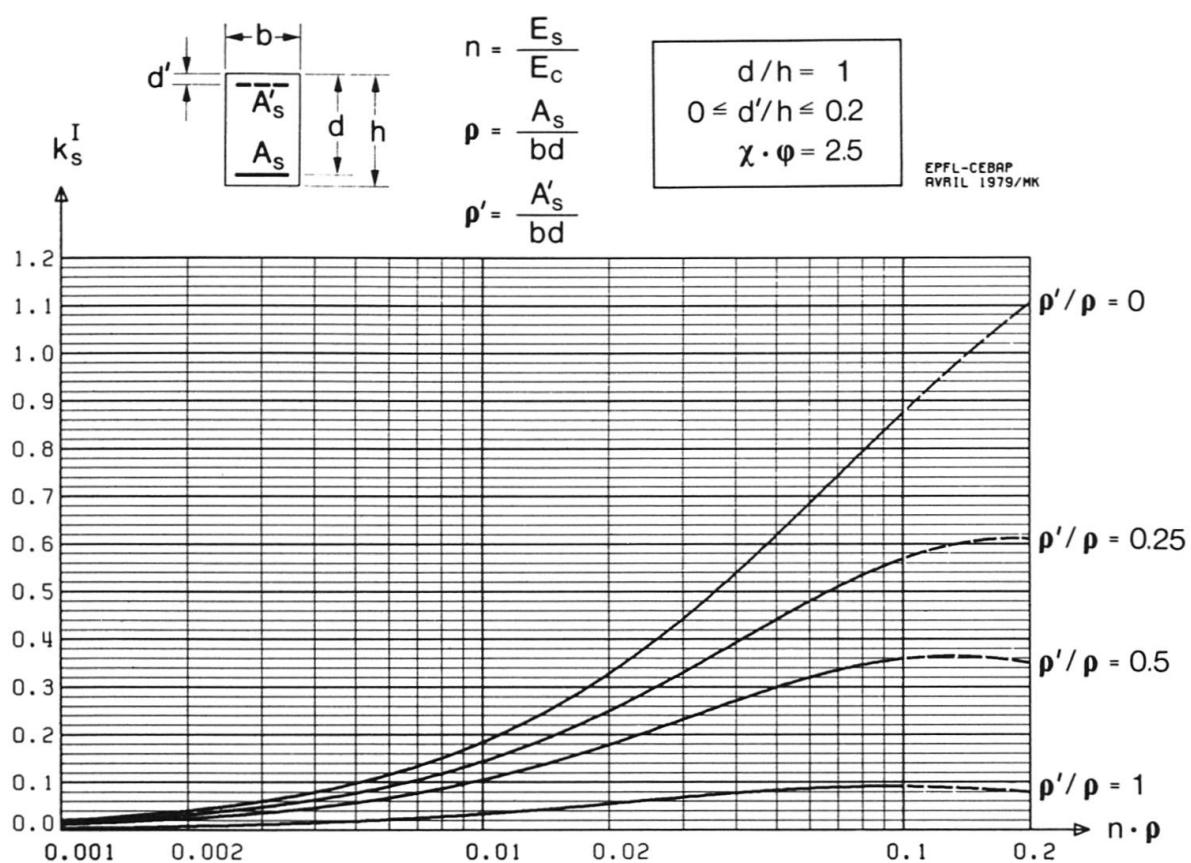
the coefficient  $k_s$  becomes:

$$k_s = \frac{\frac{nA_A}{I_B} \cdot y \cdot d \cdot (1 + \chi \cdot \varphi)}{D}$$

*Note: The coefficients  $k_A$ ,  $k_\varphi$ ,  $k_s$  are dimensionless and can be tabulated for different types of sections. We give here, as an example, the graphical representation of each of the coefficients for a rectangular section. These curves were established by means of a program developed at the Institute of Reinforced and Prestressed Concrete (IBAP) of the Swiss Federal Institute of Technology in Lausanne.*









### 3. CALCULATION OF THE DEFORMATIONS BY INTEGRATION

The deformations can be obtained by double integration of the mean curvature  $\kappa_m$  (eq. 2.6) along the element, respecting the end conditions.

The deformation  $a$  (displacement, rotation) of an element can be obtained by the virtual work theorem:

$$a = \int \kappa_m \cdot \bar{M} \cdot dx \quad (3.1)$$

The development of the calculation is as follows:

From the basic flexural stiffnesses (uncracked sections, effect of the reinforcement neglected), one calculates the bending moments  $M_c$ . The basic curvatures  $\kappa_c = M_c/EI$  then enable us to calculate the curvatures  $\kappa_1$  in state I and  $\kappa_2$  in state II<sub>o</sub>. The mean curvatures are defined by (eq. 2.6):

$$\kappa_m = (1-c) \cdot \kappa_1 + c \cdot \kappa_2$$

with (eq. 1.5)

$$c = 1 - \beta_1 \cdot \beta_2 \cdot \frac{\sigma_{sr}^2}{\sigma_{s2}^2}$$

For statically determinate structures, the calculation of the mean curvature ends here. For statically indeterminate structures, however, the calculation becomes iterative due to the redistribution of moments (cracking, creep). The mean flexural stiffnesses

$$EI_m = \frac{M_c}{\kappa_m}$$

make it possible to recalculate the bending moments  $M$ . The cycle previously described is then carried out again. The iteration is completed when the difference in flexural stiffnesses of two consecutive cycles is judged acceptable.

The mean stiffnesses thus obtained enable us to calculate deformation  $a$  (eq. 3.1) by numerical integration.

The calculation is tedious and requires in most cases the use of the computer. This difficulty is the reason for research for simplified calculation models.

### 4. PRACTICAL CALCULATION OF DEFLECTIONS : BILINEAR METHOD

In order to determine the probable deflection  $a$ , the theorem of virtual work is applied:

$$a = \int \kappa_m \cdot \bar{M} \cdot dx$$

The difficulty consists, as we saw in the previous chapter, in the calculation of the mean curvatures:

$$\kappa_m = (1-c) \cdot \kappa_1 + c \cdot \kappa_2$$

The bilinear method, which is based on the fact that the deflection-load relationship, in the serviceability state, can be approached by a bilinear function, enables us to avoid this difficulty by reducing the essential of the calculation to that of a single section, the determinant section. The latter is defined as the section where the deflection is calculated, in mid-span, or the end restraint section for a cantilever.

The simplifications made are situated at two levels, one concerning the coefficient  $c$ , the other the deflections  $a_1$  and  $a_2$ .

#### 4.1 Simplification of coefficient $c$

For a given load level, coefficient  $c$  defined by eq. 1.5

$$c = 1 - \beta_1 \cdot \beta_2 \cdot \frac{\sigma_{sr}^2}{\sigma_{s2}^2}$$

varies along the length of the element.

The following simplifications, considered acceptable,

- the stress  $\sigma_{sr}$  is assumed constant and equal to the stress  $\sigma_{sr}$  in the determinant section:  $\sigma_{sr} = \text{constant}$ ;
- the stress  $\sigma_{s2}$  is assumed constant and equal to the geometrical mean of the stresses  $\sigma_{sr}$  and  $\sigma_{s2} \equiv \sigma_{sD}$  in the determinant section:

$$\sigma_{s2} = \sqrt{\sigma_{sr} \cdot \sigma_{sD}}$$

lead to

$$c = 1 - \beta_1 \cdot \beta_2 \cdot \frac{\sigma_{sr}}{\sigma_{sD}} = \text{const. for a given load level}$$

Let us replace the ratio  $\sigma_{sr}/\sigma_{sD}$  by the ratio of the corresponding moments:

$$\frac{\sigma_{sr}}{\sigma_{sD}} = \frac{M_r}{M_D}$$

$M_D$  being the moment calculated in the determinant section and  $M_r$  the cracking moment in the same section, as defined in the Model-Code-clause 16.2.1:

$$M_r = W_1 \cdot (f_{ct} - \frac{N}{A_1}) \quad (4.1)$$

with

$f_{ct} = f_{ct0,05}$  for the prevention of damage;

$= f_{ctm}$  for the calculation of camber;



$A_1$  - idealized section, taking the reinforcement into account (state I);

$W_1$  - modulus of inertia of the idealized section with respect to the neutral axis (state I).

The final expression for the simplified coefficient  $c$  is:

$$c = 1 - \beta_1 \cdot \beta_2 \cdot \frac{\frac{M_r}{M_D}}{(4.2)}$$

#### 4.2 Bilinear method

The probable deflection is given by:

$$\begin{aligned} a &= \int_{\kappa_m} \cdot \bar{M} \cdot dx \\ &= \int ((1-c) \cdot \kappa_1 + c \cdot \kappa_2) \cdot \bar{M} \cdot dx \end{aligned}$$

By introducing the simplifications indicated in § 4.1, the coefficient  $c$  given by eq. (4.2) becomes independent of  $x$  and can be placed in front of the integral:

$$a = (1-c) \cdot \int \kappa_1 \cdot \bar{M} \cdot dx + c \cdot \int \kappa_2 \cdot \bar{M} \cdot dx$$

As

$$\begin{aligned} a_1 &= \int \kappa_1 \cdot \bar{M} \cdot dx \quad - \text{ deflection in state I} \\ a_2 &= \int \kappa_2 \cdot \bar{M} \cdot dx \quad - \text{ deflection in state II}_0 \end{aligned}$$

we obtain

$$a = (1-c) \cdot a_1 + c \cdot a_2 \quad (4.3)$$

It is obvious that this relationship is bilinear (see fig. 4.1). The deflections  $a_1$  in state I and  $a_2$  in state  $II_0$  represent the extremes of the probable deflection  $a$ .

These deflections  $a_1$  and  $a_2$  can be calculated in an exact manner from the curvatures  $\kappa_1$  and  $\kappa_2$  defined by eq. (2.2) and (2.3).

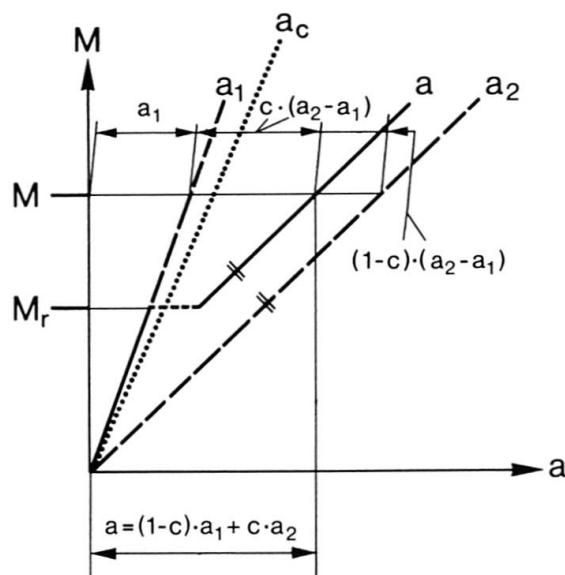


Fig. 4.1 Bilinear relationship deflection - moment

Due to the very distribution of the reinforcement and that of the virtual moment  $\bar{M}$ , the variation of the correction coefficients  $k$  along the element can be neglected and one obtains successively for the deflections  $a_1$  and  $a_2$ :

$$a_1 = \int k_{10} + k_{1\phi} + k_{1s} \cdot \bar{M} \cdot dx = \int (k_{A1} \cdot k_c + k_{A1} \cdot k_{\phi 1} \cdot \varphi \cdot k_{cg} + k_{s1} \cdot \left| \frac{\varepsilon_{cs}}{d} \right|) \cdot \bar{M} \cdot dx$$

$$= \int (k_{A1} \cdot k_c + k_{A1} \cdot k_{\phi 1} \cdot \varphi \cdot k_{cg} + k_{s1} \cdot \left| \frac{\varepsilon_{cs}}{d} \right|) \cdot \bar{M} \cdot dx$$

$$a_1 \approx k_{A1} \cdot a_c + k_{A1} \cdot k_{\phi 1} \cdot \varphi \cdot a_{cg} + k_{s1} \cdot \left| \frac{\varepsilon_{cs}}{d} \right| \cdot \int |\bar{M}| \cdot dx \quad (4.4a)$$

and similarly

$$a_2 \approx k_{A2} \cdot a_c + k_{A2} \cdot k_{\phi 2} \cdot \varphi \cdot a_{cg} + k_{s2} \cdot \left| \frac{\varepsilon_{cs}}{d} \right| \cdot \int |\bar{M}| \cdot dx \quad (4.4b)$$

Notation:

$a_c$  - basic deflection (section of concrete only);

$a_{cg}$  - basic deflection due to the permanent loads;

$\bar{M}$  - virtual moment in the statically indeterminate system;

$k$  - coefficient of correction of the determinant section.



Notes: 1) This method can be improved, but at the cost of simplicity, by introducing, for example, for coefficient  $c$ :

$$c = (1 - \beta_1 \cdot \beta_2 \cdot \frac{\alpha_M \cdot M_r}{M_D}) \cdot \alpha_p$$

where

$\alpha_M$  - modifies the point of intersection of the two lines;

$\alpha_p$  - modifies the slope in state  $II_o$ .

Coefficients  $\alpha_M$  and  $\alpha_p$  could be determined and tabulated as functions of the end fixity and of the distribution of the reinforcement.

2) A possible redistribution of the moments (Model-Code clause 8.3.2) that will influence moment  $M_D$  must be taken into account.

Summary of the bilinear method for the calculation of deflections:

In order to calculate the probable deflection  $a$ , it is necessary to determine successively:

1. the basic deflection  $a_c$ ;
2. the extremes  $a_1$  and  $a_2$  (eq. 4.4);
3. the cracking moment  $M_r$  (eq. 4.1);
4. the maximum moment  $M_D$ ;
5. the probable deflection  $a$  (eq. 4.3).

It is obvious that all these values must be calculated for the same load.

Extension to the calculation of the deflection of slabs:

In order to determine the probable deflection  $a$  of a reinforced or prestressed concrete slab, the method described above is applied according to the following procedure:

1. Calculation of  $a_c$  by means of a classical elastic method. At this point one could introduce the effect of the anisotropy, which, however, is negligible in most cases.
2. Calculation of  $\max m_x$ ,  $\max m_y$  in the considered field. The section in which the greatest positive moment acts is determinant (this section does not necessarily coincide with the spot where the basic deflection  $a_c$  was established): the coefficients of correction as well as the cracking moment are determined for this section.
3. Calculation of  $a_1$ ,  $a_2$  and of the probable deflection  $a$ .

Notes: 1)  $m_r$ ,  $\rho$ ,  $\rho'$  are introduced per unit of width.

2) Tests are at present being carried out at IBAP, which should allow verification of this procedure, especially of the influence of long term effects.

## 5. ESTIMATION OF DEFLECTIONS

In the majority of cases in practice, at the preliminary design stage especially, the practising engineer is only interested in a simple estimation of the probable deflection. To this effect, we propose a simplified method, based on the bilinear method.

In general, the probable deflection due to the permanent loads is given by (see § 4)

$$a = a_{cg} \cdot (k_o + k_t \cdot \varphi) \quad (5.1)$$

with

$a_{cg}$  - basic deflection due to the permanent loads, calculated with  
 $EI = E_c I_c$ ;

$k_o$  - coefficient of correction taking into account the reinforcement and cracking for the calculation of the immediate deflection:

$$k_o = k_{A1} + c \cdot (k_{A2} - k_{A1}) \quad (5.2)$$

$k_t$  - coefficient of correction taking into account the reinforcement, cracking and creep for the calculation of the part of the deflection due to creep:

$$k_t = k_{A1} \cdot k_{\varphi 1} + c \cdot (k_{A2} \cdot k_{\varphi 2} - k_{A1} \cdot k_{\varphi 1}). \quad (5.3)$$

By fixing the following parameters:

$$d/h = 0,9 \quad d'/h = 0,1 \quad \chi \cdot \varphi = 2$$

coefficients  $k_o$  and  $k_t$  that are indispensable to the calculation can be tabulated as functions of the following parameters:

$$n \cdot \rho \quad \rho'/\rho \quad c$$

Knowing the basic deflection  $a_{cg}$ , the cracking moment  $M_r$  and the working moment  $M_D$  in the determinant section, one calculates the coefficient  $c$  and one estimates the percentages of reinforcement  $\rho$  and  $\rho'$  in the same section. These parameters  $c$ ,  $\rho$  and  $\rho'$  enable then to read in a table the coefficients  $k_o$  and  $k_t$  from which one determines the probable deflection  $a$ .

The calculation is thereby greatly simplified, but still provides a reasonable estimation of the deflection.



## CONCLUSIONS

The definition of calculation models composed of two parts, one acting in state I and the other in state II<sup>o</sup>, enabled us to deal with the problem of deformations due to bending by means of an exact method. The successive introduction of a number of simplifications led us to a bilinear method for the calculation of deflections and then to a simplified form of this same method, allowing a rapid estimation of deflection.

For practical applications, it is indispensable to have aids to ease the calculation of the coefficients of correction  $k$ . One can refer for this to [3] or to the CEB/FIP "Cracking and Deformation" Manual, at present being prepared.

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