Stresses and Deformations of Composite Members

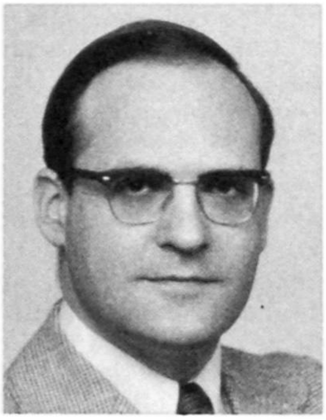
Contraintes et déformations dans une section composée

Spannungen und Dehnungen in einem Verbundquerschnitt

Amin GHALI
Professor of Civil Engineering
University of Calgary
Calgary, AB, Canada

Renaud FAVRE
Professor
Swiss Fed. Inst. of Technology
Lausanne, Switzerland

Amin Ghali, born 1928, received his B.Sc. and M.Sc. degrees in civil engineering from Cairo University in 1950 and 1954 and Ph.D. degree from Leeds University, England in 1957.



R. Favre, born 1934, received his diploma in Zurich in 1957. He worked in design offices and with a contractor in the field of concrete dams, bridges, and buildings. Since 1973, he is Professor of concrete structures at the Swiss Fed. Inst. of Technol. in Lausanne. He is also consultant of Wolf and Partners.

SUMMARY

Analysis of strain and stress in composite cross sections due to the effects of temperature variation, creep and shrinkage of concrete and relaxation of prestressed steel is performed using one general approach. Sections composed of one or more types of concrete or of concrete and structural steel are considered. The concrete parts are assumed to be reinforced with prestressed and non-prestressed steels. The loss of tension in prestressed steel need not be evaluated separately; the analysis accounts for the effects of changes in forces in the steels. The magnitude of prestress loss, if required, can be determined as an additional result of the same analysis.

RÉSUMÉ

L'analyse des déformations et contraintes dans une section composée en tenant compte du fluage, du retrait, de l'effet de la température et de la relaxation de l'acier de précontrainte, est présentée à l'aide d'une méthode générale. Cette section peut être composée de différents bétons et d'armatures active et passive et d'acier de construction. Les pertes de précontraintes ne doivent pas être évaluées séparément mais résultent de l'analyse globale.

ZUSAMMENFASSUNG

Es wird eine generelle Methode dargelegt, um in einem Verbundquerschnitt die Dehnungen und Spannungen zu ermitteln, unter Berücksichtigung des Kriechens, Schwindens, eines Temperaturgradientes und der Spannstahlrelaxation. Es können Querschnitte mit verschiedenen Betonqualitäten und -alter, schlaffer und vorgespannter Armierung und Konstruktionsstahl untersucht werden. Das Resultat liefert die zeitlichen Entwicklungen in den verschiedenen Materialien, unter anderem die Vorspannverluste, welche nicht gesondert betrachtet werden müssen.

1. INTRODUCTION

This paper is concerned with prediction of the stress and deformation of composite members made of one or more concrete parts of different quality or age, or made of structural steel and concrete. The concrete parts are provided with non-prestressed and prestressed reinforcement.

The stress and strain distributions over a composite cross section vary with time due to creep and shrinkage of concrete and relaxation of prestressed steel and due to the changes in temperature distribution. The redistribution of stresses between the steel section, the prestressed and non-prestressed reinforcement and the concrete which develop gradually with time are accounted for in the analysis. Thus, the loss of tension in prestressed steel or the amount of compression picked up by the non-prestressed steel need not be calculated separately; the analysis accurately accounts for the effect of prestress losses and the magnitude of the losses if desired can be determined as an additional result of the same analysis.

The temperature distribution over the cross section of composite structures is generally nonlinear. The same procedure mentioned above will be used to determine the stress and strain caused by temperature change which varies in magnitude over the depth in an arbitrary fashion.

In a partially prestressed member, the stress and strain distribution after cracking and the extent of cracking largely depend upon the stress distribution that exists immediately before cracking. The above-mentioned analysis is applied to determine the stress and strain distribution before cracking accounting for the time-dependent effects of creep, shrinkage and relaxation of prestressed steel. Analysis of stress and strain caused by subsequent loading producing cracking is treated in a separate paper [6].

Once the strain distribution over a section is known, it is simple to find the axial strain and curvature and use these to calculate the displacements at any point on the structure. Numerical examples are included.

The present paper is limited to composite members for which the elongations or end rotations are not restrained by the supports or by continuity with other members. For such a statically determinate structure, creep and shrinkage of concrete, relaxation of prestressed steel or temperature variations affect the distribution of stresses over a composite cross section but do not change the reactions and the stress resultants (values of axial force, shear or bending on the section). Analysis of the change of stress resultants due to these effects on continuous beams and other statically indeterminate structures is treated elsewhere [7].

2. CREEP OF CONCRETE

A uniaxial stress $\sigma(t_o)$ introduced on concrete at time t_o and sustained without change in magnitude produces at time t a strain given by:

$$\varepsilon_c(t) = \frac{\sigma(t_o)}{E_c(t_o)} [1 + \varphi(t,t_o)] \qquad (1)$$

where $E_c(t_o)$ is the modulus of elasticity of concrete at time t_o. The quantity outside the square brackets is the instantaneous strain; $\varphi(t,t_o)$ is the creep coefficient, the ratio of creep occurring during the period $(t - t_o)$ to the instantaneous strain.

In many practical applications, it is necessary to calculate the strain caused by a stress increment which is introduced gradually from zero at time t_o to a full value $(\Delta\sigma_c)$ at time t. The corresponding strain increment at time t may be calculated by:

$$\Delta\varepsilon_c = \frac{\Delta\sigma_c}{E_c(t_o)} [1 + \chi\ \varphi(t,t_o)] \tag{2}$$

where χ is the aging coefficient, a value smaller than 1 used as multiplier of the creep coefficient to account for the fact that the stress increment is not sustained during the full period $(t - t_o)$ and thus must produce smaller creep [11].

In a prestressed concrete section, prestressing at time t_o produces a stress $\sigma(t_o)$ of which a part $(\Delta\sigma_c)$ is gradually lost by the effects of creep, shrinkage and relaxation. Equations 1 and 2 may be used to find the strain at time t due to $\sigma(t_o)$ and due to $(\Delta\sigma_c)$, respectively; superposition gives the total strain. The modulus of elasticity of concrete $E_c(t_o)$ depends upon the properties of the material and the age of concrete at time t_o. The creep coefficient $\varphi(t,t_o)$ depends upon the concrete properties, the humidity and temperature of the ambient air and the cross section dimensions. Guidance on the values of E_c and φ to be used in practice is available [1, 3]. The aging coefficient χ is generally a value between 0.6 and 0.9. Naturally, its value depends upon the shape of the time function representing the magnitude of the stress increment during the period $(t - t_o)$. χ depends also upon variation of E_c with the age of concrete at loading and the variation of the creep coefficient φ with the age at loading and the length of the period for which the stress is sustained.

For derivation of numerical values of χ, Bazant [2] assumes that $(\Delta\sigma_c)$ varies at the same rate as that of the relaxation that would occur when a strain is introduced and maintained constant. With this assumption, it is possible to use graphs or tables [2, 7, 9, 10] to obtain the χ-value.

The aging coefficient is employed to calculate the product $(\chi\varphi)$. Because φ is rarely determined with high accuracy, there is no justification to seek high accuracy for χ.

3. REDUCED RELAXATION

A tendon stretched between two fixed points gradually loses a part of its initial stress. The loss of stress in such a constant-length relaxation test is referred to as the *intrinsic relaxation*. For a given steel quality and duration of test, the absolute value of the intrinsic relaxation increases rapidly with the increase of the ratio:

$$\lambda = \frac{\sigma_{po}}{f_{ptk}} \tag{3}$$

where σ_{po} is the initial stress and f_{ptk} is the characteristic tensile strength. Two tendons with the same initial stress σ_{po}, one in a constant-length relaxation test and the other in a prestressed concrete member exhibit different relaxation values. In the later tendon, the loss of tension due to creep and shrinkage of concrete has the same effect as if the initial stress were small in the constant-length relaxation test. Thus a *reduced relaxation* value should be used in prediction of the time-dependent effects. The reduced relaxation is given by

$$\Delta\bar{\sigma}_{pr} = \chi_r \, \Delta\sigma_{pr} \tag{4}$$

where $(\Delta\sigma_{pr})$ is the intrinsic relaxation, a value usually provided by suppliers of prestressed steel; χ_r is a reduction factor (usually close to 0.7) which can be calculated by the equation [8]:

$$\chi_r = e^{(-6.7 + 5.3\lambda)\Omega} \tag{5}$$

where

$$\Omega = \frac{\Delta\sigma_{ps} - \Delta\sigma_{pr}}{\sigma_{po}} \tag{6}$$

where $(\Delta\sigma_{ps})$ is the change in stress in the prestressed steel during the period $(t - t_o)$ due to the combined effect of creep, shrinkage and relaxation and $(\Delta\sigma_{pr})$ is the intrinsic relaxation in the same period. The value $(\Delta\sigma_{ps})$ is generally not known a priori, because it depends upon the reduced relaxation. Iteration is required: the value of the reduction factor χ_r is estimated (for example 0.7) and is later adjusted if necessary.

4. SIGN CONVENTION

An axial force N and a stress σ are positive when tensile. Positive strain ε indicates elongation; thus, shrinkage ε_{cs} is generally a negative quantity. A bending moment M is positive when it produces tension at the bottom fibre; the corresponding curvature ψ is also positive. The symbol P indicates the absolute value of the prestress force. Δ represents an increment or decrement when positive or negative, respectively. Thus, the loss of tensile stress in prestressed steel $\Delta\sigma_{ps}$ is generally a negative quantity. The y-coordinate defines the location of any fibre from a reference point 0; y is positive for fibres below 0.

5. STRESS AND STRAIN DUE TO TEMPERATURE

Temperature distribution over the cross section of members exposed to weather is generally nonlinear [4]. Analysis of stresses and deformations due to creep, shrinkage and relaxation can be performed in the same way as the analysis of the effect of temperature. Consider a cross section of a *statically-determinate* member subjected to a rise of temperature whose magnitude varies over the depth of the section in an arbitrary fashion (Fig. 1a and b). The hypothetical strain that would occur at any fibre if it were free to expand

$$\varepsilon_f = \alpha_t \, T \tag{7}$$

where α_t = coefficient of thermal expansion per degree; T = temperature rise at any fibre at a distance y below an arbitrary reference point 0.

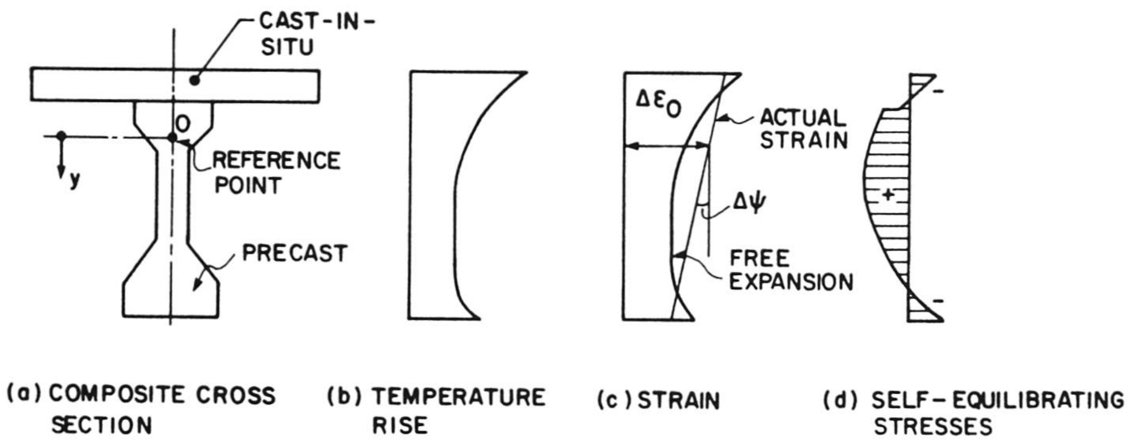


Fig. 1 Analysis of stress and strain in the cross section of a statically-determinate member subjected to nonlinear temperature distribution.

If this strain is artificially prevented, the stress at any fibre in the restrained condition will be:

$$\sigma_{restrained} = -E\,\varepsilon_f \tag{8}$$

where E is the modulus of elasticity, which may vary in different parts of the cross section.

The resultant of this stress may be represented by a normal force ΔN at the reference point 0 and a bending moment ΔM given by:

$$\Delta N = \int \sigma_{restrained}\, dA \tag{9}$$

$$\Delta M = \int \sigma_{restrained}\, y\, dA \tag{10}$$

Release the artificial restraint by application of a normal force $(-\Delta N)$ at 0 and a bending moment $(-\Delta M)$, producing axial strain and curvature (Fig. 1c):

$$\begin{Bmatrix} \Delta\varepsilon_0 \\ \Delta\psi \end{Bmatrix} = \frac{1}{E_{ref}\,(A\,I - B^2)} \begin{bmatrix} I & -B \\ -B & A \end{bmatrix} \begin{Bmatrix} -\Delta N \\ -\Delta M \end{Bmatrix} \tag{11}$$

The corresponding stress at any fibre

$$\Delta\sigma = E\,[\Delta\varepsilon_o + (\Delta\psi)\,y] \tag{12}$$

E_{ref} is an arbitrary reference elasticity modulus; A is the cross section area of a transformed section composed of the area of each part of the actual section multiplied by its modulus of elasticity and divided by E_{ref}; B and I are first and second moments of the transformed section about an axis through the reference point 0. When 0 is chosen at the centroid, B is zero and Eq. 11 simplifies to the more familiar forms:

$$\Delta\varepsilon_0 = \frac{-\Delta N}{E\ A} \qquad (13)$$

$$\Delta\psi = \frac{-\Delta M}{E\ I} \qquad (14)$$

The actual stress due to temperature (Fig. 1d) is the sum of $\sigma_{restrained}$ and $\Delta\sigma$ (Eqs. 8 and 12). In a statically-determinate structure, temperature variation produces no change in reaction or in internal forces; hence the stresses shown in Fig. 1d must be self-equilibrating.

Equations 11 and 12 are based on the assumption that a plane cross section remains plane. In the following sections of this paper, composite cross sections will be replaced in the analysis by transformed sections composed of the area of concrete plus α times the area of steel; with α being the ratio of the modulus of elasticity of steel to that of concrete. The modulus of elasticity of concrete will vary when considering the time dependent effects of aging and creep; hence, the position of the centroid of the transformed section will vary with time. For this reason, Equation 11 for which 0 is an arbitrarily chosen fixed reference point is more suitable than Equations 13 and 14.

6. TIME-DEPENDENT STRESS AND STRAIN

Consider any of the composite sections in Fig. 2 subjected at time t_o to a prestressing force P, a bending moment M and an axial force N at an arbitrary reference point 0. It is required to find the strain, the curvature and the stresses in concrete and steel at time t_o, immediately after prestressing and at a later time t. Assumed to be known are: the cross section dimensions and reinforcement areas, the magnitudes of P, N and M, the moduli of elasticity of the steel types. In addition, values related to the properties of concrete in various parts are assumed known: $E_c(t_o)$ the modulus of elasticity at time t_o, $\varepsilon_{cs}(t,t_o)$ the shrinkage that would occur during the period (t_o to t) if the fibres were free, the creep coefficient $\varphi(t,t_o)$ and the aging coefficient $\chi(t, t_o)$.

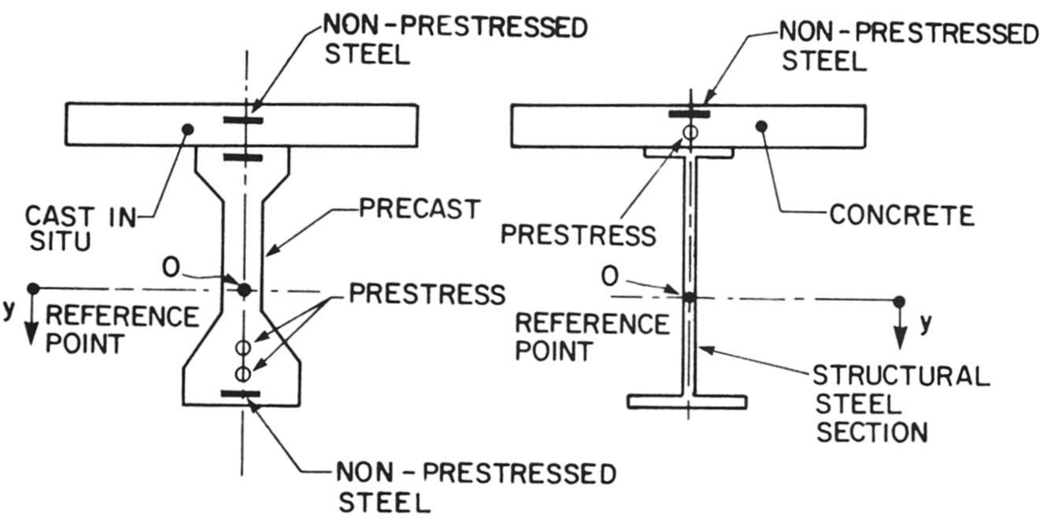


Fig. 2 Composite cross sections

The reduced relaxation value of the prestressed steel ($\Delta\bar{\sigma}_{pr}$) during the period ($t - t_o$) is also assumed to be known.

6.1 Instantaneous Stress and Strain at Time t_o

Combine N and M and the prestressing force into an equivalent normal force at the reference point 0 and a bending moment:

$$\begin{Bmatrix} N \\ M \end{Bmatrix}_{equivalent} = \begin{Bmatrix} N - \Sigma P_i \\ M - \Sigma P_i \, y_{psi} \end{Bmatrix} \qquad (15)$$

where P_i and y_{psi} are the initial prestress force and the y coordinate of the ith steel layer.

The instantaneous axial strain and curvature immediately after prestressing may be calculated by (see Eq. 11):

$$\begin{Bmatrix} \varepsilon_0(t_o) \\ \psi(t_o) \end{Bmatrix} = \frac{1}{E_{ref}\,(A\,I - B^2)} \begin{bmatrix} I & -B \\ -B & A \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix}_{equivalent} \qquad (16)$$

where A, B and I are the area and its first and second moments of a transformed section about an axis through 0. The transformed section is composed of the area of concrete in each part plus the reinforcement in any layer multiplied by a modular ratio α given by

$$\alpha_i = \frac{E_{si}}{E_{ref}} \quad \text{or} \quad \frac{E_{ci}(t_o)}{E_{ref}} \quad \text{or} \quad \frac{\bar{E}_{ci}(t,t_o)}{E_{ref}} \qquad (17)$$

The subscript i refers to a concrete part or to a reinforcement layer; $E_{ci}(t_o)$ is the modulus of elasticity of concrete in part i at age t_o; E_{ref} is an arbitrary reference modulus of elasticity which may be chosen equal to $E_c(t_o)$ for one of the concrete parts; $\bar{E}_{ci}(t, t_o)$ is the age-adjusted elasticity modulus of concrete (see Eq. 24).

The instantaneous strain and stress in concrete at any fibre

$$\varepsilon_c(t_o) = \varepsilon_0(t_o) + \psi(t_o)\, y \qquad (18)$$

$$\sigma_c(t_o) = E_{ci}(t_o) \;\; \varepsilon_c(t_o) \qquad (19)$$

The instantaneous stress in the non-prestressed steel

$$\sigma_{ns}(t_o) = E_{ns}\, [\varepsilon_0(t_o) + \psi(t_o)\, y_{ns}] \qquad (20)$$

When prestressing is employed, the stress in the prestressed steel immediately after transfer

$$\sigma_{ps}(t_o) = (\sigma_{ps})_{initial} + E_{ps}\, [\varepsilon_0(t_o) + \psi(t_o)\, y_{ps}] \qquad (21)$$

where $(\sigma_{ps})_{initial}$ is the stress in prestressed steel before transfer. The second term in this equation represents the instantaneous loss of tension due to shortening of concrete.

When post-tensioning is employed, the stress in prestressed steel immediately before and after transfer are the same,

$$\sigma_{ps}(t_o) = (\sigma_{ps})_{initial} \tag{22}$$

(The loss due to anchor setting is excluded in the present discussion.)

6.2 Changes in Strain and Stress During the Period $(t - t_o)$

Let the strain change at any concrete fibre due to creep and shrinkage be artificially restrained by application of a stress:

$$\sigma_{restrained} = -\bar{E}_c(t,t_o)[\varphi(t,t_o)\ \varepsilon_c(t_o) + \varepsilon_{cs}] \tag{23}$$

ε_{cs} is the free shrinkage during the period $(t - t_o)$; $\bar{E}_c(t,t_o)$ is the age-adjusted elasticity modulus of concrete.

$$\bar{E}_c(t,t_o) = \frac{E_c(t_o)}{1 + \chi\varphi(t,t_o)} \tag{24}$$

$\bar{E}_c(t, t_o)$ represents the magnitude of the stress increment which when gradually introduced during a period $(t - t_o)$ produces a unit strain at the end of the period. The product $(-\bar{E}_c\, \varphi\, \varepsilon_c)$ in Eq. 23 represents the stress necessary to prevent creep; substitution for ε_c from Eq. 18 and integration gives the resultants of this stress:

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{creep} = -\sum_{j=1}^{m} \left\{ \bar{E}_c\ \varphi \begin{bmatrix} A_c & B_c \\ B_c & I_c \end{bmatrix} \begin{Bmatrix} \varepsilon_0(t_o) \\ \psi(t_o) \end{Bmatrix} \right\}_j \tag{25}$$

The subscript j refers to the jth concrete part of the composite section and m is the total number of concrete parts. A_{cj}, B_{cj} and I_{cj} are the area and its first and second moments of the jth part about an axis through 0. $\bar{E}_{cj}$ and φ_j are the age-adjusted modulus of concrete and creep coefficient for the same part. Similarly, the product $(-\bar{E}_c\ \varepsilon_{cs})$ in Eq. 23 represents the stress necessary to prevent shrinkage; assuming uniform shrinkage over each concrete part, the resultants of this stress are:

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{shrinkage} = -\sum_{j=1}^{m} \left\{ \bar{E}_c\ \varepsilon_{cs} \begin{Bmatrix} A_c \\ B_c \end{Bmatrix} \right\}_j \tag{26}$$

The forces necessary to prevent the strain due to the relaxation of prestressed steel

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{relaxation} = \sum_{i=1}^{n} \begin{Bmatrix} A_{ps} & \Delta\bar{\sigma}_{pr} \\ A_{ps} & \Delta\bar{\sigma}_{pr} & y_{ps} \end{Bmatrix}_i \tag{27}$$

The subscript i refers to the ith prestressed steel layer and A_{psi} is its cross section area; n is the number of prestressed steel layers.

The forces necessary to prevent creep, shrinkage and relaxation is the sum of Eqs. 25, 26 and 27

$$\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix} = \begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{creep} + \begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{shrinkage} + \begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{relaxation} \quad (28)$$

The artificial restrain is now eliminated by application of the forces $\{-\Delta N, -\Delta M\}$ on an age-adjusted transformed composite section resulting in the following changes in axial strain and in curvature:

$$\begin{Bmatrix}\Delta\varepsilon_0\\ \Delta\psi\end{Bmatrix} = \frac{1}{\bar{E}_{ref}\,(\bar{A}\,\bar{I} - \bar{B}^2)} \begin{bmatrix}\bar{I} & -\bar{B}\\ -\bar{B} & \bar{A}\end{bmatrix} \begin{Bmatrix}-\Delta N\\ -\Delta M\end{Bmatrix} \quad (29)$$

where $\bar{A}$, $\bar{B}$ and $\bar{I}$ are the area of the age-adjusted transformed section and its first and second moments about an axis through the reference point 0; $\bar{E}_{ref}$ is an arbitrary reference modulus which may be conveniently chosen equal to the age-adjusted modulus of elasticity for concrete in one of the parts. The age-adjusted transformed section is composed of the area of concrete and steel multiplied by appropriate α-value calculated by Eq. 17.

The stress increments which develop during the period $(t - t_o)$ in concrete, in non-prestressed steel and in prestressed steel are respectively given by:

$$\Delta\sigma_c = \sigma_{restrained} + \bar{E}_{cj}\,(\Delta\varepsilon_o + y\,\Delta\psi) \quad (30)$$

$$\Delta\sigma_{ns} = E_{ns}\,(\Delta\varepsilon_0 + y_{ns}\,\Delta\psi) \quad (31)$$

$$\Delta\sigma_{ps} = \Delta\bar{\sigma}_{pr} + E_{ps}\,(\Delta\varepsilon_0 + y_{ps}\,\Delta\psi) \quad (32)$$

y is the distance from the reference point to the concrete fibre or the steel layer considered; the subscript j refers to the concrete part concerned and the subscripts ns and ps refer to the non-prestressed and the prestressed steels, respectively.

Equation 32 gives $\Delta\sigma_{ps}$ the change in stress in prestressed steel which is commonly referred to as the loss due to creep shrinkage and relaxation.

7. EXAMPLE 1

Figure 3 shows a cross section composed of a precast pre-tensioned beam (part 1) and a cast-in-site slab (part 2). Find the distributions of stress and strain immediately after prestressing. Also find the changes in stress and strain occurring between prestressing and casting of the slab and after a long period. The following data are given:

Ages of the precast beam at time of prestressing, t_1 = 3 days and at time of casting of the deck slab, t_2 = 60 days; the stress and strain are required at $t_3 = \infty$. The prestress force P = 4100 kN; the bending moment due to self-weight of the precast beam introduced at the time of prestressing, M_1 = 1400 kN-m; additional bending moment introduced at age t_2 representing weight of the slab plus superimposed dead load, M_2 = 1850 kN-m. Moduli of elasticity of concrete of the precast beam at ages 3 and 60 are: $E_{c1}(3) = 25$ GPa and $E_{c1}(60) = 37$ GPa.

The composite action occurring during the first three days after casting of the slab will be here ignored. Consider that age $t_2 = 60$ days for the precast beam corresponds to age 3 days of the slab at which time the modulus of elasticity of the deck, $E_{c2}(3) = 23$ Gpa.

Creep and aging coefficients and free shrinkage values are (chosen according to MC-78 [3]):

Concrete Part 1: $(60,3) = 1.20$; $(\infty,3) = 2.30$; $(\infty, 60) = 2.27$; $\chi(60,3) = 0.86$; $\chi(\infty,60) = 0.80$; $\varepsilon_{cs}(60,3) = -57 \times 10^{-6}$; $\varepsilon_{cs}(\infty,60) = -205 \times 10^{-6}$

Concrete Part 2: $(\infty,3) = 2.40$; $\chi(\infty,3) = 0.78$; $\varepsilon_{cs}(\infty,3) = -269 \times 10^{-6}$

Reduced relaxation $\Delta\bar{\sigma}_{pr} = -85$ MPa of which -15 MPa occurs in the first 57 days; $E_{ps} = E_{ns} = 200$ GPa.

The dimensions and properties of areas of the concrete and steel in the two parts are given in Fig. 3.

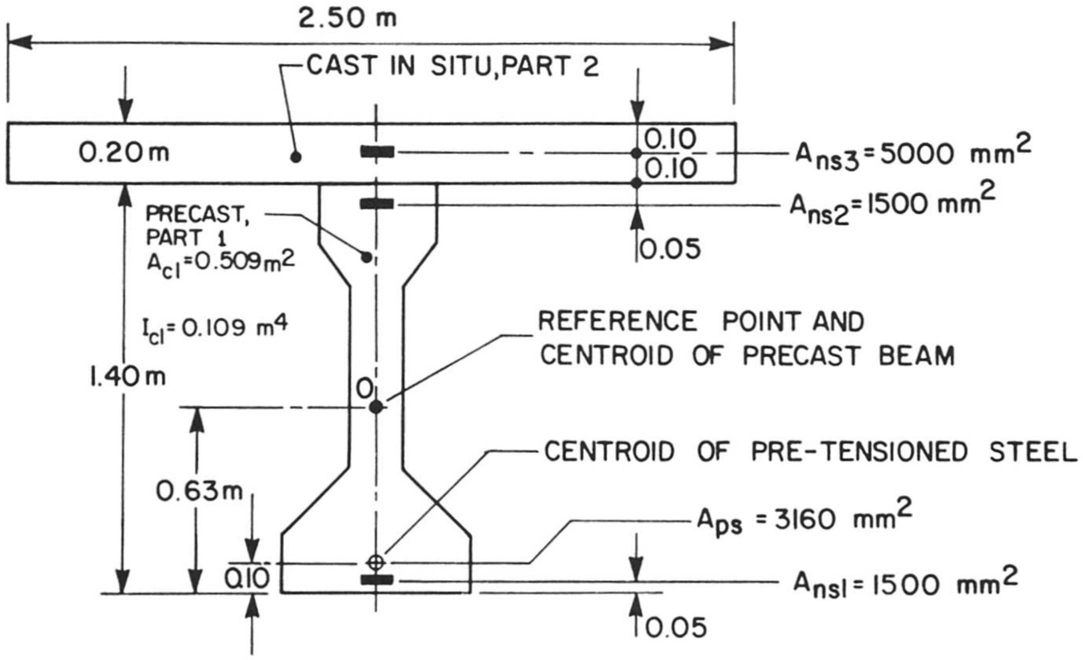


Fig. 3 Composite section considered in Example 1

The analysis is in four parts:

(a) *Stress and Strain Immediately After Prestressing*

Properties of the transformed section of the precast beam using $E_{ref} = E_{ci}(3) = 25$ GPa and choosing the reference point 0 at the centroid of concrete are:

$A = 0.5583$ m^2; $B = 0.0117$ m^3; $I = 0.1264$ m^4.

The prestress force and moment introduced at t_o are equivalent to (Eq. 15):

$$\begin{Bmatrix} N \\ M \end{Bmatrix}_{equivalent} = \begin{Bmatrix} -4100 \times 10^3 \\ 1400 \times 10^3 - 4100 \times 10^3 \times 0.53 \end{Bmatrix} = \begin{Bmatrix} -4100 \text{ kN} \\ -773 \text{ kN-m} \end{Bmatrix}$$

Instantaneous axial strain and curvature at t_1 = 3 days (Eq. 16)

$$\begin{Bmatrix} \varepsilon_0(t_1) \\ \psi(t_1) \end{Bmatrix} = \frac{1}{25 \times 10^9 [0.5583(0.1264) - (0.0117)^2]} \begin{bmatrix} 0.1264 & -0.0117 \\ -0.0117 & 0.5583 \end{bmatrix} \begin{Bmatrix} -4100 \times 10^3 \\ -773 \times 10^3 \end{Bmatrix}$$

$$= 10^{-6} \begin{Bmatrix} -289 \\ -218 \text{ m}^{-1} \end{Bmatrix}$$

The distributions of strain and stress are shown in Fig. 4a.

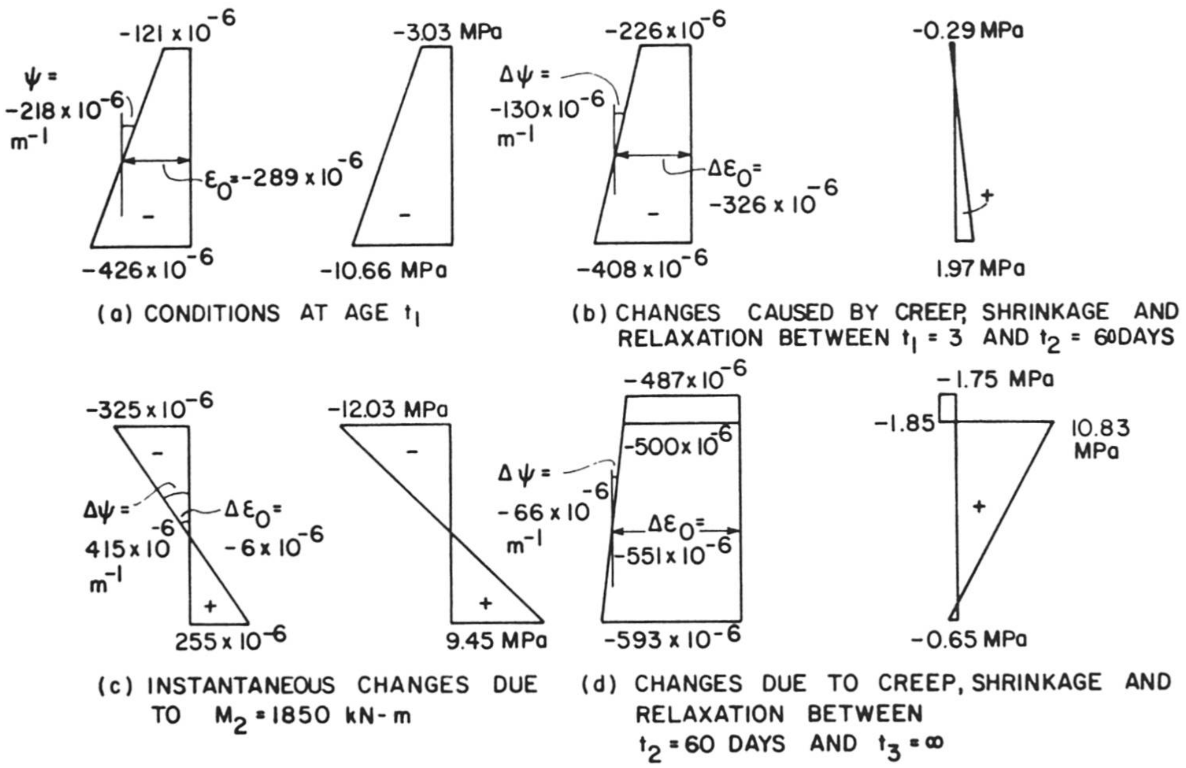


Fig. 4 Analysis of strain and stress in the composite cross section of Example 1

(b) *Changes in Stress and Strain Between t_1 = 3 Days and t_2 = 60 Days*

The age-adjusted elasticity modulus of concrete for part 1 (Eq. 24)

$$\bar{E}_c(60, 3) = \frac{25 \times 10^9}{1 + 0.86(1.20)} = 12.30 \text{ GPa}$$

Artificial stress to restrain creep and shrinkage (Eq. 23)

$$(\sigma_{restrained})_{top} = -12.3 \times 10^9 [1.2(-121 \times 10^{-6}) - 57 \times 10^{-6}] = 2.487 \text{ MPa}$$

$$(\sigma_{c\ restrained})_{bot} = -12.3 \times 10^9 (1.2(-426 \times 10^{-6}) - 57 \times 10^{-6}] = 6.985 \text{ MP}$$

Artificial forces to restrain creep, shrinkage and relaxation (Eqs. 25-27)

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{creep} = -12.30 \times 10^9 \times 1.2 \begin{bmatrix} 0.5090 & 0 \\ 0 & 0.1090 \end{bmatrix} \begin{Bmatrix} -289 \\ -218 \end{Bmatrix} 10^{-6} \begin{Bmatrix} 2171 \text{ kN} \\ 351 \text{ kN-m} \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{shrinkage} = -12.3 \times 10^9 (-57 \times 10^{-6}) \begin{Bmatrix} 0.5090 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 357 \text{ kN} \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{relaxation} = \begin{Bmatrix} 3160 \times 10^{-6}(-15 \times 10^6) \\ 3160 \times 10^{-6}(0.53)(-15 \times 10^6) \end{Bmatrix} = \begin{Bmatrix} -47 \text{ kN} \\ -25 \text{ kN-m} \end{Bmatrix}$$

The total restraining forces (Eq. 28)

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix} = 10^3 \begin{Bmatrix} 2171 + 357 - 47 \\ 351 + 0 - 25 \end{Bmatrix} = \begin{Bmatrix} 2481 \text{ kN} \\ 326 \text{ kN-m} \end{Bmatrix}$$

The properties of the age-adjusted transformed section are calculated using $\bar{E}_{ref} = \bar{E}_c(60, 3) = 12.3$ GPa giving for the precast part: $\bar{A} = 0.6092 \text{ m}^2$; $\bar{B} = 0.0238 \text{ m}^3$; $\bar{I} = 0.1443 \text{ m}^4$.

Removal of the artificial restraining forces results in the following changes in axial strain and curvature (Eq. 29)

$$\begin{Bmatrix} \Delta\varepsilon_0(t_2, t_1) \\ \Delta\psi(t_2, t_1) \end{Bmatrix} = 10^{-6} \begin{Bmatrix} -326 \\ -130 \text{ m}^{-1} \end{Bmatrix}$$

The distribution of strain and of stress changes during the period t_1 to t_2 are plotted in Fig. 4b. The stress change is calculated by Eq. 30. Superposition of Figs. 4a and 4b gives the strain and stress at $t_2 = 60$ days.

(c) *Instantaneous Increments of Stress and Strain at $t_2 = 60$ Days*

The bending moment M = 1850 kN-m is applied before hardening of the slab, at which time the properties of the transformed precast section are (using $E_{ref} = E_c(60) = 37$ GPa): $A = 0.5423 \text{ m}^2$; $B = 0.0079 \text{ m}^3$; $I = 0.1207 \text{ m}^4$. Substitution in Eq. 16 with N = 0 and M = 1850 kN-m gives the instantaneous increments of axial strain and curvature (Fig. 4c):

$$\begin{Bmatrix} \Delta\varepsilon_0(t_2) \\ \Delta\psi(t_2) \end{Bmatrix} = 10^{-6} \begin{Bmatrix} -6 \\ 415 \text{ m}^{-1} \end{Bmatrix}$$

(d) *Changes in Stress and Strain During the Period $t_2 = 60$ to $t_3 = \infty$*

The age-adjusted elasticity moduli for the precast beam and the slab are (Eq. 24):

$$\bar{E}_{c1}(\infty, 60) = \frac{37 \times 10^9}{1 + 0.80\ (2.27)} = 13.14 \text{ GPa}$$

$$\bar{E}_{c2}(\infty,3) = \frac{23 \times 10^9}{1 + 0.78\ (2.40)} = 8.01 \text{ GPa}$$

The stresses in Figs. 4a and 4b are introduced at various ages; however, for simplicity, combine the stress in the two figures and assume that the combined stress is introduced when the precast beam is 3 days old. The coefficient for creep during the period 60 to ∞ is equal to $[\varphi(\infty,3) - \varphi(60,3) = 2.30 - 1.20 = 1.10$. The stress in Fig. 4c is introduced when the precast beam is 60 days; the creep coefficient to be used for this stress is $\varphi(\infty,60) = 2.27$.

Artificial stress to restrain creep and shrinkage in the precast section (Eq. 23):

$$(\Delta\sigma_{c\ \text{restrained}})_{\text{top}} = -\ 13.14 \times 10^9 [1.1(-121-226)10^{-6} + 2.27(-\ 325 \times 10^{-6}) + (-\ 205 \times 10^{-6})] = 17.403 \text{ MPa}$$

$$(\Delta\sigma_{c\ \text{restrained}})_{\text{bot}} = -\ 13.14 \times 10^9 [1.1(-426-408)10^{-6} + 2.27(255 \times 10^{-6}) + (-\ 205 \times 10^{-6})] = 7.142 \text{ MPa}$$

In the deck slab, only shrinkage strain needs to be restrained by a constant stress:

$$\sigma_{c\ \text{restrained}} = -\ 8.01 \times 10^9 (-\ 269 \times 10^{-6}) = 2.155 \text{ MPa}$$

Forces to restrain creep and shrinkage and relaxation (Eqs. 25-27)

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{\text{creep}} = -\ 13.14 \times 10^9 \begin{bmatrix} 0.509 & 0 \\ 0 & 0.109 \end{bmatrix} \begin{Bmatrix} 1.10(-289-326) + 2.27(-6) \\ 1.10(-218-130) + 2.27(415) \end{Bmatrix} 10^{-6} = \begin{Bmatrix} 4616 \text{ kN} \\ -801 \text{ kN-m} \end{Bmatrix}$$

The deck slab is not included in the above equation because no stress is applied on the slab before the period considered.

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{\text{shrinkage}} = -13.14\text{x}10^9(-205\text{x}10^{-6}) \begin{Bmatrix} 0.509 \\ 0 \end{Bmatrix} - 8.01\text{x}10^9(-269\text{x}10^{-6}) \begin{Bmatrix} 0.495 \\ -0.4307 \end{Bmatrix} = \begin{Bmatrix} 2437 \text{ kN} \\ -928 \text{ kN-m} \end{Bmatrix}$$

The values 0.495 m^2 and -0.4307 m^3 employed in this equation represent the area of concrete section and its first moment about an axis through 0 for part 2 (see Fig. 3).

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{\text{relaxation}} = \begin{Bmatrix} 3160 \times 10^{-6}(-\ 70 \times 10^6) \\ 3160 \times 10^{-6}(0.53)(-\ 70 \times 10^6) \end{Bmatrix} = \begin{Bmatrix} -221 \text{ kN} \\ -117 \text{ kN-m} \end{Bmatrix}$$

The total restraining forces

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix} = 10^6 \begin{Bmatrix} 4616 + 2437 - 221 \\ -801 - 928 - 117 \end{Bmatrix} = \begin{Bmatrix} 6832 \text{ kN} \\ -1846 \text{ kN-m} \end{Bmatrix}$$

Properties of the transformed composite section age-adjusted for the period t_2 to t_3 are (using $E_{ref} = \bar{E}_{c1}(\infty, 60) = 13.14$ GPa); $\bar{A} = 0.9806\ m^2$; $\bar{B} = -0.3064\ m^3$; $\bar{I} = 0.4290\ m^4$.

Eliminate the artificial restraint by application of $\{-\Delta N, -\Delta M\}$ on the age-adjusted transformed section giving the following changes in axial strain and curvature by Eq. 29 (Fig. 4d).

$$\begin{Bmatrix} \Delta\varepsilon_0(t_3, t_2) \\ \Delta\psi(t_3, t_2) \end{Bmatrix} = 10^{-6} \begin{Bmatrix} -551 \\ -66\ m^{-1} \end{Bmatrix}$$

The change in stress during the period t_2 to t_3 is calculated by Eq. 30 and plotted in Fig. 4d.

The total strain and stress at $t_3 = \infty$ is the sum of the values in the four parts of Fig. 4; see Fig. 5.

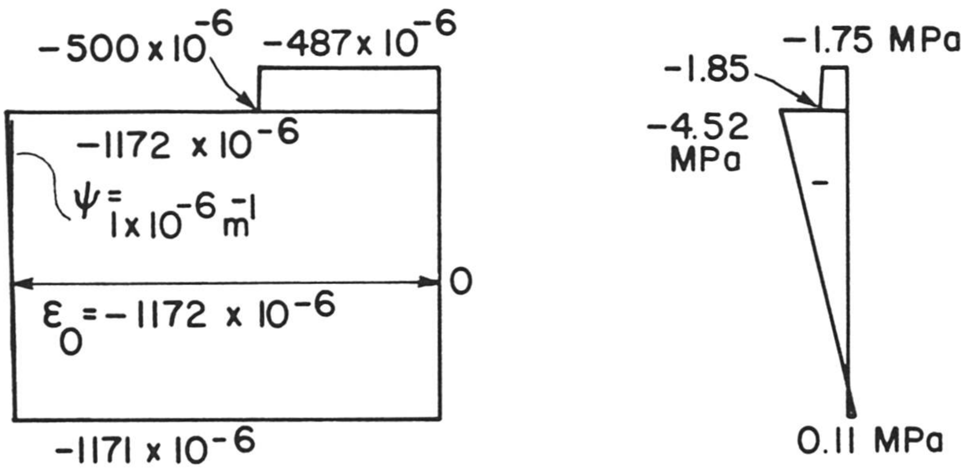


Fig. 5 Strain and stress at time $t_3 = \infty$ in the composite cross section of Fig. 3

Changes in Stress in Prestressed Steel

In each of the four stages considered above the stress increments in the prestressed steel may be determined by Eq. 32. This gives: (a) Instantaneous change at prestress transfer, $\Delta\sigma_{ps} = -80.9$ MPa; (b) increment between $t_1 = 3$ days and $t_2 = 60$ days, $\Delta\sigma_{ps} = -94.0$ MPa; (c) instantaneous change at t_2, $\Delta\sigma_{ps} = +42.8$ MPa; (d) increment between $t_2 = 60$ days and $t_3 = \infty$, $\Delta\sigma_{ps} = -187.2$ MPa.

The total change in stress in the prestressed steel is -319.3 MPa; this is a loss of 1009 kN from the initial tension. The tension remaining in the prestressed steel at $t_3 = \infty$ is 3091 kN. The resultant normal force on the concrete at the same instant is = -1894 kN (= integral of stress in Fig. 5 over the concrete area). The difference between the absolute values of these two quantities is 1197 kN, which represents the compression picked up by the non-prestressed steel.

This large force indicates that the presence of the non-prestressed steel cannot be ignored. The force in each layer of the non-prestressed steel may be calculated by multiplication of the strain at each layer from Fig. 5 by $(E_{ns} A_{ns})$; this gives the following forces in the three layers of non-prestressed steel from top to bottom, respectively: -494, -352 and -351 kN and the sum of the three forces is -1197 kN. This can serve as a check on equilibrium.

8. DISPLACEMENTS

When the axial strain ε_0 and the curvature ψ are known at a number of sections, the displacements can be determined by integration. If for a straight member the variation of ε_0 and ψ are assumed parabolic, the following equations may be employed to calculate the change in length and the transverse deflection at mid-point with respect to the ends:

$$\Delta \ell = \frac{\ell}{6} [1 \quad 4 \quad 1] \begin{Bmatrix} \varepsilon_{01} \\ \varepsilon_{02} \\ \varepsilon_{03} \end{Bmatrix} \tag{33}$$

$$\delta = \frac{\ell^2}{96} [1 \quad 10 \quad 1] \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{Bmatrix} \tag{34}$$

where ℓ is the length of the member; the subscripts 1 and 3 refer to sections at the two ends and 2 refers to the section at the middle.

9. EXAMPLE 2

Assuming that the cross section analyzed in Example 1 is the middle section of a simple beam 30 m span, what is the central deflection at time t_2 = 60 days, just before casting of the deck slab?

The curvature at mid-span is $10^{-6}(-218-130) = -348 \times 10^{-6}\ m^{-1}$. Apply Eq. 34 and ignore the curvature at the ends:

$$\delta = \frac{(30)^2}{96} (10)(-348 \times 10^{-6}) = -33 \text{ mm}$$

The minus sign indicates a camber.

10. CONCLUSIONS

The method presented gives the strain and stress distributions in composite or non-composite prestressed cross-sections without the need of separate calculations of prestress losses due to creep, shrinkage and relaxation. These time-dependent effects are accurately accounted for by the analysis which satisfies compatibility of strain at all reinforcements and equilibrium of forces in the steel and concrete with forces applied on the section. Use of the method minimizes the approximation in the analysis and limits the source of errors to the uncertainty in material properties.

Presence of non-prestressed steel greatly influences the stresses and the deformations and hence must not be ignored. The analysis can be done by hand, but for repetitive calculations, use of a small computer may be more convenient; ref. [5] includes a computer program which can be used for the analysis of non-cracked and cracked composite sections.

NOTATION

A	area
B	first moment of area
E	modulus of elasticity
$\bar{E}$	age-adjusted modulus of elasticity employed when the stress is gradually introduced
I	second moment of area
M	bending moment
N	normal force
P	absolute value of prestressing force
t	time or age of concrete in days
α	ratio of modulus of elasticity of one concrete part or a steel type to a reference elasticity modulus
α_t	coefficient of thermal expansion
χ	aging coefficient smaller than 1 used as multiplier to for creep due to a gradually introduced stress
χ_r	relaxation reduction factor
Δ	increment
ε	normal strain
φ	creep coefficient = ratio of creep to instantaneous strain
ψ	curvature
σ	normal stress

Subscripts

c	concrete
cs	shrinkage
f	free, unstrained
ns	non-prestressed steel
0	reference point
pr	relaxation of prestressed steel
ps	prestressed steel
0, 1, 2	instants of time

REFERENCES

1. AMERICAN CONCRETE INSTITUTE (ACI) COMMITTEE 209, "Prediction of Creep, Shrinkage and Temperature Effects in Concrete Structures", *ACI Publication*, SP-76, 1982, ACI, Detroit, Michigan, pp. 193-300.

2. BAZANT, Z.P., "Prediction of Concrete Creep Effects Using Age-Adjusted Effective Modulus Method", *American Concrete Institute Journal*, Proceedings V. 69, No. 4, April 1972, pp. 212-217.

3. COMITÉ EURO-INTERNATIONAL DU BÉTON (CEB) - FÉDÉRATION INTERNATIONALE DE LA PRÉCONTRAINTE (FIP), *Model Code for Concrete Structures*, 1978 (MC-78), CEB, 6 rue Lauriston, F-75116 Paris.

4. ELBADRY, M.M and A. GHALI, "Nonlinear Temperature Distribution and its Effects on Bridges", *International Association of Bridge and Structural Engineering Proceedings*, p. 66/86, Periodica 3/1983.

5. GHALI, A. and ELBADRY, M.M., *Manual of Computer Program CRACK*, Research Report No. CE85-1, 1985, Department of Civil Engineering, The University of Calgary, Alberta, Canada.

6. GHALI, A. and ELBADRY, M.M., "Cracking of Composite Prestressed Cross Sections", to be published.

7. GHALI, A. and R. FAVRE, *Concrete Structures: Stresses and Deformations*, Chapman and Hall, London and New York, 1986, 350 pp.

8. GHALI, A. and J. TREVINO, "Relaxation of Steel in Prestressed Concrete", *Prestressed Concrete Institute Journal*, Vol. 30, No. 5, Sept./Oct. 1985, pp. 82-94.

9. FAVRE, R., BEEBY, A.W., FALKNER, H., KOPRNA, M. and SCHIESSEL, P., *CEB design Manual on Cracking and Deformations*, Ecole Polytechnique de Lausanne, Switzerland, 1985.

10. NEVILLE, A.M., DILGER, W.H. and BROOKS, J.J., *Creep of Plain and Structural Concrete*, Construction Press, London and New York, 1983, 361 pp.

11. TROST, H., "Auswirkungen des Superpositionspringzips auf Kriech - und Relaxations - Probleme bei Beton und Spannbeton, *Beton und Stahlbetonbau*, V. 62, No. 10, 1967, pp. 230 - 238 and No. 11, 1967, pp. 261-269.