Geometric Nonlinear Analysis of Structures by Discrete Energy Method

Analyse géométrique non linéare de structures, à l'aide de la méthode d'énergie discrète

Geometrisch nicht-lineare Berechnung von Tragwerken mittels der Methode der diskreten Energie

SUBHASH C. PATODI
Lecturer
S.V.R. College of Engineering and Technology
Surat, India

DHIRENDRA N. BURAGOHAIN
Professor of Civil Engineering
Indian Institute of Technology
Powai, Bombay, India

SUMMARY
The paper presents a special form of finite difference scheme for geometrically nonlinear analysis of one and two dimensional problems. The method uses energy principles to derive a set of nonlinear algebraic equations which are solved by using Newton-Raphson iterative procedure. A study of large deflection and stability behaviour of various structural problems is presented and results are compared with available exact and approximate solutions.

RÉSUMÉ
L'article présente une méthode particulière par différences finies, pour l'analyse géométrique non linéaire de problèmes à une et deux dimensions. Des équations non linéaires sont établies, puis résolues au moyen de la méthode d'itération de Newton-Raphson. Le comportement de structures du point de vue de la stabilité et des grandes déformations est présenté et les résultats sont comparés avec des solutions disponibles, exactes et approchées.

ZUSAMMENFASSUNG
Der Artikel zeigt ein spezielles Differenzen-Schema für die geometrisch nichtlineare Berechnung von ein- und zweidimensionalen Problemen. Die Methode macht von Energie-Prinzipien Gebrauch, um ein System von nichtlinearen Gleichungen aufzustellen, welche mit dem Newton-Raphson Iterationsverfahren gelöst werden. Das Verformungs- und Stabilitätsverhalten verschiedener Tragwerke werden untersucht und die Ergebnisse mit genauen Methoden und Näherungsverfahren verglichen.

1. INTRODUCTION

Geometric nonlinearity results in two important classes of problems which are well known to the structural engineer - the large deflection problem and the problem of structural stability. Problems which involve either large deformation or elastic stability have become important with the increased use of very thin structural elements. In some cases, such as ship and air craft structures, an elastic large deflection theory is of special interest, since the structural components undergo deflections of the order of their thickness before the onset of plasticity.

A thorough analysis of such structural problems has become possible only with the development of modern analytical techniques requiring the use of digital computers. Basically, there are two distinct numerical methods for quantitative solution to such problems: (1) Finite difference method and (2) Finite element method.

The traditional finite difference approach, known as direct substitution method, involves applying the finite difference technique directly to the derived partial differential equations using conventional molecule schemes. A second approach, known as energy method, is to write expressions for strain energies, and to model differential quantities by difference quantities in the energy expression. Minimization of the potential energy expressions yields a set of algebraic equations.

The energy method is superior to the direct substitution method because it yields a symmetric and positive definite coefficient matrix, and has to satisfy only the displacement boundary conditions.

In the classical finite element method, the element strain energy is expressed in terms of the nodal displacements of the element. Such a process necessarily increases the number of degrees of freedom at a node and requires handling of comparatively large size element matrices which finally results into more computer storage and solution time particularly in nonlinear analysis.

Thus the energy method with finite difference appears to be a economical alternative to the finite element method since it retains the advantageous properties of symmetry and positive definiteness, yet it needs fewer degrees of freedom per node to describe the deformed configuration of the structure compared to the finite element method.

The finite difference energy method is well established in the linear and nonlinear analysis of plate and shell structures [1,2]. A comprehensive discussion of various alternative forms of this method has been presented by Noor and Schnobrich [2]. However, most of the approaches using the finite difference energy concept are found to be boundary dependent.

In this paper a special form of finite difference energy method is presented, and is called as discrete energy method (DEM), in which additional degrees of freedom are introduced in terms of the nodal displacements to make the formulation independent of the boundary condition. The development of a discrete energy method for large deflection analysis of plates and shells has been earlier reported by Buragohain and Patodi [3]. The same approach is utilized here to develop tangential stiffness matrix for a beam and a arch problem and is applied to study the large deflection and stability behaviour of various one and two dimensional problems.

2. LARGE DISPLACEMENT ANALYSIS

If $\{\Psi\}$ represents the vector of the sum of the sum of the internal and external forces, we can derive the equilibrium equation by following the principle of virtual work and can finally write [4]

$$\{\Psi\} = \int [B]^T \{\sigma\} dV - \{R\} = 0 \tag{1}$$

where $\{R\}$ represents the external forces due to imposed loads and $[B]$ is the strain displacement coefficient matrix and relates strains $\{\varepsilon\}$ to displacements $\{\delta\}$ as

$$\{\varepsilon\} = [B]\{\delta\} \tag{2}$$

and $\{\sigma\}$ is the stress matrix related to strains $\{\varepsilon\}$ through elasticity matrix $[D]$ by

$$\{\sigma\} = [D]\{\varepsilon\} \tag{3}$$

Appropriate variations of $\{\Psi\}$ due to changes $d\{\delta\}$ finally results into [4]

$$\begin{aligned} d\{\Psi\} &= [K_T]\, d\{\delta\} \\ &= ([K_L] + [K_N] + [K_\sigma])\, d\{\delta\} \end{aligned} \tag{4}$$

where $[K_T]$ is the total stiffness matrix and is the sum of linear stiffness matrix $[K_L]$, initial displacement matrix $[K_N]$ and initial stress matrix $[K_\sigma]$ and is known as tangential stiffness matrix, formulation of which requires calculation of various linear and nonlinear strain displacement coefficient matrices.

2.1 Beam Formulation

The geometry of the beam is characterized by the axial coordinate y and vertical coordinate z. The cross-sectional area is A, the extreme fibres are situated at $Z = \pm t/2$, where t is the depth of the beam, and behaviour is constant with x, see Fig. 1. In accordance with Von Karman large deflection theory the strain at any point of the beam can be obtained from [4]

$$\{\varepsilon\} = v_{,y} + \frac{1}{2} w_{,y}^2 + w_{,yy} \tag{5}$$

where v and w are the displacements in the y and z directions respectively and a 'comma' denotes derivatives with respect to the subscript.

For convenience, total strain vector for a beam is written as

$$\{\varepsilon\} = \left\{\begin{array}{c} v_{,y} \\ \hline w_{,yy} \end{array}\right\} + \frac{1}{2}\left\{\begin{array}{c} w_{,y}^{2} \\ \hline 0 \end{array}\right\} = \left\{\begin{array}{c} \varepsilon_p^L \\ \hline \varepsilon_b \end{array}\right\} + \left\{\begin{array}{c} \varepsilon_p^N \\ \hline 0 \end{array}\right\} = \left\{\begin{array}{c} \varepsilon_p \\ \hline \varepsilon_b \end{array}\right\} \qquad (6)$$

in which the first column represents linear terms and the second gives nonlinear contribution. Here subscript, 'p' stands for in-plane, 'b' for bending and superscript, 'L' stands for linear and 'N' for nonlinear or large deflection contribution.

Since only straight derivatives are present in the strain vector, strain energy U due to bending and extensional forces is only present and can be defined as

$$\begin{aligned} U &= \frac{1}{2}\int \{\varepsilon\}^T [\,D\,]\{\varepsilon\}\, dV \\ &= \frac{1}{2}\int \{\delta\}^T [\,B\,]^T [\,D\,][\,B\,]\{\delta\}\, dV \\ &= \frac{1}{2}\int \{\delta\}^T [\,K\,]\{\delta\}\, dV \qquad (7) \end{aligned}$$

where $[\,K\,]$ is the stiffness matrix and is defined as

$$[\,K\,] = \int [\,B\,]^T [\,D\,][\,B\,]\, dV \qquad (8)$$

Equation (7) with the help of equation (6) can be written as

$$U = \frac{1}{2} \left\{\begin{array}{c} \varepsilon_p \\ \hline \varepsilon_b \end{array}\right\}^T \left[\begin{array}{c|c} D_p & 0 \\ \hline 0 & D_b \end{array}\right] \left\{\begin{array}{c} \varepsilon_p \\ \hline \varepsilon_b \end{array}\right\} dV \qquad (9)$$

where D_p and D_b are the extensional and flexural rigidities of the beam respectively.

To estimate the strain energy or to calculate the strain displacement relation matrices, the continuum is divided into a set of beam elements. The w-nodes form the regular finite difference grid pattern and v-nodes are placed in the middle of two w-nodes in the y direction. At the boundaries, the displacement v and the rotation $w_{,y}$ $(= \theta_x)$ are introduced to make the formulation independent of the boundary conditions. Thus three types of elements are found necessary, two boundary elements A2 and A3 and one intermediate element A1 representing for all the elements coming between A2 and A3, refer to Fig. 2.

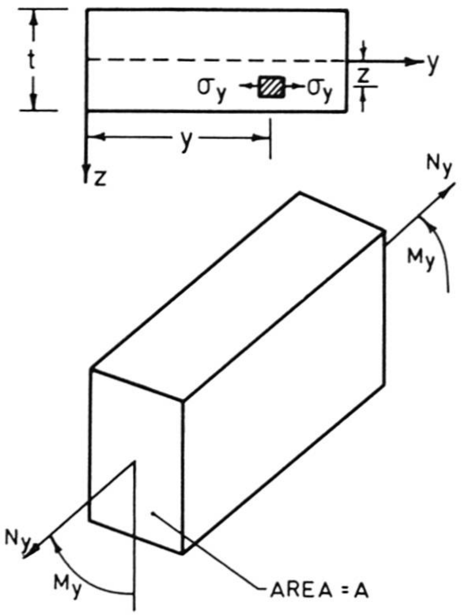

Fig.1 Beam geometry and stresses

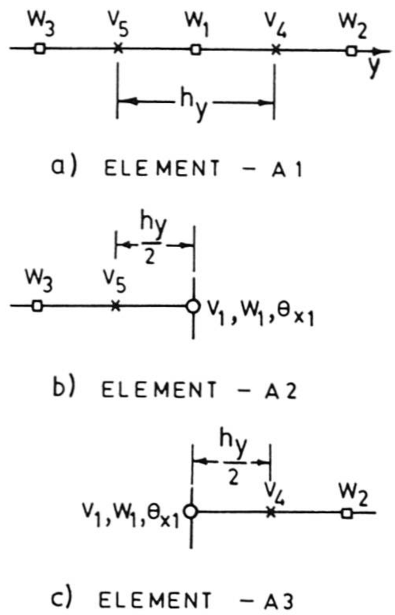

Fig.2 Contribution areas for a beam

Element - A1

Referring to Fig. 2a and equation (6), we can define strains in terms of displacements by replacing strain derivatives by their finite difference equivalents.

$$\{\varepsilon_p^L\} = \{v_{,y}\} = \begin{bmatrix} \frac{1}{h_y} & \frac{-1}{h_y} \end{bmatrix} \begin{Bmatrix} v_4 \\ v_5 \end{Bmatrix} = [B_p^L]\{\delta_p\} \tag{10}$$

$$\{\varepsilon_b\} = \{w_{,yy}\} = \begin{bmatrix} \frac{-2}{h_y^2} & \frac{1}{h_y^2} & \frac{1}{h_y^2} \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = [B_b^L]\{\delta_b\} \tag{11}$$

and

$$\{\varepsilon_p^N\} = \frac{1}{2}\{w_{,y}^2\} = \frac{1}{2}[w_{,y}]\{w_{,y}\} = \frac{1}{2}[A]\{\theta\} . \tag{12}$$

Rotation $\{w_{,y}\}$ is considered at v node and is assumed to vary linearly in y over the element. Thus derivative of w can be related to the bending displacement vector $\{\delta_b\}$ by a coefficient matrix $[G]$ as

$$\{\theta\} = \{w_{,y}\} = [G]\{\delta_b\} \tag{13}$$

where $$[G] = \begin{bmatrix} \frac{-2y}{h_y^2} & \frac{(1+2y/h_y)}{2h_y} & \frac{-(1-2y/h_y)}{2h_y} \end{bmatrix}$$

Finally the nonlinear strain displacement matrix $[B_b^N]$ can be obtained from [4]

$$[B_b^N] = [A][G] . \tag{14}$$

<u>Element - A2</u>

Referring to Fig. 2b and equation (6), we can write

$$\{\varepsilon_p^L\} = \{v_{,y}\} = \begin{bmatrix} \frac{2}{h_y} & \frac{-2}{h_y} \end{bmatrix} \begin{Bmatrix} v_1 \\ v_5 \end{Bmatrix} = [B_p^L]\{\delta_p\} \tag{15}$$

$$\{\varepsilon_b\} = \{w_{,yy}\} = \begin{bmatrix} \frac{-2}{h_y^2} & \frac{2}{h_y} & \frac{2}{h_y^2} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{x1} \\ w_3 \end{Bmatrix} = [B_b^L]\{\delta_b\} \tag{16}$$

and

$$\{\theta\} = \{w_{,y}\} = \begin{bmatrix} \frac{-2y}{h_y^2} & (1+\frac{2y}{h_y}) & \frac{2y}{h_y^2} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{x1} \\ w_3 \end{Bmatrix} \tag{17}$$

$$= [G]\{\delta_b\} \tag{18}$$

<u>Element - A3</u>

Referring to Fig. 2c and equation (6), we can write

$$\{\varepsilon_p^L\} = \{v_{,y}\} = \begin{bmatrix} \frac{2}{h_y} & \frac{-2}{h_y} \end{bmatrix} \begin{Bmatrix} v_4 \\ v_1 \end{Bmatrix} = [B_p^L]\{\delta_p\} \tag{19}$$

$$\{\varepsilon_b\} = \{w_{,yy}\} = \begin{bmatrix} \frac{-2}{h_y^2} & \frac{2}{h_y^2} & \frac{-2}{h_y} \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ \theta_{x1} \end{Bmatrix} = [B_b^L]\{\delta_b\} \tag{20}$$

and

$$\{\theta\} = \{w_{,y}\} = \begin{bmatrix} \frac{-2y}{h_y^2} & \frac{2y}{h_y^2} & (1-\frac{2y}{h_y}) \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ \theta_{x1} \end{Bmatrix} \tag{21}$$

Once the strain displacement relation matrices $[B_p^L]$, $[B_b^L]$ and $[B_b^N]$ are known, element tangential stiffness matrix can be obtained from [4]

$$[K_T] = [K_L] + [K_N] + [K_\sigma]$$

$$\underset{(5\times5)}{[K_T]} = \int \left[\begin{array}{c|c} [B_p^L]^T [D_p][B_p^L] & [B_p^L]^T [D_p][B_b^N] \\ \hline [B_b^N]^T [D_p][B_p^L] & [B_b^L]^T [D_b][B_b^L] + \\ & [B_b^N][D_p][B_p^N]^T + \\ & [G]^T [S][G] \end{array} \right] dV \tag{22}$$

where [S] is known as stress matrix and for a beam problem is given by

$$[S] = N_y \tag{23}$$

It can be noted here that the use of the modified finite difference scheme results into constant $[B_p^L]$ and $[B_b^L]$ matrices whereas the nonlinear strain displacement matrix $[B_b^N]$ vary linearly within the element and hence calculation of nonlinear stiffness matrices requires numerical integration.

2.2 Modification for Shallow Arch

Following the shallow arch theory of Marguerre [5], total strain vector for a large deflection problem can be defined as

$$\begin{aligned}\{\varepsilon\} &= \left\{\begin{array}{c} v_{,y} \\ \hline w_{,yy} \end{array}\right\} - \left\{\begin{array}{c} w/R_y \\ \hline 0 \end{array}\right\} + \frac{1}{2}\left\{\begin{array}{c} w^2_{,y} \\ \hline 0 \end{array}\right\} \\ &= \left\{\begin{array}{c} \varepsilon_p^L \\ \hline \varepsilon_b \end{array}\right\} + \left\{\begin{array}{c} \varepsilon_p^N \\ \hline 0 \end{array}\right\} = \left\{\begin{array}{c} \varepsilon_p \\ \hline \varepsilon_b \end{array}\right\}\end{aligned} \tag{24}$$

where v is the tangential displacement along circumferential direction, w is the normal displacement and R_y is the radius of curvature in y direction.

The change in $\{\varepsilon_p^L\}$ can be taken into account by changing $[B_p^L]$ matrix. For example, for element - A1, we can write

$$\begin{aligned}\{\varepsilon_p^L\} &= \{v_{,y}\} - \{w/R_y\} \\ &= \left[\frac{1}{h_y} \quad \frac{-1}{h_y} \quad \frac{-1}{R_y}\right] \begin{Bmatrix} v_4 \\ v_5 \\ w_1 \end{Bmatrix} = [B_p^L]\{\delta_p\}\end{aligned} \tag{25}$$

Now the element tangent stiffness matrix is generated by using this new definition of $[B_p^L]$. Computational procedure and other things remain same as for a beam problem.

3. STABILITY ANALYSIS

If the displacements, but not stresses, are such that the influence of the initial displacements can be ignored, i.e. the nonlinear stiffness matrix $[K_N] = 0$, then equation (4) can be written as

$$d\{\Psi\} = ([K_L]\lambda + [K_\sigma])\, d\{\delta\} \tag{26}$$

If initial stress matrix $[K_\sigma]$ is evaluated from a linear solution at a certain load level and the stresses are assumed to vary linearly according to some load parameter λ, then a linear eigenvalue buckling analysis results characterized by the equation

$$d\{\Psi\} = ([K_L]\lambda + [K_\sigma])\, d\{\delta\} = 0 \tag{27}$$

From this λ can be obtained by solving a typical eigenvalue problem. The lowest eigenvalue gives the critical stress σ_{cr} which can be related to plate constants by a constant factor C as

$$\sigma_{cr} = C\,\frac{\pi^2 D}{L^2 t} \tag{28}$$

where D is the plate flexural elastic constant and L is the width of the plate.

This approach can give physically significant answers only if the elastic solution gives such deformations that the large deformation matrix $[K_N]$ is identically zero. If the initial displacements are significant and $[K_N]$ is included then such problems are investigated using the full tangential stiffness matrix, i.e. when $[K_T]\, d\{\delta\}$ is identically zero, neutral equilibrium is obtained.

An equilibrium position for an elastic system is stable when the total potential energy evaluated at the equilibrium position is a relative minimum. This requires the second variation of potential energy to be positive definite. Thus to detect instability, the sign of $|[K_T]|$, must be noted. Bifurcation occurs when $|[K_T]| = 0$ and the position is unstable when $|[K_T]| < 0$.

4. ILLUSTRATIVE EXAMPLES

The following applications are intended to demonstrate the capability of the suggested discrete energy formulation.

4.1 Beam Problem

The method is applied to a variety of large deflection beam problems. Newton-Raphson iterative method [4] with incremental loading is used for the solution of nonlinear algebraic equations and 'Euclidean norm' displacement criterion [6] is used with one per cent cut off for checking the convergence. A 20 inch long beam of square cross-section, 1/8 in x 1/8 in, is considered for analysis. The modulus of elasticity is $E = 30 \times 10^6$ psi. The beam is divided into 10 equal parts which result into 11 elements, 25 number of unknowns and a band width of 6.

The first beam problem is a fixed beam subjected to a central point load. Only half of the beam is analysed. Load deflection response is presented in Fig. 3 . A total of 6 load increments are used and for each level of loading Newton-Raphson iteration is used. DEM requires 21 iterations for convergence with time/cycle as 5 seconds on EC 1030 Computer. Deflection values compare well with the exact values obtained by Frisch-Fay [7].

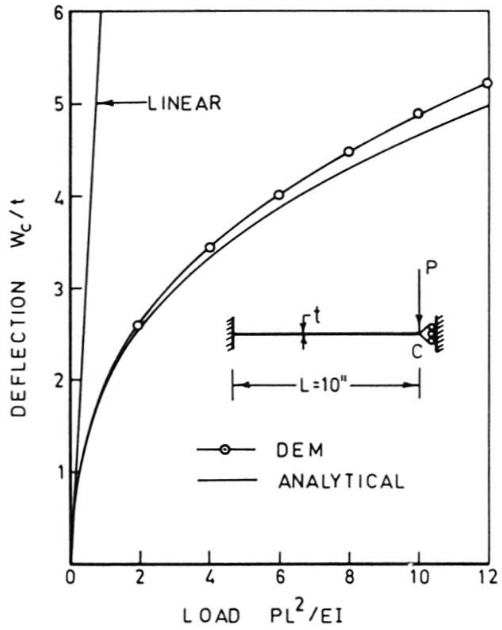

Fig. 3 Clamped beam under central point load

Next, a hinged beam subjected to a central point load is analysed. DEM requires a total of 18 iterations for final convergence. Again time/cycle is 5 seconds. Comparison of central deflection with the results obtained by Wood [8], by using finite element method (10 elements), is shown in Fig. 4 .

Again the same idealization, geometry and sectional properties are used for analysing a cantilever beam problem subjected to an end load. A total of 16 iterations are required for reaching the final load value of 3.0 with time/cycle as 5 seconds. The results are presented in Fig. 5 .

It is clear from the graphs presented that the discrepancy between the deflection values obtained for beam problems is less than 5 per cent with the finite element and analytical results.

4.2 Shallow Circular Arch Problem

A shallow circular arch fixed at the end and subjected to a central point load is analysed based on discrete energy formulation. 12 elements are used and only half of the arch is analysed. The full tangential stiffness matrix is used to detect instability. The load deflection response is shown in Fig. 6 together with the critical load value. Results are compared with those obtained by Marcal [9] and with Gjelsvik's experimental results [10]. DEM shows a slightly higher value of the critical load compared to the experimental value while Marcal obtained a lower bound solution by using finite element method with 16 elements.

4.3 Shallow Cylindrical Shell Problem

Shallow shell discrete energy formulation described elsewhere [3] is used here to analyse a clamped shallow cylindrical shell subjected to uniform normal pressure. 4 x 4 discretization is used and only one quarter is analysed due to two way symmetry which results into 105 unknowns.

Load deflection behaviour obtained by using DEM is presented in Fig. 7. Value of the critical load at which determinant becomes zero is also given. The critical load value obtained is comparable with the solutions obtained by Gallagher [11], Dhatt [12] and Morin [13].

4.4 Plate Problems

Here bifurcation instability analysis of a plate loaded purely in its own plane is presented. As lateral deflections, w, are produced only due to inplane action, the small deflection theory gives an exact solution and only initial stability is considered.

A fixed square plate is first analysed for uniform compression in one direction and then it is analysed for uniform compression from two directions. Due to symmetry only quarter of the plate is discretized. 4 x 4 DEM discretization results into only 45 unknowns as only lateral displacements are considered.

The lowest eigenvalue gives the critical stress. Values of factor C obtained by using DEM for a fixed square plate under two loading conditions are compared with the values obtained by Kapur and Hartz [14] using finite element method and also with exact values [15]. The comparison presented in Table 1 shows very good agreement.

Table - 1 Comparison of Factor C for a Fixed Plate

Method	Number of unknowns	Compression in one direction		Compression in two directions	
		Value of C	Error in percentage	Value of C	Error in percentage
1 DEM (4 x 4)	45	10.340	+2.68	5.551	+4.44
2 FEM (4 x 4)	75	9.782	-2.86	5.160	-2.92
3 Exact [15]	-	10.070	-	5.315	-

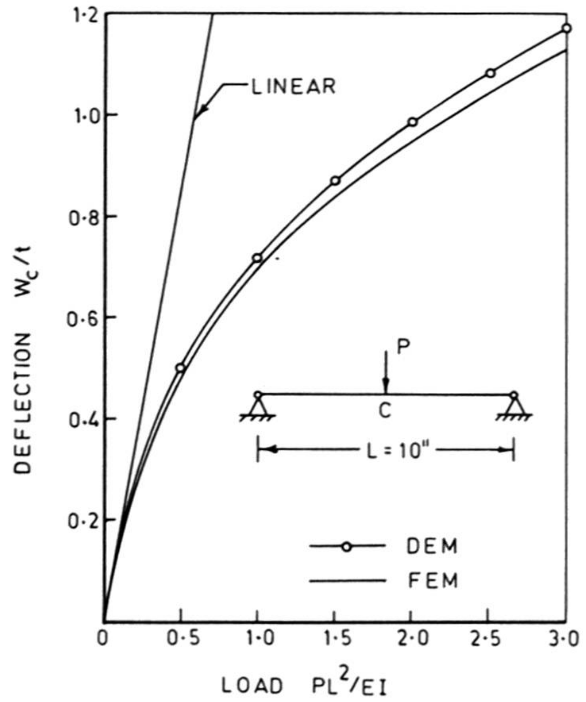

Fig.4 Hinged beam under central point load

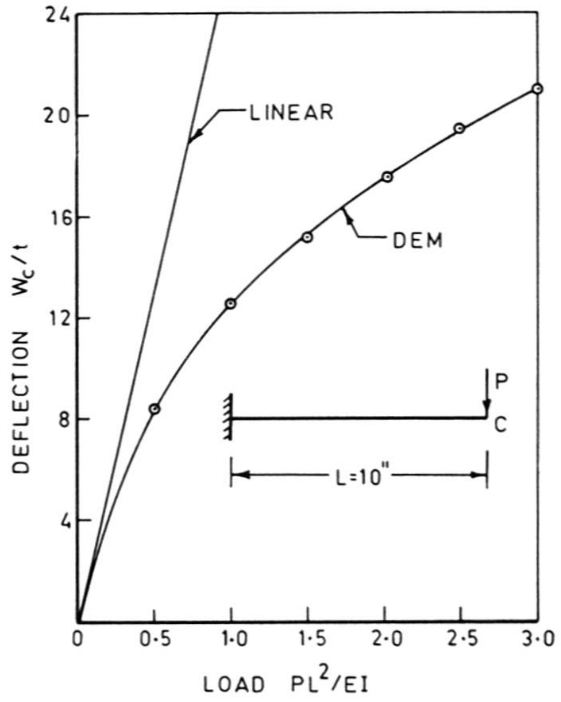

Fig.5 Cantilever under point load

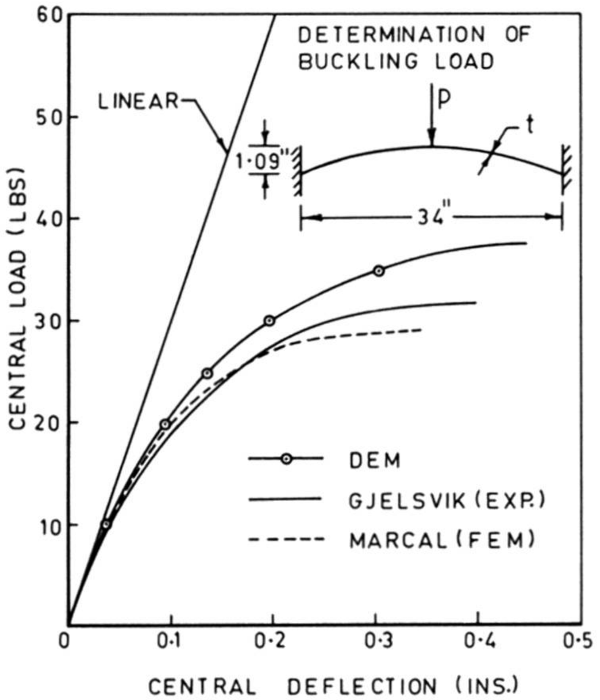

Fig.6 Geometry and load deflection graph - shallow circular arch

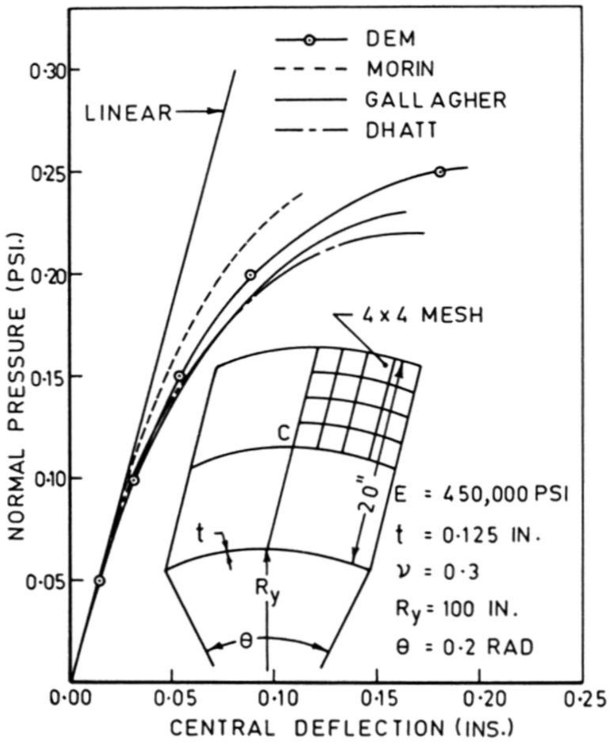

Fig.7 Clamped shallow cylindrical shell under UDL

5. CONCLUSIONS

By using modified finite difference approach, in conjunction with minimum energy principle, a simplified and systematic method of derivation has been discussed for development of tangent stiffness matrices for geometrically nonlinear structural problems.

As the number of unknown displacements to be evaluated are less and since numerical integrations are avoided for calculating element inplane and bending linear stiffness matrices, computationally the present method is faster than the equivalent finite element scheme and requires less computer core storage.

Finally, on the basis of the numerical results presented here, as well as the results presented by the authors earlier [3,16], it can be stated that whereever applicable the proposed formulation can hope to achieve computational economy without much loss of accuracy because of fewer degrees of freedom and fewer computations.

REFERENCES

1. BUSHNELL, D., ALMROTH, B.O. and BROGAN, F.: Finite difference energy method for nonlinear shell analysis, Computers and Structures, vol. 1, 1971, pp 361 - 387.
2. NOOR, A.K. and SCHNOBRICH, W.C.: On improved finite difference discretization procedure, Proc. of the International Conference on Variational Methods in Engineering, Univ. of Southampton, 1972, p 12/1-12/50.
3. BURAGOHAIN, D.N. and PATODI, S.C.: Large deflection analysis of plates and shells by discrete energy method, Computers and Structures (To appear shortly)
4. ZIENKIEWICZ, O.C.: The finite element method in engineering science, McGraw-Hill Book Co., 1971.
5. TIMOSHENKO, S. and KRIEGER, W.S.: Theory of plates and shells, McGraw-Hill Book Co., 1959.
6. BERGAN, P.G. and CLOUGH, R.W.: Convergence criterion for iterative processes, J. of AIAA, vol. 10, 1972, pp 1107 - 1108.
7. FRISCH-FAY, R.: Flexible bars, Butterworths, London, 1962.
8. WOOD, R.D.: The application of finite element methods to geometrically nonlinear structural analysis, Ph.D. Dissertation, Univ. of Wales, Swansea, C/PH/20/73, 1973.
9. MARCAL, P.V.: The effect of initial displacements on problems of large deflection and stability, ARPEA 54, Div. of Engg., Brown Univ., Providence, R.I., 1967.
10. GJELSVIK, A. and BODNER, S.R.: The energy criterion and snap buckling of arches, J. of Engg. Mech. Div. ASCE, vol. 88, 1962, pp 87 - 134.
11. GALLAGHER, R.H.: The finite element method in shell stability analysis, Computers and Structures, vol. 3, 1973, pp 543 - 557.
12. DHATT, G.S.: Instability of thin shells by the finite element method, Proc. Symp. Int. Assoc. of Shell Structures, Vienna, 1970.
13. MORIN, N.: Nonlinear analysis of thin shells, Report R70-43, Dept. of Civil Engg., MIT, 1970.
14. KAPUR, K.K. and HARTZ, B.J.: Stability of plates using the finite element method, J. of Engg. Mech. Div., ASCE, vol. 92, 1966, pp 177 - 195.
15. TIMOSHENKO, S. and GERE, J.M.: Theory of elastic stability, McGraw-Hill Book Co., 1961.
16. PATODI, S.C.: Geometric nonlinear analysis of plates and shells by discrete method, Ph.D. Thesis, Indian Institute of Technology, Bombay, India, 1976.