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Influence of Joists on the Lateral Buckling of I Beams

Influence de poutres secondaires sur le déversement latéral de poutres en double té

Einfluss von Querbalken auf die Kippstabilität von I Trägern

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Introduction

LEBELLE [1] has studied a large number of problems relating to lateral buckling of beams. When the connections between the joists and the main beams of a floor or roof system are such that an angle of twist of the main beams entails an equal angle of rotation of the ends of the joists, the flexural rigidity of the joists hampers lateral buckling of the beams.

The purpose of the present paper is to evaluate the resulting increase of the buckling load. Lebelle does not deal with this problem. TAYLOR and OJALVO [2] do; they present their results in the form of graphs, while in the present paper solutions are obtained in the form of equations; Taylor and Ojalvo do not mention the influence of the level of the point of application of the load with respect to the centroid on the magnitude of the critical load; although this effect is not large in most practical cases, it is not negligible.

We discuss the influence of rigidly connected transverse beams on the lateral stability of the main prismatic *I* beams of a rectangular grid system, which is supposed to be elastic. We assume that both ends of each main beam rotate freely about both principal axes of its cross-section, but that twisting of the ends is prevented by the supports.

We further assume that:

1. The restraining action of the transverse beams consists only in reactive couples and does not include horizontal forces perpendicular to the main beams. This situation obtains when two or more identical and identically loaded main beams have identical restraints and thus could buckle simultaneously.
2. Each main beam has a vertical plane of symmetry and its loading acts in that plane.
3. The moment of inertia of the cross-section of the main beam about the horizontal axis through its centroid is much greater than the moment of inertia about the vertical centre line. Hence we neglect vertical deflections with respect to horizontal displacements.

For the meaning of the notations used, see page 156.

Local Torsional Restraint at Mid-Span-Negligible Warping Rigidity-Pure Bending

The main beam is subjected to equal bending couples M at the ends (Fig. 2b). If, for a certain value of M, a slightly deflected and twisted form of equilibrium becomes possible, the rotation ϕ_0 of the cross-section of the beam at mid-span gives rise to a reactive couple $A\phi_0$. The cross-section shown in Fig. 2a is the one at mid-span. The couple $A\phi_0$, due to the existence of a transverse beam, is kept in equilibrium by two couples $\frac{A\phi_0}{2}$, applied by the forklike supports at the ends of the beam.

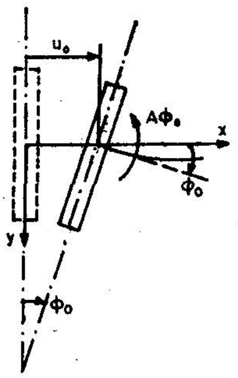

Fig. 2a.

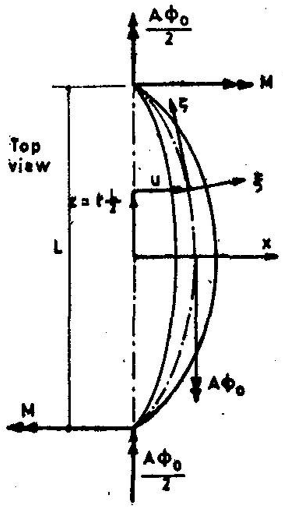

Fig. 2b.

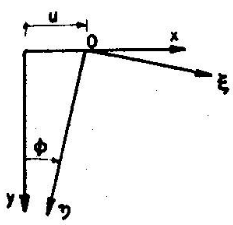

Fig. 2c.

The differential equations for horizontal bending and for twisting of the beam, as written by TIMOSHENKO [3] and taking account of the moment $\frac{A\phi_0}{2}$, are, for the half-beam with positive coordinates z:

$$B\frac{d^2u}{dz^2} = M_\eta = -M\sin\phi \cong -M\phi \tag{1}$$

$$C\frac{d\phi}{dz} = M_\zeta = \frac{A\phi_0}{2}\cos(z\zeta) + M\cos(x\zeta) = \frac{A\phi_0}{2} + M\frac{du}{dz} \tag{2}$$

By differentiating equation (2) with respect to z and eliminating $\frac{d^2u}{dz^2}$ by means of equation (1), we obtain:

$$C\frac{d^2\phi}{dz^2} = M\frac{d^2u}{dz^2} = -\frac{M^2}{B}\phi \text{ or } \frac{d^2\phi}{dz^2} + \frac{4\lambda^2}{L^2}\phi = 0 \tag{3}$$

The general solution of this differential equation is

$$\phi = K_1\sin\frac{2\lambda}{L}z + K_2\cos\frac{2\lambda}{L}z \qquad (z \geqq 0)$$

The conditions at the ends of the half-beam considered are:

1) $\phi = 0$ at $z = \frac{L}{2}$, which yields: $K_1 \sin\lambda + K_2 \cos\lambda = 0$ (4)
2) on account of the symmetry:

$\frac{du}{dz} = 0$ at $z = 0$, which is equivalent to

$$C\frac{d\phi}{dz} = \frac{A\phi_0}{2} \text{ at } z=0, \text{ or to } C\cdot\frac{2\lambda}{L}K_1 = \frac{AK_2}{2}, \text{ or to } -K_1\cdot\frac{2\lambda C}{L} + K_2\cdot\frac{A}{2} = 0 \qquad (5)$$

Lest equation (3) have only the trivial solution $\phi \equiv 0$ and the original shape of the beam be its only form of equilibrium, the determinant of the coefficients of K_1 and K_2 in (4) and (5) must be zero: $\frac{A}{2}\sin\lambda + \frac{2\lambda C}{L}\cos\lambda = 0$. The lowest value of λ satisfying this condition is the critical value of λ. Hence λ_c is the smallest root of the equation

$$tg\,\lambda_c = -\frac{4C}{AL}\cdot\lambda_c = -\frac{\lambda_c}{\alpha} \qquad (6)$$

The critical value of the bending moment in pure bending then is

$$M_c = \frac{2\lambda_c}{L}\sqrt{BC} \qquad (7)$$

A simple diagram (Fig. 3) shows that λ_c always lies in the interval $+\frac{\pi}{2}$, $+\pi$. When there is no torsional restraint of the main beam at mid-span, $A = 0$, $\lambda_c = \frac{\pi}{2}$ and $M_c = \frac{\pi}{L}\sqrt{BC}$. This is the well-known result for buckling of a beam of narrow rectangular cross-section, subjected to pure bending.

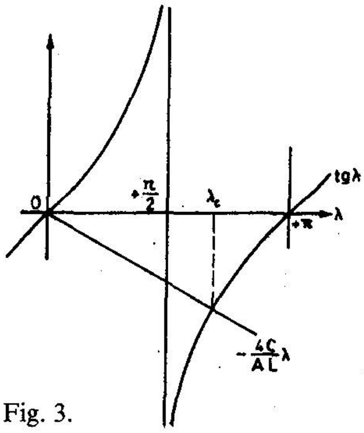

Fig. 3.

If, on the other hand, the rigidity of the transverse beam at the centre of the main beam increases indefinitely:
$A \to \infty$, $\lambda_c \to \pi$ and $M_c \to \frac{2\pi}{L}\sqrt{BC}$, as pointed out by TIMOSHENKO [3].

For all finite values of A or α, the solution is easily found by means of equations (6) and (7).

Comments about Equations 6 and 7

1. Equation (3) and hence equations (6) and (7) are strictly correct only in the case of cross-sections having two axes of symmetry. When there is only one axis of

symmetry, a fraction of the torque is taken up by the bending stresses, because these act in fibres that are twisted helicoidally. A term representing the derivative with respect to z of that part of the torque must be introduced into equation (3):

$$C\frac{d^2\phi}{dz^2} = -\frac{M^2}{B}\phi + c_y M\frac{d^2\phi}{dz^2}$$

(for the meaning of c_y, see ref. 4, p. 395).

$c_y = 0$ and formula (7) is not affected in the case of rolled I beams and of all other steel or concrete I beams having two planes of symmetry.

Steel or concrete beams that are symmetrical with respect to a vertical plane only have in most practical cases their widest flange acting in compression. Formula (7) should be conservative and may be applied with confidence for such beams, since, according to STÜSSI and DUBAS (ref. 4, p. 397) the term $c_y M\frac{d^2\phi}{dz^2}$ increases the critical load when the shear centre 0 is located in the compressive zone of the cross-section.

In the rare cases of beams having their shear centre in the tensile part of the cross-section, formula (7) overestimates the critical load, since, according to the same authors, the effect of the term $c_y M\frac{d^2\phi}{dz^2}$ that we have neglected is to decrease the critical load in those cases.

2. Neglecting the vertical deflections with respect to the horizontal displacements results in a slight underestimation of the critical load (ref. 4, p. 385).

3. The two above remarks also apply to the cases studied below and, more specifically, to the critical loads computed by means of (17) and of the appropriate equation among the equations (16), (21), (23), (26) and (28).

Local Torsional Restraint at Mid-Span-Negligible Warping Ridigity-Uniform Load

When the beam buckles, the deformation is again impeded by the restraining couple $A\phi_0$ at mid span (Fig. 4b), the shear centre 0 of the cross-section with coordinate z moves horizontally an amount u and lowers slightly, and the point of

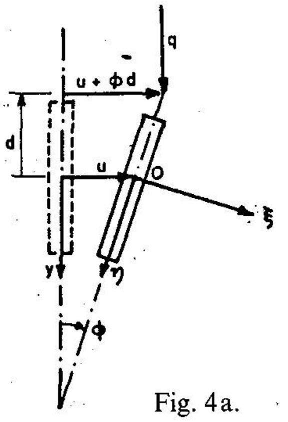

Fig. 4a.

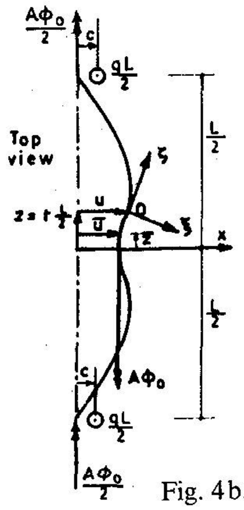

Fig. 4b.

application of the uniform load q moves horizontally an amount $u+\phi d$ and lowers a little more than 0 (Fig. 4a). Each end support again provides a torsional couple $\frac{A\phi_0}{2}$, but, besides, a couple of magnitude $c\cdot\frac{qL}{2}=q\int_0^{L/2}(u+\phi d)\,dz$ necessary to offset the horizontal deviation of the load q.

All bending and twisting moments appearing in the following equations are moments acting in the displaced cross-section with coordinate z and taken about an axis *through 0* whose direction is indicated by the subscript.

$$B\frac{d^2u}{dz^2}=M_\eta=-M_x\cdot\sin\phi\cong -M_x\cdot\phi=-\frac{q}{2}\left(\frac{L^2}{4}-z^2\right)\cdot\phi \tag{8}$$

$$C\tfrac{d\phi}{dz}=M_\zeta=M_x\cdot\cos(x\zeta)+M_z\cdot\cos(z\zeta)=M_x\cdot\sin(z\zeta)+M_z=M_x\cdot\tfrac{du}{dz}+M_z \tag{9}$$

We find M_z as the resulting moment about the z-axis shifted towards 0 of all the forces acting between 0 and the middle of the beam, and in the cross-section at mid-span, taking into account the following facts:

1) Half of the restraining couple $A\phi_0$ at mid-span acts on the part of the beam considered.
2) The bending moment at mid-span does not contribute to M_z.
3) There is no shear force at mid-span.
4) At any point $\bar{z}$ between 0 and mid-span, where the horizontal displacement of the shear centre is $\bar{u}$ and the angle of twist is $\bar{\phi}$, the load $q.d\bar{z}$ acts at a distance $u-\bar{u}-\bar{\phi}d$ to the left of 0.

$$M_z=\frac{A\phi_0}{2}+\int_0^z q(u-\bar{u}-\bar{\phi}d)\,d\bar{z}$$

Equation (9) becomes

$$C\frac{d\phi}{dz}=\frac{A\phi_0}{2}+q\int_0^z(u-\bar{u}-\bar{\phi}d)\,d\bar{z}+\frac{q}{2}\left(\frac{L^2}{4}-z^2\right)\frac{du}{dz} \tag{10}$$

We differentiate with respect to z:

$$C\frac{d^2\phi}{dz^2}=-q\phi d+qz\frac{du}{dz}+\frac{q}{2}\left(\frac{L^2}{4}-z^2\right)\frac{d^2u}{dz^2}-qz\frac{du}{dz}$$

and use (8) to eliminate $\frac{d^2u}{dz^2}$:

$$C\frac{d^2\phi}{dz^2}=-q\phi d-\frac{q^2}{4B}\left(\frac{L^2}{4}-z^2\right)^2\phi$$

This equation can be written in the form

$$\phi''+\phi[\varepsilon\delta+\varepsilon^2(1-t^2)^2]=0 \tag{11}$$

The solution must satisfy the following boundary conditions:

1) $\phi = 0$ at the support ($t = 1$)
2) $\frac{du}{dz} = 0$ at mid-span; (10) shows that this requires

$$C\frac{d\phi}{dz} = \frac{A\phi_0}{2} \text{ at } t = 0, \text{ or } C\phi'(0) = \frac{AL}{4}\cdot\phi(0), \text{ or } \phi'(0) = \alpha\cdot\phi(0)$$

Equation (11) further shows that $\phi''(1) = 0$.

Instead of trying to solve equation (11), we shall now use the Ritz energy method, as developed by TIMOSHENKO [3].

The increase ΔU in the strain energy of one half of the main beam and one half of the corresponding transverse beam(s) while the system moves from the loaded, unbuckled configuration to the loaded, buckled configuration is given by the expression

$$\Delta U = \frac{1}{2B}\int_0^{L/2} M_\eta^2\,dz + \frac{1}{2C}\int_0^{L/2} M_\zeta^2\,dz + \frac{1}{2}\left(A\phi_0\cdot\frac{\phi_0}{2}\right)$$

$$= \frac{1}{2B}\int_0^{L/2} M_x^2\phi^2\,dz + \frac{C}{2}\int_0^{L/2}\left(\frac{d\phi}{dz}\right)^2 dz + \frac{A}{4}\phi_0^2 \tag{12}$$

The original equilibrium configuration changes from stable to unstable when ΔU is equal to the work ΔT done by the load q during lateral buckling. As explained in LEBELLE'S paper (ref. 1, p. 792), the work done by the load whilst the shear centre 0 is lowered owing to the horizontal curvature combined with the twisting of the beam, would be

$\frac{1}{B}\int_0^{L/2} M_x^2\phi^2\,dz$, if the load acted at the shear centre 0. Because the load acts at distance d above the shear centre, there is an additional lowering (Fig. 5) by the amount $d(1-\cos\phi) \cong d\frac{\phi^2}{2}$, and the additional work is $q\int_0^{L/2} d\frac{\phi^2}{2}dz$. Hence

$$\Delta T = \frac{1}{B}\int_0^{L/2} M_x^2\phi^2\,dz + \frac{qd}{2}\int_0^{L/2}\phi^2\,dz \tag{13}$$

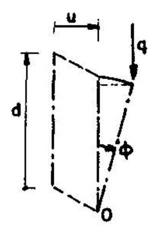

Fig. 5.

The energy equation $\Delta T = \Delta U$ determining the critical load q_c becomes

$$\frac{1}{2B}\int_0^{L/2} M_x^2\phi^2\,dz + \frac{q_c d}{2}\int_0^{L/2}\phi^2\,dz = \frac{C}{2}\int_0^{L/2}\left(\frac{d\phi}{dz}\right)^2 dz + \frac{A}{4}\phi_0^2$$

Substituting $M_x = \frac{q_c}{2}\left(\frac{L^2}{4} - z^2\right) = \frac{q_c L^2}{8}(1 - t^2)$ and introducing other symbols defined under the heading "Notations" (page 156), we obtain the integral equation

$$\varepsilon_c^2 \int_0^1 (1-t^2)^2 \phi^2 \, dt + \varepsilon_c \delta \int_0^1 \phi^2 \, dt = \int_0^1 \phi'^2 \, dt + \alpha \phi_0^2 \tag{14}$$

If all conceivable functions $\phi(t)$ that satisfy the boundary conditions were introduced into equation (14), the one yielding the lowest value of ε_c and hence of q_c would represent the true buckling configuration and would yield the correct value of ε_c and q_c. We will, however, obtain a fair approximation for ε_c by any reasonable choice for the function $\phi(t)$ that satisfies the boundary conditions.

We assume for ϕ the expression $\phi(t) = \phi_0(1 + at + bt^2 + ct^3)$. The boundary conditions $\phi(1) = 0$ and $\phi'(0) = \alpha . \phi(0)$, together with the condition $\phi''(1) = 0$ deduced from equation (11), enable us to find the coefficients a, b and c as functions of α. We introduce the expression

$$\phi(t) = \phi_0[1 + \alpha t - \tfrac{3}{2}(1+\alpha)t^2 + \tfrac{1}{2}(1+\alpha)t^3] \quad (0 \leqq t \leqq 1) \tag{15}$$

thus obtained into equation (14), and, after performing the calculations, find

$$\frac{\varepsilon_c^2}{132}(161\alpha^2 + 1620\alpha + 5287) + \varepsilon_c \delta(2\alpha^2 + 18\alpha + 51) - 21(\alpha+1)(\alpha+6) = 0 \tag{16}$$

This quadratic equation has a positive and a negative root. It is easy to compute the positive root ε_c and the buckling load

$$q_c = 16\varepsilon_c \frac{\sqrt{BC}}{L^3} \tag{17}$$

The negative root of equation (16) gives the magnitude of the load which, applied upward at the same point, would produce lateral buckling of the beam. The negative root is, understandably, equal to minus the positive root of equation (16), written with the sign of δ reversed. The same comment applies to the negative root of similar quadratic equations to follow.

It is possible to refine the procedure that led to equation (16) by adding a fourth power term to the polynomial assumed for $\phi(t)$, by writing the coefficients as functions of a free parameter in order that $\phi(t)$ satisfy the three known conditions, by finding ε_c as a function of this parameter, and by adjusting the parameter in such a way as to make ε_c a minimum.

The computations would be very laborious and they would probably not be worth-while, as is indicated by the fact that replacement of the third power term by a fourth power term in expression (15) yields values of ε_c, which are hardly different from, but generally a fraction of 1% higher than the positive root of equation (16). That this equation may be used with confidence is further shown by its containing previously known results (ref. 1, p. 791) obtained by another method for the particular case

$$\alpha = 0, \ \delta = 0 \text{ or } \delta \neq 0$$

Local Torsional Restraint at Mid-Span-Uniform Load

Buckling in One Half-Wave

We refer again to Fig. 4, but now suppose that the beam has flanges and that the warping rigidity $C_1 = EC_w$ is not negligible.

Differential equation (8) remains valid. A term representing the part of the torque taken by the couple of shear forces in the flanges has to be added to the first member of equations (9) and (10):

$$C\frac{d\phi}{dz} - C_1\frac{d^3\phi}{dz^3} = M_\zeta = \frac{A\phi_0}{2} + q\int_0^z (u - \bar{u} - \bar{\phi} d)\, d\bar{z} + \frac{q}{2}\left(\frac{L^2}{4} - z^2\right)\frac{du}{dz}$$

Differentiating with respect to z and using (8) to eliminate $\frac{d^2u}{dz^2}$, we find:

$$C\frac{d^2\phi}{dz^2} - C_1\frac{d^4\phi}{dz^4} = -q\phi d + \frac{q}{4}\left(\frac{L^2}{4} - z^2\right)\frac{d^2u}{dz^2} = -q\phi d - \frac{q^2}{4B}\left(\frac{L^2}{4} - z^2\right)^2\phi$$

which can be written in the form:

$$-\beta\phi'''' + \phi'' + \phi[\varepsilon\delta + \varepsilon^2(1 - t^2)^2] = 0 \qquad (18)$$

Four boundary conditions must be satisfied:

1) $\phi(1) = 0$.
2) At the support, the curvature $\frac{d^2u}{dz^2} + h_1\frac{d^2\phi}{dz^2}$ of the upper flange (Fig. 1) and the curvature $\frac{d^2u}{dz^2} - h_2\frac{d^2\phi}{dz^2}$ of the lower flange in the plane of the flanges are zero, and consequently $\phi''(1) = 0$.

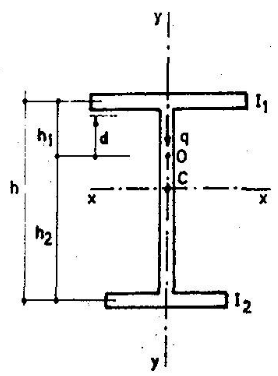

Fig. 1.

3) Owing to symmetry, buckling does not change the direction of the flanges at mid-span:

$$\frac{du}{dz} + h_1\frac{d\phi}{dz} = 0 \text{ and } \frac{du}{dz} - h_2\frac{d\phi}{dz} = 0;\text{ hence } \phi'(0) = 0$$

4) In the cross-section adjacent to the transverse beam, the torque

$$M_\zeta = C\frac{d\phi}{dz} - C_1\frac{d^3\phi}{dz^3} \text{ is } \frac{A\phi_0}{2}. \text{ Since } \frac{d\phi}{dz} = 0:$$

$$-\frac{8C_1}{L^3}\cdot\phi'''(0) = \frac{A}{2}\phi(0), \text{ or } -\beta\cdot\phi'''(0) = \alpha\cdot\phi(0)$$

Equation (18), together with the conditions 1) and 2), shows that $\phi''''(1) = 0$.

In order to apply the energy method, we must supplement expression (12) for ΔU with the strain energy resulting from the differential bending of the flanges in their plane. Excepting the general horizontal bending of the beam, the curvature of the upper and lower flanges is $h_1\frac{d^2\phi}{dz^2}$ and $-h_2\frac{d^2\phi}{dz^2}$ respectively. The corresponding strain energy for the half-beam is

$$\int_0^{L/2}\left|\frac{EI_1}{2}\left(h_1\frac{d^2\phi}{dz^2}\right)^2 + \frac{EI_2}{2}\left(h_2\frac{d^2\phi}{dz^2}\right)^2\right|dz = \frac{E}{2}\left(h_1^2 I_1 + h_2^2 I_2\right)\cdot\int_0^{L/2}\left(\frac{d^2\phi}{dz^2}\right)^2 dz =$$

$$= \frac{C_1}{2}\int_0^{L/2}\left(\frac{d^2\phi}{dz^2}\right)^2 dz$$

and the counterpart of expression (14) of the energy equation $\Delta T = \Delta U$ is here

$$\varepsilon_c^2\int_0^1 (1-t^2)^2\phi^2\,dt + \varepsilon_c\delta\int_0^1 \phi^2\,dt = \int_0^1 \phi'^2\,dt + \beta\int_0^1 \phi''^2\,dt + \alpha\phi_0^2 \tag{19}$$

We use a fifth power polynomial to describe approximately the unknown buckling configuration. The four boundary conditions, together with the fact that $\phi''''(1) = 0$, enable us to find the five coefficients in the polynomial as functions of $\mu = \frac{\alpha}{6\beta}$:

$$\phi(t) = \phi_0\left[1 - \tfrac{1}{4}(5-2\mu)t^2 - \mu t^3 + \tfrac{5}{16}(1+2\mu)t^4 - \tfrac{1}{16}(1+2\mu)t^5\right] \quad 0 \leqq t \leqq 1 \tag{20}$$

Introducing (20) into (19), we obtain after tedious calculations:

$$\frac{\varepsilon_c^2}{416}(1612\mu^2 + 88340\mu + 2243159) + 2\varepsilon_c\delta(4\mu^2 + 180\mu + 3455) =$$

$$= 22(4\mu^2 + 40\mu + 775) + 495\beta(1+2\mu)(85+2\mu) \tag{21}$$

One finds the buckling load q_c by substituting into equation (17) the one positive root of equation (21).

When $\mu = 0$, equation (21) is very nearly equivalent to the formula for ε_c given by LEBELLE (ref. 1, p. 791) for beams without torsional restraint at mid-span; there is a slight difference in the term representing the influence of the warping rigidity.

It is important to note that equation (21) is *not* valid when $\beta = 0$ (no warping rigidity), for the third boundary condition used in the process which led to equation (21) is not compatible with the second boundary condition obtaining when $\beta = 0$, unless $\phi(0)$ be zero, and there is no restraint that keeps $\phi(0)$ zero whenever $\beta = 0$.

Buckling in Two Half-Waves

Equation (21) gives the critical load that causes the main beam to buckle in one half-wave, which is symmetrical with respect to its mid-point, as sketched in Fig. 4. It is, however, conceivable that a transverse beam of great, but finite stiffness may prevent twisting of the main beam at mid-span during buckling.

In order to evaluate the critical load associated with the buckling mode sketched in Fig. 6 we describe this new buckling configuration approximately by means of the function

$$\phi(t) = a(8t - 20t^3 + 15t^4 - 3t^5) \qquad (22)$$

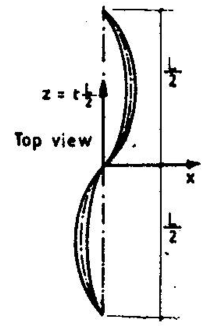

Fig. 6.

whose coefficients are such that:

1) as before: $\phi(1) = 0$; $\phi''(1) = 0$; $\phi''''(1) = 0$.
2) $\phi(0) = 0$; $\phi''(0) = 0$, thereby reflecting the absence of a twisting movement of the cross-section and the presence of inflection points in the flanges at mid-span.

When expression (20) is written with $\alpha\phi_0$ as a new, finite parameter while ϕ_0 vanishes, its second derivative is not zero at $t = 0$. Consequently, it does *not* represent the buckling mode pictured in Fig. 6, and expression (22) may therefore yield a lower critical load. We substitute it for ϕ in equation (19), in which $\alpha\phi_0^2$ is now zero. The positive root of the resulting quadratic equation is

$$16\varepsilon_c = 67{,}05\,(\sqrt{1 + 10\beta + 0{,}04479\,\delta^2} - 0{,}2116\,\delta) \qquad (23)$$

For high values of α and small values of β, formula (23) turns out to give lower values of ε_c than equation (21). The transverse beam is then stiff enough to induce the main beam to buckle in two half-waves, and further stiffening of the transverse beam would not increase the buckling load. The ε_c to be used in (17) is always the lowest resulting either from (21) or from (23).

Remarks

Instead of assuming expression (22) to define the buckling configuration, we could use a sixth power polynomial that satisfies the same five conditions, and, moreover, the condition $\phi''''(0) = 0$ derived from equation (18). Although it accords more completely with our information about the form of buckling, it yields, rather surprisingly, higher and therefore poorer values of the critical load.

When expression (15), which we used for beams with negligible warping rigidity, is rewritten with $\alpha\phi_0$ as a new, finite parameter while ϕ_0 is allowed to vanish, it does represent the buckling mode depicted in Fig. 6. Consequently, equation (16) remains valid even when the transverse beam is very stiff.

Local Torsional Restraint at Mid-Span-Effect of the Various Parameters

Table 1 illustrates the influence of the various parameters on the magnitude of the critical load. The numerical values appearing in the table may occur in practical cases. The values of $16\varepsilon_c$ in the column $\beta = 0$ were calculated by means of equation (16), the other ones by means of (21) or (23). The more heavily circumscribed figures are associated

Table 1

α	δ	$\beta = 0$	$\beta = 0{,}011$	$\beta = 0{,}1$
		$16\varepsilon_c$		
0	−0,2	30,5	31,0	33,9
	0	28,4	28,8	31,8
	+0,2	26,4	26,9	29,8
5	−0,2	54,2	70,8	66,4
	0	51,9	67,8	64,1
	+0,2	49,6	64,9	61,9
13	−0,2	61,9	73,5	90,7
	0	59,4	70,6	88,2
	+0,2	56,9	67,9	85,7
∞	−0,2	69,1	73,5	97,7
	0	66,4	70,6	94,8
	+0,2	63,8	67,9	92,0

with buckling in two half-waves. For instance, if $\beta = 0{,}011$, the critical load does not increase when α augments from 13 to infinity.

It is seen from the table that local torsional restraint at mid-span improves the stability substantially. Values of α of the order of magnitude of 10 are easily achieved in practice, and such restraint doubles or more than doubles the buckling load.

The critical load decreases from 4 to 7% when δ varies from 0 to +0,2. For a prestressed concrete *I* beam, $\delta = 0{,}2$ may mean, depending on the proportions of the cross-section, that the load acts on the upper flange.

Uniform Torsional Restraint-Negligible Warping Rigidity-Uniform Load

When the beam buckles, the deformation is hindered by a continuous restraining couple of magnitude $A_1\phi$ per unit length of the beam. The top view in Fig. 4b has to be replaced by Fig. 7. Each end support provides a torsional couple of magnitude $A_1\int_0^{L/2}\phi dz$ and a couple $c\cdot\frac{qL}{2}$.

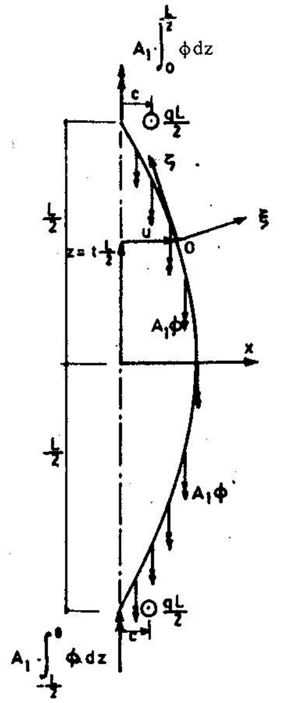

Fig. 7.

Equations (8) and (9) remain valid. Remembering that, owing to symmetry, there is no torque and no shear force in the cross-section at mid-span of the beam, we find M_z at 0 as the sum of the restraining moments $A_1\bar{\phi}d\bar{z}$ between 0 and mid-span, and of the moments $q(u-\bar{u}-\bar{\phi}d)d\bar{z}$ for the same portion of the beam:

$$M_z = A_1\int_0^z \bar{\phi}d\bar{z} + q\cdot\int_0^z (u-\bar{u}-\bar{\phi}d)\,d\bar{z}$$

Equation (10) becomes here:

$$C\frac{d\phi}{dz} = A_1\cdot\int_0^z \bar{\phi}\cdot d\bar{z} + q\cdot\int_0^z (u-\bar{u}-\bar{\phi}d)\,d\bar{z} + \frac{q}{2}\left(\frac{L^2}{4}-z^2\right)\frac{du}{dz} \tag{24}$$

Differentiation with respect to z and elimination of $\frac{d^2u}{dz^2}$ by means of (8) give the differential equation:

$$C\frac{d^2\phi}{dz^2} = A_1\phi - q\phi d + \frac{q}{2}\left(\frac{L^2}{4}-z^2\right)\frac{d^2u}{dz^2} = A_1\phi - q\phi d - \frac{q^2}{4B}\left(\frac{L^2}{4}-z^2\right)^2\phi$$

which is easily transformed into:

$$\phi'' + \phi[\varepsilon\delta - \alpha_1 + \varepsilon^2(1-t^2)^2] = 0 \quad (25)$$

The boundary conditions are:

1) $\phi(1) = 0$;
2) at mid-span: $\frac{du}{dz} = 0$; equation (24) shows that this implies $\frac{d\phi}{dz} = 0$ or $\phi'(o) = 0$.

Comparison of the differential equations (25) and (11), and of the boundary conditions pertaining to these equations shows that the mathematical formulation of the present problem is identical with that for the case of local torsional restraint at midspan, provided that in the latter we substitute the constant $\varepsilon\delta - \alpha_1$ for $\varepsilon\delta$, and zero for α. To obtain an approximate answer to the present problem, we need only make the same substitutions in equation (16). Hence we find the critical value of ε as the positive root of the equation

$$\frac{5287}{132}\varepsilon_c^2 + 51(\varepsilon_c\delta - \alpha_1) - 126 = 0$$

The value of $16\varepsilon_c$ to be used in expression (17) for the critical load q_c is

$$16\varepsilon_c = 28{,}4\left(\sqrt{1 + 0{,}405\alpha_1 + 0{,}1288\delta^2} - 0{,}359\delta\right) \quad (26)$$

Uniform Torsional Restraint-Uniform Load

If the warping rigidity $C_1 = EC_w$ is not negligible, we find the differential equation governing ϕ by adding the terms $-C_1\dfrac{d^3\phi}{dz^3}$ and $-\beta\cdot\phi''''$ to the first member of the equations (24) and (25), respectively. The new equation is

$$-\beta\cdot\phi'''' + \phi'' + \phi[\varepsilon\delta - \alpha_1 + \varepsilon^2(1-t^2)^2] = 0 \quad (27)$$

The boundary conditions are:

1) $\phi(1) = 0$; $\phi''(1) = 0$; $\phi'(o) = 0$, as in the case of local torsional restraint at mid-span.
2) $\phi'''(o) = 0$, resulting from the modified equation (24) and from the fact that $\frac{du}{dz} = 0$ at $t = 0$.

The differential equation (27) and the four boundary conditions are exactly the same as equation (18) and the appurtenant boundary conditions for the case of local torsional restraint at mid-span, if in the latter we replace $\varepsilon\delta$ by $\varepsilon\delta - \alpha_1$ and α by zero. Consequently we need only make the same changes in equation (21), which thus becomes:

$$\frac{2243159}{416}\varepsilon_c^2 + 6910(\varepsilon_c\delta - \alpha_1) - 17050 - 42075\beta = 0$$

Solving for ε_c we obtain

$$16\varepsilon_c = 28{,}4\left(\sqrt{1 + 0{,}405\alpha_1 + 2{,}468\beta + 0{,}1298\delta^2} - 0{,}36\delta\right) \quad (28)$$

For $\alpha_1 = 0$, this result is identical with known results referred to previously.

Formula (28) is valid also for $\beta = 0$ and indeed is then very nearly the same as formula (26), because this time the boundary conditions holding when $\beta \neq 0$ are not incompatible with those holding when $\beta = 0$.

Lateral buckling in two half-waves always requires a higher load than the load calculated with expression (28).

Value of A or A_1

One Transverse Beam at Mid-Span

When two main beams are connected by a single secondary beam of flexural rigidity $E_t I_t$ and length l (Fig. 8a), the stiffness of the restraint is

$$A = \frac{6E_t I_t}{l}$$

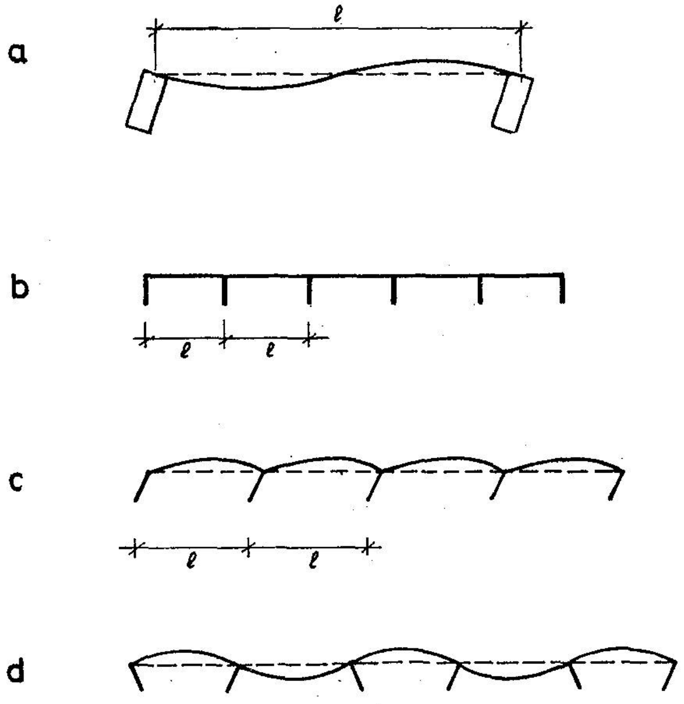

Fig. 8.

When a continuous cross-beam joins a number of principal beams (Fig. 8b), the stiffness of the restraint is

$A = \dfrac{12E_t I_t}{l}$ for all the intermediate main beams, and

$A = \dfrac{6E_t I_t}{l}$ for the first and the last one. The equations developed above give correct values of ε_c or λ_c only if, L and d being the same for all the principal beams, q (or M in the case of pure bending), B, C and C_1 for the first and the last beam are each half as great as the common value of the corresponding parameter for the intermediate beams. If *all* the beams are identical, the first and last one carrying only half as much load as the other ones, the theory produces approximate, but safe values of the critical load.

Prof. P. Dubas has justly pointed out that lateral buckling of the main beams according to Fig. 8d is also possible, that in this case the values of A given above must be divided by 3 and the main beams do not deflect horizontally at the junctions with the secondary beams, and that the latter stabilizing effect may conceivably not quite compensate for the former unfavourable circumstance, thus leading to a lower critical load. Horizontal restraint of the beams delays lateral buckling quite efficiently, especially if the compressive flange is restrained. For instance, according to the German specifications (ref. 5, p. 24), lateral buckling is impossible when complete horizontal restraint without torsional restraint is applied at least half as high above the axis of the beam as the load. In actual practice, the joists are normally placed upon the upper flange of the main beams. Thus it seems highly unlikely that the mode of failure depicted in Fig. 8d would occur under a lower critical load than the one inducing all the main beams to rotate in the same direction.

The above expressions for A may be used to calculate α and ε or λ, provided that the connection between the transverse and each principal beam is such that any angle of twist, clockwise or counterclockwise, of the principal beam entails a rotation of the end of the secondary beam of exactly the same magnitude. If the beams are precast, reinforced or prestressed concrete beams, it may be more difficult to prevent opening than to prevent closing of the right angle between main and transverse beam (Fig. 8c). In that case, it is reasonable to estimate the stiffness of the restraint for the intermediate beams by means of the expression $A = \frac{3E_t I_t}{l}$, $E_t I_t$ again being the flexural rigidity of the cross-beam, and to assume that the first and the last principal beam are not restrained.

Joists Distributed Along the Span

The stiffness A_1 of the continuous torsional restraint is to be calculated with the appropriate expression given above for A, in which $E_t I_t$ now represents the flexural rigidity of the joists per unit length of the main beam.

Effect of Continuous Restraint and of Local Restraint at More Than One Point of the Span

Assuming, for example, that $\beta = 0$, we learn from table 1 that, for local torsional restraint at mid-span,

$$16\varepsilon_c = 51{,}9 \text{ when } \alpha = 5,\ \delta = 0, \text{ and that}$$
$$16\varepsilon_c = 49{,}6 \text{ when } \alpha = 5,\ \delta = 0{,}2.$$

If the single cross-beam were replaced by numerous identical joists having together the same moment of inertia and uniformly distributed over the span of the principal beam, the new buckling load would be given by formula (26), with

$$A_1 = \frac{A}{L} \text{ or } \alpha_1 = \frac{A_1 L^2}{4C} = \frac{AL}{4C} = \alpha = 5.$$

The resulting values are

$$16\varepsilon_c = 49{,}4 \text{ when } \delta = 0,$$
$$16\varepsilon_c = 47{,}4 \text{ when } \delta = 0{,}2.$$

If, on the other hand, the stiffness of the single secondary beam were spread evenly over the central half of the main beam and if equivalent uniform torsional restraint were provided in the outer quarters of the span, A_1 in formula (26) would be $A_1 = \frac{A}{\frac{L}{2}}$ and α_1 would be $\alpha_1 = 2\alpha = 10$, with the following result:

$$16\varepsilon_c = 63{,}8 \text{ when } \delta = 0,$$
$$16\varepsilon_c = 61{,}8 \text{ when } \delta = 0{,}2.$$

Corresponding figures for a higher value of α are:

$16\varepsilon_c = 59{,}4$ when $\alpha = 13$, $\delta = 0$,
$16\varepsilon_c = 56{,}9$ when $\alpha = 13$, $\delta = 0{,}2$, torsional restraint being provided at mid-span only by a cross-beam with flexural rigidity $E_t I_t$, and
$16\varepsilon_c = 71{,}1$ when $\alpha_1 = 13$, $\delta = 0$,
$16\varepsilon_c = 69{,}1$ when $\alpha_1 = 13$, $\delta = 0{,}2$, $E_t I_t$ being distributed over many joists placed uniformly along the span.

Comparison of the above sets of figures suggests that when lateral buckling of the principal beam is counteracted by three or more joists, more or less equally spaced along the span, a fair approximation of the critical load will result from the assumption that the total stiffness of the joists is spread evenly over the span of the main beam.

Uniformly distributed joists stabilize the beam more efficiently than a single secondary beam at mid-span, whose moment of inertia is equal to the total moment of inertia of all the joists, when $\alpha = 13$; joists spread over the length of the span and of total rigidity $E_t I_t$ are almost as efficient as a single cross-beam of rigidity $E_t I_t$, when $\alpha = 5$.

The buckling load of beams without torsional restraint is multiplied by about 2,3 to 3, depending on β and δ, when an infinitely stiff transverse beam is rigidly connected to them at mid-span. It stands to reason, and expressions (26) and (28) confirm that the critical load increases indefinitely when the stiffness of uniformly distributed joists is increased indefinitely.

In prestressed concrete buildings, precast joists whose cross-section has the shape of a wide inverted U are often used. They are considerably stiffer horizontally than vertically. When their ends are connected to a flange of the principal beams in such a way that they participate in any horizontal rotation of the flange, the joists hamper lateral buckling of the main beams quite efficiently. The writer intends to elaborate on this subject in another paper.

Strength of the Rigid Connection Between Principal Beam and Cross-Beam

In stability problems, the question how strong — as opposed to how stiff — a restraining member has to be is always a moot point.

In the case of the joint between a cross-beam and a main beam, the simple answer that it must have the same moment capacity as the cross-beam itself is unsatisfactory. For one thing the positive moment capacity of the transverse beam may be considerably different from the negative moment capacity, at least for concrete beams. Actually there is no clear-cut answer. Some degree of arbitrariness is involved in every solution, including the procedure presented here. It pertains to the case of principal beams of narrow rectangular cross-section ($\beta = 0$), tied together by one secondary beam at mid-span.

Lebelle (ref. 1, p. 800) has obtained the differential equation for the angle of twist ϕ of a beam subjected to loads q and q_1, that act at the centroid $C \equiv 0$ of the rectangular cross-section in the plane of maximum and minimum flexural rigidity, respectively:

$$\phi'' + \frac{q^2 L^6}{4 \times 64 BC}(1 - t^2)^2 \phi + \frac{q q_1 L^6}{4 \times 64 BC}(1 - t^2)^2 = 0$$

$$\text{or } \phi'' + \varepsilon^2 (1 - t^2)^2 \phi + \varepsilon^2 \frac{q_1}{q}(1 - t^2)^2 = 0$$

This second order differential equation with variable coefficients and with a term not containing the unknown function ϕ obviously has $\phi = -\dfrac{q_1}{q}$ as a particular solution. Lebelle (ref. 1, p. 788) has obtained the general solution of the complete equation in the form of a power series in the independent variable t. When all terms whose coefficients of the powers of t contain ε^6 or higher powers of ε are neglected, the general solution is

$$\phi = K_1 \left| 1 - \frac{\varepsilon^2}{2} t^2 + \left(\frac{\varepsilon^2}{6} + \frac{\varepsilon^4}{24}\right) t^4 - \left(\frac{\varepsilon^2}{30} + \frac{7\varepsilon^4}{180}\right) t^6 + \frac{13\varepsilon^4}{840} t^8 - \frac{7\varepsilon^4}{2700} t^{10} + \frac{\varepsilon^4}{3960} t^{12} \right|$$
$$+ K_2 \left| t - \frac{\varepsilon^2}{6} t^3 + \left(\frac{\varepsilon^2}{10} + \frac{\varepsilon^4}{120}\right) t^5 - \left(\frac{\varepsilon^2}{42} + \frac{13\varepsilon^4}{1260}\right) t^7 + \frac{41\varepsilon^4}{7560} t^9 - \frac{31\varepsilon^4}{23100} t^{11} + \frac{\varepsilon^4}{6552} t^{13} \right|$$
$$- \frac{q_1}{q}$$

$$\text{or } \phi = K_1 \left| 1 + \varepsilon^2 \left(-\frac{t^2}{2} + \frac{t^4}{6} - \frac{t^6}{30} \right) + \varepsilon^4 \left(\frac{t^4}{24} - \frac{7t^6}{180} + \frac{13t^8}{840} - \frac{7t^{10}}{2700} + \frac{t^{12}}{3960} \right) \right.$$
$$\left. + K_2 \right| t + \varepsilon^2 \left(-\frac{t^3}{6} + \frac{t^5}{10} - \frac{t^7}{42} \right) + \varepsilon^4 \left(\frac{t^5}{120} - \frac{13t^7}{1260} + \frac{41t^9}{7560} - \frac{31t^{11}}{23100} + \frac{t^{13}}{6552} \right) \Bigg| - \frac{q_1}{q}$$

in which K_1 and K_2 are integration constants.

The boundary conditions of our problem are the same as those associated with equation (11):

$$\phi(1) = 0 \text{ and } \phi'(o) = \alpha \cdot \phi(o)$$

Since $\phi(o) = K_1 - \frac{q_1}{q}$ and $\phi'(o) = K_2$, the second boundary condition requires $K_2 = \alpha\left(K_1 - \frac{q_1}{q}\right)$.

The first condition becomes

$$\phi(1) = K_1\left(1 - \frac{11}{30}\varepsilon^2 + \frac{6617}{415800}\varepsilon^4\right) + \alpha\left(K_1 - \frac{q_1}{q}\right)\left(1 - \frac{19}{210}\varepsilon^2 + \frac{12161}{5405400}\varepsilon^4\right) - \frac{q_1}{q} = 0$$

Solving this equation for K_1, we are able to calculate the angle of twist at mid-span:

$$\phi(o) = K_1 - \frac{q_1}{q} = \frac{q_1}{q} \cdot \frac{\frac{11}{30}\varepsilon^2 - \frac{6617}{415800}\varepsilon^4}{1 + \alpha - \frac{\varepsilon^2}{210}(77 + 19\alpha) + \frac{\varepsilon^4}{5405400}(86021 + 12161\alpha)}$$

We now assume that the transverse load q_1 is due to an accidental out of plumb ω of the principal beam (Fig. 9) and hence is equal to $q_1 = q \sin \omega \cong q\omega$, so that $\frac{q_1}{q} = \omega$.

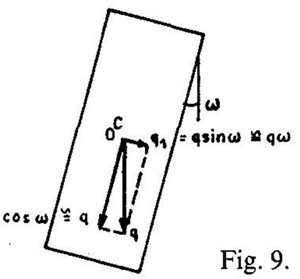

Fig. 9.

Noting that the moment applied by the cross-beam to the main beam is $A \cdot \phi(o)$, we obtain for this moment the expression:

$$A \cdot \phi(o) = \frac{11}{30}\omega A \varepsilon^2 \cdot \frac{1 - 0{,}0434\varepsilon^2}{1 + \alpha - \varepsilon^2(0{,}367 + 0{,}0905\alpha) + \varepsilon^4(0{,}0159 + 0{,}00225\alpha)} \tag{29}$$

In this formula $\varepsilon^2 = \frac{q^2 L^6}{256BC}$, in which q is not the critical load, but the working load, and, more specifically: the sum of dead load and superimposed load if the main beam did not carry its own weight when junction with the cross-beam was achieved, but the superimposed load only if the junction was achieved while the principal beam carried its own weight.

The designer has to assume a value for ω, for example 1% or 2%.

The joint between a cross-beam tying two principal beams together and each of these must have a positive and negative moment capacity calculated by means of expression (29). The joint between each span of a continuous transverse beam and each intermediate beam of a series of principal beams must be able to resist a negative as well as a positive moment, whose magnitude is half of that given by formula (29).

Formula (29) gives finite values, even when A and α are infinite.

Conclusions

(Subject to the reservations explained in the "Comments about equations 6 and 7")

Equations (6) and (7) give the critical moment for a beam of narrow rectangular cross-section, stiffened by local torsional restraint at mid-span and subjected to pure bending.

The uniform load causing lateral buckling is given by expression (17), in which ε_c is

— the positive root of equation (16) in the case of a beam of negligible warping rigidity, torsionally restrained by a cross-beam at mid-span;

— the positive root of equation (21), but never greater than the value resulting from expression (23), in the case of a beam with warping rigidity, torsionally restrained at mid-span;

— given by formula (28), when the beam has uniform torsional restraint.

Torsional restraint substantially increases the critical load.

The moment capacity required for the rigid connection between principal beams and secondary beams may be estimated with the help of formula (29).

Safety and Economy Aspects of the Problem

There have been failures of beams due to lateral buckling, most often during erection, while they were suspended from cranes and while their ends could rotate horizontally about the point of suspension, but also after erection, under the combined influence of their own weight and the superimposed load. In most cases the designers had not investigated the danger of lateral buckling; they simply had overlooked the problem.

If many other girders managed to carry their total load, in spite of not having been checked for lateral buckling, it is due undoubtedly to the fortunate circumstance that very few girders have no torsional restraint at all. Most principal beams are connected with each other by means of secondary beams or by joists or by a floor or by roof elements. Even when the designer does not consciously detail the connections with the purpose of providing torsional restraint to the principal beams, they normally possess some stiffness. This has probably saved many beams from failure, even though the designer was not aware of it.

Of course, a designer should not rely on chance and good luck to ensure the safety of a structure. The present paper provides a rational basis for the quantitative evaluation of the stabilizing influence of torsional restraint. Provided that the designer pays some attention to the details of the connections between slender main girders and secondary structural elements which are needed anyway, the formulas developed in this paper enable him to ascertain, without increasing the cross-section of the principal beams, that they are not in danger of buckling laterally, and thus to avoid the concomitant expenditure.

Notations

All Pertaining to the Main Beam

L span length.

I_x, I_y moment of inertia of the cross-section about the principal axes $x - x$ and $y - y$, respectively, through the centroid C (Fig. 1); $I_x >> I_y$.

I_1, I_2 moment of inertia of the upper and the lower flange, respectively, about the centre line $y - y$.

d vertical distance between the point of application of the uniform load q and the shear centre 0; d is positive when the load acts above the shear centre; the location of the shear centre results from the relation $h_1 I_1 = h_2 I_2$.

$B = EI_y$ horizontal flexural rigidity.

$C = GJ$ torsional rigidity; Poisson's ratio η in the relation $G = \dfrac{E}{2(1+\eta)}$ may be taken equal to 0,3 for steel and to 0,2 for prestressed or reinforced concrete.

$C_1 = EC_w$ warping rigidity; $C_w = \dfrac{h^2 I_1 I_2}{I_1 + I_2} = h_1^2 I_1 + h_2^2 I_2$; for a beam with two planes of symmetry: $h_1 = h_2 = \dfrac{h}{2}$ and $C_w = \dfrac{h^2}{2} I_1 = \dfrac{h^2}{2} I_2$.

A stiffness of the local torsional restraint at mid-span.

A_1 stiffness, per unit length of the main beam, of the continuous torsional restraint

$$\varepsilon = \frac{qL^3}{16\sqrt{BC}} \text{ or } \lambda = \frac{L}{2} \cdot \frac{M}{\sqrt{BC}},$$

$$\delta = 4\frac{d}{L}\sqrt{\frac{B}{C}} \quad \beta = \frac{4C_1}{L^2 C}$$

$$\alpha = \frac{AL}{4C} \text{ or } \alpha_1 = \frac{A_1 L^2}{4C}:$$

dimensionless parameters representing respectively: the intensity of the load, the level of its point of application, the warping rigidity of the cross-section, and the stiffness of the torsional restraint

$$\mu = \frac{\alpha}{6\beta}.$$

The subscript c denotes the critical value of the load q or M, or of the parameter ε or λ.

The superscript $'$ denotes differentiation with respect to t, with $z = t\frac{L}{2}$ (Fig. 2b).

u and ϕ horizontal displacement of the shear centre and angle of twist, respectively, of the cross-section with coordinate z (Fig. 2c).

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Summary

Equations are developed which allow the easy calculation of the critical load causing lateral buckling of I beams without or with warping rigidity, which are stiffened either by local torsional restraint at mid-span or by uniform torsional restraint. The restraint may be provided by a cross beam or by joists, rigidly connected with the main beams. The strength necessary for the rigid connection is also discussed.

Résumé

L'auteur établit des équations qui permettent un calcul aisé de la charge critique provoquant le déversement latéral de poutres avec ou sans ailes, dont la stabilité est accrue par la présence d'une poutre secondaire à mi-portée ou de poutres secondaires disposées sur toute sa longueur et assemblée(s) rigidement à la poutre principale. L'auteur discute aussi la résistance requise pour le nœud rigide.

Zusammenfassung

Der Verfasser entwickelt Gleichungen, mit denen die Kippbelastung für I-Träger, deren Stabilität von einem Querbalken in der Mitte der Spannweite oder von über die ganze Länge der Spannweite angeordneten Querbalken gewährleistet wird, leicht zu berechnen ist. Querbalken und Hauptträger müssen steif miteinander verbunden sein. Der für diese steifen Knoten erforderliche Widerstand wird ebenfalls erörtert.

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