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A Curved Plate Element for the Analysis of Thin, Thick and Sandwich Plates

Un élément de plaque courbe pour le calcul de plaques minces, épaisses et sandwich

Ein gekrümmtes Plattenelement zur Berechnung dünner, dicker und Sandwichplatten

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Introduction

An extensive effort has been invested in the construction of plate bending finite elements which include shear [1-6]. This has been done in order to obtain more realistic elements for the analysis of structures which experience significant transverse shear deformations such as thick plates, sandwich plates and cellular plate structures.

The elements derived, however, are either rectangular, triangular or quadrilateral in shape; the triangular and quadrilateral shapes being very useful because of their ability to idealize more general shapes. General finite element formulations for the analysis of arbitrarily curved homogenous thick shells have also been presented [7-8].

Herewith is reported the formulation of a curved quadratic isoparametric plate element for the analysis of thin, thick and sandwich plates. It is assumed in the derivation that plane cross-sections of the plate remain plane during deformation, but not necessarily normal to the deformed reference surface of the plate. This assumption violates the Kirchoff's normality hypothesis used in classical thin plate theory, and permits the plate to experience transverse shear deformations. It is also assumed that the transverse normal stress is zero.

These assumptions are identical to the approximations made by AHMAD *et al.* [7], in formulating the curved quadratic superparametric thick shell element. The same approximations were adopted by Too [8] in his version of Ahmad's thick shell element, in which reduced numerical integration was used to eliminate spurious shear effects and to improve element performance.

However, being general thick shell elements, Ahmad/Too elements involve more degrees of freedom than are required for the complete specification of plate bending. Also, since they were derived from three-dimensional elements, they involve numerical integration with respect to the thickness co-ordinate of the shell. This can be avoided by defining the constitutive relationship in terms of stress resultants as is done below, thereby saving considerably on the computing cost.

Formulation of the Quadratic Isoparametric Plate Element

The details of the formulation of the quadratic isoparametric plate element are given elsewhere [9], and only the essential steps are reiterated here.

Fig. 1 defines the deformation of the plate, which is described by the transverse deflection w and the rotations ϑ_x and ϑ_y of the normal to the reference xy-plane. Fig. 2 shows the quadratic isoparametric plate element with nodes numbered in a clockwise sense. The displacement of a typical node i has three components which comprise the transverse deflection w_i and normal rotations ϑ_{xi} and ϑ_{yi}. These may be listed as a vector

$$\{\delta_i\} = \{w_i, \vartheta_{xi}, \vartheta_{yi}\}^T \tag{1}$$

and the element displacements as a vector

$$\{\delta\} = \{\delta_1, \delta_2, \dots \delta_8\}^T \tag{2}$$

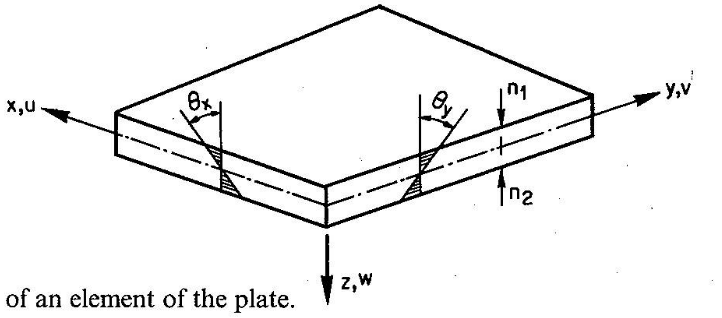
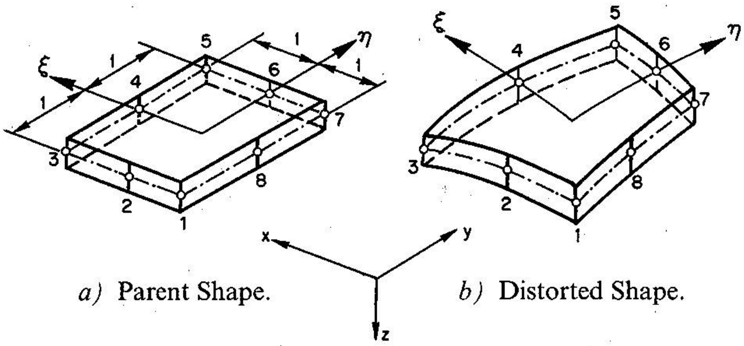


Fig. 1. Deformations of an element of the plate.



a) Parent Shape. *b)* Distorted Shape.

Fig. 2. The quadratic isoparametric plate bending element:

The displacements within the element are then defined in terms of the nodal displacements by the following equations

$$\begin{aligned} u &= -z\vartheta_x = -z\sum_{i=1}^{8} N_i\vartheta_{xi} \\ v &= -z\vartheta_y = -z\sum_{i=1}^{8} N_i\vartheta_{yi} \\ w &= \sum_{i=1}^{8} N_i w_i \end{aligned} \tag{3}$$

in which the shape functions N_i are:

$$N_i = \tfrac{1}{4}(1+\xi_0)(1+\eta_0)(\xi_0+\eta_0-1) \tag{4a}$$

at the corner nodes and

$$\begin{aligned}\xi_i &= 0,\ N_i=\tfrac{1}{2}(1-\xi^2)(1+\eta_0)\\ \eta_i &= 0,\ N_i=\tfrac{1}{2}(1-\eta^2)(1+\xi_0)\end{aligned} \tag{4b}$$

at the midside nodes;

$$\text{where}\quad \xi_0=\xi\xi_i,\ \eta_0=\eta\eta_i \tag{4c}$$

The same shape functions are used to define the x- and y-coordinates at any point within the element and the corresponding thickness of the plate at that point in terms of their nodal values. Thus

$$x=\sum_{i=1}^{8} N_i x_i,\ y=\sum_{i=1}^{8} N_i y_i,\ t=\sum_{i=1}^{8} N_i t_i \tag{5}$$

This definition enables the element to take up curved plan shapes and to have a parabolically varying thickness. Hence, any arbitrary geometry can be closely approximated.

The constitutive relationship for the element is given by [9]

$$\begin{Bmatrix} M_x\\ M_y\\ M_{xy}\\ Q_x\\ Q_y\end{Bmatrix}=[D]\sum_{i=1}^{8}\begin{bmatrix} 0 & -\dfrac{\delta N_i}{\delta x} & 0\\ 0 & 0 & -\dfrac{\delta N_i}{\delta y}\\ 0 & -\dfrac{\delta N_i}{\delta y} & -\dfrac{\delta N_i}{\delta x}\\ \dfrac{\delta N_i}{\delta x} & -N_i & 0\\ \dfrac{\delta N_i}{\delta y} & 0 & -N_i\end{bmatrix}\begin{Bmatrix} w_i\\ \vartheta_{xi}\\ \vartheta_{yi}\end{Bmatrix} \tag{6a}$$

$$\text{or}\ \{M\}=[D]\sum_{i=1}^{8}[B_i]\{\delta_i\}=[D][B]\{\delta\}=[S]\{\delta\} \tag{6b}$$

in which the matrix $[D]$ is the 5×5 property matrix, whose coefficients constitute the usual bending, twisting, shearing and coupling rigidities of the plate [9].

The stiffness matrix is obtained by numerical integration, using a 2×2 Gaussian integration grid, from the following equation [9]

$$[k]=\iint[B]^T[D][B]\,dx\,dy=\sum_{i=1}^{8}\sum_{j=1}^{8}\iint[B_i]^T[D][B_j]\,dx\,dy \tag{7}$$

It will be noted that for the exact integration of equation (7) a 3×3 Gaussian integration grid is required. But as explained in reference [9] a lower order 2×2 grid has been adopted in order to eliminate spurious shear effects and to improve element performance. This also offers a significant saving in computing effort.

Design Stresses

Equations (6) yield the stress resultants at any point within the element. But for the purposes of design equation (6) is usually evaluated at the nodal points to give the stress resultants at the nodes. This is the conventional approach of evaluating design stresses.

It has been observed that elements with reduced integration yield exceptionally accurate stresses at the Gaussian integration points, but that, in the case of thin plates, the shear stress resultants calculated at the nodes are poor [8-10]. However, in the case of the quadratic isoparametric plate element presented here, if the stress resultants at the nodes are bilinearly extrapolated from those calculated at the Gaussian integration points, then the shear stress resultants are always good irrespective of the magnitude of the shearing rigidity of the plate [9]. This important result follows from the fact that the lower order 2×2 Gaussian integration grid adopted actually implies smoothing of the components of the strain matrix $[B]$ by a least squares bilinear fit [9, 10]. Consequently, the unwanted spurious straining modes associated with the higher order shear strain terms appearing in equation (6) are filtered out, resulting in a more efficient element.

This method of obtaining nodal stresses from those already calculated at the Gaussian integration points will be referred to as the extrapolation method.

Numerical Results

In order to verify the accuracy of the formulation, a uniformly loaded square simply supported sandwich plate of side a, and having flexural and shear rigidities of D and $\frac{100D}{a^2}$ respectively, is analysed. Results of the maximum deflection, shearing

Table 1. Analysis of simply supported square sandwich plate having Poisson's ratio $\nu = 0.3$

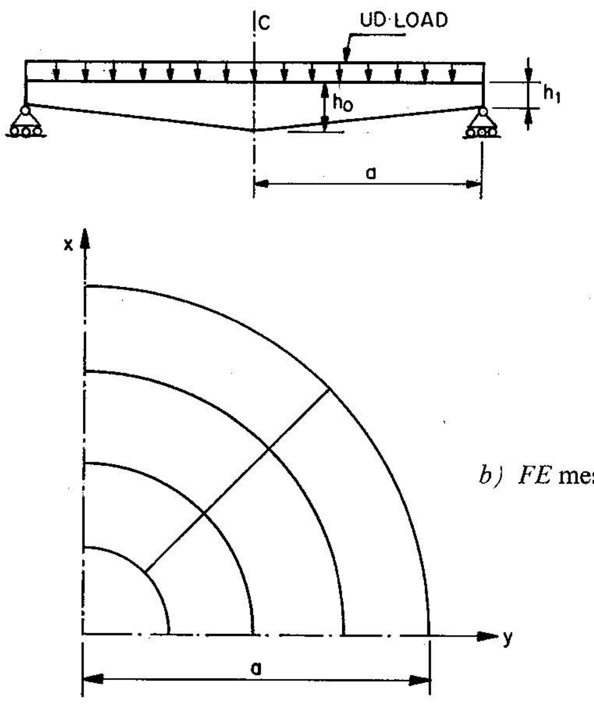
FE mesh (symmetric quater)	Maximum central deflection	Maximum bending moment	Maximum twisting moment	Maximum edge shear
1×1	0.00464	0.0626 (0.0678)	0.0357 (0.0382)	0.625 (0.412)
2×2	0.00480	0.0495 (0.0516)	0.0337 (0.0351)	0.406 (0.339)
3×3	0.00480	0.0485 (0.0495)	0.0332 (0.0340)	0.371 (0.339)
4×4	00.0480	0.0482 (0.0488)	0.0330 (0.0335)	0.0358 (0.338)
5×5	0.00480	0.0481 (0.0484)	0.0328 (0.0332)	0.0351 (0.338)
Theory [11]	0.00480	0.0479	0.0325	0.338
Multiplier	$\frac{qa^4}{D}$	qa^2	qa^2	qa

force, bending and twisting moment are given in Table 1. The stress resultants given in Table 1 are calculated using both the conventional and extrapolation (shown in brackets) method. The close agreement between the two should be noted. It is seen that as the mesh is refined the results converge rapidly to the theoretical values given by PLANTEMA [11].

Table 2 gives the central deflection of a circular plate of variable thickness analysed using the finite element mesh shown in Figure 3. The results are in good agreement with theoretical values [12].

Table 2. Analysis of a simply supported circular plate of variable thickness. $\nu = 0.3$.

Thickness ratio (h_0/h_1)	Central deflection $\frac{wEh_0^3}{qa^4}$ for $\frac{a}{h_0}$ =			
	5	20	40	Thin plate solution [12]
1.0	0.764	0.734	0.727	0.738
1.5	1.300	1.261	1.249	1.26



a) Plate cross-section.

b) FE mesh in a symmetric quater.

Fig. 3. Circular plate of variable thickness:

Table 3 involves a study of the shear stress resultants in a uniformly loaded square simply supported plate and how they are influenced by the method of sampling them. It is seen that the shear stress resultants calculated using the extrapolation method are independent of the shear rigidity, which is consistent with theoretical findings [11]. However, those predicted by the conventional method are only satisfactory when the shear rigidity is small and deteriorate considerably as the shear rigidity increases. For this reason, the shear stress resultants should, as far as possible, be calculated by extrapolation.

Table 3. Shear stress resultants in a square simply supported plate analysed using 5 × 5 mesh. $\nu = 0.3$. Solution B is by extrapolation and *A* by the conventional method

Shear rigidity $\frac{Sa^2}{D}$	Method	Edge shearing force $\frac{Q_y}{qa}$ at $x=$ 0	0.1*a*	0.2*a*	0.3*a*	0.4*a*	0.5*a*
20	*A*	0	0.172	0.253	0.304	0.331	0.339
	B	0.007	0.176	0.254	0.303	0.329	0.338
100	*A*	0	0.176	0.261	0.314	0.342	0.351
	B	0.007	0.176	0.254	0.303	0.329	0.338
500	*A*	0	0.200	0.301	0.364	0.399	0.410
	B	0.005	0.176	0.254	0.303	0.329	0.338
2500	*A*	0	0.317	0.501	0.618	0.685	0.706
	B	0.002	0.176	0.254	0.303	0.329	0.338
35000	*A*	0	2.229	3.727	4.740	5.325	5.518
	B	0.006	0.176	0.257	0.302	0.328	0.338
All values	Theory [11, 12]	0					0.338

Conclusion

A curved quadratic isoparametric plate bending element which involve shear has been presented for the elastic analysis of arbitrarily shaped plates. From the numerical solutions presented, it is evident that the element, which is formulated according to the isoparametric concept, offers demonstrated advantages in reproducing transverse shear deformability and in modelling curved geometries. The element is applicable to the analysis of not only thin, thick and sandwich plates, but also cellular and voided bridge decks that can be idealized by an equivalent homogeneous material.

Practical Application and Scope

The formulation presented extends the finite element method to plate structures which are arbitrarily curved in plan. Examples of such structures include the curved bridge decks which occur frequently in the design of modern highways and highway interchanges, where the designer has to fit his structure into the environment with the least disturbance to amenities and services.

Due to its accuracy, the quadratic isoparametric plate element presented is probably one of the best plate bending elements available. It is easy to formulate numerically and can represent curved plate boundaries and reproduce transverse shear deformations. It works well whether shear deformations are significant or not and can adequately simulate not only thin, thick and sandwich plates, but also cellular and voided bridge decks that can be idealized by an equivalent homogeneous material. These features make the use of this element a very attractive proposition for bridge designers.

When compared with ordinary triangular or rectangular plate bending elements, which have no midside nodes, the quadratic isoparametric plate bending element involves a relatively high number of degrees of freedom. However, this is more than compensated for by its excellent performance and superior rate of convergence.

The application of the plate element presented is restricted to situations where there are no membrane forces acting in the plane of the plate. However, the formulation could be modified to include membrane action, and thereby extend its applicability to folded plates and flat shell structures.

Nomenclature

a	length of side of a square plate.
D	flexural rigidity of isotropic plate.
h, t	plate thickness.
M_x, M_y, M_{xy}	bending and twisting moments.
q	distributed transverse loading.
Q_x, Q_y	transverse shearing forces.
S	transverse shear rigidity of isotropic plate.
u, v, w	components of displacement parallel to x-, y- and z-axes.
x, y, z	rectangular co-ordinates.
$\vartheta_x \vartheta_y$	normal rotations of plate cross-section.
ξ, η	local natural dimensionless co-ordinates.
ν	Poisson's ratio.
$\{M\}$	stress resultants vector.
$\{\delta\}$	displacement vector.
$[B]$	matrix connecting strains and displacements of an element.
$[D]$	property matrix.
$[k]$	stiffness matrix.
$[N]=[N_1, N_2, ...]$	shape functions matrix.
$[S]$	stress matrix.

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Summary

A finite element formulation based on the isoparametric concept is presented for a curved plate element which includes shear, for the analysis of thin, thick and sandwich plates. Numerical examples are presented which demonstrate the applicability of the formulation to practical problems.

Résumé

L'auteur présente une méthode d'éléments fins basée sur le concept isoparamétrique et tenant compte du cisaillement, pour un élément de plaque courbe servant au calcul de plaques minces, épaisses et sandwich. Des exemples numériques montrent l'application de cette théorie à des problèmes pratiques.

Zusammenfassung

Für ein gekrümmtes Plattenelement, das auch Schubbeanspruchung einschliesst, wird eine auf isoparametrischem Konzept beruhende Methode der finiten Elemente zur Berechnung dünner, dicker und Sandwichplatten aufgezeigt. Numerische Beispiele zeigen die Anwendbarkeit der Formulierung auf praktische Probleme.