Zeitschrift: IABSE publications = Mémoires AIPC = IVBH Abhandlungen

Band: 34 (1974)

Artikel: Generalization of the finite strip method for solid continua

Autor: Glück, Jacob / Gellert, Menachem

DOI: https://doi.org/10.5169/seals-26273

Generalization of the Finite Strip Method for Solid Continua

Généralisation de la méthode des bandes finies pour les continus solides

Verallgemeinerung der Finite-Streifen-Methode für Festkörperkontinua

JACOB GLÜCK MENACHEM GELLERT
Department of Structures, Faculty of Civil Engineering
Technion-Israel Institute of Technology

Introduction

The object of this paper is to present a more general formulation of the finite strip method for solid continua, and discusses the cases where the use of the present approach is more advantageous compared with the finite element method.

In principle the finite element method in its present stage of development may be applied for analysis of any kind of plane or space structure. The problem which may arise is more of an economic kind as computer time, data preparation, interpretation of results etc. It is obvious that the finite element approach i.e. a discretization in three dimensions for the general case, is unavoidable for structures with geometric or physical irregularities in all three directions.

In a broad of structures especially in civil engineering the irregularities are not in all three directions for space structures nor in two directions for plane structures. For these cases the use of the finite element seems not to be the natural method of solution; it would be more rational to apply a method where the discretization is made only for the direction where irregularities appear. For the case of a three-dimensional body having irregularities in one direction, the basic element will be in the general case a curved plate with finite width and for two-directional irregularities a bar with deformable cross section, generally bounded with curved line segments.

This procedure leads essentially to the reduction of a three-dimensional problem to a two- or one-dimensional one. This idea was successfully applied for two dimensional problems [1], [2] where by use of Fourier series expansion the problem was reduced to a one-dimensional one.

The formulation of the problem will be done for a three-dimensional body using the variational approach based on the principle of minimum total potential energy, but may be extended also to other energetic principles like generalized functionals by Lagrange multipliers or based on complementary energy or Reissner's principle [3]. For practical problems the most important case is that where the three-dimensional problem is reduced to a one-dimensional one, leading to the solution of a system of linear differential equations with variable coefficients. For such system well known procedures are known. The development here will be made for this case only. Small displacements and linear elasticity are assumed. The present approach will be exemplified for a plate in bending by neglection of the shear force energy.

Formulation of the Problem

The total potential energy of an elastic body may be expressed as

$$\pi = \int_V \left(\tfrac{1}{2} E^{ijkl} \epsilon_{ij} \epsilon_{kl} - \bar{F}^i u^i\right) dV - \int_{S_\sigma} \bar{T}^i u^i \, dS, \tag{1}$$

where E^{ijkl} = elastic constants, ϵ_{ij} = strain tensor component, $\bar{F}^i$ = prescribed body force component, u^i = displacement, V = volume, $\bar{T}^i$ = prescribed surface traction and S_σ = portion of S over which the surface tractions are prescribed.

The strain-displacement relation for small displacements has the form

$$\epsilon_{ij} = \tfrac{1}{2}\left(u^i_{,j} + u^j_{,i}\right). \tag{2}$$

When a solid is divided into a finite number of discrete strips V_n, the total potential energy may be written as

$$\pi = \sum_{n=1}^{m} \left[\int_{V_n} \left(\tfrac{1}{2} E^{ijkl} \epsilon_{ij} \epsilon_{kl} - \bar{F}^i u^i\right) dV_n - \int_{S_{\sigma_n}} \bar{T}^i u^i \, dS_n\right]. \tag{3}$$

The strips will be supposed to be along the x_1 axis, and with a finite surface into the x_2, x_3 plane. The unknown displacements may be expressed in this case as discrete values in the corners of the finite surface, and are functions along the x_1 axis only.

$$\boldsymbol{u}(x_1, x_2, x_3) = \boldsymbol{A}(x_1, x_2, x_3)\, \boldsymbol{q}(x_1), \tag{4}$$

where $\boldsymbol{A}$ = matrix containing the interpolation functions, and $\boldsymbol{q}$ = vector of displacements function at the corners of the finite surface.

The strain vector may be expressed now in the form

$$\varepsilon = \boldsymbol{B}(x_1, x_2, x_3)\,\boldsymbol{q} + \boldsymbol{C}(x_1, x_2, x_3)\,\boldsymbol{q}', \tag{5}$$

where

$$\boldsymbol{q}' = \frac{d\boldsymbol{q}}{dx_1}. \tag{6}$$

In the case of theory of plates and shells where shear strain energy is neglected Eq. (5) includes also second derivatives of $\boldsymbol{q}$ with respect to x_1.

The bounding surfaces of a strip are defined by planes of the form $x_1 = \text{const.}$ and surfaces $x_2 = f(x_3)$. For $x_1 = \text{const.}$ the matrix $\boldsymbol{A}$ will be written in the form $\boldsymbol{A}_b(x_2, x_3)$, the load vector $\overline{\boldsymbol{T}}_b(x_2, x_3)$ and the unknown displacement in the form $\boldsymbol{q}_b$. For the surfaces $x_2 = f(x_3)$ by assuming that x_2 and x_3 may be curvilinear, i. e.

$$x_2 = g(S), \tag{7}$$

$$x_3 = h(S) \tag{8}$$

the matrix $\boldsymbol{A}$ will be written in the form $\boldsymbol{A}_s(x_1, S)$ and the load vector $\overline{\boldsymbol{T}}_s(x_1, S)$.

With the above mentioned notations Eq. (3) may be written as

$$\pi = \sum_{n=1}^{m} \iiint \left[\tfrac{1}{2}(\boldsymbol{B}\boldsymbol{q} + \boldsymbol{C}\boldsymbol{q}')^T \boldsymbol{E}(\boldsymbol{B}\boldsymbol{q} + \boldsymbol{C}\boldsymbol{q}') - \overline{\boldsymbol{F}}^T \boldsymbol{A}\,\boldsymbol{q}\right] dx_1\,dx_2\,dx_3$$
$$-\iint \overline{\boldsymbol{T}}_s^T \boldsymbol{A}_s \boldsymbol{q}_s\,dx_1\,ds - \iint \overline{\boldsymbol{T}}_b^T \boldsymbol{A}_b \boldsymbol{q}_b\,dx_2\,dx_3. \tag{9}$$

Denoting with

$$\boldsymbol{C}_{00}(x_1) = \iint \boldsymbol{B}^T \boldsymbol{E}\,\boldsymbol{B}\,dx_2\,dx_3, \tag{10}$$

$$\boldsymbol{C}_{01}(x_1) = \boldsymbol{C}_{10}^T(x_1) = \iint \boldsymbol{B}^T \boldsymbol{E}\,\boldsymbol{C}\,dx_2\,dx_3, \tag{11}$$

$$\boldsymbol{C}_{11}(x_1) = \iint \boldsymbol{C}^T \boldsymbol{E}\,\boldsymbol{C}\,dx_2\,dx_3, \tag{12}$$

$$\boldsymbol{Q}^T(x_1) = \iint \overline{\boldsymbol{F}}^T \boldsymbol{A}\,dx_2\,dx_3 + \int \overline{\boldsymbol{T}}_s^T \boldsymbol{A}_s\,ds, \tag{13}$$

$$\boldsymbol{P}^T = \iint \overline{\boldsymbol{T}}_b^T \boldsymbol{A}_b\,dx_2\,dx_3. \tag{14}$$

Eq. (9) may be expressed in the form

$$\pi = \sum_{n=1}^{m} \int \left[\tfrac{1}{2}(\boldsymbol{q}^T \boldsymbol{C}_{00}\boldsymbol{q} + \boldsymbol{q}^T \boldsymbol{C}_{01}\boldsymbol{q}' + \boldsymbol{q}'^T \boldsymbol{C}_{10}\boldsymbol{q} + \boldsymbol{q}'^T \boldsymbol{C}_{11}\boldsymbol{q}') - \boldsymbol{Q}^T \boldsymbol{q}\right] dx_1 - \boldsymbol{P}^T \boldsymbol{q}_b. \tag{15}$$

Not all unknown $\boldsymbol{q}$ are independent. Denoting with $\boldsymbol{d}$ the vector of independent unknowns the following relations may be written

$$\sum_{n=1}^{m} \boldsymbol{q} = \boldsymbol{J}\boldsymbol{d}, \tag{16}$$

$$\sum_{n=1}^{m} \boldsymbol{q}_b = \boldsymbol{J}\boldsymbol{d}_b, \tag{17}$$

where $\boldsymbol{J}$ is a transformation matrix.

Denoting
$$\boldsymbol{K}_{ij}(x_1) = \boldsymbol{J}^T \sum_{n=1}^{m} \boldsymbol{C}_{ij} \boldsymbol{J}, \quad i=0,1;\ j=0,1, \tag{18}$$

$$\boldsymbol{L}^T = \sum_{n=1}^{m} \boldsymbol{Q}^T \boldsymbol{J}, \tag{19}$$

$$\boldsymbol{X}^T = \sum_{n=1}^{m} \boldsymbol{P}^T \boldsymbol{J}, \tag{20}$$

$$\boldsymbol{d}' = \frac{d(\boldsymbol{d})}{dx_1}, \tag{21}$$

$$\boldsymbol{d}'' = \frac{d^2(\boldsymbol{d})}{dx_1^2}. \tag{22}$$

Eq. (15) may be expressed as function of the unknown vector $\boldsymbol{d}$ as follows:

$$\pi = \int [\tfrac{1}{2}(\boldsymbol{d}^T \boldsymbol{K}_{00} \boldsymbol{d} + \boldsymbol{d}^T \boldsymbol{K}_{01} \boldsymbol{d}' + \boldsymbol{d}'^T \boldsymbol{K}_{10} \boldsymbol{d} + \boldsymbol{d}'^T \boldsymbol{K}_{11} \boldsymbol{d}') - \boldsymbol{L}^T \boldsymbol{d}]\, dx_1 - \boldsymbol{X}^T \boldsymbol{d}_b. \tag{23}$$

Equating to zero of the variation of Eq. (23) and some integration by parts yields

$$\begin{aligned}\delta \pi = \int \delta \boldsymbol{d}^T [(\boldsymbol{K}_{00} - \boldsymbol{K}'_{10}) \boldsymbol{d} + (\boldsymbol{K}_{01} - \boldsymbol{K}_{10} - \boldsymbol{K}'_{11}) \boldsymbol{d}' - \boldsymbol{K}_{11} \boldsymbol{d}'' - \boldsymbol{L}]\, dx_1 \\ + \delta \boldsymbol{d}_b^T (\boldsymbol{K}_{10b} \boldsymbol{d}_b + \boldsymbol{K}_{11b} \boldsymbol{d}'_b - \boldsymbol{X}) = 0,\end{aligned} \tag{24}$$

where
$$\boldsymbol{K}'_{ij} = \frac{d\boldsymbol{K}_{ij}}{dx_1}, \tag{25}$$

$$\boldsymbol{K}_{ijb} = \boldsymbol{K}_{ij\,(x_1 = const)}. \tag{26}$$

For existence of Eq. (24) for any variation δd^T the following equations must be satisfied:

$$-\boldsymbol{K}_{11} \boldsymbol{d}'' + (\boldsymbol{K}_{01} - \boldsymbol{K}_{10} - \boldsymbol{K}'_{11}) \boldsymbol{d}' + (\boldsymbol{K}_{00} - \boldsymbol{K}'_{10}) \boldsymbol{d} = \boldsymbol{L}, \tag{27}$$

$$\boldsymbol{K}_{11b} \boldsymbol{d}'_b + \boldsymbol{K}_{10b} \boldsymbol{d}_b = \boldsymbol{X}. \tag{28}$$

The Euler Eq. (27) represents a system of linear differential equations with variable coefficients with unknown vector function $\boldsymbol{d}$. Eq. (28) represents the natural boundary conditions which are mechanical conditions, the left side representing the interior forces at $x_1 = \text{const}$ and the right side the exterior in the region where exterior forces are prescribed.

Denoting
$$\boldsymbol{S}_2(x_1) = -\boldsymbol{K}_{11}, \tag{29}$$

$$\boldsymbol{S}_1(x_1) = \boldsymbol{K}_{01} - \boldsymbol{K}_{10} - \boldsymbol{K}'_{11}, \tag{30}$$

$$\boldsymbol{S}_0(x_1) = \boldsymbol{K}_{00} - \boldsymbol{K}'_{10} \tag{31}$$

the final form of Eq. (27) is obtained

$$\boldsymbol{S}_2 \boldsymbol{d}'' + \boldsymbol{S}_1 \boldsymbol{d}' + \boldsymbol{S}_0 \boldsymbol{d} = \boldsymbol{L}. \tag{32}$$

As example the plate bending problem with neglection of the shear strain energy (see Fig. 1) will be considered. By initial integration along x_3 the

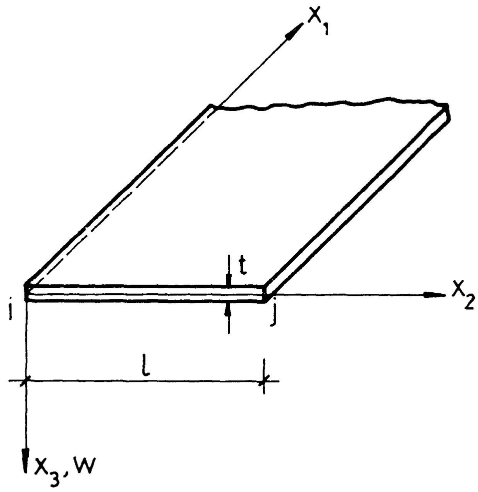

Fig. 1.

problem transforms to a two-dimensional one. The vector of unknowns for a single strip will be:

$$q(x_1) = \{w_i, \theta_i, w_j, \theta_j\}^T, \tag{33}$$

where w_i and θ_i are the vertical displacement and rotation functions respectively along the ridge i, and w_j and θ_j are similar but for ridge j.

The displacements w and θ may be expressed as function of the ridge displacements as follows:

$$w = w_i - \frac{x_2}{l}\theta_i l + \frac{x_2^2}{l^2}(3w_j - 3w_i + 2\theta_i l + \theta_j l) + \frac{x_2^3}{l^3}(2w_i - 2w_j - \theta_i l - \theta_j l), \tag{34}$$

$$\theta = -\frac{\partial w}{\partial x_2} = \theta_i - \frac{2x_2}{l}\left(\frac{3w_j}{l} - \frac{3w_i}{l} + 2\theta_i + \theta_j\right) - \frac{3x_2^2}{l^2}\left(\frac{2w_i}{l} - \frac{2w_j}{l} - \theta_i - \theta_j\right). \tag{35}$$

The matrix $\boldsymbol{A}$ in this case is a function of x_2 only

$$A = \left\{1 - 3\frac{x_2^2}{l^2} + 2\frac{x_2^3}{l^3} \;\middle|\; -x_2 + \frac{2x_2^2}{l} - \frac{x_2^3}{l^2} \;\middle|\; \frac{3x_2^2}{l^2} - \frac{2x_2^3}{l^3} \;\middle|\; \frac{x_2^2}{l} - \frac{x_2^3}{l^2}\right\}. \tag{36}$$

The generalized strain vector $\boldsymbol{\varepsilon}$ may be expressed as function of the curvatures

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x_1^2} \\ -\dfrac{\partial^2 w}{\partial x_2^2} \\ -2\dfrac{\partial^2 w}{\partial x_1 \partial x_2} \end{Bmatrix}. \tag{37}$$

Denoting with w', θ', $\boldsymbol{q}'$ the first derivative and with w'', θ'', $\boldsymbol{q}''$ the second derivative of w, θ and q respectively with respect to x_1, the components of the vector $\boldsymbol{\varepsilon}$ may be expressed in the form

$$-\frac{\partial^2 w}{\partial x_1^2} = -w_i'' + \frac{x_2}{l}\theta_i'' l - \frac{x_2^2}{l^2}(3w_j'' - 3w_i'' + 2\theta_i'' l + \theta_j'' l)$$

$$-\frac{x_2^3}{l^3}(2w_i'' - 2w_j'' - \theta_i'' l - \theta_j'' l), \tag{38}$$

$$-\frac{\partial^2 w}{\partial x_2^2} = -2\left(\frac{3w_j}{l^2} - \frac{3w_i}{l^2} + \frac{2\theta_i}{l} + \frac{\theta_j}{l}\right) - \frac{6x_2}{l}\left(\frac{2w_i}{l^2} - \frac{2w_j}{l^2} - \frac{\theta_i}{l} - \frac{\theta_j}{l}\right), \tag{39}$$

$$-2\frac{\partial^2 w}{\partial x\,\partial y} = 2\theta_i' - \frac{4x_2}{l}\left(\frac{3w_j'}{l} - \frac{3w_i'}{l} + 2\theta_i' + \theta_j'\right) - \frac{6x_2^2}{l^2}\left(\frac{2w_i'}{l} - \frac{2w_j'}{l} - \theta_i' - \theta_j'\right). \tag{40}$$

Eq. (37) may now be expressed in the form

$$\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{q} + \boldsymbol{C}\boldsymbol{q}' + \boldsymbol{D}\boldsymbol{q}'', \tag{41}$$

where $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are matrices, the components of which are functions of x_2 only.

$$\boldsymbol{B} = \left[\begin{array}{c:c:c:c} 0 & 0 & 0 & 0 \\ \frac{6}{l^2} - \frac{12x_2}{l^3} & -\frac{4}{l} + \frac{6x_2}{l^2} & -\frac{6}{l^2} + \frac{12x_2}{l^3} & -\frac{2}{l} + \frac{6x_2}{l^2} \\ 0 & 0 & 0 & 0 \end{array}\right], \tag{42}$$

$$\boldsymbol{C} = \left[\begin{array}{c:c:c:c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{12x_2}{l^2} - \frac{12x_2^2}{l^3} & 2 - \frac{8x_2}{l} + \frac{6x_2^2}{l^2} & -\frac{12x_2}{l^2} + \frac{12x_2^2}{l^3} & -\frac{4x_2}{l} + \frac{6x_2^2}{l^2} \end{array}\right], \tag{43}$$

$$\boldsymbol{D} = \left[\begin{array}{c:c:c:c} -1 + \frac{3x_2^2}{l^2} - \frac{2x_2^3}{l^3} & x_2 - \frac{2x_2^2}{l} + \frac{x_2^3}{l^2} & -\frac{3x_2^2}{l^2} + \frac{2x_2^3}{l^3} & -\frac{x_2^2}{l} + \frac{x_2^3}{l^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{44}$$

The total potentional energy of the plate may be expressed as follows:

$$\pi = \sum_{n=1}^{m} \iint \left(\tfrac{1}{2}\boldsymbol{\varepsilon}^T \boldsymbol{E}\boldsymbol{\varepsilon} - p w\right) dx_1\, dx_2 - \int (\bar{Q}_s w + \bar{M}_s \theta)\, dx_1 - \int (\bar{Q}_b w_b - \bar{M}_b w_b')\, dx_2, \tag{45}$$

where p = the normal load on the plate, Q_s and M_s = shear force and bending moment respectively acting on the boundaries x_2 = const. where forces are prescribed, Q_b and M_b = shear forces and bending moments respectively acting on the boundaries x_1 = const. where forces are prescribed, and $\boldsymbol{E}$ = the elasticity matrix.

Substituting in Eq. (45) the expressions for $\boldsymbol{\varepsilon}$, w, θ, w_b and w_b' as given above yields

$$\pi = \sum_{n=1}^{m} \iint \left[\tfrac{1}{2}(\boldsymbol{B}\boldsymbol{q} + \boldsymbol{C}\boldsymbol{q}' + \boldsymbol{D}\boldsymbol{q}'')^T \boldsymbol{E}(\boldsymbol{B}\boldsymbol{q} + \boldsymbol{C}\boldsymbol{q}' + \boldsymbol{D}\boldsymbol{q}'') - p\boldsymbol{A}\boldsymbol{q}\, dx_1\, dx_2\right.$$

$$- \int \left(\bar{Q}_s \boldsymbol{A}_s - \bar{M}_s \frac{\partial \boldsymbol{A}_s}{\partial x_2}\right)\boldsymbol{q}\, dx_1 - \int \boldsymbol{A}\,(\bar{Q}_b \boldsymbol{q}_b - \bar{M}_b \boldsymbol{q}_b')\, dx_2. \tag{46}$$

Denoting

$$\boldsymbol{C}_{00} = \int \boldsymbol{B}^T \boldsymbol{E}\,\boldsymbol{B}\, dx_2, \tag{47}$$

$$\boldsymbol{C}_{01} = \boldsymbol{C}_{10}^T = \int \boldsymbol{B}^T \boldsymbol{E}\,\boldsymbol{C}\, dx_2, \tag{48}$$

$$\boldsymbol{C}_{02} = \boldsymbol{C}_{20}^T = \int \boldsymbol{B}^T \boldsymbol{E}\,\boldsymbol{D}\, dx_2, \tag{49}$$

$$\boldsymbol{C}_{11} = \int \boldsymbol{C}^T \boldsymbol{E}\,\boldsymbol{C}\, dx_2, \tag{50}$$

$$\boldsymbol{C}_{12} = \boldsymbol{C}_{21}^T = \int \boldsymbol{C}^T \boldsymbol{E}\,\boldsymbol{D}\, dx_2, \tag{51}$$

$$\boldsymbol{C}_{22} = \int \boldsymbol{D}^T \boldsymbol{E}\,\boldsymbol{D}\, dx_2, \tag{52}$$

$$\boldsymbol{Q}^T(x_1) = \int p\,\boldsymbol{A}\, dx_2 + \bar{Q}_s \boldsymbol{A}_s \overline{M}_s \frac{\partial \boldsymbol{A}_s}{\partial x_2}, \tag{53}$$

$$\boldsymbol{P}^T = \int \bar{Q}_b \boldsymbol{A}\, dx_2, \tag{54}$$

$$\boldsymbol{M}^T = \int \overline{M}_b \boldsymbol{A}\, dx_2. \tag{55}$$

For non-anisotropic plates, matrices $\boldsymbol{C}_{01}$, $\boldsymbol{C}_{10}$, $\boldsymbol{C}_{12}$, $\boldsymbol{C}_{21}$ vanish. In this case Eq. (46) may be written in the form:

$$\pi = \sum_{n=1}^{m} \int [\tfrac{1}{2}(\boldsymbol{q}^T \boldsymbol{C}_{00}\boldsymbol{q} + \boldsymbol{q}^T \boldsymbol{C}_{02}\boldsymbol{q}'' + \boldsymbol{q}'^T \boldsymbol{C}_{11}\boldsymbol{q}' + \boldsymbol{q}''^T \boldsymbol{C}_{20}\boldsymbol{q} + \boldsymbol{q}''^T \boldsymbol{C}_{22}\boldsymbol{q}'') - \boldsymbol{Q}^T \boldsymbol{q}]\, dx_1 - \boldsymbol{P}^T \boldsymbol{q}_b + \boldsymbol{M}^T \boldsymbol{q}_b'. \tag{56}$$

The expressions for the matrices $\boldsymbol{C}_{00}$, $\boldsymbol{C}_{02}$, $\boldsymbol{C}_{11}$, $\boldsymbol{C}_{20}$ and $\boldsymbol{C}_{22}$ for the case of isotropic plates are given in the appendix.

The generalized displacement vector $\boldsymbol{q}$ and its derivatives may now be expressed by adding the independent displacement vector $\boldsymbol{d}$ using Eq. (16) and (17). With the notations based on Eqs. (18), (19) and (20), Eq. (56) may be expressed as follows:

$$\pi = \int [\tfrac{1}{2}(\boldsymbol{d}^T \boldsymbol{K}_{00}\boldsymbol{d} + \boldsymbol{d}^T \boldsymbol{K}_{02}\boldsymbol{d}'' + \boldsymbol{d}'^T \boldsymbol{K}_{11}\boldsymbol{d}' + \boldsymbol{d}''^T \boldsymbol{K}_{20}\boldsymbol{d} + \boldsymbol{d}''^T \boldsymbol{K}_{22}\boldsymbol{d}'' - \boldsymbol{L}^T \boldsymbol{d}]\, dx_1 - \boldsymbol{X}^T \boldsymbol{d}_b + \boldsymbol{Y}^T \boldsymbol{d}_b'. \tag{57}$$

The condition of vanishing of the variation of the total potential energy, after some integrations by parts, yields

$$\delta\pi = \int \delta\boldsymbol{d}^T [\boldsymbol{K}_{22}\boldsymbol{d}'''' + (\boldsymbol{K}_{02} - \boldsymbol{K}_{11} + \boldsymbol{K}_{20})\boldsymbol{d}'' + \boldsymbol{K}_{00}\boldsymbol{d} - \boldsymbol{L}]\, dx_1 + \delta\boldsymbol{d}_b^T[-\boldsymbol{K}_{22}\boldsymbol{d}_b''' + (-\boldsymbol{K}_{20} + \boldsymbol{K}_{11})\boldsymbol{d}_b' - \boldsymbol{X}] + \delta\boldsymbol{d}_b'^T(\boldsymbol{K}_{22}\boldsymbol{d}_b'' + \boldsymbol{K}_{20}\boldsymbol{d}_b + \boldsymbol{Y}) = 0. \tag{58}$$

From the requirement of arbitrary possible variations the following conditions result:

— equations of equilibrium expressed in the form of a system of nonhomogeneous linear differential equations with constant coefficients:

$$\boldsymbol{K}_{22}\boldsymbol{d}'''' + (\boldsymbol{K}_{02} - \boldsymbol{K}_{11} + \boldsymbol{K}_{20})\boldsymbol{d}'' + \boldsymbol{K}_{00}\boldsymbol{d} = \boldsymbol{L}. \tag{59}$$

— natural boundary conditions at $x_1 = \text{const}$

$$-\boldsymbol{K}_{22}\boldsymbol{d}_b''' + (-\boldsymbol{K}_{02} - \boldsymbol{K}_{11})\boldsymbol{d}_b' = \boldsymbol{X}, \tag{60}$$

$$-\boldsymbol{K}_{22}\boldsymbol{d}_b'' - \boldsymbol{K}_{20}\boldsymbol{d}_b = \boldsymbol{Y}. \tag{61}$$

Eq. (60) is the boundary condition for prescribed shear force at cross sections $x_1 = \text{const.}$, and Eq. (61), for prescribed bending moment at the same sections.

Solving the system of differential Eqs. (59) under boundary conditions (60), (61), the interior forces may be obtained for each strip as shown for the general approach.

The system of differential equations with variable coefficients may be solved for example using finite differences. This method of solution will lead to a system of linear algebraic equations. Essentially the accuracy of the solution and amount of numerical calculations will be approximately the same as in the case application of standard finite element approach, but the finite difference solution is less satisfactory owing to the fact, that boundary conditions are not automatically satisfied as in finite element approach. If the first derivative of the unknowns is represented by finite differences and substituted in Eq. (23), direct minimalization of the functional with respect to the discrete unknowns leads to the Euler's method. This method is similar to the finite element method and the coefficient matrix is of the same order of magnitude. Automatic fulfillment of mechanical boundary conditions is achieved.

Another way for solution of the system of differential equations may be obtained by means of infinite power series. A new non-dimensional variable will be chosen:

$$\xi = \frac{x_1}{x_{1max}}, \qquad |\xi| \leqq 1, \tag{62}$$

where x_{1max} is an arbitrary length. Expanding the right side of Eq. (32) into an infinite power series

$$\boldsymbol{L} = \boldsymbol{N}\boldsymbol{\xi}, \tag{63}$$

where

$$\boldsymbol{N} = \begin{bmatrix} N_{1,0} & N_{1,1} & \dots & N_{1,\infty} \\ N_{2,0} & N_{2,1} & \dots & N_{2,\infty} \\ N_{m,0} & N_{m,1} & \dots & N_{m,\infty} \end{bmatrix}, \tag{64}$$

$$\boldsymbol{\xi} = \{1, \xi, \xi^2, \xi^3, \dots \xi^\infty\}^T. \tag{65}$$

m being the number of unknown displacement functions. The solution of Eq. (32) may be selected also in the form of an infinite power series

$$\boldsymbol{d} = \boldsymbol{G}_0 \boldsymbol{\xi}, \tag{66}$$

where $\boldsymbol{G}_0$ is an $m x \infty$ coefficient matrix to be determined.

Differentiating Eq. (66) results

$$\boldsymbol{d}' = \boldsymbol{G}_1 \boldsymbol{\xi}, \tag{67}$$

$$\boldsymbol{d}'' = \boldsymbol{G}_2 \boldsymbol{\xi}, \tag{68}$$

where the element (i, j) of the matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ respectively have the form

$$G_{1_{i,j}} = \frac{1}{x_{1max}} j \, G_{0_{i,j+1}}, \tag{69}$$

$$G_{2_{i,j}} = \frac{1}{x_{1max}^2} (j+1) j \, G_{0_{i,j+2}}. \tag{70}$$

Substituting Eqs. (63), (66), (67) and (68) into Eq. (32) the following equation will be obtained:

$$\boldsymbol{S}_2\,\boldsymbol{G}_2\,\xi + \boldsymbol{S}_1\,\boldsymbol{G}_1\,\xi + \boldsymbol{S}_0\,\boldsymbol{G}_0\,\xi = \boldsymbol{N}\xi. \tag{71}$$

Generally the elements of matrices $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ and $\boldsymbol{S}_0$ may be functions which can be expanded into power series. The following relation may be written

$$\boldsymbol{S}_k\,\boldsymbol{G}_k = \sum_{n=0}^{\infty} \xi^n\,\overline{\boldsymbol{T}}_{k,n}, \quad (k = 0, 1, 2). \tag{72}$$

The elements of the matrices $\overline{\boldsymbol{T}}_{k,n}$ are constants. For convenience the columns of these matrices will be shifted to right with n steps, and zeros will be placed in the corresponding n columns. The new matrices will be denoted $\boldsymbol{T}_{k,n}$ and the following notation will be introduced

$$\boldsymbol{T}_k = \sum_{n=0}^{\infty} \boldsymbol{T}_{k,n}. \tag{73}$$

Multiplying on both sides by ξ, Eq. (72) may now be written

$$\boldsymbol{S}_k\,\boldsymbol{G}_k\,\xi = \boldsymbol{T}_k\,\xi. \tag{74}$$

Denoting

$$\boldsymbol{M} = \boldsymbol{T}_2 + \boldsymbol{T}_1 + \boldsymbol{T}_0. \tag{75}$$

Eq. (71) yields

$$\boldsymbol{M}\xi = \boldsymbol{N}\xi. \tag{76}$$

This equation must exist separately for each power of ξ which means that the respective columns in the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ must be equal.

$$\boldsymbol{M}_j = \boldsymbol{N}_j. \tag{77}$$

Eq. (77) is a recursive equation which relates the coefficients $G_{0_{i,n}}$ $(n \geqq 3)$ with the coefficients of lower index. For example for $j = 1$

$$\boldsymbol{M}_1 = \boldsymbol{N}_1 \tag{78}$$

is a system of m equations which relates the coefficients $G_{0_{i,3}}$, $G_{0_{i,2}}$ and $G_{0_{i,1}}$ assuming that the last two are known. Solving this system the coefficients $G_{0_{i,4}}$ may now be calculated from the system

$$\boldsymbol{M}_2 = \boldsymbol{N}_2. \tag{79}$$

The procedure is continued until the coefficients $G_{i,n}$ are small enough to be neglected, remembering that $0 \leqq |\xi| \leqq 1$.

Since the constants $G_{0_{i,1}}$ and $G_{0_{i,2}}$ are not known the recursive equation must be solved for all homogeneous cases ($\boldsymbol{N} = \boldsymbol{0}$) assuming one of the constants equal to 1 while all other coefficients equal zero. Let us denote with $\boldsymbol{H}^{i1}$ the matrix $\boldsymbol{G}_0$ which is obtained by assuming $G_{0_{i,1}} = 1$ and all other constants with index smaller than 3 equals zero, and with $\boldsymbol{H}^{i2}$ the matrix $\boldsymbol{G}_0$ obtained by assuming $G_{0_{i,2}} = 1$ and all other constants with index smaller than 3 equals zero. Similarly the coefficients obtained in the same way for the first derivative of $\boldsymbol{d}$ with respect to x_1 will be denoted by $\boldsymbol{R}^{i1}$ respective $\boldsymbol{R}^{i2}$

$$H_{i,j}^{k\,1} = G_{0_{i,j}}, \tag{80}$$

$$R_{i,j}^{k\,1} = \frac{j}{x_{1\,max}} G_{0_{i,j+1}} \tag{81}$$

with $G_{0_{k,1}} = 1$, $G_{0_{k,2}} = G_{0_{i,2}} = G_{0_{i,1}} = 0$ and $1 \leqq i < k$ and $k < i \leqq m$.

$$H_{i,j}^{k\,2} = G_{0_{i,j}}, \tag{82}$$

$$R_{i,j}^{k\,2} = \frac{j}{x_{1\,max}} G_{0_{i,j+1}} \tag{83}$$

with $G_{0_{k,2}} = 1$, $G_{0_{k,1}} = G_{0_{i,1}} = G_{0_{i,2}} = 0$ and $1 \leqq i < k$ and $k < i \leqq m$.

A particular solution will be obtained assuming all the constants $G_{0_{i,1}}$ and $G_{0_{i,2}}$ vanish and solving the nonhomogeneous Eqs. (77). Denoting the corresponding matrix $\boldsymbol{G}_0$ by $\boldsymbol{H}^0$ and the coefficients of the first derivative of $\boldsymbol{d}$ by $\boldsymbol{R}^0$ the unknown vector $\boldsymbol{d}$ and its first derivative may be written in the form

$$\boldsymbol{d} = \sum_{i=1}^{m} (G_{0_{i,1}} \boldsymbol{H}^{i1} + G_{0_{i,2}} \boldsymbol{H}^{i2}) \xi + \boldsymbol{H}^0 \xi,$$

$$\boldsymbol{d}' = \sum_{i=1}^{m} (G_{0_{i,1}} \boldsymbol{R}^{i1} + G_{0_{i,2}} \boldsymbol{R}^{i2}) \xi + \boldsymbol{R}^0 \xi.$$

For solution of the unknowm coefficients $G_{0_{i,1}}$ and $G_{0_{i,2}}$, $2m$ equations will be necessary. These equations are obtained for the boundary conditions at two sections $x_1 = \text{const}$. The boundary conditions may be geometric or mechanical and may be different for each ridge. The equations are obtained by substituting the values of $x_1 = \text{const}$ ($\xi = \text{const}$) in the corresponding row of $\boldsymbol{d}$ in Eq. (84) or in Eq. (28), using both Eqs. (84) and (85). It is obvious that in order to avoid rigid body motions a minimum number of geometric boundary consitions will be necessary. Knowing the coefficients $G_{0_{i,1}}$ and $G_{0_{i,2}}$ the vector d is known. The displacement vector $\boldsymbol{q}$ may then be calculated from Eq. (16) and the displacement from Eq. (4) and the strains from Eq. (5). Finally the stresses will be calculated from $\boldsymbol{\sigma} = \boldsymbol{E}\,\boldsymbol{\varepsilon}$.

An important class of problems leads to a system of differential equations with constant coefficients.

A particular and very frequent case is the one where boundary conditions enable the use of orthogonal functions as Fourier series.

The solution by differential equations may also be very advantageous in the case of specific problems where no boundary conditions have to be satisfied in x_1 direction as in the case of axisymmetric problems.

In conclusion, the present approach may be applied successfully and economically for a broad class of practical problems encountered especially in civil engineering.

Appendix

$$
C_{00} = \frac{E\,t^3}{12\,(1-\mu^2)} \begin{bmatrix} \frac{12}{l^3} & -\frac{6}{l^2} & -\frac{12}{l^3} & -\frac{6}{l^2} \\ & \frac{4}{l} & \frac{6}{l^2} & \frac{2}{l} \\ & & \frac{12}{l^3} & \frac{6}{l^2} \\ & \text{Sym.} & & \frac{4}{l} \end{bmatrix},
$$

$$
C_{11} = \frac{E\,t^3}{24\,(1+\mu)} \begin{bmatrix} \frac{24}{5\,l} & -\frac{2}{5} & -\frac{24}{5\,l} & -\frac{2}{5} \\ & \frac{8\,l}{15} & \frac{2}{5} & -\frac{2\,l}{15} \\ & & \frac{24}{5\,l} & \frac{2}{5} \\ & \text{Sym.} & & \frac{8\,l}{15} \end{bmatrix},
$$

$$
C_{02} = C_{20}^T = \frac{\mu\,E\,t^3}{12\,(1-\mu^2)} \begin{bmatrix} -\frac{6}{5\,l} & \frac{1}{10} & \frac{6}{5\,l} & \frac{1}{10} \\ \frac{11}{10} & -\frac{2\,l}{15} & -\frac{1}{10} & \frac{l}{30} \\ \frac{6}{5\,l} & -\frac{1}{10} & -\frac{6}{5\,l} & -\frac{1}{10} \\ \frac{1}{10} & \frac{l}{30} & -\frac{11}{10} & -\frac{2\,l}{15} \end{bmatrix},
$$

$$
C_{22} = \frac{E\,t^3}{12\,(1-\mu^2)} \begin{bmatrix} \frac{13\,l}{35} & -\frac{11\,l^2}{210} & \frac{9\,l}{70} & \frac{13\,l}{420} \\ & \frac{l^3}{105} & -\frac{13\,l^2}{420} & -\frac{l^3}{140} \\ & & \frac{13\,l}{35} & \frac{11\,l^2}{210} \\ & \text{Sym.} & & \frac{l^3}{105} \end{bmatrix}.
$$

References

1. CHEUNG, Y. K.: Finite strip method of analysis of elastic slabs. Proc. Am. Soc. Civ. Eng., *94*, EM6, 1365–1378, 1968.
2. ZIENKIEWICZ, O. C.: The Finite Element Method in Engineering Science. McGraw-Hill, 1971.
3. WASHIZU, K.: Variational Methods in Elasticity and Plasticity. Pergamon Press, 1968.

Summary

The present paper is a more general formulation of the finite strip method for solid continua, and discusses the advantages compared with finite element method. The formulation is done for a three-dimensional body using the variational approach based on the principle of minimum total potential energy. As an application the particular case where the three-dimensional problem is reduced to an one-dimensional one is presented in more detail, leading to a system of linear differential equations with variable coefficients for which an infinite series solution is proposed. Small displacements and linear elasticity are assumed. The one-dimensional case is exemplified for a plate in bending by neglecting shear force energy.

Résumé

Ce travail présente un énoncé plus général de la méthode des bandes finies pour les continus solides, et discute les avantages comparés à la méthode des éléments finis. Cet énoncé est étudié pour un corps à trois dimensions en utilisant le calcul des variations basé sur le principe du minimum de l'énergie potentielle totale. Comme application on présente en détail le cas particulier où le problème à trois dimensions est réduit à un problème unidimensionnel, conduisant à un système d'équations différentielles linéaires avec coefficients variables pour lequel une solution aux séries infinies est proposée. On admet de petits déplacements et une élasticité linéaire. On montre comme exemple le cas unidimensionnel d'une plaque soumise à la flexion en négligeant l'énergie des efforts tranchants.

Zusammenfassung

Der vorliegende Artikel ist eine allgemeinere Formulierung der Finite-Streifen-Methode für Festkörperkontinua und bespricht deren Vorteile, verglichen mit der Finite-Elemente-Methode. Die Formulierung erfolgt anhand eines dreidimensionalen Körpers unter Anwendung der Variationsnäherung, die auf dem Prinzip der minimalen potentiellen Energie beruht. Als Anwendung wird ein spezieller Fall näher behandelt, bei dem das dreidimensionale Problem auf ein eindimensionales reduziert wird; es führt auf ein System linearer Differentialgleichungen mit variablen Koeffizienten, für welches eine Lösung mit infiniten Reihen vorgeschlagen wird. Kleine Deformationen und lineare Elastizität werden vorausgesetzt. Der eindimensionale Fall wir anhand einer Biegeplatte erläutert, wobei die Querkraftenergie vernachlässigt wird.