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Lateral Buckling of Tapered Beams

Flambage latéral de poutres à section réduite

Kippen von sich verjüngenden Balken

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1. List of Symbols

b_f	flange breadth
d	depth of beam
E	Young's modulus
G	shear modulus
I_y	minor second moment of area
J	torsion constant
M_{cr}	critical moment
r	reduction in stability due to taper
$R^2 = \frac{L^2 G J}{E \Gamma}$	slenderness parameter
t_f	flange thickness
t_w	web thickness
Z	section modulus
$\alpha = \frac{Z_0}{Z_1}\left[\left(\frac{d_1}{d_0}\right)^3 \left(\frac{b_{f0}}{b_{f1}}\right)^3 \left(\frac{t_{f0}}{t_{f1}}\right)^2\right]^{1/2}$	taper parameter
Γ	warping constant
γ_e	lateral buckling coefficient
η	$\frac{J \text{ for whole section}}{J \text{ for flanges only}}$
ν	ratio of minimum to maximum value of tapered dimension(s)
σ	bending stress in beam
σ_m	maximum bending stress in beam

2. Introduction

In cases where beams are required to support a given system of loads which produces a varying distribution of moment within the span a more economical use of material may be obtained if a tapered member is employed, curtailment of the section to suit the bending moment distribution producing a more favourable pattern of stress. Tapered members may also be preferred in certain circumstances for aesthetic reasons. However, the reduction in section properties may also render the member more susceptible to failure by lateral buckling. Therefore, before such sections can safely be employed, a check on their lateral stability is necessary. Although adequate lateral support may be available in the completed structure, during erection the beam may be required to support considerable loads when little or even no bracing is provided.

A review of several methods of calculating stresses in tapered beams has been given by O'CONNER [1], whilst VICKERY [2] has presented an extension of the simple plastic theory to cover portal frames fabricated from tapered members. In his tests, VICKERY observed that failure of the frames was often precipitated by flexural-torsional buckling.

An analysis of the lateral buckling of a narrow rectangular section with linearly tapering depth loaded by a uniform moment has been presented by LEE [3]. For a centrally loaded beam, MASSEY [4] has given a solution for a section with linearly varying lateral flexural and torsional rigidities. Both of these quantities were allowed to vary independently but the effects of warping were not included. In their paper outlining the background to the revised B.S. 153, KERENSKY, FLINT and BROWN [5] have given some consideration to the buckling of members with curtailed flanges, curtailment of either flange breadth or flange thickness being considered. However, the solutions given were only approximate and further study of the subject was suggested.

Experiments on tapered I and channel sections tested as cantilevers supporting either a tip load or a uniform load have been reported by KREFIELD, BUTLER and ANDERSON [6]. Their test program considered both depth taper and flange breadth taper and from the results the parameter $\alpha = \frac{Z_0}{Z_1}\left(\frac{b_{f0}\,d_1}{b_{f1}\,d_0}\right)^{3/2}$, where Z = section modulus, b_f = flange breadth, d = depth and suffices 0 and 1 relate to root and tip respectively, was suggested as a governing para-

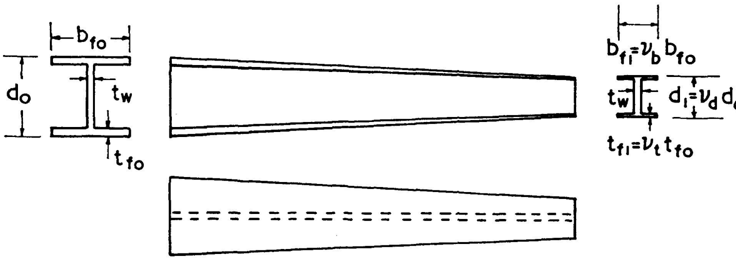


Fig. 1. Tapered I-section.

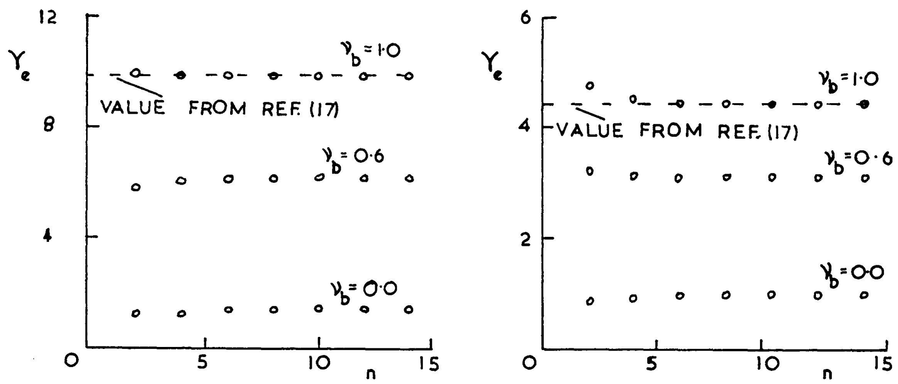
meter for the reduced lateral stability of tapered cantilevers. Extensions of this work to cover beam columns [7] and the lateral bracing of tapered cantilevers [8] have also been reported.

This paper presented the results of a theoretical study of the lateral buckling of tapered beams. Linear curtailment of either flange thickness, flange breadth, web depth or all three quantities as illustrated in Fig. 1 is considered and reduction formulae for use with existing methods of determining the stability of uniform members proposed.

3. Method of Solution

The results presented in this paper have all been obtained from a Finite Element analysis of lateral buckling. Details of the Finite Element formulation of stability problems have been given by a number of authors [9–12] and will not be repeated here. In the present work the element first proposed for the analysis of torsional-flexural stability problems by BARSOUM and GALLAGHER [13] and subsequently used extensively by the writer [14–16] has been employed. Earlier papers [13–16] have demonstrated the high degree of accuracy that may be obtained with the use of only a small number of these elements.

However, all of these previous studies related to uniform members and it was considered prudent that the present study should begin with an examination of the method's convergence characteristics when applied to the buckling of non uniform members. Since computing time increases rapidly with an increase in the number of elements employed, it was necessary to determine the minimum number of elements consistent with reasonable accuracy. Fig. 2 shows two series of results for a cantilever with a single load at the tip. Two extremes of length have been included together with three values of flange breadth taper ν_b. The results are presented in a form which relates γ_e, where γ_e is defined in Eq. (1), to n, the number of elements.



a) Warping important $R^2 = 4$ b) Warping unimportant $R^2 = 550$

Fig. 2. Convergence of finite element results for the buckling of tapered end loaded cantilevers.

$$M_{cr} = \frac{\gamma_e}{L}(E\,I_y\,G\,J)^{1/2}, \tag{1}$$

where M_{cr} = maximum moment in the beam i.e. moment at the root,
$E\,I_y$ = minor flexural rigidity,
$G\,J$ = torsional rigidity,
L = span.

For the purposes of the analysis the tapered member was replaced by a stepped member, the dimensions and therefore the geometrical properties of each element being those of its mid point.

Exact results are only available for the two cases involving a uniform beam [17] and for each of these Fig. 2 shows that the method rapidly converges. For the non uniform beams convergence is almost as rapid, the use of eight elements always giving answers that are accurate to at least one per cent. Several other examples including different types and levels of loading and different forms of taper have also been studied and in all cases convergence of the results was similar to that illustrated in Fig. 2. Thus it was decided that the tapered beam could adequately be represented by eight elements.

For depth tapered members additional terms appear in the torsional equilibrium equation due to the inclination of the flanges (18). The finite element model, since it replaces the actual member by an elastic line possessing the appropriate flexural, warping and torsional stiffness, neglects the effect of

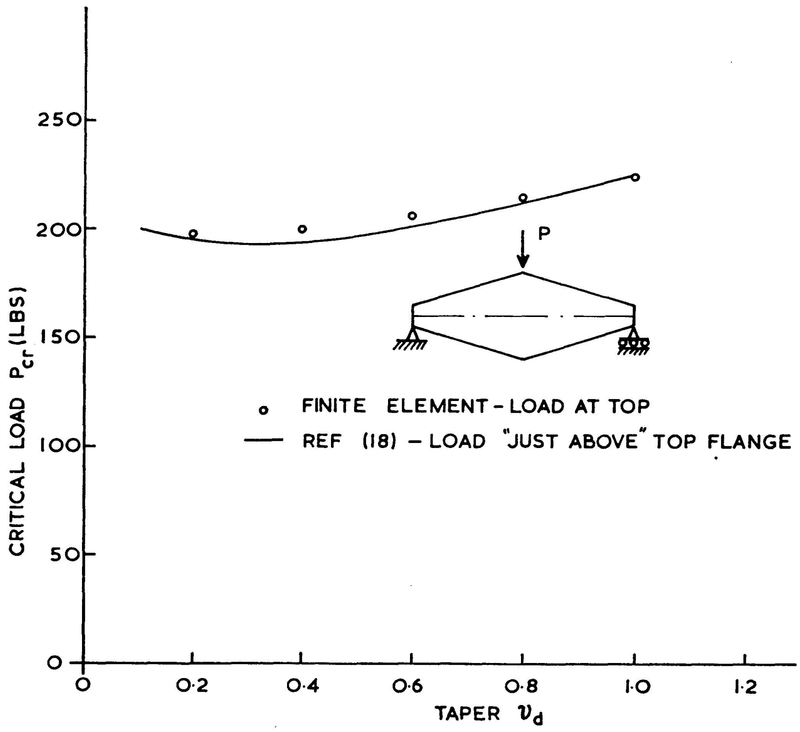


Fig. 3. Comparison of results for depth tapered section.

these additional terms and an additional check upon its ability to represent depth tapered members was therefore conducted.

Fig. 3 compares the author's results with a set obtained from a numerical solution of the complete differential equations by KITIPORNCHAI and TRAHAIR [18]. Eight elements were used to represent one half of the beam, and since part of the discrepancy between the two solutions is due to the fact that the load was applied to the top flange whereas ref. [18] specifies a load acting "just above" the top flange agreement seems perfectly acceptable.

The computer programs employed were such that only the beam's dimensions, material properties, loading and support conditions and type and degree of taper needed to be input. The average dimensions and hence the geometrical properties of each element were calculated automatically within the program which then proceeded to calculate the critical load and the associated buckling mode.

4. Lateral Buckling of Tapered Cantilevers

A cantilever loaded only at its free end presents a good example of a member subjected to a rapidly varying bending moment distribution. Fig. 4 contrasts the distribution of bending stress in a cantilever of uniform section with that in several types of tapered beams (for the example shown flange breadth curtailment produces an almost identical stress distribution to flange thickness curtailment whilst tapering breadth, flange thickness and depth gives results very similar to those for depth taper alone). Since the efficiency of the section is proportional to the area under the curve, Fig. 4 clearly shows the tapered sections to be more economic in their use of material. It is also of interest to

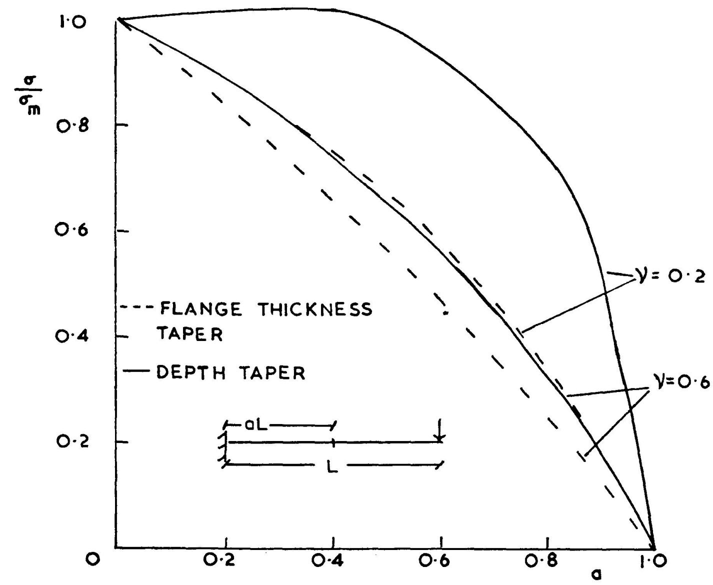


Fig. 4. Variation of stress with length for tapered cantilever (12 × 4 × 16.5 UB section).

Table 1a. Buckling Parameters for tapered cantilevers – end load, flange breadth taper

	Values of γ_e										
Value of ν	1.0	0.8		0.6		0.4		0.2		0.0	
Value of η	—	1.92	1.15	1.92	1.15	1.92	1.15	1.92	1.15	1.92	1.15
Value of R^2		*Top flange loading*									
550	4.18	2.58	3.55	2.97	2.90	2.34	2.24	1.65	1.50	0.96	0.57
310	4.22	3.61	3.58	2.99	2.91	2.35	2.24	1.65	1.50	0.96	0.57
140	4.27	3.65	3.60	3.01	2.93	2.36	2.24	1.66	1.50	0.97	0.56
96	4.29	3.66	3.62	3.03	2.94	2.37	2.24	1.67	1.50	0.97	0.56
62	4.28	3.67	3.61	3.04	2.93	2.38	2.23	1.68	1.49	0.98	0.56
35	4.21	3.65	3.56	3.04	2.90	2.40	2.21	1.69	1.48	0.99	0.56
15	3.84	3.38	3.29	2.88	2.72	2.33	2.12	1.68	1.44	1.03	0.57
4	3.42	3.04	2.95	2.64	2.46	2.21	1.95	1.68	1.39	1.07	0.58
		Shear centre loading									
550	4.40	3.78	3.72	3.11	3.04	2.44	2.34	1.70	1.56	0.98	0.58
310	4.54	3.88	3.84	3.18	3.11	2.48	2.38	1.72	1.58	0.93	0.58
140	4.84	4.09	4.08	3.34	3.26	2.58	2.48	1.78	1.63	1.01	0.59
96	5.03	4.24	4.21	3.46	3.38	2.64	2.53	1.81	1.66	1.03	0.60
62	5.32	4.45	4.44	3.59	3.50	2.72	2.64	1.86	1.70	1.05	0.61
35	5.82	4.82	4.84	3.87	3.76	2.89	2.79	1.95	1.78	1.08	0.62
15	6.93	5.66	5.61	4.48	4.35	3.33	3.15	2.20	1.95	1.19	0.65
4	9.78	7.89	7.80	6.10	5.95	4.39	4.18	2.76	2.49	1.43	0.76
		Bottom flange Loading									
550	4.60	3.92	3.89	3.23	3.18	2.51	2.43	1.75	1.62	1.00	0.59
310	4.81	4.09	4.05	3.35	3.30	2.59	2.51	1.79	1.66	1.02	0.60
140	5.27	4.42	4.39	3.60	3.52	2.76	2.66	1.89	1.74	1.05	0.62
96	5.57	4.65	4.62	3.76	3.68	2.86	2.76	1.95	1.80	1.08	0.63
62	6.04	5.01	4.97	4.01	3.91	3.02	2.91	2.03	1.88	1.12	0.65
35	6.85	5.60	5.58	4.43	4.38	3.29	3.18	2.17	2.01	1.17	0.68
15	8.90	7.09	7.06	5.51	5.42	3.99	3.85	2.56	2.34	1.34	0.74
4	14.91	11.67	11.60	8.70	8.58	6.02	5.91	3.60	3.32	1.72	0.91

note that linear curtailment of either flange breadth or flange thickness produces an almost linear decrease in major second moment of area with distance along the beam. The cantilever is clearly one type of beam where tapered sections may efficiently be employed and for this reason it has been used as the basis for the present study. Later sections of the paper will show how the

Table 1b. Buckling Parameters for tapered cantilevers – end load, flange thickness taper

		Values of γ_e									
Value of ν	1.0	0.8		0.6		0.4		0.2		0.0	
Value of η	—	1.92	1.15	1.92	1.15	1.92	1.15	1.92	1.15	1.92	1.15
Value of R^2		*Top flange loading*									
550	4.18	3.85	3.74	3.52	3.28	3.18	2.79	2.83	2.28	2.38	1.73
310	4.22	3.87	3.74	3.53	3.24	3.18	2.72	2.82	2.20	2.37	1.66
140	4.27	3.89	3.72	3.53	3.15	3.16	2.56	2.80	2.02	2.35	1.54
96	4.29	3.89	3.67	3.51	3.06	3.14	2.44	2.78	1.78	2.33	1.46
62	4.28	3.86	3.59	3.47	2.92	3.09	2.28	2.73	1.77	2.30	1.37
35	4.21	3.75	3.43	3.33	2.69	2.97	2.07	2.64	1.62	2.25	1.27
15	3.84	3.36	3.04	3.01	2.35	2.64	1.81	2.39	1.43	2.10	1.16
4	3.42	3.01	2.78	2.68	2.27	2.42	1.91	2.21	1.61	1.99	1.35
		Shear centre loading									
550	4.40	4.07	3.98	3.73	3.54	3.38	3.07	3.00	2.57	2.43	1.97
310	4.54	4.18	4.10	3.83	3.63	3.46	3.14	3.06	2.62	2.47	2.00
140	4.84	4.43	4.35	4.03	3.83	3.63	3.29	3.20	2.73	2.57	2.08
96	5.03	4.58	4.53	1.17	3.96	3.75	3.40	3.30	2.82	2.64	2.14
62	5.32	4.84	4.76	4.39	4.17	3.94	3.57	3.47	2.95	2.74	2.25
35	5.82	5.28	5.19	4.78	4.53	4.27	3.85	3.74	3.18	2.93	2.39
15	6.93	6.30	6.13	5.67	5.34	5.04	4.52	4.36	3.37	3.38	2.82
4	9.78	8.86	8.66	7.98	7.56	7.09	6.37	6.15	5.21	4.68	3.96
		Bottom flange loading									
550	4.60	4.25	4.17	3.90	3.74	3.54	3.27	3.15	2.77	2.65	2.15
310	4.81	4.45	4.36	4.07	3.90	3.68	3.41	3.27	2.88	2.74	2.25
140	5.27	4.86	4.77	4.44	4.24	4.00	3.70	3.53	3.12	2.94	2.45
96	5.57	5.15	5.03	4.69	4.52	4.21	3.90	3.71	3.31	3.07	2.59
62	6.04	5.57	5.55	5.06	4.85	4.54	4.23	3.98	3.60	3.28	2.81
35	6.85	6.28	6.29	5.71	5.52	5.11	4.86	4.47	4.08	3.66	3.20
15	8.90	8.11	8.07	7.31	7.24	6.53	6.36	5.66	5.42	4.56	4.20
4	14.91	13.70	13.67	12.41	12.36	11.05	11.01	9.52	9.41	7.52	7.35

results obtained for cantilevers may be used to estimate the stability of tapered beams subjected to other forms of loading and provided with other types of support.

Numerical values of the coefficient γ_e as defined in Eq. (1) are given in Tables 1 and 2 respectively for tapered cantilevers subjected to an end load

Table 1c. Buckling Parameters for tapered cantilevers – end load, depth taper

		Values of γ_e									
Value of ν	1.0	0.8		0.6		0.4		0.2		0.0	
Value of η	—	1.92	1.15	1.92	1.15	1.92	1.15	1.92	1.15	1.92	1.15
Value of R^2		*Top flange loading*									
550	4.18	4.17	4.20	4.16	4.23	4.15	4.25	4.13	4.28	4.12	4.30
310	4.22	4.22	4.25	4.22	4.30	4.22	4.34	4.22	4.38	4.22	4.41
140	4.27	4.31	4.35	4.34	4.43	4.37	4.51	4.40	4.58	4.42	4.65
96	4.29	4.35	4.39	4.41	4.50	4.46	4.60	4.51	4.70	4.55	4.79
62	4.28	4.39	4.45	4.49	4.61	4.58	4.76	4.67	4.89	4.75	5.01
35	4.21	4.40	4.40	4.58	4.68	4.76	4.94	4.93	5.20	5.08	5.42
15	3.84	4.19	4.24	4.40	4.92	4.99	5.22	5.47	5.64	5.85	6.11
4	3.42	3.78	3.86	4.30	4.53	5.08	5.35	6.26	6.70	7.73	8.12
		Shear centre loading									
550	4.40	4.35	4.39	4.30	4.37	4.24	4.35	4.19	4.34	4.14	4.32
310	4.54	4.48	4.53	4.42	4.50	4.36	4.48	4.30	4.46	4.24	4.43
140	4.84	4.75	4.82	4.68	4.78	4.60	4.74	4.53	4.71	4.44	4.68
96	5.03	4.93	5.00	4.84	4.96	4.76	4.91	4.68	4.87	4.59	4.84
62	5.32	5.20	5.29	5.10	5.23	5.00	5.17	4.90	5.12	4.78	5.07
35	5.82	5.67	5.76	5.54	5.69	5.42	5.60	5.29	5.51	5.14	5.41
15	6.93	6.74	6.79	6.54	6.67	6.36	6.53	6.16	6.39	5.90	6.25
4	9.78	9.52	9.61	9.24	9.39	8.95	9.12	8.62	8.88	8.32	8.64
		Bottom flange loading									
550	4.60	4.51	4.56	4.42	4.50	4.33	4.45	4.24	4.39	4.15	4.33
310	4.81	4.69	4.75	4.59	4.67	4.48	4.60	4.33	4.53	4.26	4.45
140	5.27	5.10	5.18	4.95	5.05	4.90	4.93	4.67	4.83	4.48	4.71
96	5.57	5.36	5.44	5.19	5.31	5.01	5.15	4.82	5.02	4.81	4.87
62	6.04	5.80	5.93	5.56	5.75	5.33	5.52	5.10	5.32	4.85	5.12
35	6.85	6.52	6.78	6.22	6.50	5.91	6.19	5.58	5.90	5.24	5.58
15	8.90	8.37	8.41	7.68	8.13	7.34	7.44	6.76	6.93	6.16	6.40
4	14.91	13.83	14.01	12.71	12.81	11.17	11.36	10.18	10.32	8.67	8.98

or a uniform load. Three different levels of loading have been considered as recent work [19] has shown that for short cantilevers the effect of altering the level of application of the loading is considerable. The results cover a series of values of the torsional parameter $R^2 = \frac{L^2 G J}{\sum E \Gamma}$ where $E\Gamma$ is the warping rigidity, and are listed for five different values of each of the taper parameters ν.

Table 1d. Buckling Parameters for tapered cantilevers – end load, flange breadth, flange thickness and web depth taper

		Values of γ_e									
Value of ν	1.0	0.8		0.6		0.4		0.2		0.0	
Value of η	—	1.92	1.15	1.92	1.15	1.92	1.15	1.92	1.15	1.92	1.15
Value of R^2		*Top flange loading*									
550	4.18	3.27	3.17	2.41	2.22	1.59	1.35	0.82	0.58	0.24	0.29
310	4.22	3.30	3.18	2.42	2.22	1.59	1.34	0.83	0.58	0.24	0.29
140	4.27	3.34	3.20	2.45	2.22	1.61	1.34	0.84	0.58	0.25	0.29
96	4.29	3.35	3.20	2.46	2.22	1.62	1.34	0.84	0.58	0.25	0.29
62	4.28	3.36	3.18	2.48	2.20	1.64	1.33	0.85	0.58	0.25	0.30
35	4.21	3.35	3.11	2.50	2.17	1.67	1.32	0.86	0.59	0.25	0.30
15	3.84	3.12	2.89	2.48	2.05	1.74	1.29	0.92	0.60	0.26	0.30
4	3.42	2.91	2.67	2.40	1.95	1.80	1.27	1.01	0.64	0.28	0.32
		Shear centre loading									
550	4.40	3.41	3.33	2.48	2.32	1.62	1.39	0.83	0.59	0.24	0.07
310	4.54	3.49	3.41	2.53	2.36	1.64	1.41	0.84	0.59	0.24	0.07
140	4.84	3.65	3.58	2.62	2.44	1.68	1.44	0.85	0.60	0.24	0.07
96	5.03	3.79	3.70	2.68	2.50	1.71	1.46	0.86	0.61	0.24	0.07
62	5.32	3.96	3.87	2.79	2.58	1.75	1.50	0.87	0.62	0.25	0.07
35	5.82	4.26	4.16	2.96	2.77	1.83	1.58	0.90	0.64	0.25	0.07
15	6.93	5.05	4.85	3.41	3.13	2.10	1.74	0.99	0.68	0.26	0.07
4	9.78	6.92	6.71	4.56	4.11	2.63	2.22	1.16	0.80	0.28	0.08
		Bottom flange loading									
550	4.60	3.53	3.46	2.55	2.40	1.65	1.43	0.84	0.60	0.24	0.29
310	4.81	3.66	3.59	2.62	2.47	1.68	1.46	0.85	0.61	0.24	0.29
140	5.27	3.93	3.86	2.77	2.60	1.75	1.52	0.87	0.62	0.25	0.29
96	5.57	4.12	4.04	2.86	2.70	1.79	1.56	0.88	0.63	0.25	0.30
62	6.04	4.42	4.32	2.91	2.84	1.85	1.62	0.90	0.65	0.25	0.30
35	6.85	4.87	4.91	3.28	3.16	1.97	1.74	0.93	0.68	0.25	0.30
15	8.90	6.31	6.04	4.11	3.77	2.36	2.01	1.06	0.75	0.26	0.31
4	14.91	9.88	9.86	6.01	5.82	3.18	2.87	1.28	0.92	0.29	0.34

KERENSKY et al. [5] have shown that for tapered beams a parameter additional to those that govern the lateral stability of uniform beams affects the magnitude of the reduction in stability due to taper. This parameter is the ratio of the torsional rigidity of the whole section to the torsional rigidity of the beam's flanges, η, each quantity relating to the maximum section. A study

Table 2a. Buckling Parameters for tapered cantilevers – uniform load, flange breadth taper

		Values of γ_e									
Value of ν	1.0	0.8		0.6		0.4		0.2		0.0	
Value of η	—	1.92	1.15	1.92	1.15	1.92	1.15	1.92	1.15	1.92	1.15
Value of R^2		*Top flange loading*									
550	6.84	6.14	6.09	5.43	5.33	4.68	4.55	3.87	3.69	2.91	2.52
310	6.91	6.20	6.31	5.47	5.36	4.71	4.56	3.88	3.68	2.92	2.50
140	7.01	6.29	6.26	5.53	5.40	4.76	4.57	3.91	3.68	2.93	2.48
96	7.04	6.31	6.21	5.55	5.40	4.78	4.57	3.93	3.67	2.94	2.47
62	7.07	6.33	6.21	5.58	5.39	4.81	4.55	3.95	3.65	2.95	2.45
35	6.98	6.27	6.17	5.62	5.34	4.82	4.51	4.02	3.62	2.97	2.42
15	6.71	6.17	5.96	5.51	5.19	4.79	4.39	3.98	3.53	3.01	2.37
4	6.59	6.02	5.80	5.42	5.04	4.71	4.30	4.02	3.47	3.11	2.36
		Shear centre loading									
550	7.47	6.67	6.63	5.87	5.78	5.03	4.90	4.13	3.96	3.08	2.68
310	7.82	6.96	6.92	6.09	6.00	5.20	5.06	4.25	4.06	3.14	2.72
140	8.57	7.56	7.51	6.56	6.47	5.55	5.39	4.48	4.27	3.28	2.82
96	9.94	7.94	7.90	6.83	6.73	5.76	5.63	4.62	4.41	3.39	2.87
62	9.77	8.51	8.47	7.33	7.21	6.11	5.81	4.85	4.64	3.48	2.97
35	10.94	9.47	9.41	8.06	8.01	6.66	6.37	5.25	5.00	3.68	3.12
15	13.64	11.64	11.61	9.83	9.65	7.98	7.74	6.15	5.81	4.23	3.46
4	20.58	17.48	17.34	14.46	14.21	11.51	11.23	8.55	8.12	5.52	4.37
		Bottom flange loading									
550	8.08	7.19	7.16	6.30	6.22	5.39	5.26	4.39	4.23	3.25	2.82
310	8.67	7.67	7.65	6.68	6.61	5.67	5.56	4.60	4.41	3.30	2.93
140	9.98	8.76	8.68	7.55	7.42	6.34	6.17	5.05	4.85	3.64	3.16
96	10.83	9.41	9.38	8.05	7.97	6.71	6.57	5.33	5.12	3.81	3.29
62	12.14	10.52	10.46	8.91	8.79	7.36	7.18	5.77	5.56	4.06	3.50
35	14.48	12.36	12.29	10.41	10.25	8.49	8.23	6.33	6.19	4.52	3.82
15	19.98	15.88	15.98	13.96	13.88	11.18	11.01	8.41	8.18	5.63	4.82
4	35.28	29.49	29.45	23.99	23.88	18.77	18.57	13.71	13.42	8.59	7.21

of the values of η for all beam and column I sections in the British handbook has been made from which extreme values of 1.103 and 1.917 have been obtained. The average value is about 1.35, low values usually indicating compact column type sections and high values deep beam type sections. Therefore Tables 1 and 2 contain results for two extreme values of η, 1.15 and 1.92. The

Table 2b. Buckling Parameters for tapered cantilevers – uniform load, flange thickness taper

		Values of γ_e									
Value of ν	1.0	0.8		0.6		0.4		0.2		0.0	
Value of η	—	1.92	1.15	1.92	1.15	1.92	1.15	1.92	1.15	1.92	1.15
Value of R^2		*Top flange loading*									
550	6.84	6.45	6.30	6.05	5.75	5.65	5.17	5.25	4.57	4.81	3.93
310	6.91	6.50	6.32	6.09	5.72	5.66	5.09	5.24	4.44	4.79	3.77
140	7.01	6.60	6.31	6.10	5.61	5.65	4.87	5.20	4.15	4.74	3.46
96	7.04	6.59	6.28	6.09	5.51	5.62	4.73	5.17	3.97	4.71	3.30
62	7.07	6.54	6.19	6.03	5.35	5.56	4.50	5.10	3.73	4.64	3.00
35	6.98	6.44	6.01	5.90	5.06	5.42	4.17	4.97	3.42	4.53	2.86
15	6.71	6.11	5.65	5.57	4.64	5.14	3.79	4.74	3.14	4.36	2.68
4	6.59	5.95	5.47	5.42	4.61	5.00	3.93	4.68	3.43	4.39	3.07
		Shear centre loading									
550	7.47	7.06	6.98	6.65	6.45	6.23	5.87	5.79	5.28	5.30	4.64
310	7.82	7.38	7.26	6.95	6.71	6.48	6.09	6.01	5.46	5.49	4.76
140	8.57	8.05	7.94	7.54	7.28	7.01	6.59	6.48	5.87	5.89	5.10
96	9.04	8.49	8.35	7.92	7.64	7.35	6.90	6.78	6.15	6.15	5.31
62	9.77	9.15	9.01	8.52	8.21	7.90	7.40	7.24	6.56	6.54	5.68
35	10.94	10.12	10.01	9.51	9.18	8.79	8.24	8.04	7.30	7.23	6.30
15	13.64	12.71	12.49	11.78	11.38	10.85	10.13	9.90	8.94	8.83	7.72
4	20.58	19.14	18.82	17.73	17.19	16.33	15.50	14.89	13.49	13.24	11.96
		Bottom flange loading									
550	8.08	7.65	7.57	7.23	7.02	6.77	6.43	6.30	5.85	5.76	5.20
310	8.67	8.20	8.12	7.72	7.50	7.23	6.90	6.71	6.26	6.14	5.58
140	9.98	9.41	9.32	8.84	8.63	8.26	7.93	7.63	7.20	6.95	6.41
96	10.83	10.21	10.16	9.57	9.42	8.91	8.62	8.23	7.80	7.48	6.95
62	12.14	11.40	11.38	10.70	10.54	9.95	9.62	9.19	8.77	8.33	7.81
35	14.48	13.61	13.52	12.57	12.72	11.64	11.51	10.89	10.48	9.83	9.36
15	19.98	18.71	18.68	17.36	17.35	16.14	15.99	14.82	14.56	13.16	12.98
4	35.28	33.18	33.08	31.01	30.98	28.74	28.67	26.29	26.05	23.55	23.36

results presented in this paper are not, of course, limited merely to these sections listed in the handbook; the survey was conducted merely to obtain an indication of the range of values of η likely to be encountered in practice.

In general, the reductions in stability caused by taper are larger for sections possessing low values of η than for sections for which η is large, the differences

Table 2c. Buckling Parameters for tapered cantilevers – uniform load, depth taper

		Values of γ_e									
Value of ν	1.0	0.8		0.6		0.4		0.2		0.0	
Value of η	—	1.92	1.15	1.92	1.15	1.92	1.15	1.92	1.15	1.92	1.15
Value of R^2		*Top flange loading*									
550	6.84	6.85	6.88	6.85	6.93	6.86	6.98	6.87	7.03	6.88	7.08
310	6.91	6.93	6.98	6.96	7.06	7.00	7.14	7.02	7.22	7.05	7.29
140	7.01	7.10	7.14	7.18	7.29	7.26	7.44	7.34	7.59	7.43	7.72
96	7.04	7.17	7.20	7.29	7.41	7.41	7.61	7.54	7.79	7.83	7.98
62	7.07	7.24	7.28	7.43	7.55	7.62	7.83	7.82	8.10	8.01	8.36
35	6.98	7.28	7.32	7.58	7.74	7.90	8.14	8.24	8.56	8.58	8.98
15	6.71	7.23	7.26	7.88	7.89	8.41	8.59	9.14	9.34	9.77	10.12
4	6.59	7.18	7.21	7.99	8.13	9.08	9.41	10.58	11.08	12.51	13.13
		Shear centre loading									
550	7.47	7.39	7.45	7.32	7.41	7.26	7.38	7.19	7.34	7.10	7.31
310	7.82	7.71	7.78	7.63	7.74	7.57	7.70	7.47	7.65	7.38	7.61
140	8.57	8.45	8.52	8.32	8.45	8.21	8.37	8.10	8.30	7.98	8.23
96	9.04	8.90	8.96	8.76	8.86	8.62	8.78	8.48	8.70	8.27	8.62
62	9.77	9.54	9.67	9.37	7.55	9.21	9.44	9.05	9.33	8.89	9.22
35	10.94	10.71	10.86	10.49	10.70	10.28	10.55	10.06	10.39	9.84	10.25
15	13.64	13.36	13.38	13.16	13.17	12.68	12.86	12.34	12.60	11.98	12.33
4	20.58	20.09	20.18	19.58	19.64	19.04	19.39	18.46	18.62	17.81	17.98
		Bottom flange loading									
550	8.08	7.88	8.00	7.77	7.88	7.63	7.78	7.48	7.65	7.35	7.55
310	8.67	8.40	8.54	8.23	8.39	8.05	8.24	7.87	8.08	7.68	7.93
140	9.98	9.52	9.76	9.30	9.51	9.03	9.29	8.75	9.03	8.47	8.77
96	10.83	10.30	10.54	9.96	10.26	9.62	9.92	9.33	9.61	8.98	9.30
62	12.14	11.65	11.78	11.23	11.38	10.81	11.00	10.31	10.52	9.82	10.02
35	14.48	13.83	14.02	13.21	13.42	12.55	12.76	11.92	12.13	11.21	11.45
15	19.98	18.88	18.99	17.98	18.00	16.81	16.96	15.68	15.86	14.49	14.69
4	35.28	33.24	33.44	31.31	31.37	28.88	28.93	26.44	26.64	23.63	23.74

becoming more noticeable for highly tapered members. The exception to this is, as might be expected, the case of depth taper for which the reverse is true. When all three dimensions (b_f, t_f, and d) are tapered the effect of flange curtailment predominates.

Table 2d. Buckling Parameters for tapered cantilevers – uniform load, flange thickness, flange breadth and web depth taper

	Values of γ_e										
Value of ν	1.0	0.8		0.6		0.4		0.2		0.0	
Value of η	—	1.92	1.15	1.92	1.15	1.92	1.15	1.92	1.15	1.92	1.15
Value of R^2		*Top flange loading*									
550	6.84	5.77	5.64	4.73	4.48	3.68	3.34	2.59	2.13	1.31	1.48
310	6.91	5.82	5.67	4.76	4.49	3.70	3.33	2.60	2.13	1.31	1.48
140	7.01	5.91	5.71	4.84	4.50	3.76	3.33	2.63	2.12	1.32	1.49
96	7.04	5.94	5.72	4.86	4.51	3.77	3.32	2.65	2.11	1.33	1.50
62	7.07	5.97	5.71	4.90	4.49	3.84	3.32	2.69	2.12	1.34	1.51
35	6.98	5.98	5.66	4.96	4.45	3.91	3.31	2.75	2.12	1.37	1.54
15	6.71	5.92	5.47	5.02	4.25	4.02	3.29	2.92	2.16	1.42	1.60
4	6.59	5.88	5.46	5.20	4.40	4.42	3.44	3.30	2.36	1.60	1.68
		Shear centre loading									
550	7.47	6.22	6.12	5.03	4.81	3.87	3.54	2.68	2.23	1.33	0.60
310	7.82	6.47	6.35	5.19	4.95	3.96	3.61	2.73	2.25	1.33	0.60
140	8.57	6.99	6.86	5.53	5.27	4.17	3.77	2.82	2.32	1.36	0.60
96	9.04	7.32	7.18	5.76	5.46	4.31	3.89	2.89	2.36	1.37	0.60
62	9.77	7.85	7.69	6.15	5.79	4.54	4.04	2.99	2.44	1.40	0.60
35	10.94	8.66	8.48	6.65	6.31	6.83	4.37	3.13	2.56	1.44	0.61
15	13.64	10.65	10.41	8.01	7.60	5.75	5.14	3.62	2.91	1.57	0.64
4	20.58	15.82	15.46	11.64	11.02	7.96	7.18	4.67	3.72	1.79	0.71
		Bottom flange loading									
550	8.08	6.66	6.56	5.34	5.12	4.05	3.75	2.77	2.33	1.35	1.50
310	8.67	7.09	6.97	5.62	5.40	4.23	3.89	2.86	2.40	1.36	1.52
140	9.98	7.98	7.87	6.20	5.96	4.57	4.20	3.03	2.53	1.40	1.55
96	10.83	8.59	8.46	6.62	6.32	4.79	4.41	3.13	2.60	1.42	1.57
62	12.14	9.50	9.38	7.22	6.88	5.19	4.73	3.31	2.73	1.45	1.60
35	14.48	11.17	10.96	8.31	7.96	5.77	5.33	3.57	2.99	1.51	1.65
15	19.98	15.02	14.93	10.98	10.60	7.29	6.93	4.32	3.73	1.70	1.75
4	35.28	25.44	25.23	18.21	17.48	11.68	10.82	6.27	5.36	2.00	2.01

Fig. 5, which shows a plot of the reduction in stability r against the degree of taper ν for end load and flange breadth taper, shows that the effect of the parameter η is, however, less than that of certain other factors, notably the value of R^2 (a measure of the importance of warping) and the level of applica-

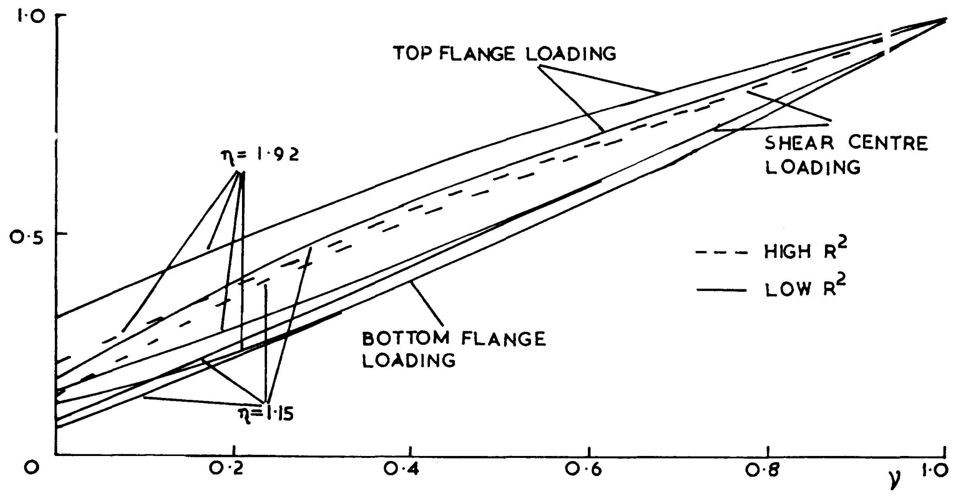


Fig. 5. Reductions in stability caused by linear flange breadth curtailment.

tion of the loading. Since the torsion constants J and the warping constant Γ, where J and Γ are defined in Eqs. (2), will be affected to different degrees by the variation in geometry, variations in r with R^2 are to be expected.

$$J = \tfrac{1}{3}[2\,b_f\,t_f^3 + (d-t_f)\,t_w^3],$$
$$\Gamma = \frac{t_f\,(d-t_f)^2\,b_f^3}{24}. \tag{2}$$

For all cases considered in this section of the paper, loading on one or other flange was found to produce larger reductions in stability than shear centre loading, top flange loading usually being the critical case although for flange breadth taper, bottom flange loading was the more severe. The width of the "band" into which the results shown in Fig. 5 fall is typical of that obtained for other types of taper and also for the four cases involving uniformly loaded beams.

5. Reduction Formulae

From the results of their tests KREFIELD et al. [6] suggested that reductions in lateral stability due to taper were approximately dependent upon the parameter $\alpha = \frac{Z_0}{Z_1}\left(\frac{b_{f0}\,d_1}{b_{f1}\,d_0}\right)^{3/2}$. This parameter was derived from the $\frac{L\,d}{b_f\,t_f}$ formula for lateral buckling first proposed by DE VRIES [20] and does not allow for the possibility of flange thickness taper t_f. KREFIELD et al. showed that if their results were plotted in a form relating r the reduction in stability due to taper to α, then as shown in Figs. 6 and 7 which are taken from their original paper, for each type of loading, the experimental points fall within a sufficiently narrow band for a line to be drawn through them. The suggested reduction formulae were:

$$r = \frac{7+\alpha}{5+3\alpha} \quad \text{for an end load,}$$
$$r = \frac{4+\alpha}{3+2\alpha} \quad \text{for a uniform load,} \qquad (3)$$

the scatter in the results being attributed to experimental error.

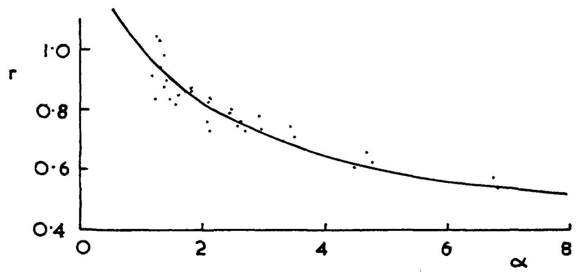


Fig. 6. Experimental results from ref. [6], end load.

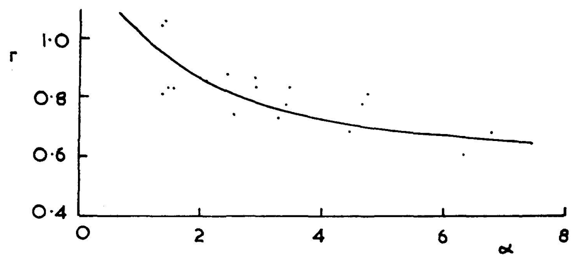


Fig. 7. Experimental results from ref. [6], uniform load.

It has been found that the present solutions (Tables 1 and 2) agree very well with Eqs. (3) and moreover, that the effects of linear flange thickness curtailment may also be included in the reduction formulae if α is redefined as:

$$\alpha = \frac{Z_0}{Z_1}\left[\left(\frac{d_1}{d_0}\right)^3\left(\frac{b_{f0}}{b_{f1}}\right)^3\left(\frac{t_{f0}}{t_{f1}}\right)^2\right]^{1/2}. \qquad (4)$$

As an additional check on the accuracy of this approach several other solutions have been obtained for beams possessing arbitrarily chosen types and degrees of linear taper and the results compared with Eqs. (3) and (4). Figs. 8 and 9 show, for two extreme values of R^2, that the reduction formulae give very reasonable predictions of the stability of tapered cantilevers loaded with either an end load or a uniform load providing this load is applied at the level of the

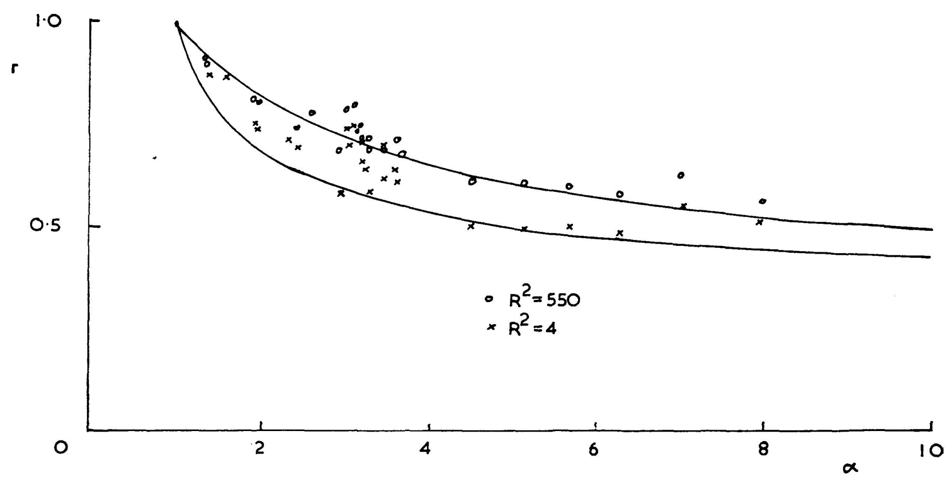


Fig. 8. Comparison of reduction formulae with numerical results for arbitrarily tapered cantilevers – end load.

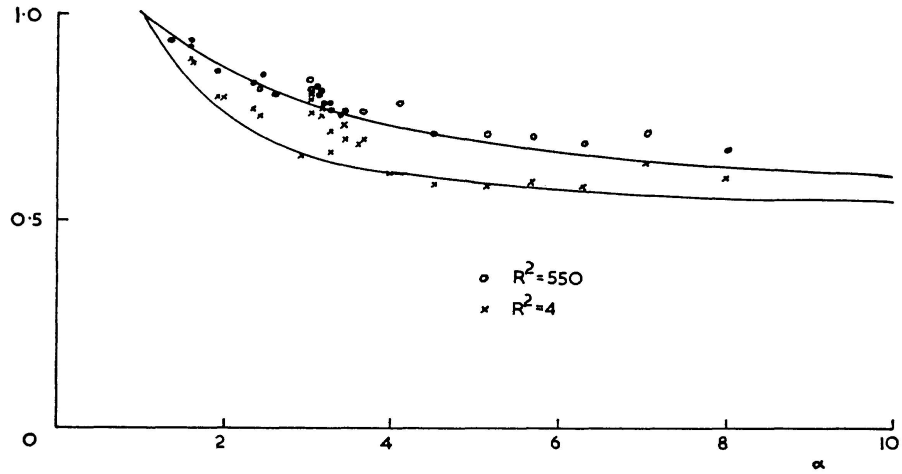


Fig. 9. Comparison of reduction formulae with numerical results for arbitrarily tapered cantilevers – uniform load.

shear centre. Also included in each figure is an approximate lower bound line, the equations of which are:

$$r = \frac{4+\alpha}{3+3\alpha} \quad \text{for an end load,}$$

$$r = \frac{1+\alpha}{2\alpha} \quad \text{for a uniform load.} \tag{5}$$

Similar sets of results have been obtained for top and bottom flange loading. For the former case Eqs. (3) now provide an approximate lower bound i.e. reductions in stability are smaller, whilst for the latter case some results fall slightly below the lower bound lines (Eqs. (5)) which now approximate to average curves. It should be noted, however, that the above statements apply only for low values of R^2 i.e. for short beams, since for high values the effects of flange loading are slight. It is also of interest to note that whereas the work of KREFIELD et al. was based on the use of the $\frac{L\,d}{b_f\,t_f}$ approximate formula for determining lateral buckling strength [20] the present work uses the more accurate Eq. (1).

6. Extension to Other Types of Loading and Conditions of Support

The effect of providing lateral bracing at the tip of the cantilever on the reductions in stability caused by taper has been extensively studied and complete sets of results similar to those of Tables 1 and 2 have been obtained for the following two conditions:

Lateral deflexion prevented at the tip, complete fixity at the tip. Providing the correct value of the critical load for the similar uniform beam was used [19], in all cases, the reductions in stability caused by taper were found to be either the same as or less than those for the identical unbraced beams. Therefore the reduction formulae given in the previous section are equally applicable to cantilevers supported laterally at the tip.

Using the set of arbitrarily chosen sections previously employed to check the accuracy of the reduction formulae, further series of results for variously loaded, simply supported beams have been obtained. In each case the numerical results have been plotted against the taper parameter α and the best line passed through the points by means of a least squares curve fit. The resulting curves are compared in Fig. 10 with similarly obtained curves for cantilevers and also with Eqs. (3). For both cases of transverse load, three different levels of loading have been considered and the results indicate that this factor is less

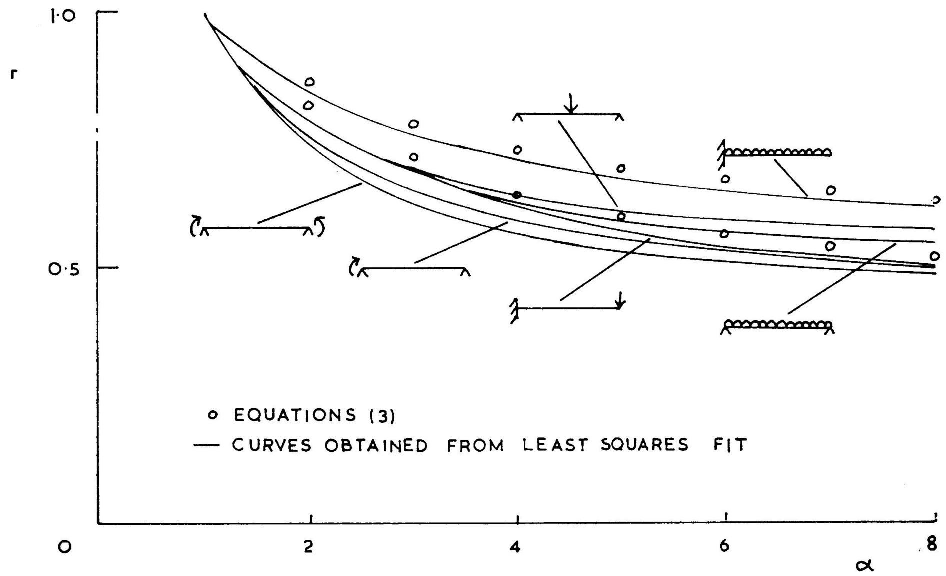


Fig. 10. Effect of loading and support conditions on reduction curve.

Calculate the critical load for a similar uniform member using the methods of ref. (20)

↓

Determine $\alpha = \frac{Z_0}{Z_1}\left[\left(\frac{d_1}{d_0}\right)^3\left(\frac{b_{f0}}{b_{f1}}\right)^3\left(\frac{t_{f0}}{t_{f1}}\right)^2\right]^{1/2}$

↓

Calculate the reduction in stability due to taper as

$$r = \frac{7+\alpha}{5+3\alpha}$$

Fig. 11. Procedure for estimating the critical load for a tapered beam.

important for simply supported beams than for cantilevers, the degree of scatter in the complete set of results being only of the order of that shown previously in Figs. 8 and 9, which relate to shear centre loading only.

Fig. 10 suggests that variations in the type of loading and support conditions do not have a pronounced affect upon the r, α relationship. Furthermore, the original reduction formula (Eq. (3)) for end loaded cantilevers of KREFIELD et al. [6] appears to provide a very reasonable indication of the stability of tapered members when subjected to any of the loading systems considered herein. A rapid and fairly accurate estimate of the critical load of a tapered beam may therefore be obtained by the process outlined in Fig. 11.

7. Conclusions

Results have been presented for the lateral buckling of I-section cantilevers having various forms of linear taper. By plotting the numerical results in a suitable form, the accuracy of the reduction formulae proposed in reference [6] have been confirmed and their scope extended to include flange thickness taper. It has also been shown that the reductions in stability caused by taper are not unduly affected by variations in loading and support conditions and a simple process for calculating rapid estimates of the critical load for tapered beams has been advanced.

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Summary

The paper describes a theoretical study of the lateral buckling of tapered I-section beams. Various forms of taper are considered and cantilevers are studied in detail. The numerical results confirm the dependence of the reductions in stability due to taper on the single geometrical parameter first suggested by KREFIELD et al. [6] and also extend its scope. Finally, the influence of variations in loading and support conditions are examined and a simple procedure for rapidly estimating the critical load for tapered beams presented.

Résumé

Le travail décrit une étude théorique du flambage latéral de poutres réduites, à section double té. On considère différentes formes de réductions et des poutres en porte-à-faux sont étudiées en détail. Les résultats numériques confirment la dépendance de la diminution de stabilité due à la réduction sur le seul paramètre géométrique qui d'abord fut proposé par KREFIELD et autres [6] et ils étendent également leur but. Finalement, l'influence des variations de charge et des conditions de support sont examinées et on présente une procédure simple en vue d'une estimation rapide de la charge critique pour des poutres réduites.

Zusammenfassung

Die Arbeit beschreibt eine theoretische Untersuchung über das Kippen von Balken mit I-Profilen, deren Querschnittshöhe mit der Balkenlänge abnimmt. Verschiedene Arten von Verjüngungen werden betrachtet und Kragarme im einzelnen untersucht. Die numerischen Ergebnisse bestätigen die Abhängigkeit der Stabilitätsverminderung infolge der Verjüngung vom einzigen geometrischen Parameter, wie es zuerst von KREFIELD und anderen [6] vorgeschlagen wurde, und erweitern auch deren Rahmen. Schliesslich werden der Einfluss der Laständerung und die Auflagebedingungen untersucht und ein einfaches Verfahren zur raschen Beurteilung der kritischen Belastung für verjüngte Träger vorgelegt.