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Calculation of Response and Loading of Highway Bridges from Modal Coordinates

Calcul de l'influence et de la charge de ponts d'autoroutes moyennant coordonnées modales

Berechnung des Einflusses und der Belastung von Autobahnbrücken mittels modaler Koordinaten

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Introduction

The force distribution on a highway bridge due to a moving vehicle is time-dependent both because of the progression of the vehicle across the bridge and because the loads transmitted from the vehicle itself are time varying. The latter are associated with vehicle response on its own suspension springs and tyres, and with disturbances from pavement irregularity and bridge deflection. Studies of bridge dynamic problems have been made both theoretically and experimentally by a number of workers and a partial list of recent work on bridge behaviour under the passage of heavy vehicles is to be found in a paper by VELETSOS and HUANG [1].

For theoretical analysis of response under vehicle loading a highway bridge is usually idealised as a lumped mass beam model [1]. When the response is then calculated in terms of natural modal coordinates it is normally found that dynamic effects are important only in the lower order modes, and it is customary therefore to include in the analysis only the coordinates corresponding to these modes. However, while the essentially dynamic nature of response (as distinct from static behaviour under the instantaneous external load-distribution) is evident only in the lower order modes, this does not imply that the quasi-static response in the higher order modes is necessarily negligible. A method suggested by WILLIAMS [2] takes account of quasi-static response in higher order modes and it is adapted here to provide a means for obtaining greater accuracy in the calculation of bridge deflections than is customarily achieved by direct coordinate transformation from the modes included in the dynamic analysis. A procedure is also derived for the calculation of dynamic loading, and in particular the loading actions of bending moment and shear force, by the method of force summation [3], which again is more accurate than that obtained by direct coordinate transformation.

In the following, the terms "bridge" and "bridge model" are used interchangeably. Notations are defined where they first appear and are also listed alphabetically in an Appendix for convenience of reference.

1. Formulation of the Problem

A lumped-mass beam model for a highway bridge is shown in Fig. 1. In general the number of spans and the support conditions may differ and the structure may be either statically determinate or indeterminate.

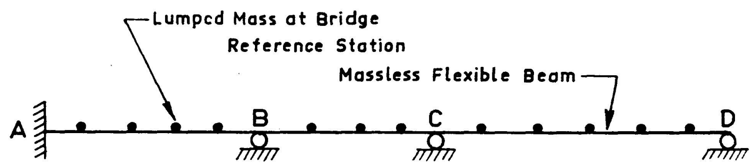


Fig. 1. Lumped mass beam model of a highway bridge.

Assuming viscous damping, the equations of motion of the bridge may be written in matrix form as

$$M\ddot{y}+C\dot{y}+Ky = P(t), \tag{1}$$

where for the reference stations at the mass points:

- M is the diagonal inertia matrix (also known as the mass matrix);
- C is the matrix of damping coefficients;
- K is the stiffness matrix;
- y is the vector of dynamic deflections;
- and P is the vector of (generalised) time-dependent forces at the reference points corresponding to the vehicle axle loads. (The axle loads do not necessarily coincide with any of the reference points at a particular instant of time and are obtained together with the bridge deflections y by solving the interacting system of Eq. (1) and the equations of motion of the vehicle.)

Single and double dots over y represent the first and second time derivatives respectively.

The vector of bridge deflections y may be written in terms of modal coordinates A as

$$y = \Phi A, \tag{2}$$

where Φ is the matrix of natural-vibration mode vectors for the bridge. The i^{th} column of Φ consists of the bridge deflections at the reference points in the i^{th} mode.

The mode-shape vectors in Φ may be so normalised that the generalised inertia matrix $\Phi^T M \Phi = I$, the unit matrix. It may then be shown that the generalised stiffness matrix $\Phi^T K \Phi$ is Ω^2 where Ω is a diagonal matrix consisting of the natural frequencies of free vibration of the bridge model. With substitution of the coordinate transformation (Eq. (2)) and pre-multiplication by Φ^T, Eq. (1) then reduces to

$$\ddot{A} + D\dot{A} + \Omega^2 A = \Phi^T P(t), \tag{3}$$

where $D = \Phi^T C \Phi$.

Eq. (3) may be solved together with the equations of motion of the vehicle to obtain the response in terms of bridge modal-coordinates and the vehicle coordinates. Only a numerical solution is generally possible.

The natural frequencies and mode shapes of free vibration of the bridge, required for the formulation of Eq. (3), are obtained by solving the eigenvalue problem corresponding to the free-vibration equation

$$M\ddot{y} + Ky = 0. \tag{4}$$

Methods for solving Eqs. (3) and (4) are available in standard textbooks on Structural Dynamics (e. g. Ref. [4]) and are not discussed here. The solution of these equations and other relevant data, including the instantaneous axle loads of the vehicle, are assumed known from previous calculations.

The flexibility matrix of the bridge is required in the procedure here and it is taken that this is known. This may, for example, be obtained by the Transfer Matrix method [5]. The stiffness matrix of the bridge is not required at any stage in the calculations and it is therefore never necessary to invert the flexibility matrix.

2. Calculation of Dynamic Deflections and Loading

2.1. General Basis

Having calculated the modal coordinates of motion at any instant according to Eq. (3), the next step may be to calculate the dynamic deflections, bending moments and shear forces in the bridge. The corresponding static values may also be required for comparison, in order to establish the dynamic effects.

By rearrangement of Eq. (1) as

$$y = F[P + (-M\ddot{y} - C\dot{y})],$$

where F denotes the bridge flexibility matrix ($F = K^{-1}$), it is evident that the dynamic deflection at a point on the bridge is the sum of deflections due to the vehicle loads and to the bridge inertia and damping forces. The generalised bridge damping and inertia forces in the i^{th} mode are given as $(-D_{i\cdot})\dot{A}$ and $\ddot{A}_i$ respectively (Eq. 3). This is consistent with a general distribution of damping in the structure and with orthogonality conditions in natural modes. It implies that response in the i^{th} mode may be caused by damping forces arising from other modes but not by inertia forces.

If all the modes of free vibration were included in the modal analysis of bridge response, the dynamic deflections as calculated by direct coordinate transformation according to Eq. (2) would be the same as would be obtained by direct solution in terms of y coordinates (Eq. (1)). In practice, however, to reduce computations only the first few modal coordinates are calculated, contributions from the higher modes being generally small. Direct coordinate transformation in such a case would imply that (a) generalised forces corresponding to the modes excluded from the analysis are considered negligibly small, both due to externally applied vehicle load and the bridge inertia and damping forces and (b) in general, deflections due to damping forces of the modes being included in the analysis do not include higher mode components.

In the alternative method proposed here, total deflections due to the axle loads and damping forces of the "included modes" are included in the calculated dynamic deflections and only the components due to the inertia and damping forces of the modes not included in the Eq. (3) analysis are neglected. In short, the inertia and damping forces due to the modes included in the Eq. (3) analysis are first calculated. Deflections due to the static application of the instantaneous axle loads and to the inertia and damping forces are then calculated separately, using the bridge flexibility matrix, and superposed to obtain the dynamic deflections. In this process calulation of deflections due to the axle loads must be considered in some detail since the axle loads need not coincide with the mass points at the instant of time under consideration.

To calculate the dynamic bending moments and shear forces by force summation [3] support reactions of the bridge are needed in addition to the known axle loads and inertia and damping forces. A procedure is developed to calculate the redundant reactions in multi-span bridges by the method of superposition, taking into account the non-coincidence of the axle loads with the mass points on the bridge.

Similar methods are then applied for the calculation of corresponding static deflections and loading for comparison.

2.2. Inertia and Damping Forces

The inertia and damping forces acting at the mass points of the bridge at any instant are given by

$$\tilde{P} = -(M\ddot{y} + C\dot{y}). \tag{5}$$

In terms of the loads associated with the modes included in the analysis, this becomes

$$\tilde{P} = -(M\Phi\ddot{A} + C\Phi\dot{A}), \tag{6}$$

where here, and in what follows, Φ represents the matrix of only those mode vectors which are included in the analysis.

$\dot{A}$ and $\ddot{A}$, as calculated during the process of numerical integration of Eq. (3), may be used in Eq. (6) to obtain the inertia and damping forces. However, in a numerical method the derivatives $\dot{A}$ and $\ddot{A}$ would have a lower order of accuracy than A; e.g. in Newmark's β method with $\beta = 1/6$, the truncation errors of A, $\dot{A}$ and $\ddot{A}$ are of the order h^4, h^3 and h^2 respectively, h being the time interval for numerical integration [6]. It would therefore be desirable to calculate the inertia and damping forces using only the coordinates A. Such a procedure is set out in the following.

The modal coordinates of motion may be thought of as consisting of two parts, due to the static application of instantaneous axle loads and due to the bridge inertia and damping forces. The vectors of these coordinates are denoted by $\bar{A}$ and $\tilde{A}$ respectively.

The axle loads $\bar{P}$ are assumed known from previous calculations. For a sprung-mass vehicle model with frictional damping in the suspension springs as shown in Fig. 2, these loads may be expressed in terms of the corresponding spring deformations [1], and their order of accuracy would therefore be the same as for the response coordinates of the bridge and the vehicle.

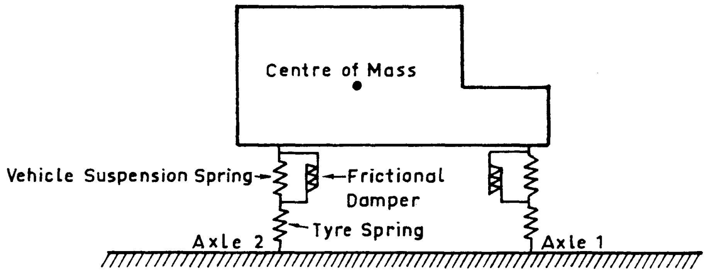


Fig. 2. Vehicle model.

The vector $\bar{A}$ of corresponding bridge deflection-coordinates may be calculated by the principle of virtual work. For the equilibrium of the bridge under the static application of instantaneous axle loads $\bar{P}$, the work done during an arbitrary virtual displacement $\delta\bar{y}$ is

$$\delta W = (\delta\bar{y})^T K\bar{y} - (\delta\bar{y}^0)^T \bar{P} = 0, \tag{7}$$

where $\bar{y}$ and $\bar{y}^0$ are the vectors of bridge deflections at the mass points and under the vehicle axles respectively due to the static application of axle loads $\bar{P}$.

With the coordinate transformations $\bar{y} = \Phi \bar{A}$ and $\bar{y}^0 = \Phi^0 \bar{A}$, Eq. (7) becomes

$$(\delta \bar{A})^T \Phi^T K \Phi \bar{A} - (\delta \bar{A})^T (\Phi^0)^T \bar{P} = 0$$

or

$$(\delta \bar{A})^T [\Omega^2 \bar{A} - (\Phi^0)^T \bar{P}] = 0, \tag{8}$$

where Φ^0 is the matrix of mode deflections under the axles and Ω^2 $(=\Phi^T K \Phi)$ is the generalised stiffness matrix as defined in Sec. 1. When the axles do not coincide with the mass points, Φ^0 is obtained by interpolation.

Since $\delta \bar{A}$ is arbitrary, the square bracket must be zero. It follows that

$$\bar{A} = (\Omega^2)^{-1} (\Phi^0)^T \bar{P}. \tag{9}$$

The vectors A and $\bar{A}$ now being known, the vector $\tilde{A}$ follows as

$$\tilde{A} = A - \bar{A}. \tag{10}$$

The inertia and damping forces are now readily calculated as

$$\tilde{P} = K \Phi \tilde{A} = M \Phi \Omega^2 \tilde{A}, \tag{11}$$

where $K\Phi = M \Phi \Omega^2$ is a consequence of Eqs. (4) and (2).

The bridge inertia and damping forces calculated by the expression (Eq. (11)) will now have the same order of accuracy as the response coordinates.

2.3. Redundant Reactions at Intermediate Supports

The support reactions are required as part of the loading for evaluation of shear-force and bending moment distribution. Redundant supports of multi-span bridges which are not located at the bridge extremeties are referred to as "intermediate redundant supports" and reactions at such supports are now considered.

A bridge with its intermediate redundant supports, if any, removed is referred to as the "basic bridge" (Fig. 3) and the corresponding flexibility matrix is called the "basic flexibility matrix", F^*. It should be noted here that a basic bridge may itself be statically indeterminate, for example as shown in Fig. 3. The support locations, e.g. B and C in Fig. 3, are included as additional reference stations in the basic bridge and the corresponding rows and columns therefore appear in F^*. The vector of displacements for the basic

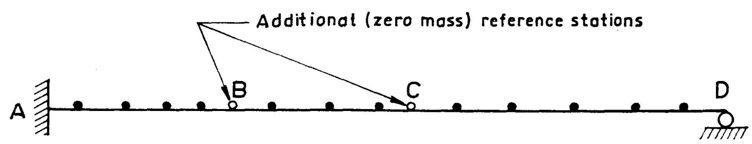


Fig. 3. Basic bridge corresponding to the bridge of figure 1.

bridge, including displacements at the redundant support locations, is denoted by y^*.

The elements of the j^{th} column, $F^*_{\cdot j}$, of F^* are the deflections at the reference stations of the basic bridge due to a unit load at station j. The corresponding deflection under a vehicle axle may be obtained by interpolation in the general case that the axle does not coincide with a reference station. By Maxwell's reciprocal theorem this is also the deflection at station j due to a unit load at the axle position. Thus the deflections at all the reference stations due to a unit load at an arbitrary axle position may be found from interpolation on each column of F^* in turn. The matrix of coefficients found in this way for each axle position at a particular instant is denoted by $\bar{F}^*$, in which the elements of the column vector $\bar{F}^*_{\cdot j}$ are the deflections of the basic bridge at its reference stations due to a unit load at the position of the j^{th} axle.

Superposition of the deflections due to vehicle axle loads and the bridge inertia and damping forces then gives the hypothetical dynamic deflections which would result in the absence of the deleted reaction forces i. e.

$$y^* = \bar{F}^* \bar{P} + F^* \tilde{P}^*, \tag{12}$$

where $\tilde{P}^*$ denotes the vector of inertia and damping forces supplemented with zero elements corresponding to the reference stations of the deleted intermediate redundant supports.

The vector of basic bridge deflections at the locations of intermediate redundant supports (e.g. stations B and C in Fig. 3) – included in y^* – is referred to as y^R. If the redundant reactions were now applied on the basic bridge, equal and opposite deflections $-y^R$ would be caused. The redundant reactions are therefore calculated as

$$R = K^R(-y^R) = -K^R y^R, \tag{13}$$

where K^R is the corresponding stiffness matrix, which is obtained by inverting the small flexibility matrix corresponding to the intermediate redundant supports, taken from F^*.

A new vector of forces is now formed by including the reactions R in place of the null elements in the vector of inertia and damping forces, $\tilde{P}^*$. This new vector is denoted by P^*.

2.4. Dynamic Response and Loading

For a bridge with no intermediate redundant supports, matrices F and F^* are identical and Eq. (12) gives the final dynamic deflections i.e. in such a case $y = y^*$.

Otherwise, with reactions R included, the dynamic deflections of the basic bridge are given by

$$y^* = \bar{F}^* \bar{P} + F^* P^* \tag{14}$$

and the deflections y of the actual bridge are then obtained from y^* by deleting the (zero) elements corresponding to the redundant-reaction stations.

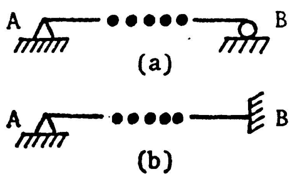
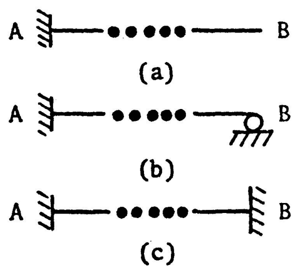
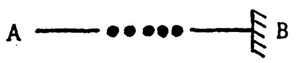
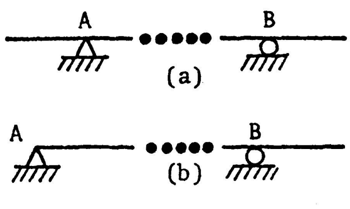
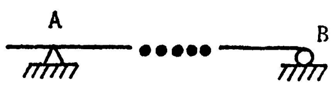
Alternatively, the dynamic deflections may be calculated as

$$y = \bar{F}\bar{P} + F\tilde{P}, \tag{15}$$

where a column $\bar{F}_{.j}$ of $\bar{F}$ represents the deflections of the actual bridge at the mass points due to a unit load at the j^{th} axle position. However, it is more convenient to calculate the deflections according to Eq. (14) as it is then not necessary to determine the additional matrix $\bar{F}$ by interpolation on each column of the matrix F.

Calculation of the dynamic bending moments and shear forces in the bridge by force summation [3] here involves evaluation of these quantities progressively from the left hand end. The additional force and moment reactions

Table 1. Outer Support Reactions Required to Calculate Bending Moments and Shear Forces in the Bridge by Force Summation Method

Case	Conditions at Bridge Outer Supports	Outer Support Reactions Required
1		Force Reaction at A (R^1); $Q^1 = 0$
2		Force Reaction at A (R^1) and moment reaction at A (Q^1)
3		*Nil*; $R^1 = Q^1 = 0$
4		Force Reaction at A (R^1) and Force Reaction at B (Q^1)
5		Force Reaction at A (R^1); $Q^1 = 0$

at the supports of the basic bridge which are necessary for this purpose are indicated in Table 1 for the various possible support conditions considered.

It is assumed here that the influence-coefficient vectors of the required reactions are known from previous calculations. These vectors are denoted as R^* and Q^*; e.g. for a bridge with both ends simply supported (Case 1, Table 1), R^* is the vector of reactions at the left hand support due to a unit load at the reference stations of the basic bridge. Since only one additional reaction is required in this case, Q^* is a null vector.

The required reactions are calculated as

$$R^1 = (R^*)^T P^* + (\bar{R}^*)^T \bar{P}$$

and

$$Q^1 = (Q^*)^T P^* + (\bar{Q}^*)^T \bar{P}, \tag{16}$$

where, when axle positions do not coincide with reference stations, $\bar{R}^*$ and $\bar{Q}^*$ are the vectors of interpolated values of reactions due to a unit load at the axle positions. For a bridge with no intermediate redundant supports, P^* is identical to $\tilde{P}$.

The dynamic bending moment and shear force at a section may now be calculated by considering equilibrium with the loads and reactions on the bridge to the left of that section.

3. Static Deflections, Bending Moments and Shear Forces

The static deflections, bending moments and shear forces are calculated in the same way as the dynamic values, with the bridge inertia and damping forces omitted.

Static deflections of the basic bridge are first calculated as

$$y_{st}^* = \bar{F}^* \bar{P}_{st}, \tag{17}$$

where $\bar{P}_{st}$ is the vector of static axle loads of the vehicle.

For a bridge with no redundant intermediate supports $y_{st} = y_{st}^*$. Otherwise the reactions at the redundant intermediate supports are calculated as

$$R_{st} = K^R(-y_{st}^R) = -K^R y_{st}^R, \tag{18}$$

where K^R is the stiffness matrix as defined in Sec. 2.2 and y_{st}^R is the vector of the static deflections at the points of intermediate redundant supports – contained in y_{st}^*, Eq. (17).

The static deflections are then calculated by an equation similar to Eq. (14), i.e.

$$y_{st}^* = \bar{F}^* \bar{P}_{st} + F^* P_{st}^*, \tag{19}$$

where the vector P_{st}^* contains the reactions R_{st} at appropriate places, other elements corresponding to forces at the mass points being zero. y_{st} may be obtained from y_{st}^* by deleting the (zero) elements corresponding to the intermediate redundant supports.

The additional static reactions required for the calculation of static bending moments and shear forces are obtained as before, i. e.

$$R_{st}^1 = (R^*)^T P_{st}^* + (\bar{R}^*)^T \bar{P}_{st}$$

and

$$Q_{st}^1 = (Q^*)^T P_{st}^* + (\bar{Q}^*)^T \bar{P}_{st}. \tag{20}$$

The static bending moments and shear forces are then calculated in the manner indicated in Sec. 2.4.

4. Illustrative Example

An example is given here to illustrate the method and to show the convergence of results obtained by including different numbers of natural modes of the bridge in the analysis. Impact factors for deflections are also compared with those obtained by direct coordinate transformation (Eq. (2)) for a one-mode and a three-mode representation of the bridge. All the results are obtained using a computer program developed for the IBM 360/50 computer at the University of New South Wales.

The response of a two-span continuous bridge with uniform distribution of mass and stiffness is calculated under the passage of a two axle sprung vehicle moving with a nominal speed of 75 ft/sec. (22.85 m/sec.) on the bridge. The actual speed fluctuates slightly about the nominal speed as a result of bridge and vehicle response, constant traction being assumed.

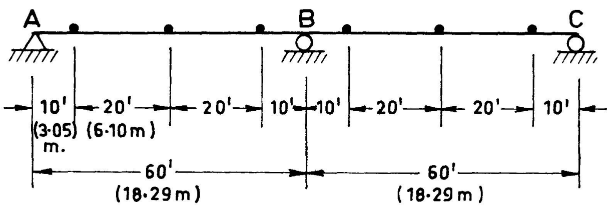


Fig. 4. Bridge model for illustrative example.

The idealised bridge model is shown in Fig. 4 and the vehicle model is shown in Fig. 2. The corresponding numerical data are as follows:

Numerical Data

1. Bridge

Length of each span:	60 ft (18.29 m)
Mass per unit length:	126.1 slugs/ft (6036 kg/m)
Stiffness (EI):	4.698×10^9 lb ft^2 (1.941×10^9 N m^2)

2. Vehicle

Static load on each axle:	32,040 lb (142,500 N)
Pitching inertia:	38,500 slugs ft^2 (52,170 kg m^2)
Axle spacing:	14 ft (4.267 m)
Normal height of centre of mass above pavement level:	6 ft (1.829 m)
Tyre stiffness at each axle:	480,000 lb/ft (7.003×10^6 N/m)
Vehicle suspension stiffness at each axle:	270,000 lb/ft (3.939×10^6 N/m)
Limiting friction at each axle:	4800 lb (21,350 N)

For ease of presentation the bridge mass is assumed lumped at only three points per span. The inertia matrix M of the bridge is then a sixth order diagonal matrix, each diagonal element being 2522.0 slugs (36,800 kg).

The natural mode shapes of the bridge with amplitudes normalised as in Sec. 1 are shown in Fig. 5.

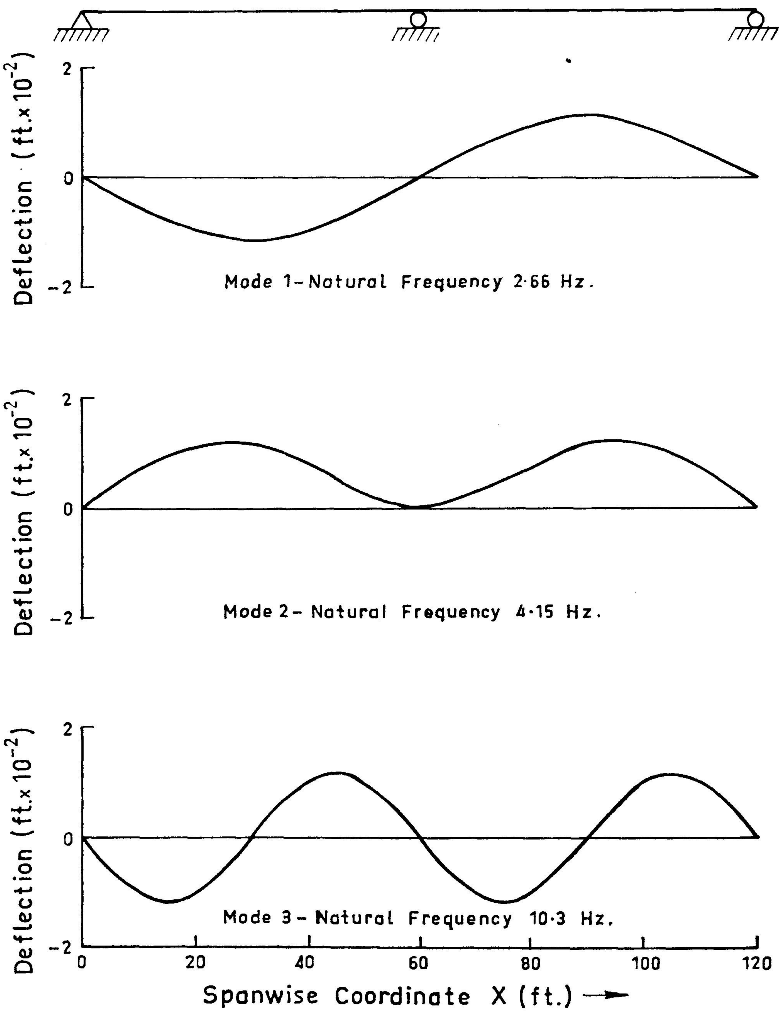


Fig. 5. Illustrative example – normalised natural mode shapes of bridge.

The basic bridge is obtained by removing the intermediate redundant support B (Fig. 4). The corresponding basic flexibility matrix F^* is the following, in which the units are ft/lb.

$$F^* \times 10^6 = \begin{bmatrix} 0.716 & 1.65 & 1.94 & 1.90 & 1.74 & 1.19 & 0.420 \\ & 4.31 & 5.34 & 5.26 & 4.88 & 3.35 & 1.19 \\ & & 7.24 & 7.36 & 6.95 & 4.88 & 1.74 \\ & & & 7.66 & 7.36 & 5.26 & 1.90 \\ & & & & 7.24 & 5.34 & 1.94 \\ & \text{Symmetric} & & & & 4.31 & 1.65 \\ & & & & & & 0.716 \end{bmatrix}.$$

(The conversion factor to convert the basic flexibility matrix to SI units is 0.0685 m/N.)

The (sixth order) flexibility matrix of the actual bridge is not required for the present calculations and is therefore not included here.

The inertia and damping forces at any instant are calculated by the procedure described in Sec. 2.2 and deflections of the basic bridge are then calculated using Eq. (12) The matrix $\bar{F}^*$ of Eq. (12) changes in accordance with the changing axle positions of the vehicle and is calculated by interpolations on the columns of the basic flexibility matrix F^* as indicated in Sec. 2.2.

The matrix K^R required to calculate the redundant reaction at support B (Eq. 13) is a single element in this example. It is the inverse of element (4,4) of the matrix F^* and is therefore 1.305×10^5 lb/ft (19.04×10^5 N/m). With the reaction at B known, the vector P^* is formulated by inserting this at the appropriate place in the vector $\tilde{P}$ of inertia and damping forces. The dynamic deflections of the bridge are then obtained using Eq. (14).

The reaction at the left hand support of the bridge is required for the calculation of dynamic bending moments and shear forces by the force summa-

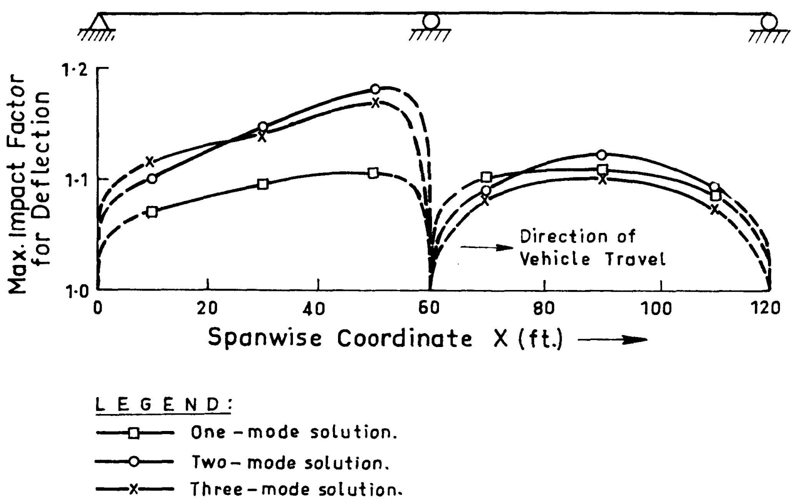


Fig. 6. Illustrative example – maximum impact-factor distribution for deflection.

tion method (Reference Case 1, Table 1). This reaction is denoted by R^1. The influence coefficient vector R^* corresponding to R^1 is obtained as

$$R^* = \{0.917 \quad 0.750 \quad 0.583 \quad 0.500 \quad 0.417 \quad 0.250 \quad 0.0833\},$$

where the braces indicate a column vector.

The vector $\bar{R}^*$ of the influence coefficients of R^1 due to a unit load at the axle positions is calculated by interpolations on R^*. The reaction R^1 is then calculated according to Eq. (16). It is then possible to calculate the dynamic bending moments and shear forces progressively from the left hand end of the bridge.

Corresponding static deflections, bending moments and shear forces are calculated in a similar manner for comparison.

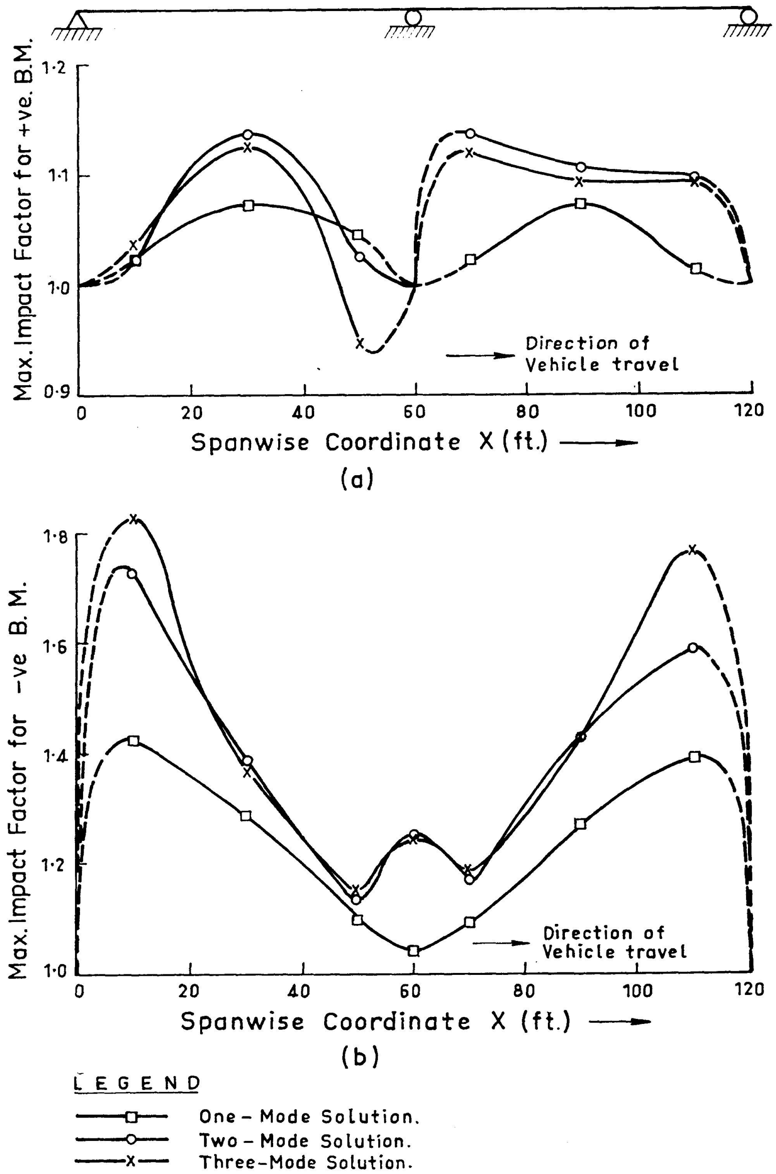


Fig. 7. Illustrative example – maximum impact factor distribution for (a) positive bending moment and (b) negative bending moment.

For the present example the bridge response has been calculated for a one-, two- and three-mode representation of the bridge. A bridge designer is usually most interested in the maximum impact factors for deflections and bending moments along the bridge. The term "maximum impact factor" is defined here for a particular point on the bridge as the ratio of the maximum dynamic value to the maximum static value at that point; and the maximum dynamic and maximum static values do not necessarily occur for the same position of the vehicle on the bridge.

The maximum impact factor curves for deflection and bending moments are plotted in Figs. 6 and 7 respectively and it will be seen that the two- and three-mode analyses by the proposed method show generally good agreement (and convergence). The corresponding curves of deflection obtained by direct coordinate transformation (Eq. (2)) are shown in Fig. 8 for one-mode and for

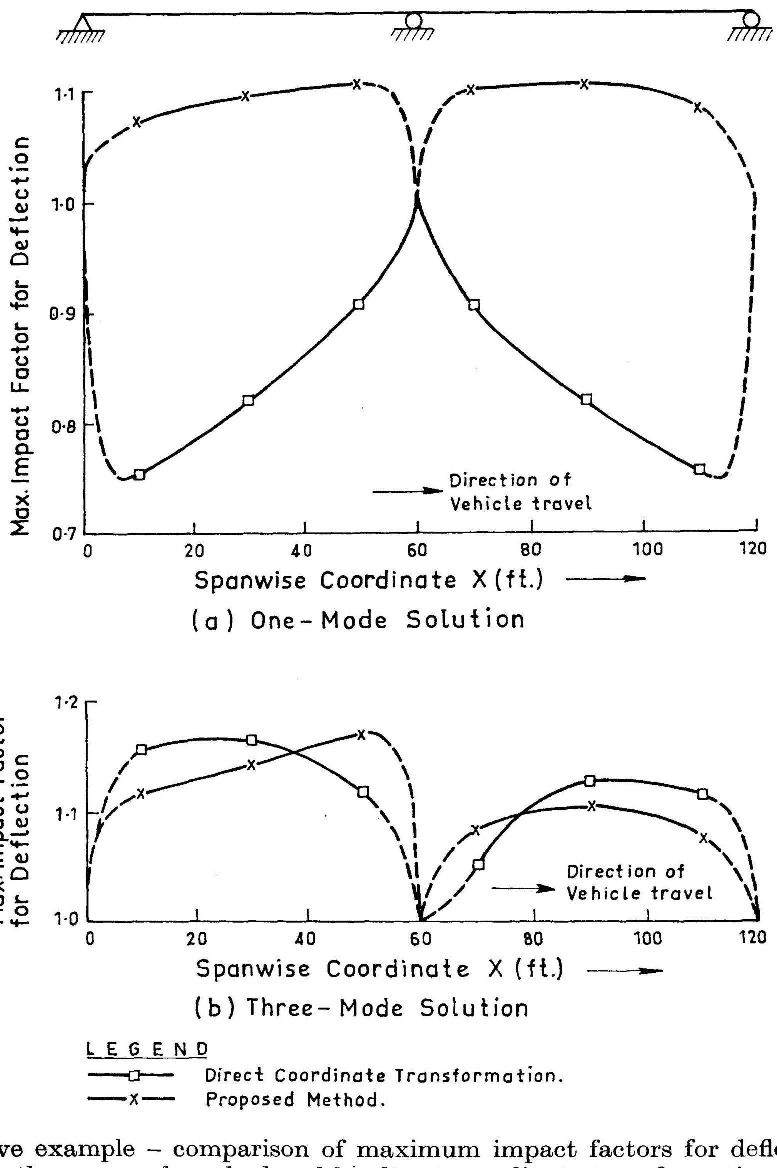


Fig. 8. Illustrative example – comparison of maximum impact factors for deflection obtained by the proposed method and by direct coordinate transformation.

three-mode representations of the bridge. In these a comparison is given with the more accurate curves of the present method, and large discrepancies are evident. The one-mode representation even shows an impact factor of less than unity everywhere.

5. Conclusions

Given that a highway bridge is idealised as a lumped-mass beam model and that its dynamic response to the passage of a vehicle is analysed in terms of a small number of natural modal coordinates, a method is presented for computing deflections, shears and bending moments which has greater accuracy than is obtained by direct coordinate transformation. The procedures are based on a method suggested by WILLIAMS [2] and the method of "force summation" [3] which are adapted to serve the present problem. In the application a "basic bridge" is defined as one with its intermediate redundant supports, if any, removed from the actual bridge. The instantaneous inertial and damping forces from the bridge structure, and the vehicle axle loads, are used in conjunction with the flexibility matrix of the basic bridge to evaluate reaction forces of the real bridge at its intermediate supports. Corresponding deflections for the actual bridge are then calculated, using the flexibility matrix of the basic bridge. Special attention is required in treating vehicle loads, since at a particular instant the axle loads may not coincide with bridge reference stations to which the flexibility matrix refers. Dynamic bending moments and shear forces are calculated from the known loadings and reaction forces by the force summation process. For comparison, corresponding calculations are then made for statically applied vehicle loads.

To illustrate the method, the response of a two-span continuous bridge is calculated under the passage of a two-axle vehicle, with analysis in terms of one-, two- and three-mode representations. When the present method is used, good agreement is obtained for maximum impact factor values for deflections and bending moments calculated by two- and three-mode representations of the bridge. However, corresponding results for deflections obtained by the usual method of direct coordinate transformation have considerable differences, especially in the single mode representation. It is shown that for comparable accuracy fewer modes need be included for the present method than are required in direct coordinate transformation.

Notations

A	Vector of modal generalised coordinates for bridge deflections.
$\bar{A}$	Vector of modal generalised coordinates for deflections due to the static application of instantaneous axle-loads.

$\tilde{A}$	Vector of modal generalised coordinates for deflections due to the static application of bridge inertia and damping forces.
C	Viscous damping matrix for the bridge corresponding to reference station deflection-coordinates.
D	Generalised damping matrix.
F	Flexibility matrix of the bridge for reference station coordinates.
F^*	Flexibility matrix of the "basic bridge".
$\bar{F}$	Matrix of bridge deflections at the reference stations due to a unit load at the instantaneous axle positions.
$\bar{F}^*$	Matrix of deflections of the basic bridge due to a unit load at the instantaneous axle positions.
h	Time-interval used for numerical integration of the equations of motion.
K	Bridge stiffness matrix for reference station coordinates.
K^R	Stiffness matrix corresponding to the locations of intermediate redundant supports in the basic bridge.
M	Diagonal inertia (or mass) matrix of the bridge.
P	Vector of generalised forces on the bridge corresponding to the axle loads.
$\bar{P}$	Vector of axle loads.
$\tilde{P}$	Vector of bridge inertia and damping forces.
P^*	Vector of forces at the reference stations of the basic bridge.
$\tilde{P}^*$	Vector of bridge inertia and damping forces and zero forces at reference stations corresponding to the deleted intermediate redundant supports in the basic bridge.
R^1, Q^1	Support reactions in addition to the intermediate redundant reactions required for the calculation of bending moments and shear forces by the force summation method.
R^*, Q^*	Influence coefficient vectors corresponding to R^1 and Q^1 for a unit load at the reference stations of the basic bridge.
$\bar{R}^*$, $\bar{Q}^*$	Influence coefficient vectors corresponding to R^1 and Q^1 for a unit load at the instantaneous axle positions on the basic bridge.
R	Vector of statically indeterminate reactions at intermediate supports.
W	Total work done by external forces and internal stresses.
y	Vector of dynamic bridge deflections at reference stations.
$\bar{y}$	Vector of bridge deflections corresponding to the static application of instantaneous axle loads.
$\bar{y}^0$	Vector of bridge deflections at the axle positions due to the static application of instantaneous axle loads.
y^R	Vector of deflections at stations corresponding to intermediate redundant supports in the basic bridge.
y^*	Vector of deflections at the reference stations of the basic bridge.
Φ	Matrix of natural mode vectors for the bridge.

Φ^0 Matrix of bridge mode deflections under the vehicle axles.

Ω Diagonal matrix of bridge natural frequencies of vibration in radians/second.

Notes:

1. "Basic bridge" refers to the bridge with the intermediate redundant supports, if any, removed.
2. The i^{th} row and the j^{th} column of a matrix (say F) are represented respectively as $F_{i.}$ and $F_{.j}$.
3. The subscript "st" with a symbol refers to the static loading condition on the bridge.

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Summary

For analysis of response under vehicle loads a highway bridge is often idealised as a massless elastic beam with an appropriate distribution of lumped masses. Equations of motion for the bridge may then be formulated in terms of natural modal (or normal) coordinates and the response calculated in terms of only a small number of these.

A method is presented here for calculating bridge dynamic response which for such a set of coordinates yields greater accuracy than is obtained by direct coordinate-transformation. Also the "force summation" concept is employed for calculation of dynamic bending moments and shear forces. A numerical example is included to demonstrate the improvement in accuracy obtained.

Résumé

Pour analyser l'influence des charges de véhicules sur un pont autoroute celui-ci est souvent idéalisé comme poutre élastique sans masse avec distribution appropriée de masses ponctuelles. Les équations de mouvement pour le pont peuvent alors être formulées en termes de coordonnées modales (ou normales) et l'influence peut être calculée en termes d'un petit nombre de ceux-ci.

On présente ici une méthode de calcul de l'influence dynamique de ponts qui pour une pareille série de coordonnées fournit une plus grande exactitude que pour une transformation directe de coordonnées. Egalement la conception de la «sommation des forces» est employée au calcul de moments de flexion dynamiques et des forces de cisaillement. Un exemple numérique est donné pour démontrer l'exactitude élevée.

Zusammenfassung

Für die Analyse des Einflusses der Fahrzeugbelastung an einer Autobahnbrücke wird diese häufig als masseloser elastischer Balken mit entsprechender Verteilung punktförmiger Massen idealisiert. Bewegungsgleichungen für die Brücke können dann in Termen natürlicher modaler (oder normaler) Koordinaten und der errechnete Einfluss in einer nur kleinen Anzahl derselben formuliert werden.

Hier wird eine Methode zur Berechnung des dynamischen Einflusses von Brücken vorgelegt, welche für eine solche Reihe von Koordinaten grössere Genauigkeit ergibt als bei direkter Koordinatentransformation. Auch das «Kraftsummierungs-Konzept» wird zur Berechnung dynamischer Biegemomente und Schubkräfte verwendet. Zum Nachweis der erhöhten Genauigkeit wird ein numerisches Beispiel angeführt.