Free vibration of curved and straight beam-slab or box-girder bridges

Free Vibration of Curved and Straight Beam-Slab or Box-Girder Bridges

Vibrations de ponts à section en T ou en caisson, courbes ou droits

Schwingungen gekrümmter oder gerader Plattenbalken- oder Kastenträgerbrücken

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Introduction

This paper describes the application of a finite strip method to the determination of the natural frequencies and modal shapes of undamped vibration of curved or straight singel-spanned bridges (Fig. 1a) made up of thin plates connected together along circumferential (longitudinal) edges.

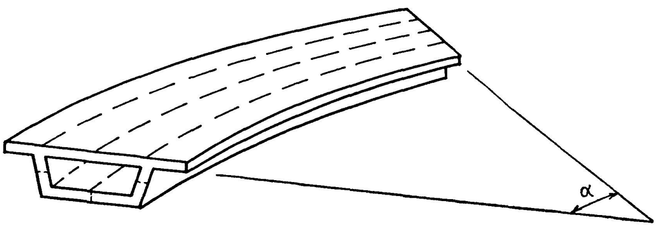


Fig. 1a. A Curved Box Bridge and It's Idealization into Strips.

The static analysis of such right bridges has been presented by CHU and DUDNIK [6], using the elasticity theory developed by GOLDBERG and LEVE [7], and by CHEUNG [3], using the finite strip approach, and very good accuracy has been demonstrated in all the numerical examples.

The static analyses of curved girder bridges have been presented by BELL and HEINZ [1], using a slope-deflection Fourier series method. However, an approximation has been introduced since the torsional and bending rigidities of each girder must be assigned somehow to account for the composite action of the plate and girder. Furthermore, the method presented is unsuitable for the analysis of box girders. Recently, the writers [2] have successfully applied the finite strip method to the static analysis of curved box girder bridges. In the present paper, the dynamic analysis of such bridges will be dealt with.

In this method, the plates are divided into strips extending from one support to the other. Displacement functions given in the form of the product of a Fourier series in the circumferential (longitudinal) direction and a simple polynomial in the transverse direction can be chosen for the displacements u, v and w and the stiffness and mass matrices of a strip can be formulated according to the usual finite element procedure. By virtue of the orthogonality of the Fourier series, all the terms of the series uncouple and only small matrices are needed for the eigenvalue solutions of each term.

The general formulation of stiffness and mass matrices has been presented in detail elsewhere [4], [5], and shall not be repeated here. Also the straight strip will simply be interpreted as a special case of the curved strip, in which the radius of curvature r is infinitely large, the subtended angle α infinitely small, and the product $r\alpha$ is equal to the span of a straight strip.

In a paper by WITTRICK and WILLIAMS [8], a similar approach is used to obtain the natural frequencies of *rectangular* stiffened plates. However, since the governing differential equations (for simply-supported case only) were solved exactly, the resulting stiffness matrix contains transcendental terms and therefore, a complicated eigenvalue solution had to be used.

Stiffness and Mass Matrices

A. Curved Interior and Exterior Webs of Box Girder

Each web is in general a part of a conical frustum (Fig. 1b), but becomes a cylindrical panel when it is in a vertical position. For such a curved surface the membrane and bending actions are coupled, and the stiffness matrix is of the sixe 8×8.

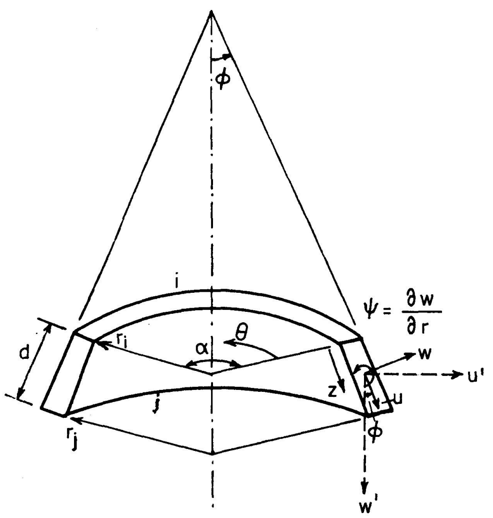


Fig. 1b. A Conical Web Strip.

Displacement functions:

$$
\begin{aligned}
u_m &= \left[\left(1-\frac{z}{d}\right) u_{im} + \left(\frac{z}{d}\right) u_{jm}\right] \sin\frac{m\pi\theta}{\alpha},\\
v_m &= \left[\left(1-\frac{z}{d}\right) v_{im} + \left(\frac{z}{d}\right) v_{jm}\right] \cos\frac{m\pi\theta}{\alpha},\\
w_m &= \left[\left(1-\frac{3z^2}{d^2}+\frac{2z^3}{d^3}\right) w_{im} + \left(z-\frac{2z^2}{d}+\frac{z^3}{d^2}\right)\psi_{im}\right.\\
&\quad \left. + \left(\frac{3z^2}{d^2}-\frac{2z^3}{d^3}\right) w_{jm} + \left(\frac{z^3}{d^2}-\frac{z^2}{d}\right)\psi_{jm}\right] \sin\frac{m\pi\theta}{\alpha}
\end{aligned} \tag{1}
$$

or

$$f = [N_m]\{\delta_m\}. \tag{1a}$$

Strain displacement relationship:

$$
\begin{Bmatrix} \epsilon_z \\ \epsilon_\theta \\ \gamma_{z\theta} \\ \chi_z \\ \chi_\theta \\ \chi_{z\theta} \end{Bmatrix}_m = \begin{Bmatrix} \dfrac{\partial u}{\partial z} \\ \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} + \dfrac{w\cos\phi + u\sin\phi}{r} \\ \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} + \dfrac{\partial v}{\partial z} - \dfrac{v\sin\phi}{r} \\ -\dfrac{\partial^2 w}{\partial z^2} \\ -\dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial^2 \theta} + \dfrac{\cos\phi}{r^2}\dfrac{\partial v}{\partial \theta} - \dfrac{\sin\phi}{r}\dfrac{\partial w}{\partial z} \\ 2\left(-\dfrac{1}{r}\dfrac{\partial^2 w}{\partial z\,\partial \theta} + \dfrac{\sin\phi}{r^2}\dfrac{\partial w}{\partial \theta} + \dfrac{\cos\phi}{r}\dfrac{\partial v}{\partial z} - \dfrac{\sin\phi\cos\phi}{r^2} v\right) \end{Bmatrix}_m = [B_m]\{\delta_m\}, \tag{2}
$$

where $\{\delta_m\}$ is equal to $\{u_{im}, v_{im}, w_{im}, \psi_{im}, u_{jm}, v_{jm}, w_{jm}, \psi_{jm}\}^T$.

Property matrix:

$$
[D] = \begin{bmatrix} K_z & K_2 & 0 & 0 & 0 & 0 \\ K_2 & K_\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{z\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_z & D_2 & 0 \\ 0 & 0 & 0 & D_2 & D_\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{z\theta} \end{bmatrix}, \tag{3}
$$

in which

$$
K_z = \frac{E_z t}{1-\nu_z\nu_\theta}, \qquad K_2 = \nu_\theta K_z, \qquad K_\theta = \frac{E_\theta t}{1-\nu_z\nu_\theta}, \qquad K_{z\theta} = G_{z\theta} t,
$$

$$
D_z = \frac{E_z t^3}{12(1-\nu_z\nu_\theta)}, \qquad D_2 = \nu_\theta D_z, \qquad D_\theta = \frac{E_\theta t^3}{12(1-\nu_z\nu_\theta)}, \qquad D_{z\theta} = \frac{G_{z\theta} t^3}{12}.
$$

The strain matrix $[B_m]$ and stiffness $[S_m]$ can be found in reference [2], while the corresponding mass matrix $[m_m]$ (8×8) is given in Appendix I.

B. Top and Bottom Flanges

The top and bottom flanges of the box girder are flat plates which are curved in plan (Fig. 1c), and therefore the membrane and bending actions can actually be uncoupled and treated separately first and then subsequently combined together. Such a formulation has been attempted and the stiffness matrix can be found in reference [2].

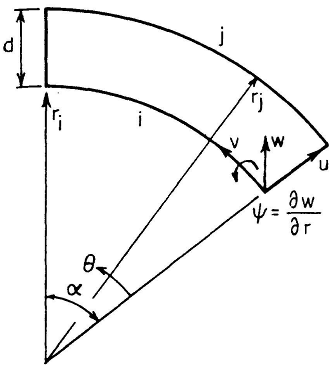


Fig. 1c. A Flange Strip.

However, it is much easier, from the programming point of view, to compute the flange strip stiffness and mass matrices directly from those of the web strip. In such cases, the angle ϕ (Fig. 1b) would be simply taken as equal to 90°.

The stiffness matrix and mass matrix of a strip are given by the following well-known relationships:

$$[S_m] = \int_A [B_m]^T [D] [B_m] \, dA, \tag{4}$$

$$[m_m] = \int_A \rho h [N_m]^T [N_m] \, dA. \tag{5}$$

Eqs. (4) and (5) refer to matrices for the local coordinate system, and such matrices must be transformed to the global coordinate system before assemblage.

The transformation matrix for a strip can be given in terms of the angle of inclination ϕ, such that

$$\{\delta_m\} = [R]\{\delta'_m\}, \tag{6a}$$

where

$$[R] = \begin{bmatrix} \sin\phi & 0 & \cos\phi & 0 \\ 0 & 1 & 0 & 0 \\ \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{6b}$$

and $\{\delta_m\}$, $\{\delta'_m\}$ are the displacements in the local and global coordinates respectively.

The transformed stiffness and mass matrices will have the form of

$$[S'_m] = [R]^T [S_m] [R] \tag{7}$$

and

$$[m'_m] = [R]^T [m_m] [R]. \tag{8}$$

Once the transformed matrices of a strip have been computed, they are assembled into overall stiffness and mass matrices in the same way as for those of a plane frame, and the resulting equation

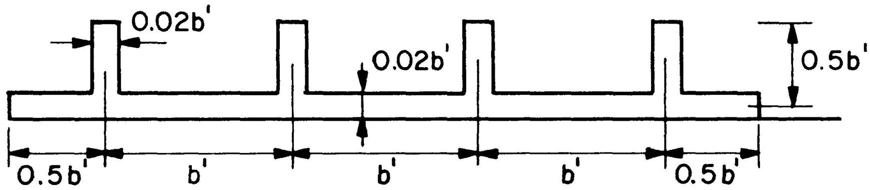
$$\{[S'_m] - \omega^2 [M'_m]\}\{\Delta'_m\} = 0 \tag{9}$$

solved by any eigenvalue solution.

Some Illustrative Numerical Examples

To illustrate the application of the preceding theory and to demonstrate its accuracy, a selection of numerical examples for both straight and curved bridges will now be presented.

Table 1. Natural Frequencies of a Stiffened Panel ($l = 6b'$) in the Range $0 < \bar{n} \leqq 0.1$

Section of the panel				
Mode Number	$\bar{n} = \frac{\omega l}{\sqrt{E/\rho}}$		Wave Number m	Type of Symmetry
	Finite Strip	Reference (8)		
1	0.0287	0.0286	1	*A*
2	0.0292	0.0291	1	*S*
3	0.0365	0.0359	1	*S*
4	0.0366	0.0362	1	*A*
5	0.0394	0.0391	1	*S*
6	0.0396	0.0395	2	*S*
7	0.0411	0.0410	2	*A*
8	0.0504	0.0504	3	*S*
9	0.0521	0.0519	3	*A*
10	0.0557	0.0555	2	*S*
11	0.0639	0.0636	1	*A*
12	0.0643	0.0641	4	*S*

In Table 1, the natural frequencies of a rectangular simply supported stiffened panel have been computed (using rectangular strips) and compared against the results of WITTRICK and WILLIAMS [8], and the two sets of results are found to be nearly identical. A total of 12 strips is used for this problem, although if symmetrical and antisymmetrical conditions were used, it would only be necessary to use 6 strips in the computation.

The circular frequencies of a curved box girder bridge (Fig. 2a) were computed by the curved strip program and the frequencies are presented in Table 2.

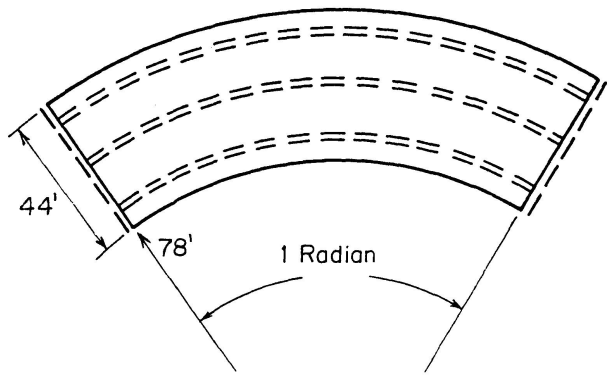


Fig. 2a. Plan of a Curved Box Girder Bridge.

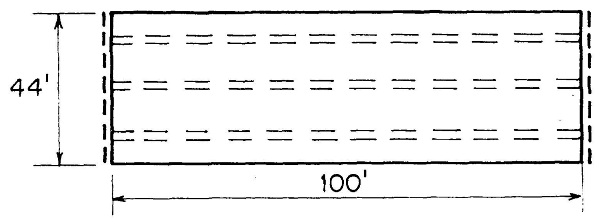


Fig. 2b. Plan of a Straight Box Girder Bridge.

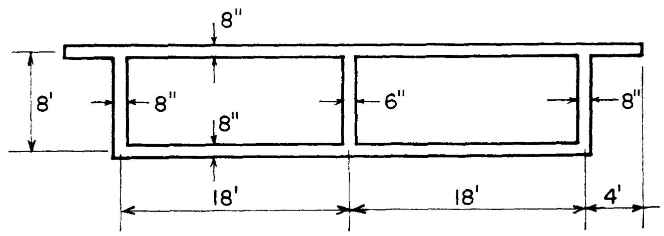


Fig. 2c. Section of the Box Girder Bridges.

Table 2. Circular Frequencies of a Curved Box Girder Bridge ($E=1$, $\nu=0.16$, $\rho=1$)

Longitudinal Mode	Circular Frequencies				
	ω_1	ω_2	ω_3	ω_4	ω_5
$m = 1$	0.002620	0.004249	0.008066	0.008501	0.008553
$m = 2$	0.008002	0.008605	0.009107	0.011427	0.013644
$m = 3$	0.009436	0.010257	0.011821	0.012246	0.020046
$m = 4$	0.010608	0.011556	0.013459	0.013523	0.025243

The modal shapes corresponding to the given frequencies are sketched in Fig. 3.

Each sketch is prepared directly from the eigenvector output which includes all the nodal displacement parameters of the box girder.

Table 3. Circular Frequencies of a Rectangular Box Girder Bridge $(E=1,\ \nu=0.16,\ \rho=1)$

Circular Frequencies (radian/sec.)	ω_1		ω_2		ω_3		ω_4		ω_5	
Longitudinal Mode	Straight Strip	Curved Strip	Straight Strip	Curved Strip	Straight Strip	Curved Strip	Straight Strip	Curved Strip	Straight Strip	Curved Strip
$m = 1$	0.003264	0.003261	0.004070	0.004067	0.007958	0.007958	0.008427	0.008425	0.009733	0.009733
$m = 2$	0.008430	0.008430	0.008531	0.008531	0.008879	0.008879	0.011322	0.011322	0.014782	0.014783
$m = 3$	0.009347	0.009347	0.010356	0.010356	0.011732	0.011732	0.012097	0.012097	0.022693	0.022693
$m = 4$	0.010585	0.010585	0.011641	0.011641	0.013183	0.013183	0.013362	0.013362	0.026011	0.026011

Table 4. Circular Frequencies of a Curved Box Girder Bridge with Concentrated Mass at Midsection of Outer Web $(E=1,\ \nu=0.16,\ \rho=1)$

Circular Frequencies	ω_1		ω_2		ω_3		ω_4		ω_5	
Longitudinal Mode	With Concentrated Mass	Without Concentrated Mass	With Concentrated Mass	Without Concentrated Mass	With Concentrated Mass	Without Concentrated Mass	With Concentrated Mass	Without Concentrated Mass	With Concentrated Mass	Without Concentrated Mass
$m = 1$	0.002283	0.002620	0.003742	0.004249	0.007965	0.008066	0.008073	0.008501	0.008503	0.008553
$m = 2$	0.008002	0.008002	0.008605	0.008605	0.009107	0.009107	0.011427	0.011427	0.013644	0.013644
$m = 3$	0.009434	0.009436	0.010016	0.010257	0.011700	0.011821	0.012240	0.012246	0.015050	0.020046
$m = 4$	0.010608	0.010608	0.011556	0.011556	0.013459	0.013459	0.013523	0.013523	0.025243	0.025243

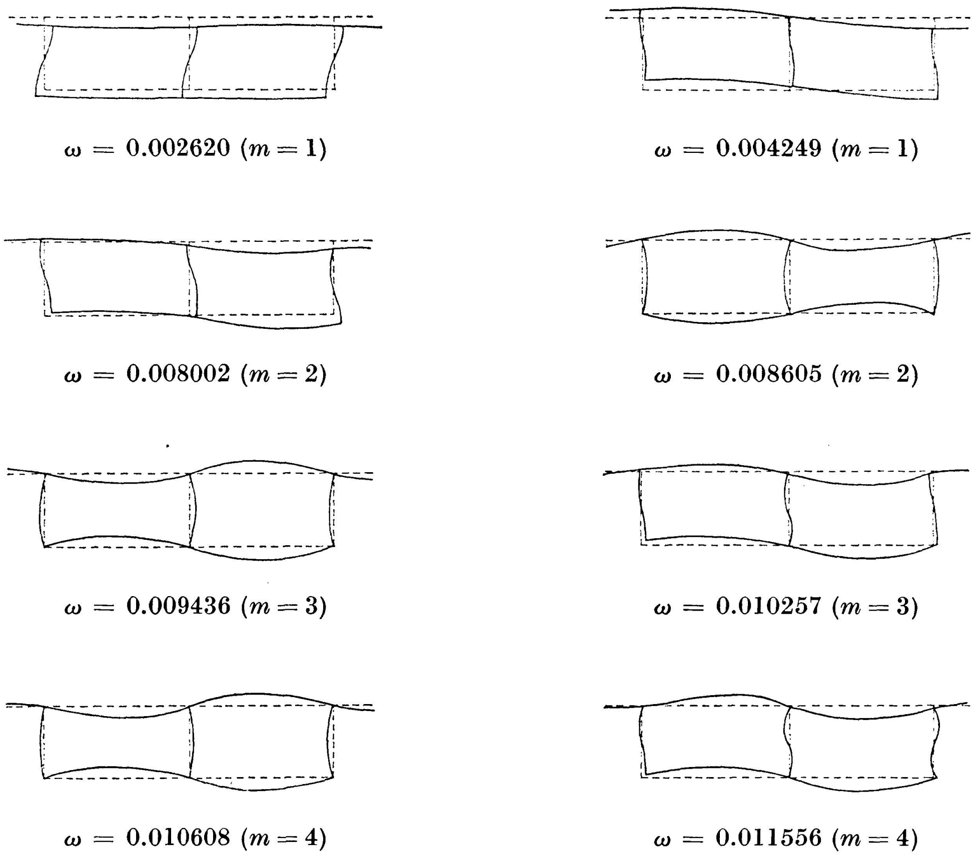


Fig. 3. Modal Shapes of a Curved Box Girder Bridge ($E = 1$, $\nu = 0.16$, $\rho = 1$).

Since the writers are unaware of any previous work on this type of structure, no comparison is offered. However, in order to test the correctness of the curved strip program, it was used to analyze a straight bridge (Fig. 2b) by assuming the subtended angle to be 0.005 radian and the mid-radius of the bridge to be 20 000 feet, so that the mid-circumferential span works out to be 100 feet. The results are then compared with those obtained from the proven straight strip program in Table 3 and the agreement has been found to be excellent for all the frequencies.

The fourth example involves the frequency analysis of the same curved box girder bridge (Example 2) with a concentrated mass attached to the top of the mid-section of the outer web. This concentrated mass, which can be due to presence of a heavy stationary vehicle, is assumed to be equal to one eighth of the total mass of the structure. From Table 4 it is possible to conclude that the additional mass will, in general, lower the natural frequencies of the bridge. However, if the concentrated mass is placed near or on a nodal line, there will be little or no effect on the frequencies. For example, no change can be observed for the frequencies which correspond to the antisymmetric modes.

Notations

a	length of the strip.
$[B]$	strain matrix.
d	width of the strip.
$[D]$	property matrix.
E_z, E_θ, ν_z $\nu_\theta, G_{z\theta}$	orthotropic material properties.
$[m]$	strip mass matrix.
$[R]$	transformation matrix.
r, θ	polar coordinates.
r_i, r_j	inner and outer radius of a strip.
$[S]$	strip stiffness matrix.
u, v, w	displacement functions.
α	subtended angle
$\{\delta\}$	displacement parameters.
ω	circular frequencies.
ρ	mass density.

Acknowledgement

The financial assistance given by the National Research Council of Canada is gratefully acknowledged.

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Appendix I

Mass Matrix of a Curved Strip

$[m_m] = \rho t \frac{\alpha}{2}$

$d\left(\frac{d}{12} S\phi + \frac{r_i}{3}\right)$			
	$d\left(\frac{d}{12} S\phi + \frac{r_i}{3}\right)$		
		$d\left(\frac{6d}{70} S\phi + \frac{13}{35} r_i\right)$	
		$d^2\left(\frac{d}{60} S\phi + \frac{11}{210} r_i\right)$	$d^3\left(\frac{d}{280} S\phi + \frac{1}{105} r_i\right)$
$d\left(\frac{d}{12} S\phi + \frac{r_i}{6}\right)$			
	$d\left(\frac{d}{12} S\phi + \frac{r_i}{6}\right)$		
		$d\left(\frac{9d}{140} S\phi + \frac{9}{70} r_i\right)$	$d^2\left(\frac{d}{60} S\phi + \frac{13}{420} r_i\right)$
		$-d^2\left(\frac{d}{70} S\phi + \frac{13}{420} r_i\right)$	$-d^3\left(\frac{d}{280} S\phi + \frac{1}{140} r_i\right)$

$S\phi = \sin\phi$.

	symmetrical		
$d\left(\frac{d}{4}S\phi+\frac{r_i}{3}\right)$			
	$d\left(\frac{d}{4}S\phi+\frac{r_i}{3}\right)$		
		$d\left(\frac{2\,d}{7}S\phi+\frac{13}{35}r_i\right)$	
		$-d^2\left(\frac{d}{28}S\phi+\frac{11}{210}r_i\right)$	$d^3\left(\frac{d}{168}S\phi+\frac{1}{105}r_i\right)$

Summary

The natural frequencies of curved and straight beam-slab or box girder bridges have been computed by the finite strip method. The bridge plates are divided into a number of curved or straight strips extending from one support to the other. By assuming suitable displacement functions for the u, v and w displacements it is possible to formulate the stiffness and mass matrices of a strip. An eigenvalue solution of the assembled overall dynamic stiffness equations will produce the desired frequencies. The stiffness and mass matrices of the straight strip can be obtained directly from those of the curved strip by changing certain variables.

Résumé

Les oscillations propres de ponts à section en T ou en caisson, courbes et droits, ont été calculées par la méthode des bandes finies. Partant d'un appui, les dalles du pont sont divisées en un nombre de bandes courbes ou droites. En supposant des fonctions de déplacement convenables pour les déplacements u, v et w on arrive à formuler les matrices de rigidité et de masse d'une bande. Une solution des valeurs propres des équations dynamiques de rigidité produit les fréquences désirées. Les matrices de rigidité et de masse des bandes droites peuvent être obtenues directement de celles des bandes courbes en changeant certaines variables.

Zusammenfassung

Die Eigenschwingungen gekrümmter und gerader Plattenbalken- oder Kastenträgerbrücken wurden mittels der Methode der finiten Elemente berechnet. Ausgehend von einem Auflager werden die Brückenplatten in eine Anzahl gekrümmter oder gerader Streifen unterteilt. Unter der Annahme passender Verschiebungsansätze für die u-, v- und w-Verschiebungen ist es möglich, die Steifigkeits- und Massmatrizen eines Streifens zu bilden. Eine Eigenwertlösung der dynamischen zusammengesetzten Steifigkeitsgleichungen liefern die Nutzfrequenzen. Die Steifigkeits- und Massmatrizen der geraden Streifen können durch Austausch gewisser Variabler direkt aus denen der gekrümmten Elemente gewonnen werden.