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Dynamic Response of Beams to a Traveling Mass

Réaction dynamique de poutres sur une masse mouvante

Dynamische Reaktion von Trägern auf eine sich bewegende Masse

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Introduction

Various studies have been made in the past in order to determine the dynamic behavior of bridge structures under the influence of moving loads. Let us mention some of the most important papers written on this subject. In 1847 R. WILLIS established the differential equation for the deflection of a single span beam, neglecting the inertia of the beam and considering the moving load as a concentrated mass [1]. An exact solution of this differential equation was given by G. STOKES in 1849 [2].

In 1905, A. N. KRYLOV [3] gave a solution assuming that the mass of the load was negligible in comparison with that of the beam. In 1923 S. P. TIMOSHENKO [4] solved the problem of a pulsating force traversing a single span beam with constant velocity. C. E. INGLIS in 1924 [5, 6], made experimental studies of the transverse vibrations of a single span bridge. A. N. LOWAN in 1935 [7], investigated the case in which the velocity of the traversing force is not constant. In 1937 SCHALLENKAMP [8] studied the case of a mass moving with constant velocity along a simply supported beam. R. S. AYRE, G. FORD, and L. S. JACOBSEN, in 1950 [9], studied the vibrations of a two-span beam under the action of a moving force of constant magnitude. In 1951, R. S. AYRE, L. S. JACOBSEN and C. S. HSU [10] published "Transverse vibration of one and two span beams under the action of a moving mass load". This investigation was experimental, with a theoretical analysis for some of the simpler cases. In 1958 RYAZANOVA [11] studied the case of a moving mass on a simply supported beam. Her approach is more sophisticated than

SCHALLENKAMP's, since it is more general, can be applied to any beam, and requires less laborious computations.

The work presented here is intended to study the problem of a mass moving with constant velocity along a beam having other than simply supported ends. As an example of such a case, a cantilever beam will be used. Thanks to the availability of the modern digital computer, it has been possible to obtain solutions with reasonable accuracy.

All computations involved were carried out at the Stanford Computation Center on an IBM 7090 computer.

Derivation of Integral-Differential Equations of Lateral Vibration of a Beam Under a Moving Mass

Let us consider a beam along which a mass particle is moving as shown in Fig. 1.

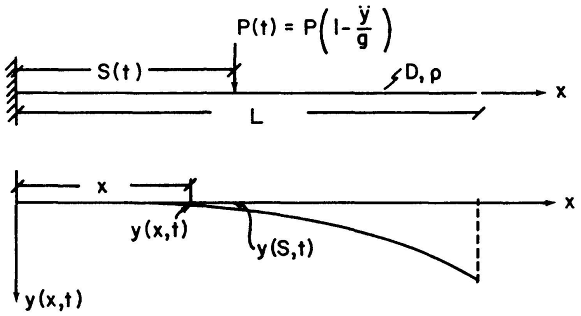

Fig. 1. Moving mass, of weight P, on a cantilever beam. The position of the moving mass is described by $S(t)$.

The basic idea [11] to be used in the derivation of equations of motion for the beam is D'Alembert's principle. To derive these equations, let us consider for the time being that a mass is moving on a massless beam. Then $j(x, S)$ represents the deflection of the beam at x under a unit load, the deflection of the beam due to the force $P(1-\ddot{y}/g)$ will be

$$y(x,t) = P\left(1-\frac{\ddot{y}}{g}\right) j(x, S), \tag{1}$$

where S is a prescribed function of time. The deflection of the beam under the point of application of the mass is obtained by substituting $x = S$ in the above equation, i.e.,

$$y(t) = P\left(1-\frac{\ddot{y}(t)}{g}\right) j(S, S). \tag{2}$$

To obtain the solution, we must solve the two differential equations (1) and (2) with two unknowns $y(t)$ and $y(x,t)$. To accomplish this we must first solve

(2) for $y(t)$ with the initial conditions prescribed only under the mass, viz: at $t=0$, $y(t)$ and $\dot{y}(t)$ are given. When the solution for $y(t)$ has been found it can be substituted into (1) and we have the general expression for the deflection of a massless beam.

Let us now study the problem of free vibrations of a beam with mass. We consider a beam with any symmetrical cross section and any arbitrary distribution of mass and stiffness, and we also allow for the existence of concentrated loads fixed to the beam. The length of the beam is divided into small elements $\Delta\eta$ and the force acting on each element is the inertia force given by $\Delta P = -\rho\ddot{y}\Delta\eta$, as illustrated in Fig. 2. The deflection of the beam under the force ΔP will be

$$\Delta y(x,t) = -j(x,\eta)\rho\ddot{y}\Delta\eta.$$

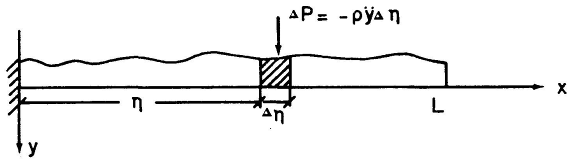

Fig. 2. Force acting on each element of the beam in the free vibration problem.

Then, by superposition the deflection due to the distributed mass in the region $0 \leqq \eta \leqq x$ is

$$y(x,t) = -\int_0^x \rho\ddot{y}(t,\eta)j(x,\eta)d\eta.$$

Similarly the deflection due to the mass elements in the region $x \leqq \eta \leqq L$ is

$$y(x,t) = -\int_x^L \rho\ddot{y}(t,\eta)j(x,\eta)d\eta.$$

Hence, the total deflection of the beam due to its mass is given by the sum of the above expressions, i.e.,

$$y(x,t) = -\int_0^x \rho\ddot{y}(t,\eta)j(x,\eta)d\eta - \int_x^L \rho\ddot{y}(t,\eta)j(x,\eta)d\eta.$$

This can be written in the form

$$y(x,t) = -\int_0^L \rho\frac{\partial^2 y(t,y)}{\partial t^2}j(x,\eta)d\eta,$$

or, if ρ is a constant,

$$y(x,t) = -\rho\int_0^L \frac{\partial^2 y(t,\eta)}{\partial t^2}j(x,\eta)d\eta. \tag{3}$$

Therefore, the deflection of the beam under a moving mass is given by the sum of (1) and (3), i.e.,

$$y(x,t) = P\left[1-\frac{1}{g}\frac{d^2y(t)}{dt^2}\right]j(x,S)-\rho\int_0^L \frac{\partial^2 y(t,\eta)}{\partial t^2} j(x,\eta)\,d\eta. \tag{4}$$

For the path of the moving mass, we let $x=S$ in the above expression and obtain

$$y(t) = P\left[1-\frac{1}{g}\frac{d^2y(t)}{dt^2}\right]j(S,S)-\rho\int_0^L \frac{\partial^2 y(t,\eta)}{\partial t^2} j(S,\eta)\,d\eta. \tag{5}$$

The solution of the problem will be obtained by solving the system of Eqs. (4) and (5) with two unknown functions $y(t)$ and $y(x,t)$. To find the solution of these equations, let us expand the function $j(x,S)$ in a Fourier series of eigenfunctions:

$$j(x,S) = \sum_{n=1}^{\infty} a_n(S)\,X_n(x). \tag{6}$$

To find the coefficients $a_n(S)$, we first write an expression for the potential energy of the beam due to a unit at the point $x=S$,

$$\Pi = \frac{D}{2}\int_0^L \left(\frac{\partial^2 j}{\partial x^2}\right)^2 dx. \tag{7}$$

With the notation $b_n = L\,m_n$, the above equation becomes

$$a_n(S) = \frac{L^4}{D}\frac{X_n(S)}{b_n^4\int_0^L X_n^2(x)\,dx}. \tag{13}$$

Substituting this coefficient into (6), it follows that

$$j(x,S) = \sum_{n=1}^{\infty}\frac{L^4}{D}\frac{X_n(S)\,X_n(x)}{b_n^4\int_0^L X_n^2(x)\,dx}. \tag{14}$$

Let us now expand $y(x,t)$ with respect to the eigenfunctions of the free vibration problem:

$$y(x,t) = \sum_{n=1}^{\infty} y_n(t)\,X_n(x). \tag{15}$$

Then the coefficients $y_n(t)$ will be found by substitution of expressions (15) and (14) into (4), which yields

$$\begin{aligned}\sum_{n=1}^{\infty} y_n(t)\,X_n(x) &= P\left\{1-\frac{1}{g}\frac{d^2y}{dt^2}\right\}\frac{L^4}{D}\sum_{n=1}^{\infty}\frac{X_n(x)\,X_n(S)}{b_n^4\int_0^L X_n^2(x)\,dx}\\ &\quad -\rho\int_0^L\left\{\frac{\partial^2}{\partial t^2}\left[\sum_{n=1}^{\infty} y_n(t)\,X_n(\eta)\right]\frac{L^4}{D}\sum_{n=1}^{\infty}\frac{X_n(x)\,X_n(\eta)}{b_n^4\int_0^L X_n^2\,dx}\right\}d\eta.\end{aligned} \tag{16}$$

From the orthogonality relation of the eigenfunctions, it follows that

$$\int_0^L \left\{ \sum_{n=1}^{\infty} \frac{d^2 y_n(t)}{dt^2} X_n(\eta) \right\} X_n(\eta)\, d\eta = \int_0^L \frac{d^2 y_n(t)}{dt^2} X_n^2(\eta)\, d\eta, \tag{17}$$

which when substituted into (16) yields

$$\begin{aligned} \sum_{n=1}^{\infty} y_n(t) X_n(x) &= \sum_{n=1}^{\infty} X_n(x) \frac{P L^4}{D} \left\{ 1 - \frac{1}{g} \frac{d^2 y}{dt^2} \right\} \frac{X_n(S)}{b_n^4 \int_0^L X_n^2\, dx} \\ &\quad - \rho \frac{L^4}{D} \sum_{n=1}^{\infty} X_n(x) \int_0^L \frac{d^2 y_n(t)}{dt^2} \frac{X_n^2(\eta)\, d\eta}{b_n^4 \int_0^L X_n^2\, dx}. \end{aligned} \tag{18}$$

From this expression the following differential equation for $y_n(t)$ is obtained:

$$\frac{d^2 y_n(t)}{dt^2} + \frac{b_n^4 a^2}{L^4} y_n(t) = \frac{P}{\rho \int_0^L X_n^2(x)\, dx} \left(1 - \frac{1}{g} \frac{d^2 y}{dt^2} \right) X_n(S), \tag{19}$$

where $a^2 = D/\rho$. Solving this differential equation with zero initial conditions, i.e., when $t = 0$, $y_n(0) = 0$, $\dot{y}_n(0) = 0$, we obtain

$$y_n(t) = \frac{P L^2}{a \rho b_n^2 \int_0^L X_n^2(x)\, dx} \int_0^t \left\{ 1 - \frac{1}{g} \frac{d^2 y}{dt^2} \right\} X_n[S(\tau)] \sin \frac{a b_n (t-\tau)}{L^2} d\tau, \tag{20}$$

where τ is a dummy variable of integration.

The deflection of the beam under the point of application of the mass is given by

$$y(t) = y(S, t) = \sum_{n=1}^{\infty} y_n(t) X_n(S). \tag{21}$$

Then, after substituting into this the expression for $y_n(t)$ given by (20), we obtain

$$y(t) = \frac{P L^2}{a \rho} \int_0^t \left\{ 1 - \frac{1}{g} \frac{d^2 y}{dt^2} \right\} \sum_{n=1}^{\infty} \frac{X_n[S(\tau)] X_n[S(t)]}{b_n^2 \int_0^L X_n^2(x)\, dx} \sin \frac{a b_n^2 (t-\tau)}{L^2} d\tau. \tag{22}$$

This is an integral-differential equation, which must be solved in order to obtain $y(t)$. When $y(t)$ is known, we substitute it into (20) and obtain the coefficients $y_n(t)$. Finally, substituting these coefficients into (15), we obtain the general expression for the deflection of the beam.

The integral-differential Eq. (22) cannot be differentiated twice with respect to t and thus reduced to an integral equation because the series

$$\sum_{n=1}^{\infty} \frac{X_n[S(\tau)] X_n[S(t)]}{b_n^2 \int_0^L X_n^2(x)\, dx} \sin \frac{a b_n^2 (t-\tau)}{L^2},$$

will be divergent after two differentiations. Hence we introduce a new variable $\varphi(t)$ such that

$$\frac{d^2 y}{dt^2} = \varphi(t), \qquad \frac{dy}{dt} = \int_0^t \varphi(\tau)\, d\tau, \qquad y(t) = \int_0^t (t-\tau)\,\varphi(\tau)\, d\tau. \tag{23}$$

Substitution of (23) into (22) gives

$$\int_0^t (t-\tau)\,\varphi(\tau)\,d\tau = \frac{P L^2}{\rho a} \int_0^t \left[1 - \frac{1}{g}\varphi(\tau)\right] \sum_{n=1}^{\infty} \frac{X_n[S(t)]\, X_n[S(\tau)]}{b_n^2 \int_0^L X_n^2(x)\,dx} \sin \frac{a b_n^2 (t-\tau)}{L^2}\, d\tau,$$

which can be rewritten as

$$\begin{aligned} &\frac{P L^2}{\rho a} \int_0^t \sum_{n=1}^{\infty} \frac{X_n[S(t)]\, X_n[S(\tau)]}{b_n^2 \int_0^L X_n^2(x)\,dx} \sin \frac{a b_n^2 (t-\tau)}{L^2}\, d\tau = \\ &\qquad \frac{P L^2}{\rho g a} \int_0^t \varphi(\tau) \sum_{n=1}^{\infty} \frac{X_n[S(t)]\, X_n[S(\tau)]}{b_n^2 \int_0^L X_n^2(x)\,dx} \sin \frac{a b_n^2 (t-\tau)}{L^2}\, d\tau + \int_0^t (t-\tau)\,\varphi(\tau)\,d\tau. \end{aligned} \tag{24}$$

Introducing the notation $F(t)$ for the left-hand side of this equation and $L(t,\tau)$ for the right-hand side, we have

$$L(t,\tau) = t - \tau + \frac{P L^2}{\rho g a} \sum_{n=1}^{\infty} \frac{X_n[S(t)]\, X_n[S(\tau)]}{b_n^2 \int_0^L X_n^2(x)\,dx} \sin \frac{a b_n^2 (t-\tau)}{L^2} \tag{25}$$

and

$$F(t) = \frac{P L^2}{\rho a} \int_0^t \sum_{n=1}^{\infty} \frac{X_n[S(t)]\, X_n[S(\tau)]}{b_n^2 \int_0^L X_n^2(x)\,dx} \sin \frac{a b_n^2 (t-\tau)}{L^2}\, d\tau. \tag{26}$$

If we let $\ddot{y} = 0$, it can be seen from (22) that $F(t)$ represents the deflection of the beam under the point of application of the moving force. In this way, Eq. (24) can be written as

$$F(t) = \int_0^t L(t,\tau)\,\varphi(\tau)\,d\tau, \tag{27}$$

which is a Volterra integral equation of the first kind [13] that cannot be reduced to an integral equation of the second kind. Using the expression for $F(t)$ together with (23) in (24), we obtain

$$y(t) = F(t) - \frac{P L^2}{g \rho a} \int_0^t \varphi(\tau) \sum_{n=1}^{\infty} \frac{X_n[S(t)]\, X_n[S(\tau)]}{b_n^2 \int_0^L X_n^2(x)\,dx} \sin \frac{a b_n^2 (t-\tau)}{L^2}\, d\tau. \tag{28}$$

The problem now consists in solving the system of equations given by (23) and (27), i.e.,

$$\begin{aligned} y(t) &= \int_0^t (t-\tau)\,\varphi(\tau)\,d\tau, \\ F(t) &= \int_0^t L(t,\tau)\,\varphi(\tau)\,d\tau, \end{aligned} \tag{29}$$

where the unknowns $y(t)$ and $\varphi(t)$ have to be determined by numerical procedure. To solve these equations by a numerical method, we will replace them approximately by a system of algebraic equations. Let us divide the interval of integration (0, t) into n equally spaced points $\tau_0, \ldots, \tau_{k-1}, \tau_k, \ldots, \tau_n$ in such a way that in the middle of the interval (τ_{k-1}, τ_k) the function $L(t,\tau)$ does not change sign. Assuming that the unknown function $\varphi(\tau)$ has a constant value in each interval, we obtain

$$\begin{aligned} F(\tau_1) &= \varphi_1 \int_0^{\tau_1} L(\tau_1, t)\,dt, \\ F(\tau_2) &= \varphi_1 \int_0^{\tau_1} L(\tau_2, t)\,dt + \varphi_2 \int_{\tau_1}^{\tau_2} L(\tau_2, t)\,dt, \\ F(\tau_3) &= \varphi_1 \int_0^{\tau_1} L(\tau_3, t)\,dt + \varphi_2 \int_{\tau_1}^{\tau_2} L(\tau_3, t)\,dt + \varphi_3 \int_{\tau_2}^{\tau_3} L(\tau_3, t)\,dt, \\ F(\tau_k) &= \varphi_1 \int_0^{\tau_1} L(\tau_k, t)\,dt + \varphi_2 \int_{\tau_1}^{\tau_2} L(\tau_k, t)\,dt + \cdots + \varphi_k \int_{\tau_{k-1}}^{\tau_k} L(\tau_k, t)\,dt. \end{aligned} \tag{30}$$

From this system of equations we can find the values of $\varphi_1, \varphi_2, \ldots, \varphi_k$ and then the values of y are obtained from

$$\begin{aligned} y_1 &= \varphi_1 \int_0^{\tau_1} (\tau_1 - t)\,dt, \\ y_2 &= \varphi_1 \int_0^{\tau_1} (\tau_2 - t)\,dt + \varphi_2 \int_{\tau_1}^{\tau_2} (\tau_2 - t)\,dt, \\ y_3 &= \varphi_1 \int_0^{\tau_1} (\tau_3 - t)\,dt + \varphi_2 \int_{\tau_1}^{\tau_2} (\tau_3 - t)\,dt + \varphi_3 \int_{\tau_2}^{\tau_3} (\tau_3 - t)\,dt, \\ y_k &= \varphi_1 \int_0^{\tau_1} (\tau_k - t)\,dt + \varphi_2 \int_{\tau_1}^{\tau_2} (\tau_k - t)\,dt + \cdots + \varphi_k \int_{\tau_{k-1}}^{\tau_k} (\tau_k - t)\,dt. \end{aligned} \tag{31}$$

When the values of $\varphi(\tau)$ and $y(\tau)$ are known, the deflection at any point of the beam is obtained from Eqs. (20) and (15).

Mass Particle Moving with Constant Velocity on a Simply Supported Beam

The eigenfunction for a simply supported beam is

$$X_n(S) = \sin n\pi \frac{x}{L}.$$

For a mass moving with constant velocity $S = Vt$, and the above expression becomes

$$X_n(S) = \sin n\pi \frac{Vt}{L}.$$

Substituting these expressions into those for $F(t)$ and $L(t,\tau)$, given by (25) and (26), we obtain [11]:

$$F(t) = 2\left(\frac{P}{\gamma L}\right)\frac{g L^2}{a\pi^2}\int_0^t \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi V t}{L} \sin\frac{n\pi V\tau}{L} \sin\frac{n^2\pi^2(t-\tau)}{L^2}\, d\tau, \tag{32a}$$

$$L(t,\tau) = (t-\tau) + 2\left(\frac{P}{\gamma L}\right)\frac{L^2}{a\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi V t}{L} \sin\frac{n\pi V\tau}{L} \sin\frac{n^2\pi^2(t-\tau)}{L^2}, \tag{32b}$$

where $\rho g = \gamma$ is the specific weight per unit length of the beam, and γL is the total weight of the beam. Integration of (32a) yields

$$F(t) = 2\left(\frac{P}{\gamma L}\right)\frac{g L^2}{a\pi^2}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi V t}{L}}{n^2}\left\{\frac{\sin\left(\frac{n\pi V t}{L}\right)+\sin\left(\frac{n^2\pi^2 a t}{L^2}\right)}{2\left(\frac{n\pi V}{L}+\frac{n^2\pi^2 a}{L^2}\right)} - \frac{\sin\left(\frac{n\pi V t}{L}\right)-\sin\left(\frac{n^2\pi^2 a t}{L^2}\right)}{2\left(\frac{n\pi V}{L}-\frac{n^2\pi^2 a}{L^2}\right)}\right\}. \tag{33}$$

while integration of (32b) between the lower limit τ_l and the upper limit τ_u yields

$$\int_{\tau_l}^{\tau_u} L(t,\tau)\,dt = (\tau_u-\tau_l)\left\{\tau-\frac{1}{2}(\tau_u+\tau_l)\right\} + 2\left(\frac{P}{\gamma l}\right)\frac{L^2}{a\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi V\tau}{L}}{n^2}$$
$$\left\{\frac{\sin\left[\left(\frac{n\pi V}{L}+\frac{n^2\pi^2 a}{L^2}\right)\tau_u-\frac{n^2\pi^2 a\tau}{L^2}\right]-\sin\left[\left(\frac{n\pi V}{L}+\frac{n^2\pi^2 a}{L^2}\right)\tau_l-\frac{n^2\pi^2 a\tau}{L}\right]}{2\left(\frac{n\pi V}{L}+\frac{n^2\pi^2 a}{L^2}\right)}\right.$$
$$\left.-\frac{\sin\left[\left(\frac{n\pi V}{L}-\frac{n^2\pi^2 a}{L^2}\right)\tau_u+\frac{n^2\pi^2 a\tau}{L^2}\right]-\sin\left[\left(\frac{n\pi V}{L}-\frac{n^2\pi^2 a}{L^2}\right)\tau_l+\frac{n^2\pi^2 a\tau}{L}\right]}{2\left(\frac{n\pi V}{L}-\frac{n^2\pi^2 a}{L^2}\right)}\right\}, \tag{34}$$

where $P/\gamma L$ is the ratio between the weight of the traveling mass and the weight of the beam. Using expressions (33) and (34) in (30), and solving this system of equations on the computer, we obtain values of $\varphi_k(\tau)$. When these are substituted into (31), we obtain the values of the deflection of the beam under the point of application of the mass.

Mass Particle Moving with Constant Velocity on a Cantilever Beam

Let us consider here the problem of a mass moving along a beam which has the eigenfunction[1])

$$X_n(\xi) = \mathrm{Ch}\, m_n \xi - \cos m_n \xi - \delta_n (\mathrm{Sh}\, m_n \xi - \sin m_n \xi),$$

where $\xi = x/L$, m_n, and δ_n are obtained from the tables [14]. To solve this problem, we must first find the corresponding expressions for $F(\tau)$ and for $\int_{\tau_l}^{\tau_u} L(t,\tau)\,dt$ and use them in (30), from which we determine the values of $\varphi_k(\tau)$. Substituting the eigenfunction $X_n(x)$ into (26) and computing the integral $\int_{\tau_l}^{\tau_u} L(t,\tau)\,dt$, we obtain after several simplifications the following expression:

$$\begin{aligned}
\int_{\tau_l}^{\tau_u} L(t,\tau)\,dt &= (\tau_u - \tau_l)\left[\tau - \frac{1}{2}(\tau_u - \tau_l)\right] + \frac{P}{\gamma L}\frac{L^2}{a}\sum_{n=1}^{\infty}\frac{X_n\left(m_n\frac{V\tau}{L}\right)}{m_n^2} \\
&\left[\frac{L^2}{a\left[m_n^2 + \left(\frac{VL}{a}\right)^2\right]}\left[\left\{X_n\left(m_n\frac{V\tau_u}{L}\right) + \cos\left(m_n\frac{V\tau_u}{L}\right) - \delta_n \sin\left(m_n\frac{V\tau_u}{L}\right)\right\}\cos\frac{a m_n^2(\tau-\tau_u)}{L^2}\right.\right. \\
&\left. - \left\{X_n\left(m_n\frac{V\tau_l}{L}\right) + \cos\left(m_n\frac{V\tau_l}{L}\right) - \delta_n \sin\left(m_n\frac{V\tau_l}{L}\right)\right\}\cos\frac{a m_n^2(\tau-\tau_l)}{L^2}\right] \\
&+ \frac{VL^3}{2a^2 m_n\left[m_n^2 + \left(\frac{VL}{a}\right)^2\right]}\left[\left\{(1-\delta_n)\exp\left(m_n\frac{V\tau_u}{L}\right)\right.\right. \\
&\left. - (1+\delta_n)\exp\left(-m_n\frac{V\tau_u}{L}\right)\right\}\sin\frac{a m_n^2(\tau-\tau_u)}{L^2} \qquad (35) \\
&\left. - \left\{(1-\delta_n)\exp\left(m_n\frac{V\tau_l}{L}\right) - (1+\delta_n)\exp\left(-m_n\frac{V\tau_l}{L}\right)\right\}\sin\frac{a m_n^2(\tau-\tau_l)}{L^2}\right] \\
&+ \frac{L^2}{2a m_n\left[-m_n + \frac{VL}{a}\right]}\left[\cos\left(m_n\frac{V\tau_u}{L} + \frac{a m_n^2(\tau-\tau_u)}{L^2}\right) - \cos\left(m_n\frac{V\tau_l}{L} + \frac{a m_n^2(\tau-\tau_l)}{L^2}\right)\right. \\
&\left. - \delta_n\left\{\sin\left(m_n\frac{V\tau_u}{L} + \frac{a m_n^2(\tau-\tau_u)}{L^2}\right) - \sin\left(m_n\frac{V\tau_l}{L} + \frac{a m_n^2(\tau-\tau_l)}{L^2}\right)\right\}\right] \\
&- \frac{L^2}{2a m_n\left[m_n + \frac{VL}{a}\right]}\left[\cos\left(-m_n\frac{V\tau_u}{L} + \frac{a m_n^2(\tau-\tau_u)}{L^2}\right)\right. \\
&- \cos\left(-m_n\frac{V\tau_l}{L} + \frac{a m_n^2(\tau-\tau_l)}{L^2}\right) \\
&\left.\left. - \delta_n\left\{\sin\left(m_n\frac{V\tau_u}{L} - \frac{a m_n^2(\tau-\tau_u)}{L^2}\right) - \sin\left(m_n\frac{V\tau_l}{L} - \frac{a m_n^2(\tau-\tau_l)}{L^2}\right)\right\}\right]\right].
\end{aligned}$$

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[1]) The solution described here is also true for other beams with different values of δ_n.

Integrating Eq. (25) from 0 to τ and comparing it with (26) we obtain

$$\int_0^\tau L(t,\tau)\,dt = \int_0^\tau L(t-\tau)\,dt + \frac{1}{g}F(\tau).$$

From this equation we conclude that we can get the expression for $F(t)$ by letting $\tau_l = 0$ in the terms contained under the summation sign in (35) which, after some transformations, gives

$$\begin{aligned} F(\tau) = \left(\frac{P}{\gamma L}\right)\frac{g L^4}{a^2}\sum_{n=1}^{\infty}\frac{X_n(\tau)}{m_n^2}\Bigg\{&\frac{X_n(\tau)}{m_n^2+\left(\frac{VL}{a}\right)^2} \\ &-\frac{2\left(\frac{VL}{a}\right)^2}{m_n^4-\left(\frac{VL}{a}\right)^4}\left[\cos(m_n\tau)-\cos\left(\frac{m_n^2 a\tau}{VL}\right)\right] \\ &+\frac{2\,\delta_n\left(\frac{VL}{a}\right)^2}{m_n^4-\left(\frac{VL}{a}\right)^4}\left[\sin(m_n\tau)-\frac{VL}{a m_n}\sin\left(\frac{m_n^2 a\tau}{VL}\right)\right]\Bigg\}. \end{aligned} \tag{36}$$

As we recall from page 224, $F(t)$ represents the deflection of the beam under the point of application of the force, and when the characterictic value m_n approaches VL/a we obtain the case of a resonance. In this case we must use L'Hôpital's rule to evaluate the following expressions:

$$\lim_{m_n\to(VL/a)}\frac{\cos(m_n\tau)-\cos\left(\frac{a m_n^2\tau}{VL}\right)}{m_n^4-\left(\frac{VL}{a}\right)^4}=\frac{2\tau\sin\left(\frac{VL}{a}\tau\right)}{4\left(\frac{VL}{a}\right)^3},$$

$$\lim_{m_n\to(VL/a)}\frac{\sin(m_n\tau)-\frac{VL}{a m_n}\sin\left(\frac{a m_n^2\tau}{VL}\right)}{m_n^4-\left(\frac{VL}{a}\right)^4}=\frac{-\tau\frac{VL}{a}\cos\left(\frac{VL}{a}\tau\right)+\sin\left(\frac{VL}{a}\tau\right)}{4\left(\frac{VL}{a}\right)^4},$$

which, when substituted into (36), give for $F(\tau)$, in the case of resonance

$$\begin{aligned} F(\tau) = \frac{P}{\gamma L}\frac{g}{2}\left(\frac{a}{V^2}\right)^2\sum_{n=1}^{\infty}X_n(\tau)\Bigg\{&X_n(\tau)+\left[\delta_n-2\tau\frac{VL}{a}\right]\sin\left(\frac{VL}{a}\tau\right) \\ &-\delta_n\tau\frac{VL}{a}\cos\left(\frac{VL}{a}\tau\right)\Bigg\}. \end{aligned} \tag{37}$$

By evaluating (35) and (36) on the computer for different times and substituting these expressions into (30), we obtain a system of equations for $\varphi_k(\tau)$. When $\varphi_k(\tau)$ has been determined, we obtain the deflection of the beam under the point of application of the mass from (31).

As an illustrative example, let us consider the case of a force and a mass moving with constant velocity along a cantilever beam as shown in Fig. 3a. Let us take $V = 120$ ft/sec,

$$L = 60 \text{ ft.}, \qquad a = \sqrt{\frac{D}{\rho}} = 20636 \frac{\text{ft.}^2}{\text{sec.}}, \qquad \frac{P}{\gamma L} = \frac{1}{2},$$

and assume that the mass particle starts moving from the built-in end. The beam is divided in ten equal parts and ten terms of the series (35) and (36) are taken into consideration. The path of the point of application of the mass as well as the path of the point of application of the force P given by (36) are illustrated in Fig. 3a, and the numerical values of the path are given in Table I. The acceleration of the mass is shown in Fig. 3b and the numerical values of the acceleration are shown in Table II.

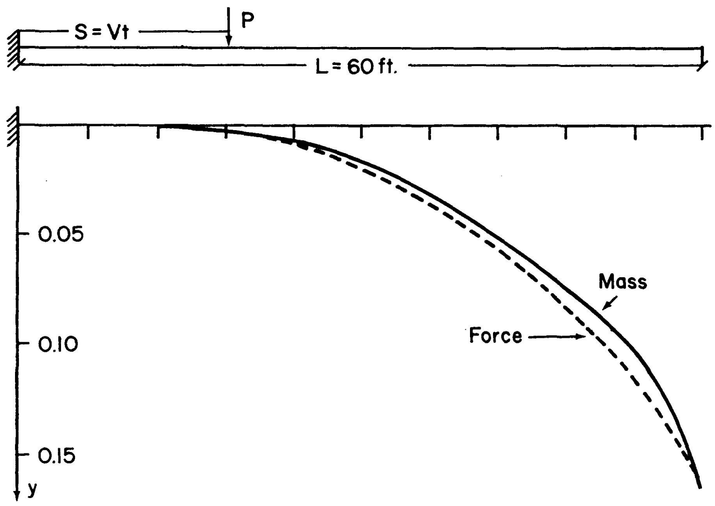

Fig. 3a. Path of the point of application of the force (in dotted line) and of the mass (in solid line) moving with constant velocity $V = 120$ ft/sec on a cantilever beam. The beam has the following properties: $L = 60$ ft., $a = \sqrt{D/\rho} = 20636$ ft²/sec., $P/\gamma L = 1/2$.

Table I. Numerical values of the deflection due to the mass and the force for different positions τ of the load on the beam described in Fig. 3a

τ	y_{force} ft.	y_{mass} ft.	τ	y_{force} ft.	y_{mass} ft.
0.1	1.223×10^{-4}	1.196×10^{-4}	0.6	3.730×10^{-2}	3.341×10^{-2}
0.2	8.634×10^{-4}	8.349×10^{-4}	0.7	5.767×10^{-2}	5.243×10^{-2}
0.3	3.227×10^{-3}	3.073×10^{-3}	0.8	8.211×10^{-2}	7.591×10^{-2}
0.4	9.241×10^{-3}	8.550×10^{-3}	0.9	1.144×10^{-1}	1.049×10^{-1}
0.5	2.067×10^{-2}	1.865×10^{-2}	1.0	1.596×10^{-1}	1.632×10^{-1}

Let us take as a second example a cantilever beam with the following characteristics: $L = 5\,m$; $a = 12.8 \times 10^{-4}\,\text{cm}^2/\text{sec}$; $P/\gamma L = 1$. The mass particle moves on the beam with $V = 10\,\text{m/sec}$. For this case, the path of the point of application of the traveling force and the traveling mass are shown in Fig. 4a and the numerical values are given in Table III. Fig. 4b represents the acceleration of the mass at chosen instants of time with the numerical values shown in Table IV.

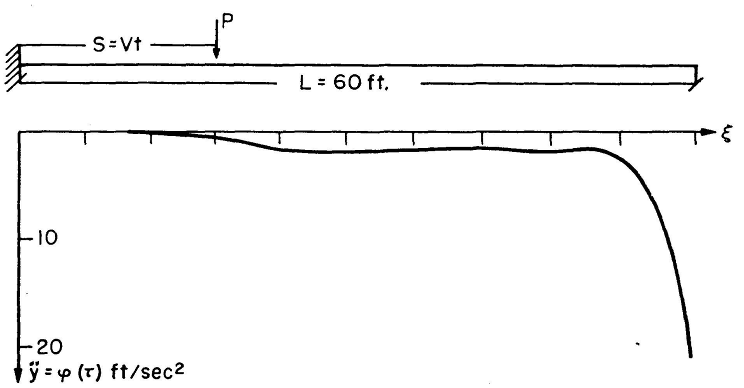

Fig. 3b. The acceleration of the mass particle for its different positions on the beam.

Table II. Numerical values of the acceleration of the mass particle for its different positions on the beam. The characteristics of the beam are described in Fig. 3a

τ	$\ddot{y}$ (ft/sec^2)	τ	$\ddot{y}$ (ft/sec^2)
0.1	0.096	0.6	1.793
0.2	0.381	0.7	1.606
0.3	0.837	0.8	1.967
0.4	1.755	0.9	2.461
0.5	1.941	1.0	20.940

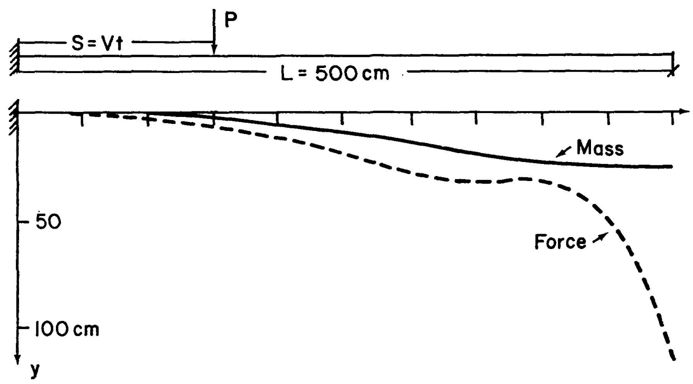

Fig. 4a. Path of the point of application of the force (in dotted line) and of the mass (in solid line) moving with constant velocity $V=10$ m/sec on a cantilever beam with the following properties: $L=5$ m, $a=\sqrt{D/\rho}=12.8\times10^4$ cm^2/sec., $P/\gamma L=1$.

The period of free vibration corresponding to the first mode of the beam in $T_1=3.5$ sec., and, to the second mode, $T_2=0.56$ sec. Since the mass crosses the beam in 0.50 sec., which is close to T_2, the effect of the second mode of free vibrations is considerable.

Table III. Numerical values of the deflection in the case of a force and mass which move on the beam whose characteristics are described in Fig. 4a

τ	y_F (cm)	y_m (cm)	τ	y_F (cm)	y_m (cm)
0.1	0.455	0.135	0.6	27.700	13.436
0.2	2.481	0.830	0.7	30.245	18.535
0.3	6.243	2.480	0.8	31.699	22.163
0.4	11.093	5.041	0.9	49.818	26.701
0.5	18.8601	8.552	1.0	112.332	26.121

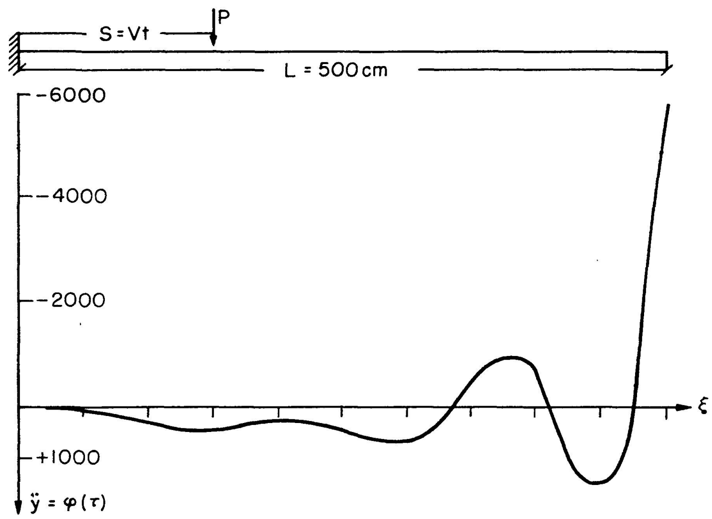

Fig. 4b. The acceleration of the mass particle for different positions on the beam. As we can see from this figure the acceleration of the mass particle at the free end of the beam is approximately $5g$, which requires some device to keep the mass attached to the beam.

Table IV. Numerical values of the acceleration of the mass particle which moves on a beam described in Fig. 4a

τ	$\ddot{y}$ (cm/sec²)	τ	$\ddot{y}$ (cm/sec²)
0.1	107.7	0.6	641.6
0.2	340.9	0.7	–468.6
0.3	422.9	0.8	–708.5
0.4	305.5	0.9	1436.0
0.5	455.5	1.0	–5530.1

In this example, the period of free vibrations corresponding to the first mode of the beam is $T_1 = 3.5$ sec. and that corresponding to the second mode is $T_2 = 0.56$ sec. Since the mass crosses the beam in 0.50 sec., which is close to T_2, the effect of the second mode of free vibrations is considerable. This explains the upward concavity of the path of the mass in the middle part of the beam. From Fig. 4a we see that there is a noticeable difference between the effect of a traveling force and that of a traveling mass, which is due mainly to the characteristic $a = \sqrt{D/\rho}$ of the beam and to the large velocity V of the mass.

As a third example let us consider a beam built-in at one end and simply supported at the other end, as illustrated in Fig. 5a, with the following characteristics: $L = 60$ ft., $a = 20636$ ft.²/sec., $P/\gamma L = 1/2$. The mass particle moves on this beam with $V = 120$ ft./sec. The paths of the point of application of the traveling mass and of the traveling force, computed by the method described in this section, are shown in Fig. 5b. From the numerical values of both paths given in Table V we see that there is a small difference between them.

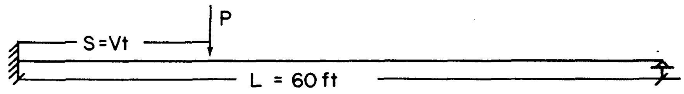

Fig. 5a

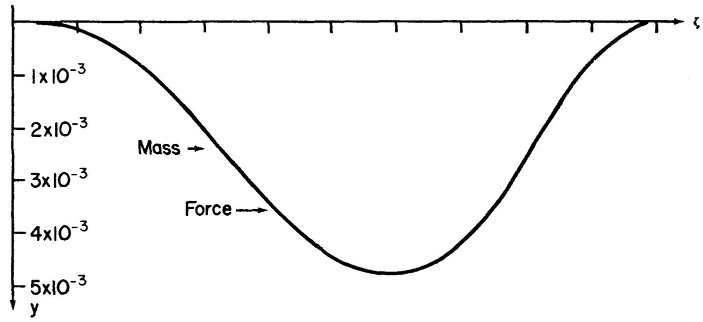

Fig. 5b.

Fig. 5. Path of the point of application of the force and of the mass moving with constant velocity $V=120$ ft/sec on a beam built-in at one end and simply supported at another end. The beam has the following characteristics:

$L=60$ ft., $a=\sqrt{D/\rho}=20636$ ft²/sec., and $P/\gamma L=1/2$.

Table V. Numerical values for the deflections of the beam described in Fig. 5a.

From here we can conclude that there is very small difference between the effects of a moving force and a moving mass.

τ	$10^3 \times y_m$ ft.	$10^3 \times y_F$	τ	$10^3 \times y_m$ ft.	$10^3 \times y_F$
0.1	0.109	0.109	0.6	4.833	4.825
0.2	0.764	0.771	0.7	4.237	4.222
0.3	2.015	2.014	0.8	2.661	2.662
0.4	3.416	3.414	0.9	0.909	0.909
0.5	4.454	4.440	1.0	0	0

Conclusions

The solution of the problem of a moving mass along a beam with arbitrary boundary conditions is obtained in this paper by an integral equation approach. This method has the advantage because by using it one can treat the problem of a mass moving with arbitrary acceleration on a beam which has arbitrary boundary conditions.

In the example illustrated in Fig. 4a one can see that there is a considerable difference between the path of the force and the path of the mass, this difference will be even more pronounced for bigger velocities. In Fig. 5b an illustration is given of a moving mass on a beam which is clamped at one end and simply supported at the other end.

This method could be applied to problems such as a moving mass on a plate resting on an elastic foundation, which has practical applications in the design of the airports. The same approach can be applied to the problem of a moving mass with arbitrary acceleration on a beam lying on an elastic foundation, which has important applications for the design of an efficient high-speed railroad transportation.

Notation

$y(x,t)$	Displacement of the beam at any point x.
$S(t)$	Abscissa of the moving load.
t	Time coordinate.
ρ	Mass per unit length of the beam.
P	Gravity force (weight of the moving mass particle).
D	Bending stiffness of the beam.
L	Length of the beam.
$y(t)=y(S,t)$	Displacement of the beam under the point of application S of the mass.
$P(t)$	Contact force acting on the beam.
$j(x,S)$	Static deflection of the beam under a unit load in place of $P(t)$.
$X_n(x)$	The n-th eigenfunction.
p_n	Angular frequency of the beam.
Π	Potential energy of the beam.
g	Acceleration of gravity.
V	Velocity of the moving mass.
γ	Specific weight per unit length of the beam.
a	$=\sqrt{D/\rho}$.
m_n^4	$=p_n^2/a^2$.
b_n	$=L m_n$.
$\varphi(t)$	$=\dfrac{d^2 y}{d t^2}$.

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Summary

The problem of a mass moving along a beam having other than simply supported ends has not been given much attention. As an example of such a case a cantilever beam will be used. An integral equation approach is used for the response of a cantilever beam to a mass particle moving with constant velocity on a beam. The obtained solution is true for the case of a concentrated mass moving with arbitrary accelerated motion on any beam whose eigenfunction has the same form as that of a cantilever beam. This method has the advantage that it can be applied to a moving mass on a plate and also to moving masses on plates or beams which rest on an elastic foundation.

Résumé

On n'a pas prêté beaucoup d'attention au problème d'une masse se mouvant le long d'une poutre ayant d'autres bouts que ceux qui sont simplement supportés. Comme exemple on utilise un porte-à-faux. Une équation intégrale approximative est utilisée pour la réaction d'un porte-à-faux sur une masse partielle se mouvant le long d'une poutre avec une vitesse constante. La solution obtenue est valable pour le cas d'une masse concentrée se mouvant d'une vitesse accélérée arbitraire sur chaque poutre dont la valeur propre a la même forme comme celle d'un porte-à-faux. Cette méthode présente l'avantage de pouvoir être appliquée à une masse se mouvant sur une dalle et de même à des masses se mouvant sur des dalles ou sur des poutres qui reposent sur une fondation élastique.

Zusammenfassung

Dem Problem einer Masse, die sich längs eines Trägers mit anderen als einfach gestützten Enden bewegt, wurde noch nicht viel Aufmerksamkeit geschenkt. Als Beispiel eines solchen Falles wird ein Kragarm gewählt. Eine näherungsweise Lösung in Form einer Integralgleichung wird für die Reaktion eines Kragarms auf ein Massenelement verwendet, welches sich mit konstanter Geschwindigkeit auf einem Träger vorwärts bewegt. Die erhaltene Lösung stimmt genau im Falle einer beliebigen beschleunigten Bewegung für jeden Träger, dessen Eigenwert mit jenem des Kragarms übereinstimmt. Die Methode bietet zudem den Vorteil, dass sie sich auch auf Massenbewegungen auf einfach oder elastisch gestützten Platten anwenden lässt.