Zeitschrift: IABSE publications = Mémoires AIPC = IVBH Abhandlungen

Band: 30 (1970)

Artikel: Importance of cell symmetry un flexural finite element method

Autor: Hrennikoff, A.

DOI: https://doi.org/10.5169/seals-23591

Importance of Cell Symmetry in Flexural Finite Element Method

Importance de la symétrie des éléments finis pour le calcul des plaques

Bedeutung der Symmetrie der endlichen Elemente für die Plattenbiegung

A. HRENNIKOFF
Sc. D., Research Professor of Civil Engineering, Emeritus, The University of British Columbia, Vancouver, B.C., Canada

General

The use of Finite Element Method for solution of plate flexure problems is based on replacement of the plate by a cell model and development of the stiffness matrices of the individual cells composing it. Out of these the computer formulates the stiffness matrix of the whole model and solves the equations for the nodal displacements. The most common shapes of cells are triangular and quadrilateral, with the nodes located at the corners of the cells. In the usual presentation these cells possess three degrees of freedom for each node: a transverse displacement and the rotations about the two co-ordinate axes in the plane of the cell, making the size of the cell's stiffness matrix 9×9 in the triangular and 12×12 in the quadrilateral cells. Cells with more than three degrees of freedom per node and with additional nodes on the sides of the cells will not be considered in the present work.

The usual procedure is to assume the deflection function of the cell in the form of a polynomial in x and y satisfying the basic biharmonic equation of plate flexure

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\,\partial y^2}+\frac{\partial^4 w}{\partial y^4}=0. \tag{1}$$

The right hand side of this equation is zero when the loads acting on the model are applied only at the nodes, which is usual.

The triangular cells are the most convenient for constructing models of non-rectangular plates or plates of irregular outline. However their non-symmetry leads to loss of precision and sometimes even to gross errors especially in flexural moments, not improved by the reduction of the mesh size. Demonstration of this proposition coupled with discussion of the necessary and desirable characteristics of the deflection polynomials is the subject of this paper.

The significance of displacement continuity on the edges of cells apart from the nodes will not be discussed here. The author's view is that the edge conformity of the adjacent cells is not a necessary condition in flexural (as well as the plane stress) elements and in this view he is supported by other investigators [1, 2].

General Deflection Polynomial

The generally used polynomials for the deflection w are composed of some or all of the following 12 terms

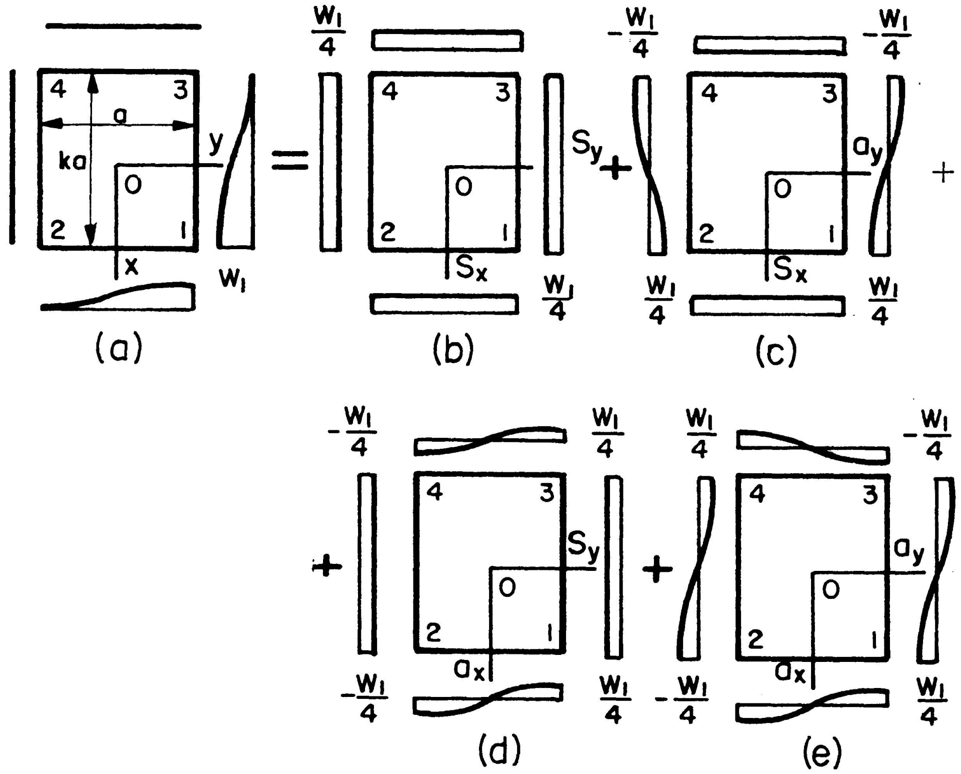
$$\begin{aligned} w = {} & A_0 + A_1 x + A_2 x^2 + A_3 x^3 + B_1 y + B_2 y^2 + B_3 y^3 + C_1 x y + C_2 x^2 y \\ & + C_3 x^3 y + C_4 x y^2 + C_5 x y^3 \end{aligned} \tag{2}$$

all of which satisfy the differential Eq. (1). The bending and torsional moments in the cell are expressed through the second derivatives of w. The three linear terms of the polynomial correspond to the three free body movements of the cell, resulting in no stresses, and the three second order terms – to the constant curvatures and torsion, the conditions which the cells must assume on infinite reduction of the mesh size, if they are to imitate faithfully the action of the plate prototype. This makes the six linear and quadratic terms compulsory irrespective of the shape of the cell. The remaining non-compulsory six terms in Eq. (2) are included or excluded depending on the type of the cell.

It may be observed that all the terms mentioned here are either symmetrical or antisymmetrical about the co-ordinate axes. The terms involving the odd powers of x and y are antisymmetrical about the y and x axes respectively, while the even power terms (including the zero power) are symmetrical.

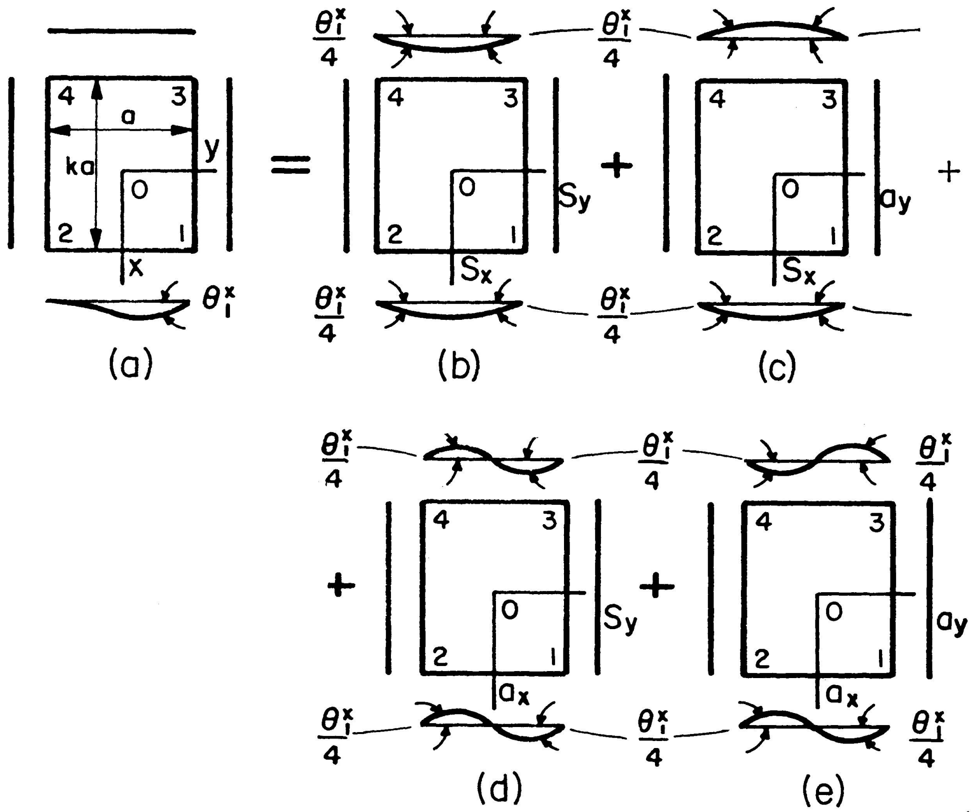
Rectangular Cell

The rectangular cell of the dimensions a by ka is referred to the axes x and y coinciding with its symmetry axes. In deriving the stiffness matrix of this cell it is sufficient to consider only the three movements of the node 1, w_1, θ_1^x and θ_1^y. It is easy to see that each of these conditions may be replaced by four symmetrical and antisymmetrical cases presented in Figs. 1, 2 and 3, and designated by the symbols s_x, a_x, s_y and a_y. In each component case the corner movements are all equal to $^1/_4$ of the total corner movement, the node 1 being always moved in the positive direction, and the other nodes – in the directions determined by the nature of the intervening axes s or a.



$\frac{W_1}{4}$
$-\frac{W_1}{4}$
4
3
a
ka
y
0
2
1
x
W_1
S_y
S_x
a_y
a_x
(a)
(b)
(c)
(d)
(e)

Fig. 1.



$\frac{\theta_1^x}{4}$
4
3
a
ka
y
0
2
1
x
θ_1^x
S_y
S_x
a_y
a_x
(a)
(b)
(c)
(d)
(e)

Fig. 2.

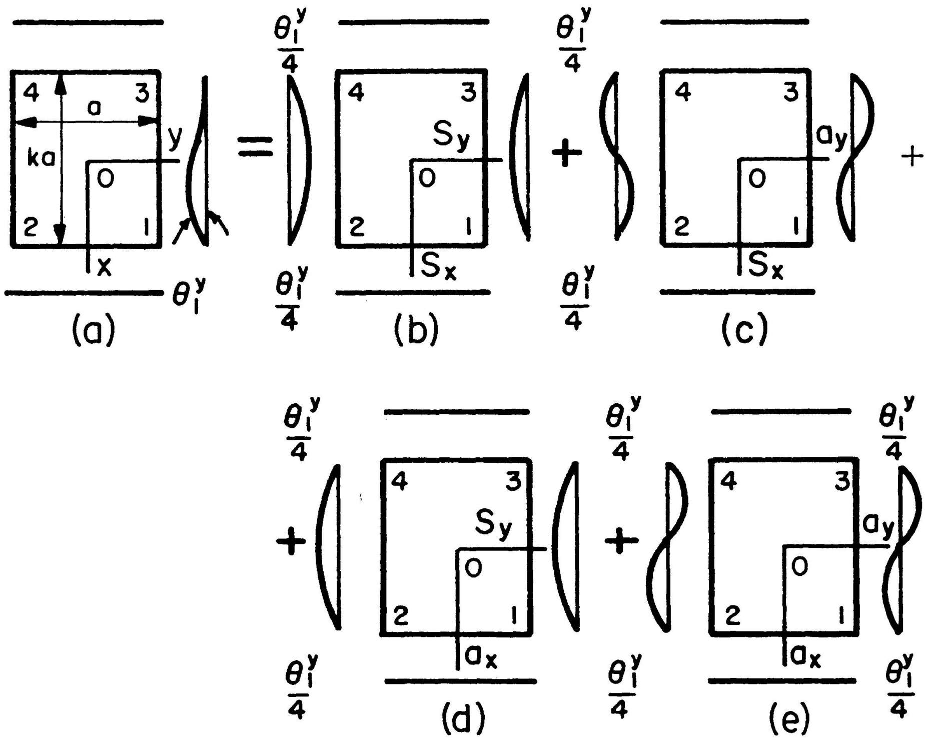


Fig. 3.

The displacement polynomials of each component case must accordingly consist of three terms, all of the same symmetry type as the case itself, i.e. $s_x s_y$, $s_x a_y$, $a_x s_y$ and $a_x a_y$. Here are the four symmetry components of the 12 term polynomial:

$$\left.\begin{aligned}
s_x s_y \ldots w_{ss} &= A_0 + A_2 x^2 + B_2 y^2, \\
s_x a_y \ldots w_{sa} &= A_1 x + A_3 x^3 + C_4 x y^2, \\
a_x s_y \ldots w_{as} &= B_1 y + B_3 y^3 + C_2 x^2 y, \\
a_x a_y \ldots w_{aa} &= C_1 x y + C_3 x^3 y + C_5 x y^3.
\end{aligned}\right\} \quad (3)$$

The partial derivatives of these polynomials are:

$$\left.\begin{aligned}
\frac{\partial w_{ss}}{\partial x} &= 2 A_2 x, & \frac{\partial w_{sa}}{\partial x} &= A_1 + 3 A_3 x^2 + C_4 y^2, \\
\frac{\partial w_{as}}{\partial x} &= 2 C_2 x y, & \frac{\partial w_{aa}}{\partial x} &= C_1 y + 3 C_3 x^2 y + C_5 y^3,
\end{aligned}\right\} \quad (4)$$

$$\left.\begin{aligned}
\frac{\partial w_{ss}}{\partial y} &= 2 B_2 y, & \frac{\partial w_{sa}}{\partial y} &= 2 C_4 x y, \\
\frac{\partial w_{as}}{\partial y} &= B_1 + 3 B_3 y^2 + C_2 x^2, & \frac{\partial w_{aa}}{\partial y} &= C_1 x + C_3 x^3 + 3 C_5 x y^2.
\end{aligned}\right\} \quad (5)$$

Each component polynomial is independent of the other three, and its coefficients are found by substituting into it and its derivatives the proper corner movements of the node 1 (or any other node) and its co-ordinates, as shown in Figs. 1, 2 and 3. Here are the resultant expressions for the three displacement fields corresponding to the transverse displacement and the two rotations of the node 1:

$$\left.\begin{aligned}
w &= \left(\frac{1}{4}+\frac{3}{4}\frac{x}{k\,a}-\frac{x^3}{k^3 a^3}+\frac{3}{4}\frac{y}{a}-\frac{y^3}{a^3}+\frac{2\,x\,y}{k\,a^2}-\frac{2\,x\,y^3}{k\,a^4}-\frac{2\,x^3\,y}{k^3 a^4}\right) w_1,\\
w &= \left(-\frac{a}{16}-\frac{x}{8\,k}-\frac{y}{8}+\frac{y^2}{4\,a}+\frac{y^3}{2\,a^2}-\frac{x\,y}{4\,k\,a}+\frac{x\,y^2}{2\,k\,a^2}+\frac{x\,y^3}{k\,a^3}\right)\theta_1^x,\\
w &= \left(\frac{k\,a}{16}+\frac{x}{8}-\frac{x^2}{4\,k\,a}-\frac{x^3}{2\,k^2 a^2}+\frac{k\,y}{8}+\frac{x\,y}{4\,a}-\frac{x^2\,y}{2\,k\,a^2}-\frac{x^3\,y}{k^2 a^3}\right)\theta_1^y.
\end{aligned}\right\} \quad (6)$$

Examining the partial polynomials in Eq. (3), it may be seen that the three terms of the polynomial w_{ss} are of the compulsory types, while the three other partial polynomials possess each only one compulsory term, thus making permissible replacement of the other terms by appropriate substitutes. For example the term $C_4\,x\,y^2$ in the polynomial $s_x\,a_y$ might be replaced by the combination $F_1\,(x^3\,y^2-x\,y^4)$ satisfying the differential equation and being of the same symmetry type. The stiffness matrix based on so modified polynomial is non-singular and would be fully suitable for solution of plate problems, but it would involve one undesirable feature; the properties of the displacements conforming to the modified polynomial would be different in the directions of the two co-ordinate axes with the result that the solution would change if the x and y axes were renamed y and x respectively. To avoid this unjustifiable non-similarity and the resultant decrease in precision, the terms of the type $x^m\,y^n$ must always be present in company with the terms $x^n\,y^m$. Thus in the example under consideration the terms $F_1\,(x^3\,y^2-x\,y^4)$ must be used together with the terms $F_2\,(x^2\,y^3-x^4\,y)$ in replacement of the part of the original polynomial $C_2\,x^2\,y+C_4\,x\,y^2$. Although the suggested alternative is quite legitimate, the original polynomial appears preferable, since its lower power terms correspond to lower, i.e. less extreme, variations of displacements and stresses within the cell. Incidentally, combinations of several terms of the same symmetry type under single parameters such as $C_2\,(x^2\,y+x^2\,y^3-x^4\,y)$ and $C_4\,(x\,y^2+x^3\,y^2-x\,y^4)$ are possible, but their advantages seem questionable.

Combinations under the same parameter of the terms of two different symmetry types, such as $C_1\,(x\,y+x^2\,y^3)$ would lead to unsymmetrical displacement pattern in symmetrical modes, as will be demonstrated presently.

Isosceles Triangle Cell

The cell (Fig. 4) is referred to the co-ordinate axes with the origin at the vertex of the triangle and the x axis placed along the axis of symmetry. The shape of the triangle is described by the aspect ratio k. The required 9 term displacement polynomial consists of the six compulsory linear and quadratic

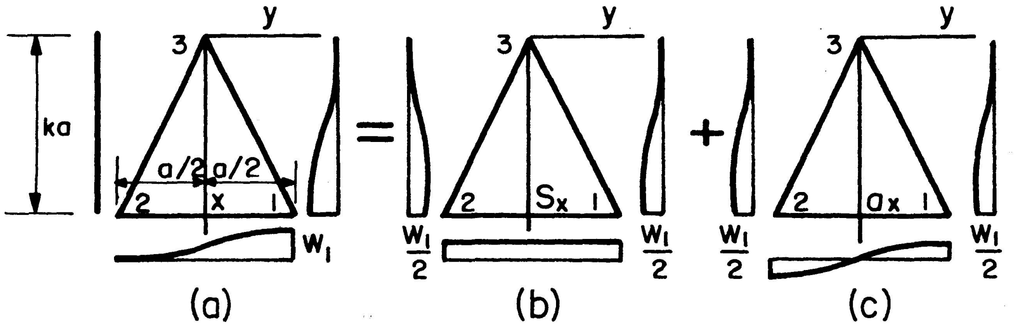


Fig. 4.

terms and the three additional terms to be chosen from the four available cubic members, $A_3 x^3$, $C_4 x y^2$, $C_2 x^2 y$ and $B_3 y^3$, the first two of which are s_x terms, and the other two $-a_x$. The question of symmetry of the terms about the y axis does not arise because the cell itself is unsymmetrical about it.

In order to form a correct polynomial it is desirable to separate the displacement modes w_1, θ_1^x and θ_1^y into their symmetrical and anti-symmetrical parts, as is done in Figs. 4, 5 and 6. The other three necessary unit movement modes w_3, θ_3^y and θ_3^x (Fig. 7) need no separation, because the first two are themselves symmetrical about the x axis and the last one – anti-symmetrical. Components of the displacement modes in Figs. 4, 5 and 6 add up to three s_x and three a_x conditions, while Fig. 7 depicts one a_x and two s_x conditions. Thus the displacement polynomial must be composed of five s_x and four a_x terms. Among the compulsory terms four are symmetrical and two – anti-

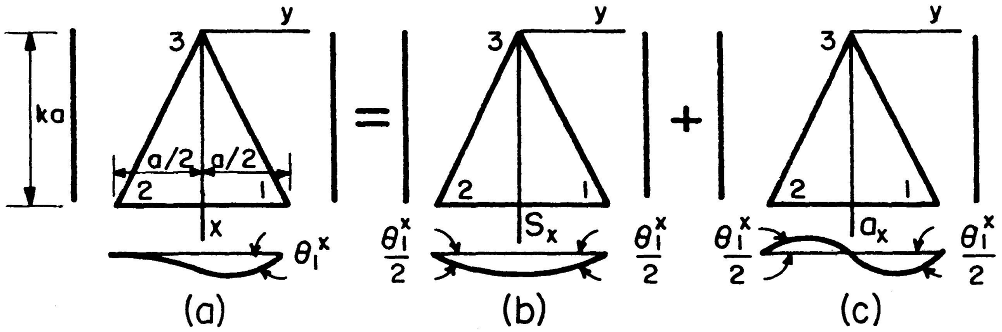


Fig. 5.

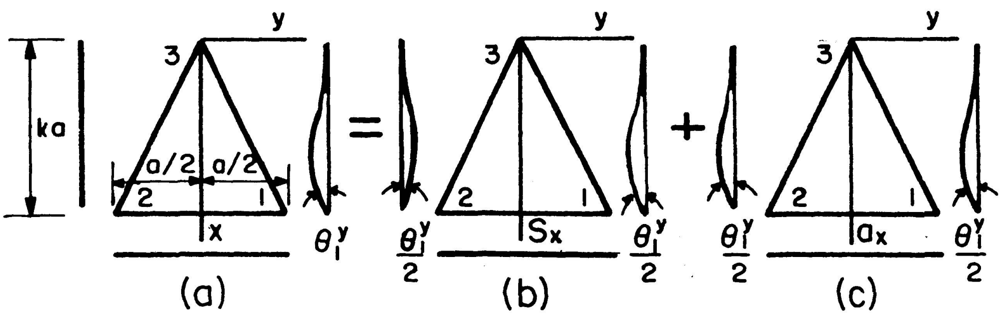


Fig. 6.

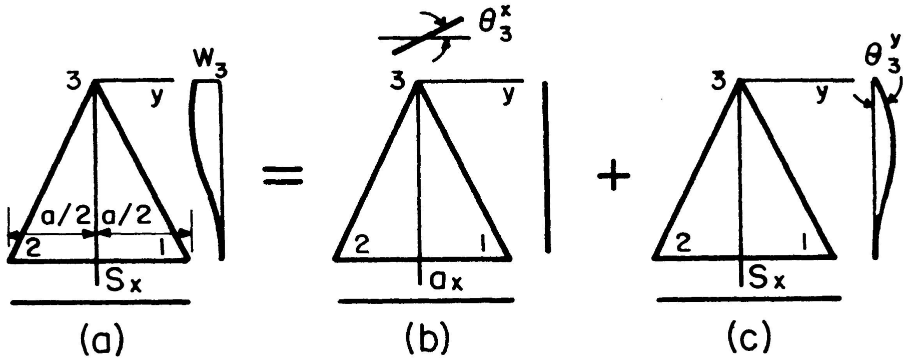


Fig. 7.

symmetrical. Therefore one more s_x term, i.e. either $A_3 x^3$ or $C_4 x y^2$, and both a_x terms $C_2 x^2 y$ and $B_3 y^3$ must be added on. Choosing arbitrarily $A_3 x^3$, the s and a parts of the displacement polynomial become

$$\left.\begin{aligned} w_s &= A_0 + A_1 x + A_2 x^2 + A_3 x^3 + B_2 y^2, \\ w_a &= B_1 y + B_3 y^3 + C_1 x y + C_2 x^2 y. \end{aligned}\right\} \tag{7}$$

Proceeding as with the rectangular cell, the six basic displacement fields are found to be

$$\left.\begin{aligned} w &= \left(\frac{3}{2k^2a^2}x^2 + \frac{3}{k a^2}x y - \frac{1}{k^3a^3}x^3 - \frac{3}{2k^2a^3}x^2 y - \frac{2}{a^3}y^3\right) w_1, \\ w &= \left(-\frac{3}{8k^2a}x^2 + \frac{2}{2a}y^2 - \frac{1}{2ka}x y + \frac{1}{4k^3a^2}x^3 + \frac{1}{4k^2a^2}x^2 y + \frac{1}{a^2}y^3\right)\theta_1^x, \\ w &= \left(\frac{1}{2ka}x^2 + \frac{1}{a}x y - \frac{1}{2k^2a^2}x^3 - \frac{1}{ka^2}x^2 y\right)\theta_1^y, \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} w &= \left(1 - \frac{3}{k^2a^2}x^2 + \frac{2}{k^3a^3}x^3\right) w_3, \\ w &= \left(y - \frac{2}{ka}x y + \frac{1}{k^2a^2}x^2 y\right)\theta_3^x, \\ w &= \left(-x + \frac{2}{ka}x^2 - \frac{1}{k^2a^2}x^3\right)\theta_3^y. \end{aligned}\right\} \tag{9}$$

It is desirable now to examine the outcome of using the combination of the s_x and a_x terms $C_2(x^2 y + x y^2)$ under the cover of a single parameter. The polynomial for θ_3^x is taken for the demonstration of the ensuing result. By using the w_a polynomial in Eq. (7), the same expression is found for the antisymmetric part of the field θ_3^x as the second of the Eqs. (9). However the presence of the term $\frac{1}{k^2a^2}x^2 y$ on this expression brings in automatically the s_x term $\frac{1}{k^2a^2}x y^2$, augmenting the w_s part of the deflection polynomial in Eq. (7). With the procedure used earlier the complete deflection field is found to be

$$w = \left(y + \frac{1}{4\,k^3\,a}\,x^2 - \frac{1}{k\,a}\,y^2 - \frac{2}{k\,a}\,x\,y - \frac{1}{4\,k^4\,a}\,x^3 + \frac{1}{k^2\,a^2}\,x^2\,y + \frac{1}{k^2\,a^2}\,x\,y^2\right)\theta_3^x. \quad (10)$$

Although the nodal displacements imposed on the cell in this mode are purely anti-symmetrical, four of the terms in its displacement polynomial are symmetrical. The stiffness matrix terms corresponding to this deflection polynomial will not conform to the requirements of symmetry and the precision of the model solution based on this matrix will naturally suffer. Thus the use of a deflection polynomial with a superfluous term combined under the same parameter with a necessary term of a different symmetry type must be avoided.

It may be observed that if an isosceles triangle cell is referred to the axes x' and y' at an angle to the axes x and y, the single term $x'^2 y'$ in the new system will give rise to both $x^2 y$ and $x\,y^2$ terms in the old system, and for this reason will be inappropriate.

Triangle Cells of Irregular Shape

A plate model may include cells in the shape of irregular triangles, such as the triangle 1–2′–3 in Fig. 8 whose node 2′ is located not far from the node 2

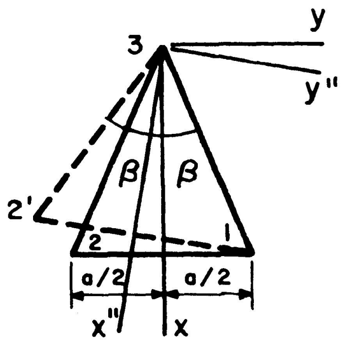


Fig. 8.

of the isosceles triangle 1–2–3, referred to the co-ordinate axes x and y, arranged as before.

Although the use of the term $C_4 x\,y^2$ in the deflection polynomial of the isosceles triangle is not permissible, the 9 term polynomial of the irregular triangle may contain either $C_2 x^2 y$ or $C_4 x\,y^2$ terms. However in view of the closeness of the node 2′ to the node 2, it is felt intuitively that the anti-symmetrical term will be the more appropriate of the two. Even better results should ensue if the cell 1–2′–3 is referred to the axes x'', y'', of which the axis x'' bisects the angle at the node 3. It is felt that of the three vertices of an irregular triangle the bisector oriented axis should be placed at the vertex enclosed between the pair of sides whose ratio is closest to unity.

Transformation of Co-ordinates

The directions of co-ordinate axes at different nodes of a cell model may be different even with cells of regular shape, and so transformation of co-ordinates becomes necessary for equalization of displacements and rotations of different cells meeting at the common nodes. The most convenient resolution of this complication requires selection of the best set of co-ordinate axes in each cell in accordance with the considerations presented above, and determination of the terms of the stiffness matrix referred to these axes. Then at each node a convenient common set of co-ordinates is chosen to which the matrix terms of all adjoining cells belonging to this particular node are converted from their individual cell axes. This conversion of the cell stiffness matrix $[K]$ in the cell co-ordinates to the matrix $[K]_c$ in the common node co-ordinates [2] is effected through the transformation matrix $[T]$ and its transpose $[T]^*$ by the equation

$$[K]_c = [T][K][T]^*. \tag{11}$$

With the use of triangular cells direct employment of a nine term deflection polynomial referred to the common axes for all cells meeting at a node would be unsatisfactory, because a cubic term $x'^2 y'$ in common co-ordinates would give rise in cell co-ordinates to both $x^2 y$ and $x y^2$ terms, one of which would be unsatisfactory and might even lead to singularity of the matrix.

Cell Symmetry

Triangular (as well as trapezoidal) cells of isosceles shape, possess only one axis of symmetry as against two in the rectangular cells. Since the deformability of space in a uni-symmetrical cell is different in the directions of the x and y axes it may be expected that the precision of plate models composed of triangular or trapezoidal cells is lower than of the rectangular cell models. The truth of this deduction is demonstrated on the examples presented below.

Examples

Models composed of rectangular and triangular cells are compared on the examples of simply supported and fixed-ended uniformly loaded square plates of Poisson's ratio 0.3. The models (Figs. 9a to f) involve the actual (or "solid") squares and the ones made of pairs of rectangular isosceles triangles in contact along the diagonals. The model (*f*) is unsymmetrical about the plate axes x and y because its contact diagonals point in the same direction over the whole plate. The four other triangular cell models have symmetrical arrangement of triangles in the four quadrants of the plate, which in the models (*b*) and (*c*) is the same in all squares, and in the other two models different in the adjacent

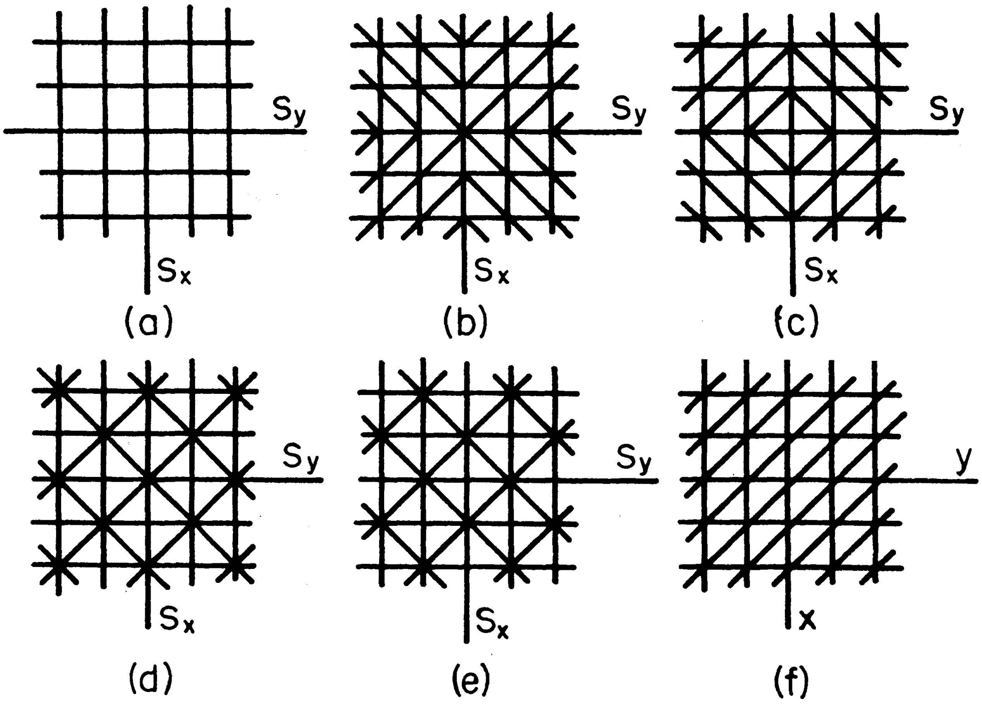


Fig. 9.

squares, with the contact diagonals pointing alternately in NE-SW and NW-SE directions.

The solutions are based on the stiffness matrix terms presented in the Appendix 1.

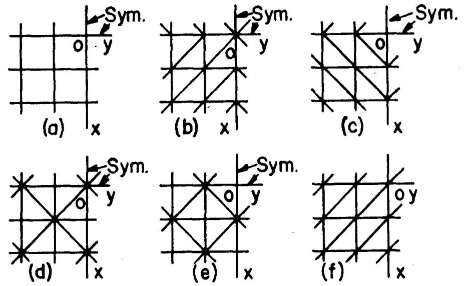
The distributed load is applied in the form of nodal concentrations coming from the areas tributary to the nodes, which means that the load of a rectangular element is divided equally between its four corners, and of a triangular one – between its three. The results of solutions are presented in Table 1.

This table contains the exact elasticity values of the central deflection and of the central bending moment M_x, as well as the bending moments M_x and M_y [2, 3] at the mid-edge of the fixed-ended plate. The precision of the finite element solutions is given by their % errors, with the plus signs corresponding to the condition when the exact value is greater than the approximate one. Models of 8×8 and 16×16 mesh were employed to indicate the convergence of the results. Plate moments were determined by dividing the nodal concentrations in the model by the tributary lengths. This method on the whole produced here better results than the deflection method.

The figures in the Table 1 show definite trends justifying, within the limits of the examples, the following deductions:

1. Of all models the ones involving the actual rectangular cells show the best precision and the best progress towards the exact values on reduction of the mesh size, in most of deflections and bending moments. The moment about the axis normal to the edge in the fixed ended plate, unlike the other moments, is inaccurate, but it improves fast as the mesh becomes finer.

Table 1. Uniformly Loaded Square Plate, Load q, Thickness t, Poisson's Ratio 0.3, % Errors of Finite Element Solutions



Finite Element Solutions		Simply Supported Plate		Fixed Ended Plate			
					Bending Moments		
		Deflection at 0, Exact $0.0443 \frac{qL^4}{Et^3}$	Bending Moment M_x at 0, Exact $0.0479\, qL^2$	Deflection at 0, Exact $0.13868 \frac{qL^4}{Et^3}$	M_x at 0 Exact $0.0231\, qL^2$	M_x at Mid-edge, Exact $-0.0513\, qL^2$	M_x at Mid-edge, Exact $-0.01539\, qL^2$
Cell Type	Mesh	% Errors		% Errors			
Squares (a)	8×8	+ 0.67%	+ 0.57%	− 2.65%	− 2.18%	+ 3.62%	+31.0 %
	16×16	+ 0.04%	+ 0.26%	− 0.41%	+ 0.06%	+ 0.83%	+16.6 %
Squares of Triangles (b)	8×8	− 4.06%	− 3.55%	−10.9 %	− 9.4 %	+13.35%	>100 %
	16×16	− 3.63%	− 2.60%	− 4.30%	− 3.35%	+16.6 %	>100 %
Squares of Triangles (c)	8×8	− 5.85%	+31.6 %	− 5.68%	+32.0 %	+ 0.22%	−45.3 %
	16×16	− 3.83%	+31.3 %	− 3.85%	+32.2 %	− 4.83%	>100 %
Squares of Triangles (d)	8×8	−14.35%	+ 5.35%	−12.5 %	− 8.88%	−18.2 %	−15.15%
	16×16	−16.0 %	− 4.85%	−14.2 %	− 4.57%	−15.88%	>100 %
Squares of Triangles (e)	8×8	−15.95%	+22.4 %	−15.3 %	+19.6 %	+19.2 %	>100%
	16×16	−16.40%	+22.5 %	−14.9 %	+22.9 %	+15.2 %	>100%
Squares of Triangles (f)	8×8	+ 0.21%	+ 0.33%	− 6.26%	− 3.67%	+ 4.05%	+ 75%
	16×16	− 0.06%	+ 0.13%	− 3.25%	− 1.19%	+ 2.10%	+ 62%

2. The unsymmetrical model (*f*) of the similarly arranged triangles comes out the next best. In fact it is better than the rectangular cell model in very precise results of the simply supported plate, but is significantly behind the leader in the fixed-ended plate.

3. Of the symmetrical triangular models two, (*b*) and (*c*), have their deflections changing in the right directions on reduction of the mesh size, but the same does not always apply to the moments, some of which are very inaccurate.

4. The symmetrical models (*d*) and (*e*) with differently oriented triangles in the adjacent squares are inaccurate and erratic both with regard to the deflections and the bending moments.

Conclusions

The following conclusions based on theoretical considerations and supported in part by the numerical results seem to be justified:

1. The terms of the deflection polynomials of the cells must conform to certain requirements, some of which are general for all types of cells, while others depend on the shape of the cell and its symmetry. Some of these requirements are compulsory, while others are only desirable for better precision.

2. Cells of triangular shape are almost always inferior in precision to rectangular cells, because they cannot imitate homogeneous isotropic elastic material. Isosceles triangle cells, similarly oriented throughout the model, such as the ones in Fig. 9 (f), are the best of the triangular cells. As their size decreases to zero the space modelled by them approaches homogeneity, while remaining anisotropic. Such cells are capable of imitating uniform flexure or torsion condition in the plate.

3. Models composed of cells of irregular triangular shape or of differently oriented isosceles triangles, may under certain conditions lead to satisfactory results in deflections. This applies particularly to some recently proposed conforming triangular cells [4]. It is not believed however that any of these models are suitable for determination of moments because the irregularity of their cell arrangement effectively precludes true imitation of a uniform stress condition in the plate.

In this connection it may also be pointed out that the common practice of selection of the centre of gravity of the cell for the assignment of stresses determined from the displacement function has no rational basis when the stress condition in the cell is non-uniform.

4. Two situations appear appropriate for use of triangular cells:

a) When the triangles are used on the edges of the plate in order to approximate its irregular shape, provided that the plate region covered by the triangles is stressed lightly.

b) When analysing circular plates or plates in the form of circular sectors, using models formed of isosceles triangles near the centre of the circle and isosceles trapezoids over most of the model.

Appendix I

Tables 2 and 3 contain nodal forces in a rectangular cell and a cell in the form of an isosceles triangle, produced by unit displacements w_1 and unit rotations θ_1^x and θ_1^y of the node 1 (see Figs. 1 to 6). The latter table presents also the corner forces caused by the unit movements of the node 3. Forces produced by the movements of the other nodes are similar and easily follow from symmetry.

Table 2. Nodal Forces in a Rectangular Cell

(Figs. 1, 2, 3)

Action $w_1 = 1$	Action $\theta_1^x = 1$	Action $\theta_1^y = 1$
$Z_1 = \left(4k+\frac{4}{k^3}+\frac{2.8}{k}-\frac{0.8\mu}{k}\right)\frac{L}{a}$	$Z_1 = \left(-2k-\frac{0.2}{k}-\frac{0.8\mu}{k}\right)L$	$Z_1 = \left(\frac{2}{k^2}+0.2+0.8\mu\right)L$
$Z_2 = \left(-4k+\frac{2}{k^3}-\frac{2.8}{k}+\frac{0.8\mu}{k}\right)\frac{L}{a}$	$Z_2 = \left(2k+\frac{0.2}{k}-\frac{0.2\mu}{k}\right)L$	$Z_2 = \left(\frac{1}{k^2}-0.2-0.8\mu\right)L$
$Z_3 = \left(2k-\frac{4}{k^3}-\frac{2.8}{k}+\frac{0.8\mu}{k}\right)\frac{L}{a}$	$Z_3 = \left(-k+\frac{0.2}{k}+\frac{0.8\mu}{k}\right)L$	$Z_3 = \left(-\frac{2}{k^2}-0.2+0.2\mu\right)L$
$Z_4 = \left(-2k-\frac{2}{k^3}+\frac{2.8}{k}-\frac{0.8\mu}{k}\right)\frac{L}{a}$	$Z_4 = \left(k-\frac{0.2}{k}+\frac{0.2\mu}{k}\right)L$	$Z_4 = \left(-\frac{1}{k^2}+0.2-0.2\mu\right)L$
$m_1^x = \left(-2k-\frac{0.2}{k}-\frac{0.8\mu}{k}\right)L$	$m_1^x = \left(\frac{4}{15k}-\frac{4\mu}{15k}+\frac{4k}{3}\right)aL$	$m_1^x = \mu a L$
$m_2^x = \left(-2k-\frac{0.2}{k}+\frac{0.2\mu}{k}\right)L$	$m_2^x = \left(-\frac{1}{15k}+\frac{\mu}{15k}+\frac{2k}{3}\right)aL$	$m_2^x = 0$
$m_3^x = \left(-k+\frac{0.2}{k}+\frac{0.8\mu}{k}\right)L$	$m_3^x = \left(-\frac{4}{15k}+\frac{4\mu}{15k}+\frac{2k}{3}\right)aL$	$m_3^x = 0$
$m_4^x = \left(-k+\frac{0.2}{k}-\frac{0.2\mu}{k}\right)L$	$m_4^x = \left(\frac{1}{15k}-\frac{\mu}{15k}+\frac{k}{3}\right)aL$	$m_4^x = 0$
$m_1^y = \left(\frac{2}{k^2}+0.2+0.8\mu\right)L$	$m_1^y = -\mu a L$	$m_1^y = \left(\frac{4}{3k}+\frac{4k}{15}-\frac{4\mu k}{15}\right)aL$
$m_2^y = \left(\frac{1}{k^2}-0.2-0.8\mu\right)L$	$m_2^y = 0$	$m_2^y = \left(\frac{2}{3k}-\frac{4k}{15}+\frac{4\mu k}{15}\right)aL$
$m_3^y = \left(\frac{2}{k^2}+0.2-0.2\mu\right)L$	$m_3^y = 0$	$m_3^y = \left(\frac{2}{3k}-\frac{k}{15}+\frac{\mu k}{15}\right)aL$
$m_4^y = \left(\frac{1}{k^2}-0.2+0.2\mu\right)L$	$m_4^y = 0$	$m_4^y = \left(\frac{1}{3k}+\frac{k}{15}-\frac{\mu k}{15}\right)aL$

Note: $L = \frac{D}{a} = \frac{E t^3}{12(1-\mu^2)a}$.

Table 3. Nodal Forces in a Triangular Cell

(Figs. 4, 5, 6 and 7)

Action $w_1 = 1$	Action $\theta_1^y = 1$	Action $\theta_3^x = 1$
$Z_1 = \left(\frac{27}{16\,k^3} + \frac{3}{2\,k} + 3\,k\right)\frac{L}{a}$ $Z_2 = \left(\frac{21}{16\,k^3} - \frac{3}{2\,k} - 3\,k\right)\frac{L}{a}$ $Z_3 = -\frac{3}{k^3}\,\frac{L}{a}$ $m_1^x = \left(-\frac{13}{32\,k^3} - \frac{1}{4\,k} - \frac{3\,k}{2} - \frac{3\,\mu}{4\,k}\right)L$ $m_2^x = \left(\frac{11}{32\,k^3} - \frac{1}{4\,k} - \frac{3\,k}{2} + \frac{\mu}{4\,k}\right)L$ $m_3^x = \left(-\frac{1}{8\,k^3} - \frac{1}{k} + \frac{\mu}{2\,k}\right)L$ $m_1^y = \left(\frac{9}{8\,k^2} + \frac{\mu}{2}\right)L$ $m_2^y = \left(\frac{7}{8\,k^2} - \frac{\mu}{2}\right)L$ $m_3^y = \frac{1}{k^2}\,L$	$Z_1 = \left(\frac{9}{8\,k^2} + \frac{\mu}{2}\right)L$ $Z_2 = \left(\frac{7}{8\,k^2} - \frac{\mu}{2}\right)L$ $Z_3 = -\frac{2}{k^2}\,L$ $m_1^x = \left(-\frac{13}{48\,k^2} - \frac{3\,\mu}{4}\right)L\,a$ $m_2^x = \left(\frac{11}{48\,k^2} + \frac{\mu}{4}\right)L\,a$ $m_3^x = -\frac{1}{12\,k^2}\,L\,a$ $m_1^y = \left(\frac{5}{6\,k} + \frac{k}{3} - \frac{\mu\,k}{3}\right)L\,a$ $m_2^y = \left(\frac{2}{3\,k} - \frac{k}{3} + \frac{\mu\,k}{3}\right)L\,a$ $m_3^y = \frac{1}{2\,k}\,L\,a$	$Z_1 = \left(-\frac{1}{8\,k^3} - \frac{1}{k} + \frac{\mu}{2\,k}\right)L$ $Z_2 = \left(\frac{1}{8\,k^3} + \frac{1}{k} - \frac{\mu}{2\,k}\right)L$ $Z_3 = 0$ $m_1^x = \left(\frac{1}{48\,k^3} + \frac{1}{6\,k} + \frac{\mu}{12\,k}\right)L\,a$ $m_2^x = \left(\frac{1}{48\,k^3} + \frac{1}{6\,k} + \frac{\mu}{12\,k}\right)L\,a$ $m_3^x = \left(\frac{1}{12\,k^3} + \frac{2}{3\,k} - \frac{2\,\mu}{3\,k}\right)L\,a$ $m_1^y = -\frac{1}{12\,k^2}\,L\,a$ $m_2^y = \frac{1}{12\,k^2}\,L\,a$ $m_3^y = 0$
Action $\theta_1^x = 1$	Action $\theta_3^y = 1$	Action $w_3 = 1$
$Z_1 = \left(-\frac{13}{32\,k^3} - \frac{1}{4\,k} - \frac{3\,k}{2} - \frac{3\,\mu}{4\,k}\right)L$ $Z_2 = \left(-\frac{11}{32\,k^3} + \frac{1}{4\,k} + \frac{3\,k}{2} - \frac{\mu}{4\,k}\right)L$ $Z_3 = \left(\frac{3}{4\,k^3} + \frac{\mu}{k}\right)L$ $m_1^x = \left(\frac{19}{192\,k^3} + \frac{1}{24\,k} + \frac{5\,k}{4} + \frac{\mu}{3\,k}\right)L\,a$ $m_2^x = \left(-\frac{17}{192\,k^3} + \frac{1}{24\,k} + \frac{k}{4} - \frac{\mu}{6\,k}\right)L\,a$ $m_3^x = \left(\frac{1}{48\,k^3} + \frac{1}{6\,k} + \frac{\mu}{12\,k}\right)L\,a$ $m_1^y = \left(-\frac{13}{48\,k^2} - \frac{3\,\mu}{4}\right)L\,a$ $m_2^y = \left(-\frac{11}{48\,k^2} - \frac{\mu}{4}\right)L\,a$ $m_3^y = -\frac{1}{4\,k^2}\,L\,a$	$Z_1 = \frac{1}{k^2}\,L$ $Z_2 = \frac{1}{k^2}\,L$ $Z_3 = -\frac{2}{k^2}\,L$ $m_1^x = -\frac{1}{4\,k^2}\,L\,a$ $m_2^x = \frac{1}{4\,k^2}\,L\,a$ $m_3^x = 0$ $m_1^y = \frac{1}{2\,k}\,L\,a$ $m_2^y = \frac{1}{2\,k}\,L\,a$ $m_3^y = \frac{1}{k}\,L\,a$	$Z_1 = -\frac{3}{k^3}\,\frac{L}{a}$ $Z_2 = -\frac{3}{k^3}\,\frac{L}{a}$ $Z_3 = \frac{6}{k^3}\,\frac{L}{a}$ $m_1^x = \left(\frac{3}{4\,k^3} + \frac{\mu}{k}\right)L$ $m_2^x = \left(-\frac{3}{4\,k^3} - \frac{\mu}{k}\right)L$ $m_3^x = 0$ $m_1^y = -\frac{2}{k^2}\,L$ $m_2^y = -\frac{2}{k^2}\,L$ $m_3^y = -\frac{2}{k^2}\,L$

Note: $L = \frac{D}{a} = \frac{E\,t^3}{12\,(1-\mu^2)\,a}$.

Notation

a	Dimension of the cell.
k	Aspect ratio of the cell.
m	Nodal moments due to unit nodal movement.
t	Plate thickness.
w	Transverse deflection.
x, y	Co-ordinate axes, co-ordinates of a point.
s_x, s_y	Symmetry axes.
a_x, a_y	Anti-symmetry axes.
Z	Transverse nodal force.
A, B, C, F	Co-efficients in the displacement polynomial.
D	Elastic constant.
E	Modulus of elasticity.
K	Stiffness matrix.
L	Elastic constant.
M	Bending moment.
T	Transformation matrix.
β	Angle.
μ	Poisson's ratio.
θ	Angle of rotation of a node.

Acknowledgement

This investigation was carried out with financial support of the National Research Council of Canada, whose contribution is gratefully acknowledged.

References

1. A. Hrennikoff: Precision of Finite Element Method in Plane Stress. Proceedings, International Association for Bridge and Structural Engineering, Zürich, Switzerland, Vol. 29-II, 1969.
2. O. C. Zienkiewicz and Y. K. Cheung: The Finite Element Method in Structural and Continuum Mechanics. McGraw-Hill Publishing Co., London.
3. S. Timoshenko: Theory of Elastic Stability. McGraw-Hill Publishing Co., New York and London.
4. K. Bell: A Refined Triangular Plate Bending Finite Element. International Journal for Numerical Methods in Engineering, Jan./March 1969.

Summary

In this study analysis is made of different terms used for deflection polynomials in the flexural finite elements of triangular and quadrilateral shapes, and the conclusion is reached that some terms are essential, while others are desirable and still others inadmissible. The triangular cells are found intrinsically inferior to the rectangular ones, especially for determination of bending moments. The theoretical expectations are confirmed by comparing the results of the finite element calculations with the values determined by the equations of elasticity.

Résumé

Dans cette étude, on examine l'influence des différents termes du polynôme de la déformée admise pour la flexion des éléments triangulaires et quadrilatères, et l'on aboutit à la conclusion que certains termes sont indispensables, tandis que d'autres sont utiles, et d'autres même inadmissibles. On a trouvé que les éléments triangulaires étaient de par leur nature inférieurs aux rectangulaires, particulièrement pour le calcul des moments de flexion. Les considérations théoriques sont confirmées par la comparaison des résultats numériques avec les valeurs obtenus par les équations d'élasticité.

Zusammenfassung

In dieser Untersuchung wurden verschiedene Terme für Durchbiegungspolynome endlicher Biegelemente von dreieckiger und viereckiger Form analysiert. Daraus folgte, daß einige Terme wesentlich, andere wünschenswert und wieder andere unzulässig sind. Die dreieckigen Elemente ergaben erheblich schlechtere Genauigkeit gegenüber den rechteckigen, insbesondere für die Bestimmung der Biegemomente. Die theoretischen Erwartungen wurden durch Vergleich mit den mittels Elastizitätsgleichungen bestimmten Werten bestätigt.