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Static Parameters of Beams on Elastic Foundation

Paramètres statiques pour des poutres sur fondation élastique

Statische Parameter von Balken auf elastischer Unterlage

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Introduction

The analysis of beams resting on elastic foundation has been developed during the second half of the past century by WINKLER [1], ZIMMERMANN [2], and SCHWEDLER [3]. HAYASHI [4] extended this type of analysis to frames and prepared a set of tables [5] facilitating the numerical calculations. New developments in this area have been initiated by UMANSKY [6], FILONENKO-BORODIC [7], and HETENEYI [8], designated as the method of initial parameters, and the method of end conditioning, respectively.

The method of initial parameters [6] forms the basis for the development of the transport matrix as shown in works of PESTEL [9], KERSTEN [10], PETERSEN [11], and others. The same approach in equation form has been introduced by BAŽANT [12] and recently restated by MIRANDA and NAIR [13].

The relationship between the transport matrix method, the flexibility method, and the stiffness method applied to the analysis of beams on elastic foundation is shown in this paper. The study is restricted to coplanar systems, consisting of straight members of constant cross-section, subjected to causes developing bending about the principal axis, normal to the system's plane. It is assumed that the material of the structure follows Hooke's law, the foundation is linearly elastic, and all deformations are small. The effect of shear and axial forces is considered negligible, but if desired adjustments may be made for these effects. The modulus of elasticity of the structure, and of the foundation are assumed to be known, and no uncertainty exists in this respect.

Sign conventions are those typical for each of the methods mentioned.

Letter symbols adopted for use in this paper are defined where they first appear, and they are arranged alphabetically in the appendix.

Differential Equation

A finite, straight bar of constant cross-section with loads, and end conditions shown in Fig. 1, is supported along its entire length l by elastic foundation of modulus k. End vectors δ = deflection, θ = slope, M = bending moment, V = shear, identified by L-, and R-subscripts, for the left, and the right end, respectively, form the corresponding state vectors.

$$[H_L] = \begin{bmatrix} \Delta_L \\ \sigma_L \end{bmatrix} = \begin{bmatrix} \delta_L \\ \Theta_L \\ M_L \\ V_L \end{bmatrix}, \qquad [H_R] = \begin{bmatrix} \Delta_R \\ \sigma_R \end{bmatrix} = \begin{bmatrix} \delta_R \\ \Theta_R \\ M_R \\ V_R \end{bmatrix}. \tag{1}$$

The geometry of the beam, and its elastic curve are defined by Fig. 1.

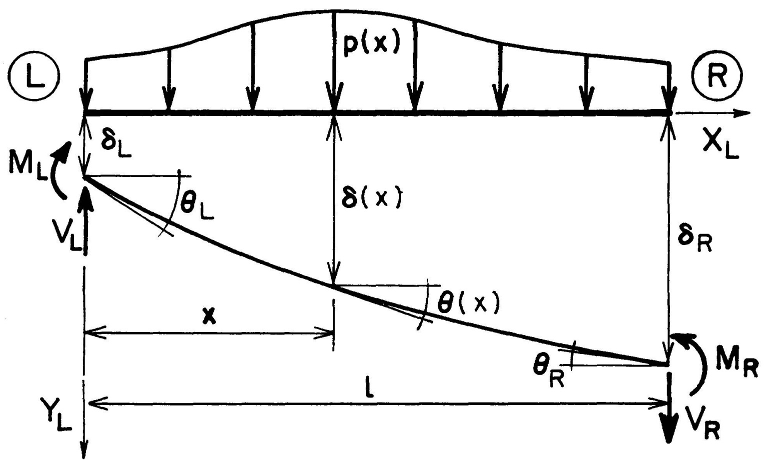


Fig. 1. Finite Beam, Elastic Foundation.

The governing differential equation in this case is:

$$EI\frac{d^4\,\delta(x)}{dx^4} + k\,\delta(x) = p(x), \tag{2}$$

in which E = beam modulus of elasticity, I = moment of inertia of the beam's cross-section, x = position coordinate, measured from L along X_L, $\delta(x)$ = deflextion at x, measured along Y_L from the initial axial of the beam, $\theta(x)$ = slope of elastic curve at the same section, and $p(x)$ = intensity of load at x.

The general solution of Eq. (2) consists of two parts,

$$\delta(u) = S(u) + L(u), \tag{3}$$

in which

$$S(u) = A\cdot a(u) + B\cdot b(u) + C\cdot c(u) + D\cdot d(u) \tag{3a}$$

is the shape function, representing the solution of the homogeneous Eq. (2), and

$$L(u) = \frac{4}{k} \int_0^u d(u-v) \cdot p(v) \cdot dv \tag{3b}$$

is the load function, representing the particular integral (Fig. 2).

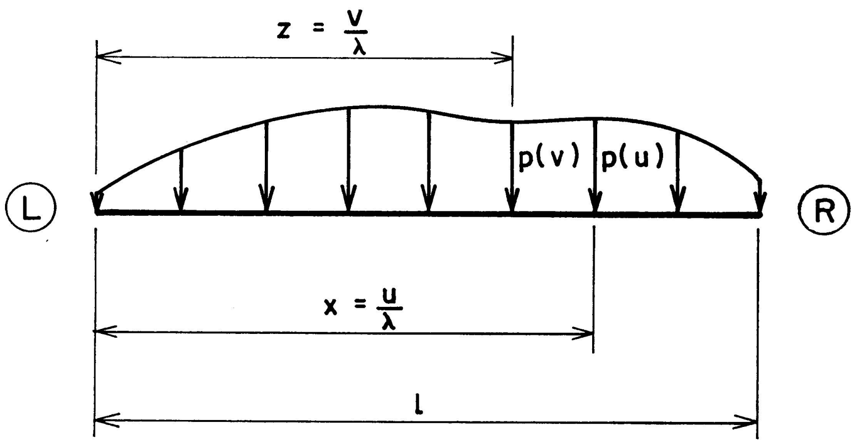


Fig. 2. General Load in Term of u and v.

New variables $$u = \lambda x, \qquad v = \lambda z \tag{4}$$

are functions of $$\lambda = \sqrt[4]{\frac{k}{4\,EI}}. \tag{5}$$

The analytical expressions of $a(u)$, $b(u)$, $c(u)$, $d(u)$ are recorded in Table 1, and designated as the static parameters. They possess certain cyclometric,

Table 1. Static Parameters

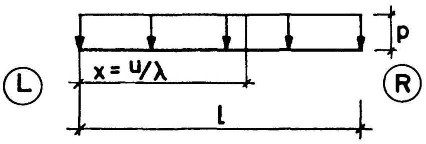
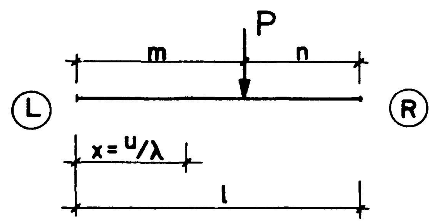
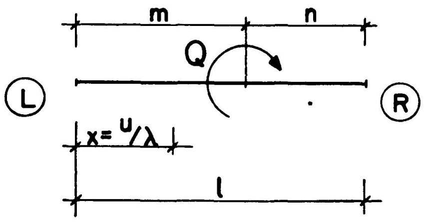
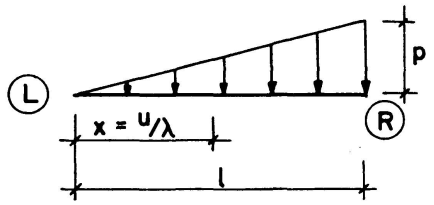
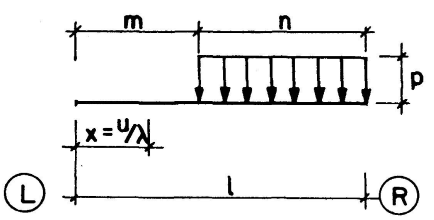
$a(\lambda x) = \cosh(\lambda x)\cos(\lambda x)$	$a(0) = 1$	$a(\lambda l) = a$
$b(\lambda x) = \dfrac{\cosh(\lambda x)\sin(\lambda x) + \sinh(\lambda x)\cos(\lambda x)}{2}$	$b(0) = 0$	$b(\lambda l) = b$
$c(\lambda x) = \dfrac{\sinh(\lambda x)\sin(\lambda x)}{2}$	$c(0) = 0$	$c(\lambda l) = c$
$d(\lambda x) = \dfrac{\cosh(\lambda x)\sin(\lambda x) - \sinh(\lambda x)\cos(\lambda x)}{4}$	$d(0) = 0$	$d(\lambda l) = d$

Table 2. Boundary Values

$S(0) = A_L = \bar{\delta}_L$	$S(\lambda l) = A_R = \bar{\delta}_R$	$L(\lambda l) = \bar{L}_{RL}$
$S'(0) = \lambda B_L = \lambda \bar{\Theta}_L$	$S'(\lambda l) = \lambda B_R = \lambda \bar{\Theta}_R$	$L'(\lambda l) = \lambda \bar{L}'_{RL}$
$S''(0) = \lambda^2 C_L = \lambda^2 \bar{M}_L$	$S''(\lambda l) = \lambda^2 C_R = \lambda^2 \bar{M}_R$	$L''(\lambda l) = \lambda^2 \bar{L}''_{RL}$
$S'''(0) = \lambda^3 D_L = \lambda^3 \bar{V}_L$	$S'''(\lambda l) = \lambda^3 D_R = \lambda^3 \bar{V}_R$	$L'''(\lambda l) = \lambda^3 \bar{L}'''_{RL}$

and cycloantimetric characteristics, useful in the evaluation of constants A, B, C, D. Since these constants two major values, depending of x ($x = 0$, $x = l$), the subscript L, or R, is used respectively as shown in Table 2. Similar is the handling of $L(u)$, and its derivates in the same table. Some special values of $L(u)$ are given in Table 3.

Table 3. Special Values of L (u)

	$u = 0$ $0 < u < \bar{\lambda}$ $u = \bar{\lambda}$	$L(0) = 0 \quad (\bar{\lambda} = \lambda l)$ $L(u) = \frac{p l^4}{4 E I} \frac{[1 - a(u)]}{\bar{\lambda}^4}$ $L(\bar{\lambda}) = \frac{p l^4}{4 E I} \frac{[1 - a]}{\bar{\lambda}^4}$
	$u < \lambda m$ $\lambda m < u < \bar{\lambda}$ $u = \bar{\lambda}$	$L(0) = 0 \quad (\bar{\lambda} = \lambda l)$ $L(u) = \frac{P l^3}{E I} \frac{d(u - \lambda m)}{\bar{\lambda}^3}$ $L(\bar{\lambda}) = \frac{P l^3}{E I} \frac{d(\lambda n)}{\bar{\lambda}^3}$
	$u < \lambda m$ $\lambda m < u < \bar{\lambda}$ $u = \bar{\lambda}$	$L(u) = 0 \quad (\bar{\lambda} = \lambda l)$ $L(u) = \frac{-Q l^2}{E I} \frac{c(u - \lambda m)}{\bar{\lambda}^2}$ $L(\bar{\lambda}) = \frac{-Q l^2}{E I} \frac{c(\lambda n)}{\bar{\lambda}^2}$
	$u = 0$ $0 < u < \bar{\lambda}$ $u = \bar{\lambda}$	$L(0) = 0 \quad (\bar{\lambda} = \lambda l)$ $L(u) = \frac{p l^4}{4 E I} \frac{[u - b(u)]}{\bar{\lambda}^5}$ $L(\bar{\lambda}) = \frac{p l^4}{4 E I} \frac{[\bar{\lambda} - b]}{\bar{\lambda}^5}$
	$u < \lambda m$ $\lambda m < u < \bar{\lambda}$ $u = \bar{\lambda}$	$L(u) = 0 \quad (\bar{\lambda} = \lambda l)$ $L(u) = \frac{p l^4}{4 E I} \frac{[1 - a(u - \lambda m)]}{\bar{\lambda}^4}$ $L(\bar{\lambda}) = \frac{p l^4}{4 E I} \frac{[1 - a(\lambda n)]}{\bar{\lambda}^4}$

Transport Matrix

With results of Tables 1 and 2, the relationship between the equivalent vectors $\bar{H}_R$ and $\bar{H}_L$ becomes

$$\bar{H}_R = \bar{t}_{RL}\bar{H}_L + \bar{l}_{RL}, \qquad \bar{H}_L = \bar{t}_{LR}\bar{H}_R + \bar{l}_{LR}. \tag{6}$$

The first algebraic form Eqs. (6) is developed in Table 4, and converted into the second one, in the same table.

$$\hat{\bar{H}}_R = \hat{t}_{RL}\hat{\bar{H}}_L, \qquad \hat{\bar{H}}_L = \hat{t}_{LR}\hat{\bar{H}}_R. \tag{7}$$

$\bar{t}_{RL}$, $\bar{t}_{LR}$ are the transport matrices of a loadless segment, and $\hat{t}_{RL}$, $\hat{t}_{LR}$ are the transport matrices, including the effect of loads in span LR. All matrices in Eqs. (6) and (7) are dimensionless, and possess following characteristics:

a) Inverse relationships

$$\bar{t}_{RL}\bar{t}_{LR} = [I], \qquad \hat{t}_{RL}\hat{t}_{LR} = [I], \tag{8}$$

b) shift relationships

$$\bar{l}_{RL} = -\bar{t}_{RL}\bar{l}_{LR}, \qquad \hat{l}_{LR} = -\hat{t}_{LR}\hat{l}_{RL}, \tag{9}$$

thus $\bar{t}_{LR}(\hat{t}_{LR})$ is the inverse of $\bar{t}_{RL}(\hat{t}_{RL})$, obtained from $\bar{t}_{RL}(\hat{t}_{RL})$ by changing the signs of b, and d; similarly, $\bar{l}_{LR}$ is obtained from $\bar{l}_{RL}$ by premultiplication. Once $\bar{t}_{RL}$-constants, and $\bar{l}_{RL}$-constants are known, Eqs. (6) and (7) are defined.

The relationship between the equivalent, state vectors $\bar{H}_R(\hat{\bar{H}}_R)$, $\bar{H}_L(\hat{\bar{H}}_L)$ and their absolute counterparts, or vice verse, is given by means of the scaling matrix $\lambda\,(\hat{\lambda})$, or the dimensioning matrix $\kappa\;(\hat{\kappa})$.

$$\begin{aligned} \bar{H}_R &= \lambda_H H_R, & \bar{H}_L &= \lambda_H H_L, \\ H_R &= \kappa_H \bar{H}_R, & H_L &= \kappa_H \bar{H}_L, \end{aligned} \tag{10}$$

$$\begin{aligned} \hat{\bar{H}}_R &= \hat{\lambda}_H \hat{H}_R, & \hat{\bar{H}}_L &= \hat{\lambda}_H \hat{H}_L, \\ \hat{H}_R &= \hat{\kappa}_H \hat{\bar{H}}_R, & \hat{H}_L &= \hat{\kappa}_H \hat{\bar{H}}_L, \end{aligned} \tag{11}$$

in which

$$\lambda_H = \begin{bmatrix} 1 & & & \\ & \dfrac{1}{\lambda} & & \\ & & -\dfrac{1}{\lambda^2 EI} & \\ & & & -\dfrac{1}{\lambda^3 EI} \end{bmatrix}, \quad (12) \qquad \kappa_H = \begin{bmatrix} 1 & & & \\ & \lambda & & \\ & & -\lambda^2 EI & \\ & & & -\lambda^3 EI \end{bmatrix} \quad (13)$$

and $\hat{\lambda}_H$, $\hat{\kappa}_H$ are obtained by extending the diagonal by one term $= 1$.

Table 4. Transport Equations

1st Form

$$\underbrace{\begin{bmatrix} \bar{\delta}_R \\ \bar{\Theta}_R \\ \hline \bar{M}_R \\ \bar{V}_R \end{bmatrix}}_{\bar{H}_R} = \underbrace{\left[\begin{array}{cc|cc} a & b & c & d \\ -4d & a & b & c \\ \hline -4c & -4d & a & b \\ -4d & -4c & -4d & a \end{array}\right]}_{\bar{t}_{RL}} \underbrace{\begin{bmatrix} \bar{\delta}_L \\ \bar{\Theta}_L \\ \hline \bar{M}_L \\ \bar{V}_L \end{bmatrix}}_{\bar{H}_L} + \underbrace{\begin{bmatrix} \bar{L}_{RL} \\ \bar{L}'_{RL} \\ \hline \bar{L}''_{RL} \\ \bar{L}'''_{RL} \end{bmatrix}}_{\bar{l}_{RL}}$$

$$\begin{bmatrix} \bar{\Delta}_R \\ \hline \bar{\sigma}_R \end{bmatrix} = \left[\begin{array}{c|c} \bar{t}_{\Delta\Delta,RL} & \bar{t}_{\Delta\sigma,RL} \\ \hline \bar{t}_{\sigma\Delta,RL} & \bar{t}_{\sigma\sigma,RL} \end{array}\right] \begin{bmatrix} \bar{\Delta}_L \\ \hline \bar{\sigma}_L \end{bmatrix} + \begin{bmatrix} \bar{l}_{\Delta,RL} \\ \hline \bar{l}_{\sigma,RL} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \bar{\delta}_L \\ \bar{\Theta}_L \\ \hline \bar{M}_L \\ \bar{V}_L \end{bmatrix}}_{\bar{H}_L} = \underbrace{\left[\begin{array}{cc|cc} a & -b & c & -d \\ 4d & a & -b & c \\ \hline -4c & 4d & a & -b \\ 4b & -4c & 4d & a \end{array}\right]}_{\bar{t}_{LR}} \underbrace{\begin{bmatrix} \bar{\delta}_R \\ \bar{\Theta}_R \\ \hline \bar{M}_R \\ \bar{V}_R \end{bmatrix}}_{\bar{H}_R} + \underbrace{\begin{bmatrix} \bar{L}_{LR} \\ \bar{L}'_{LR} \\ \hline \bar{L}''_{LR} \\ \bar{L}'''_{LR} \end{bmatrix}}_{\bar{t}_{LR}}$$

$$\begin{bmatrix} \bar{\Delta}_L \\ \hline \bar{\sigma}_L \end{bmatrix} = \left[\begin{array}{c|c} \bar{t}_{\Delta\Delta,LR} & \bar{t}_{\Delta\sigma,LR} \\ \hline \bar{t}_{\sigma\Delta,LR} & \bar{t}_{\sigma\sigma,LR} \end{array}\right] \begin{bmatrix} \bar{\Delta}_R \\ \hline \bar{\sigma}_R \end{bmatrix} + \begin{bmatrix} \bar{l}_{\Delta,LR} \\ \hline \bar{l}_{\sigma,LR} \end{bmatrix}$$

2nd Form

$$\underbrace{\begin{bmatrix} 1 \\ \hline \bar{\Delta}_R \\ \hline \bar{\sigma}_R \end{bmatrix}}_{\hat{\bar{H}}_R} = \underbrace{\left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline \bar{l}_{\Delta,RL} & \bar{t}_{\Delta\Delta,RL} & \bar{t}_{\Delta\sigma,RL} \\ \hline \bar{l}_{\sigma,RL} & \bar{t}_{\sigma\Delta,RL} & \bar{t}_{\sigma\sigma,RL} \end{array}\right]}_{\hat{\bar{t}}_{RL}} \underbrace{\begin{bmatrix} 1 \\ \hline \bar{\Delta}_L \\ \hline \bar{\sigma}_L \end{bmatrix}}_{\hat{\bar{H}}_L}$$

$$\underbrace{\begin{bmatrix} 1 \\ \hline \bar{\Delta}_L \\ \hline \bar{\sigma}_L \end{bmatrix}}_{\hat{\bar{H}}_L} = \underbrace{\left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline \bar{l}_{\Delta,LR} & \bar{t}_{\Delta\Delta,LR} & \bar{t}_{\Delta\sigma,LR} \\ \hline \bar{l}_{\sigma,LR} & \bar{t}_{\sigma\Delta,LR} & \bar{t}_{\sigma\sigma,LR} \end{array}\right]}_{\hat{\bar{t}}_{LR}} \underbrace{\begin{bmatrix} 1 \\ \hline \bar{\Delta}_R \\ \hline \bar{\sigma}_R \end{bmatrix}}_{\hat{\bar{H}}_R}$$

With help of these matrices, Eqs. (6) take the form

$$H_R = \underbrace{\kappa_H \bar{t}_{RL} \lambda_H}_{t_{RL}} H_L + \underbrace{\kappa_H \bar{l}_{RL}}_{l_{RL}}, \qquad H_L = \underbrace{\kappa_H \bar{t}_{LR} \lambda_H}_{t_{LR}} H_R + \underbrace{\kappa_H \bar{l}_{LR}}_{l_{LR}}. \tag{14}$$

Similarly, Eqs. (7) become

$$\hat{H}_R = \underbrace{\hat{\kappa}_H \hat{\bar{t}}_{RL} \hat{\lambda}_H}_{\hat{t}_{RL}} \hat{H}_L, \qquad \hat{H}_L = \underbrace{\hat{\kappa}_H \hat{\bar{t}}_{LR} \hat{\lambda}_H}_{\hat{t}_{LR}} \hat{H}_R. \tag{15}$$

Between the absolute, transport matrices holds also the relationship (8).

$$t_{RL} t_{LR} = [I], \qquad \hat{t}_{RL} \hat{t}_{LR} = [I]. \tag{16}$$

Transport Chain

Once the transport Eqs. (7) are available for a single, straight segment, their extension to the analysis of multisegment beams is accomplished by matrix multiplication.

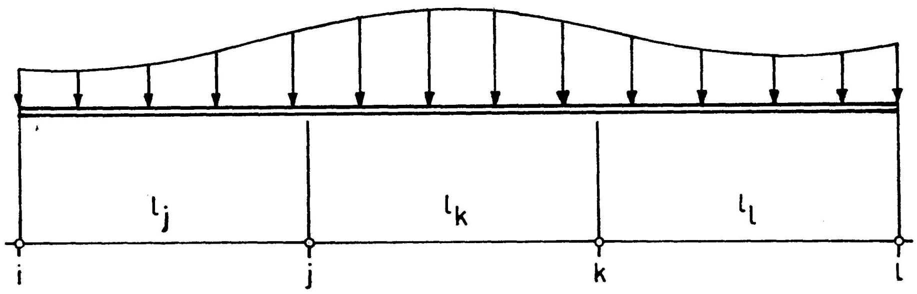


Fig. 3. Beam $i\,j\,k\,l$.

Let beam $i\,j\,k\,l$ (Fig. 3), given by its geometry (lengths of segments l_j, l_k, l_l), the moments of inertia in each span (I_j, I_k, I_l), and the modulus of elasticity E, be loaded by transverse loads, and supported by elastic foundation of modulus k.

Beginning at l, the state vector at k

$$\hat{H}_k = \hat{t}_{kl} \hat{H}_l, \tag{17a}$$

at j,

$$\hat{H}_j = \hat{t}_{jk} \hat{H}_k \tag{17b}$$

and at i,

$$\hat{H}_i = \hat{t}_{ij} \hat{H}_j. \tag{17c}$$

$\hat{t}_{ij}$, $\hat{t}_{ik}$, $\hat{t}_{kl}$ are the absolute, transport matrices of the respective spans (Table 4), and $\hat{H}_i$, $\hat{H}_j$, $\hat{H}_k$, $\hat{H}_l$ are the absolute, state vectors at the corresponding stations.

Combining Eqs. (17a, b, c) into one equation by successive substitution,

$$\hat{H}_i = \underbrace{\hat{t}_{ij} \hat{t}_{jk} \hat{t}_{kl}}_{\hat{t}_{il}} \hat{H}_l. \tag{17}$$

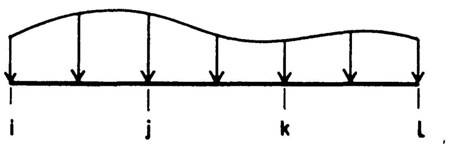
The result of the chain product in (17) is a new transport matrix $\hat{t}_{il}$, connecting directly the absolute, state vector $\hat{H}_i$ to the counterpart at l.

$$\underbrace{\bar{t}_{ij}\,\bar{t}_{jk}\,\bar{t}_{kl}}_{\hat{t}_{il}} = \left[\begin{array}{c|cc|cc} 1 & 0 & 0 & 0 & 0 \\ \hline l_{\delta,il} & t_{\delta\delta,il} & t_{\delta\Theta,il} & t_{\delta M,il} & t_{\delta V,il} \\ l_{\Theta,il} & t_{\Theta\delta,il} & t_{\Theta\Theta,il} & t_{\Theta M,il} & t_{\Theta V,il} \\ \hline l_{M,il} & t_{M\delta,il} & t_{M\Theta,il} & t_{MM,il} & t_{MV,il} \\ l_{V,il} & t_{V\delta,il} & t_{V\Theta,il} & t_{VM,il} & t_{VV,il} \end{array}\right]. \tag{18}$$

This matrix is characteristic for a given beam, and independent of the end conditions. It is designated as the transport chain, and it may be extended to any number of segments. Since there are always eight boundary values involved ($\delta_i, \theta_i, M_i, V_i, \delta_l, \theta_l, M_l, V_l$), of which four are known, and four are unknown, four equations are necessary for the solution of a given problem. The transport chain Eq. (17) provides these equations, of which only two must be solved

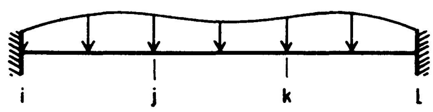
Table 5. Special cases

Free-Free Beam



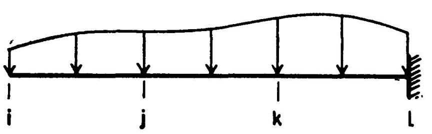
$$\begin{array}{r}\\ \\ \\ \rightarrow \\ \rightarrow\end{array}\begin{bmatrix}1\\ \delta_i\\ \Theta_i\\ 0\\ 0\end{bmatrix} = \begin{bmatrix} & & \\ & \hat{t}_{il} & \\ & & \end{bmatrix}\begin{bmatrix}1\\ \delta_l\\ \Theta_l\\ 0\\ 0\end{bmatrix}$$

Fixed-Fixed Beam



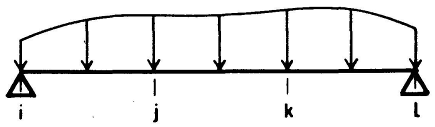
$$\begin{array}{r}\\ \rightarrow \\ \rightarrow \\ \\ \\ \end{array}\begin{bmatrix}1\\ 0\\ 0\\ M_i\\ V_i\end{bmatrix} = \begin{bmatrix} & & \\ & \hat{t}_{il} & \\ & & \end{bmatrix}\begin{bmatrix}1\\ 0\\ 0\\ M_l\\ V_l\end{bmatrix}$$

Free-Fixed Beam



$$\begin{array}{r}\\ \\ \\ \rightarrow \\ \rightarrow\end{array}\begin{bmatrix}1\\ \delta_i\\ \Theta_i\\ 0\\ 0\end{bmatrix} = \begin{bmatrix} & & \\ & \hat{t}_{il} & \\ & & \end{bmatrix}\begin{bmatrix}1\\ 0\\ 0\\ M_l\\ V_l\end{bmatrix}$$

Hinged-Hinged Beam



$$\begin{array}{r}\\ \rightarrow \\ \\ \rightarrow \\ \\ \end{array}\begin{bmatrix}1\\ 0\\ \Theta_i\\ 0\\ V_i\end{bmatrix} = \begin{bmatrix} & & \\ & \hat{t}_{il} & \\ & & \end{bmatrix}\begin{bmatrix}1\\ 0\\ \Theta_l\\ 0\\ V_l\end{bmatrix}$$

simultaneously. Four typical cases are symbolically solved in Table 5, and the starting equations are identified by $\rightrightarrows$, in each case. The handling of intermediate conditions (mechanical hinges, guides, linear springs, angular springs, linear fixities, angular fixities) developed for ordinary beam transport (9) is applicable here without modification.

Flexibilities

The end flexibility of a straight beam on elastic foundation is defined as the end deformation produced by a unit end cause, or by loads. Since the unite causes are moments and forces, the end deformations are deflections, and slopes, and the point of unit cause is the near, or the far end, sixteen unit cause flexibilities, and four load flexibilities, are required, for the formulation of a member flexibility matrix equation. Because of symmetry, and antisymmetry, only ten constants are necessary.

The derivations of the flexibility matrix follows from equations (6, rows 3, and 4 of Table 4).

$$\bar{\sigma}_R = \bar{l}_{\sigma,RL} + \bar{t}_{\sigma\Delta,RL}\bar{\Delta}_L + \bar{t}_{\sigma\sigma,RL}\bar{\sigma}_L, \qquad \bar{\sigma}_L = \bar{l}_{\sigma,LR} + \bar{t}_{\sigma\Delta,LR}\bar{\Delta}_R + \bar{t}_{\sigma\sigma,LR}\bar{\sigma}_R. \quad (19)$$

As conventional in this case,

$$\Delta_{LR} = -\bar{\Delta}_L, \qquad \Delta_{RL} = \bar{\Delta}_R \quad (20)$$

and with these changes the deflection-slope equations become

$$\begin{aligned} \Delta_{LR} &= \bar{t}^{-1}_{\sigma\Delta,RL}\bar{t}_{\sigma\sigma,RL}\bar{\sigma}_L - \bar{t}^{-1}_{\sigma\Delta,RL}\bar{\sigma}_R + \bar{t}^{-1}_{\sigma\Delta,RL}\bar{l}_{\sigma,RL}, \\ \Delta_{RL} &= \bar{t}^{-1}_{\sigma\Delta,LR}\bar{\sigma}_L - \bar{t}^{-1}_{\sigma\Delta,LR}\bar{t}_{\sigma\sigma,LR}\bar{\sigma}_R - \bar{t}^{-1}_{\sigma\Delta,LR}\bar{l}_{\sigma,LR}. \end{aligned} \quad (21)$$

In these equations,

$$\bar{t}^{-1}_{\sigma\Delta,RL} = \frac{1}{4(c^2-bd)}\begin{bmatrix} -c & d \\ b & -c \end{bmatrix}, \qquad \bar{t}^{-1}_{\sigma\Delta,LR} = \frac{1}{4(c^2-bd)}\begin{bmatrix} -c & -d \\ -b & -c \end{bmatrix} \quad (22)$$

and $\bar{t}_{\sigma\sigma,RL}$, $\bar{t}_{\sigma\sigma,LR}$, $\bar{l}_{\sigma,RL}$, $\bar{l}_{\sigma,LR}$ are submatrices defined in Table 4.

The algebraic evaluation of Eqs. (21) is shown in Table 6, and recorded in rearranged form below.

$$\begin{bmatrix} \delta_{LR} \\ \delta_{RL} \\ \hline \Theta_{LR} \\ \Theta_{RL} \end{bmatrix} = \left[\begin{array}{cc|cc} E_{LL} & E_{LR} & G_{LL} & G_{LR} \\ E_{RL} & E_{RR} & G_{RL} & G_{RR} \\ \hline -G_{LL} & G_{LR} & F_{LL} & F_{LR} \\ G_{RL} & -G_{RR} & F_{RL} & F_{RR} \end{array}\right] \begin{bmatrix} V_L \\ V_R \\ \hline M_L \\ M_R \end{bmatrix} + \begin{bmatrix} \epsilon_{LR} \\ \epsilon_{RL} \\ \hline \tau_{LR} \\ \tau_{RL} \end{bmatrix}. \quad (23)$$

Eqs. (23) define analytically the end deformation (in the flexibility sign convention), and consequently are also equations of respective elastic weights.

Table 6. Flexibilities

(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
E_{LL}	$\frac{l^3}{EJ}\frac{(bc-ad)}{\bar\lambda^3\alpha}$	$\frac{l^3}{EJ}\frac{\sinh(2\bar\lambda)-\sin(2\bar\lambda)}{2\bar\lambda^3\gamma}$	e	E_{LR}	$\frac{l^3}{EJ}\frac{d}{\bar\lambda^3\alpha}$	$\frac{l^3}{EJ}\frac{\cosh(\bar\lambda)\sin(\bar\lambda)-\sinh(\bar\lambda)\cos(\bar\lambda)}{\bar\lambda^3\gamma}$	ce
E_{RL}	$\frac{l^3}{EJ}\frac{d}{\bar\lambda^3\alpha}$	$\frac{l^3}{EJ}\frac{\cosh(\bar\lambda)\sin(\bar\lambda)-\sinh(\bar\lambda)\cos(\bar\lambda)}{\bar\lambda^3\gamma}$	ce	E_{RR}	$\frac{l^3}{EJ}\frac{(bc-ad)}{\bar\lambda^3\alpha}$	$\frac{l^3}{EJ}\frac{\sinh(2\bar\lambda)-\sin(2\bar\lambda)}{2\bar\lambda^3\gamma}$	e
G_{LL}	$\frac{l^2}{EJ}\frac{(ac+4d^2)}{\bar\lambda^2\alpha}$	$\frac{l^2}{EJ}\frac{\cosh(2\bar\lambda)-\cos(2\bar\lambda)}{2\bar\lambda^2\gamma}$	g	G_{LR}	$\frac{-l^2}{EJ}\frac{c}{\bar\lambda^2\alpha}$	$\frac{l^2}{EJ}\frac{-2\sinh(\bar\lambda)\sin(\bar\lambda)}{\bar\lambda^2\gamma}$	cg
G_{LR}	$\frac{l^2}{EJ}\frac{c}{\bar\lambda^2\alpha}$	$\frac{l^2}{EJ}\frac{2\sinh(\bar\lambda)\sin(\bar\lambda)}{\bar\lambda^2\gamma}$	$-cg$	G_{RR}	$\frac{-l^2}{EJ}\frac{(ac+4d^2)}{\bar\lambda^2\alpha}$	$\frac{l^2}{EJ}\frac{-\cosh(2\bar\lambda)+\cos(2\bar\lambda)}{2\bar\lambda^2\gamma}$	$-g$
F_{LL}	$\frac{-l}{EJ}\frac{(ab+4cd)}{\bar\lambda\alpha}$	$\frac{l}{EJ}\frac{-\sinh(2\bar\lambda)-\sin(2\bar\lambda)}{\bar\lambda\gamma}$	f	F_{LR}	$\frac{l}{EJ}\frac{b}{\bar\lambda\alpha}$	$\frac{l}{EJ}\frac{2[\cosh(\bar\lambda)\sin(\bar\lambda)+\sinh(\bar\lambda)\cos(\bar\lambda)]}{\bar\lambda\gamma}$	cf
F_{RL}	$\frac{l}{EJ}\frac{b}{\bar\lambda\alpha}$	$\frac{l}{EJ}\frac{2[\cosh(\bar\lambda)\sin(\bar\lambda)+\sinh(\bar\lambda)\cos(\bar\lambda)]}{\bar\lambda\gamma}$	cf	F_{RR}	$\frac{-l}{EJ}\frac{(ab+4cd)}{\bar\lambda\alpha}$	$\frac{l}{EJ}\frac{-\sinh(2\bar\lambda)-\sin(2\bar\lambda)}{\bar\lambda\gamma}$	f
ϵ_{LR}	$\frac{-c\bar L''_{RL}+d\bar L'''_{RL}}{\alpha}$	$\frac{1}{\gamma}\{-2\sinh(\bar\lambda)\sin(\bar\lambda)\bar L''_{RL}+[\cosh(\bar\lambda)\sin(\bar\lambda)-\sinh(\bar\lambda)\cos(\bar\lambda)]\bar L'''_{RL}\}$					
ϵ_{RL}	$\frac{c\bar L''_{LR}+d\bar L'''_{LR}}{\alpha}$	$\frac{1}{\gamma}\{2\sinh(\bar\lambda)\sin(\bar\lambda)\bar L''_{LR}+[\cosh(\bar\lambda)\sin(\bar\lambda)-\sinh(\bar\lambda)\cos(\bar\lambda)]\bar L'''_{LR}\}$				$\bar\lambda=\lambda l$	
τ_{LR}	$\frac{\bar\lambda(b\bar L''_{RL}-c\bar L'''_{RL})}{l\alpha}$	$\frac{2\bar\lambda}{l\gamma}\{[\cosh(\bar\lambda)\sin(\bar\lambda)+\sinh(\bar\lambda)\cos(\bar\lambda)]\bar L''_{RL}-\sinh(\bar\lambda)\sin(\bar\lambda)\bar L'''_{RL}\}$				$\alpha=4(c^2-bd)$ $\gamma=4\alpha=\cosh(2\bar\lambda)+\cos(2\bar\lambda)-2$	
τ_{RL}	$\frac{\bar\lambda(b\bar L''_{LR}+c\bar L'''_{LR})}{l\alpha}$	$\frac{2\bar\lambda}{l\gamma}\{[\cosh(\bar\lambda)\sin(\bar\lambda)+\sinh(\bar\lambda)\cos(\bar\lambda)]\bar L''_{LR}+\sinh(\bar\lambda)\sin(\bar\lambda)\bar L'''_{LR}\}$					

From Table 6, the following identities are observed:

$$E_{LL} = E_{RR} = e, \qquad E_{LR} = E_{RL} = c\,e,$$
$$G_{LL} = -G_{RR} = g, \quad (24\text{a}) \qquad G_{LR} = -G_{RL} = c\,g, \quad (24\text{b})$$
$$F_{LL} = F_{RR} = f. \qquad F_{LR} = F_{RL} = c\,f.$$

Constants e, g, f are the near, end flexibilities, whereas $c\,e$, $c\,g$, $c\,f$ are the far, end flexibilities (or sometimes called carry-over flexibilities). Constants ϵ_{LR}, ϵ_{RL}, τ_{LR}, τ_{RL} are the end deformations, caused by loads.

Eqs. (23), in observance of identities (24a, b), yield a new relationship of G-matrices.

$$\begin{bmatrix} \delta_{(LR)} \\ \hline \Theta_{(LR)} \end{bmatrix} = \left[\begin{array}{c|c} E_{(LR)} & G_{(LR)} \\ \hline -G^*_{(LR)} & F_{(LR)} \end{array}\right] \begin{bmatrix} V_{(LR)} \\ \hline M_{(LR)} \end{bmatrix} + \begin{bmatrix} \epsilon_{(LR)} \\ \hline \tau_{(LR)} \end{bmatrix}. \qquad (25)$$

Flexibility Chain

Once the flexibility matrix (25) is available for a single segment, the analysis of multi-segment bars is accomplished by chain overlapping.

Considering the beam $i\,j\,k\,l$ (Fig. 3), the continuity at j (any station) requires that,

$$\begin{aligned} \delta_{ji} + \delta_{jk} &= 0, \\ \theta_{ji} + \theta_{jk} &= 0. \end{aligned} \qquad (26)$$

With notation (23), and new equivalents:

$$\sum E_{jj};\ \sum G_{jj};\ \sum F_{jj};\ \sum \epsilon_j;\ \sum \tau_j$$

designating the sum of the respective, near end flexibilities and load flexibilities at j, the compatibility Eqs. (26) become typical joint, force-moment equations.

$$\left\{\begin{matrix} E_{ji} V_i + \sum E_{jj} V_j + E_{jk} V_k \\ G_{ji} M_i + \sum G_{jj} M_j + G_{jk} M_k \end{matrix}\right\} + \sum \epsilon_j = 0,$$
$$\left\{\begin{matrix} G_{ji} V_i - \sum G_{jj} V_j + G_{jk} V_k \\ F_{ji} M_i + \sum F_{jj} M_j + F_{jk} M_k \end{matrix}\right\} + \sum \tau_j = 0. \qquad (27)$$

Eqs. (27) have a general meaning, and are used as recurrence formulas.

With end conditions to be discussed later, the complete joint force-moment matrix takes the form shown in Table 7.

In symbolic form

$$\sum \Delta_{i-l} = \varphi_{\Delta\sigma, i-l}\,\sigma_{i-l} + \sum \eta_{i-l} = 0. \qquad (28)$$

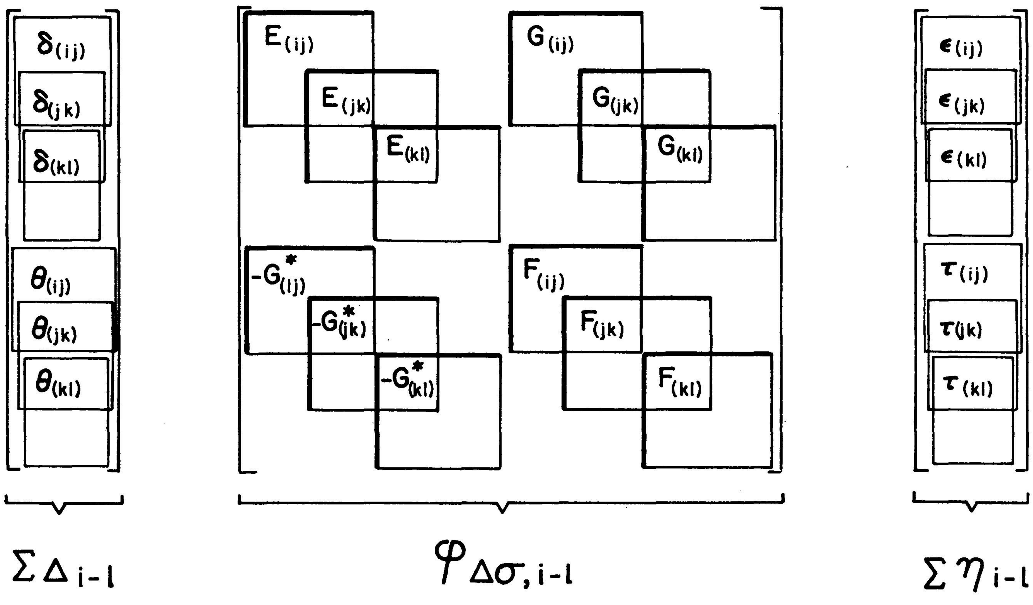
Table 7. Joint Force-Moment Matrix

$$
\begin{bmatrix} \delta_{ij} \\ 0 \\ 0 \\ \delta_{lk} \\ \hline \Theta_{ij} \\ 0 \\ 0 \\ \Theta_{lk} \end{bmatrix}
=
\left[\begin{array}{cccc|cccc}
E_{ii} & E_{ij} & & & G_{ii} & G_{ij} & & \\
E_{ji} & \sum E_{jj} & E_{jk} & & G_{ji} & \sum G_{jj} & G_{jk} & \\
 & E_{kj} & \sum E_{kk} & E_{kl} & & G_{kj} & \sum G_{kk} & G_{kl} \\
 & & E_{lk} & E_{ll} & & & G_{lk} & G_{ll} \\ \hline
-G_{ii} & G_{ij} & & & F_{ii} & F_{ij} & & \\
G_{ji} & -\sum G_{jj} & G_{jk} & & F_{ji} & \sum F_{jj} & F_{jk} & \\
 & G_{kj} & -\sum G_{kk} & G_{kl} & & F_{kj} & \sum F_{kk} & F_{kl} \\
 & & G_{lk} & -G_{ll} & & & F_{lk} & F_{ll}
\end{array}\right]
\begin{bmatrix} V_i \\ V_j \\ V_k \\ V_l \\ \hline M_i \\ M_j \\ M_k \\ M_l \end{bmatrix}
+
\begin{bmatrix} \epsilon_{ij} \\ \sum \epsilon_j \\ \sum \epsilon_k \\ \epsilon_{lk} \\ \hline \tau_{ij} \\ \sum \tau_j \\ \sum \tau_k \\ \tau_{lk} \end{bmatrix}
$$

$$
\begin{bmatrix} \sum \delta_{i-l} \\ \hline \sum \Theta_{i-l} \end{bmatrix}
=
\left[\begin{array}{c|c} E_{i-l} & G_{i-l} \\ \hline -G^{*}_{i-l} & F_{i-l} \end{array}\right]
\begin{bmatrix} V_{i-l} \\ \hline M_{i-l} \end{bmatrix}
+
\begin{bmatrix} \sum \epsilon_{i-l} \\ \hline \sum \tau_{i-l} \end{bmatrix}
$$

$$[\sum \Delta_{i-l}] = [\varphi_{\Delta\sigma, i-l}][\sigma_{i-l}] + [\sum \eta_{i-l}]$$

It is interesting to observe that $\varphi_{\Delta\sigma,i-l}$ is formed by four matrix chains, layed out diagonally like a deck of cards, with overlapping corners.



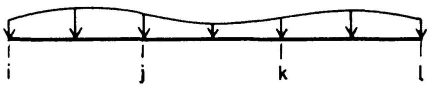
Matrices $\sum \Delta_{i-l}$, $\sum \eta_{i-l}$ are formed by column matrix chains, layed out vertically like a deck of cards, with half length overlapping.

Matrix Eq. (28) may be written for any number of segments, and is designated as the static flexibility chain.

Four typical cases solved symbolically be the transport chain method in Table 5, are also symbolically solved by the flexibility chain in Table 8.

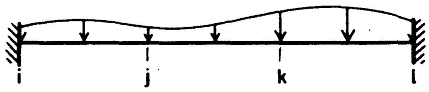
Table 8. Flexibility chain: Special cases

Free-Free Beam



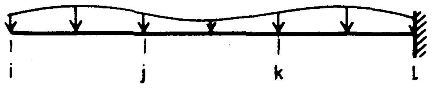
$$\left[\begin{array}{c}\delta_i\\0\\0\\\delta_l\\\hline\Theta_i\\0\\0\\\Theta_l\end{array}\right]=\left[\varphi\Delta\sigma,\, i-l\right]\left[\begin{array}{c}0\\V_j\\V_k\\0\\\hline0\\M_j\\M_k\\0\end{array}\right]+\left[\sum\eta_{i-l}\right]$$

Fixed-Fixed Beam



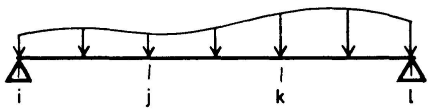
$$\left[\begin{array}{c}0\\0\\0\\0\\\hline0\\0\\0\\0\end{array}\right]=\left[\varphi\Delta\sigma,\, i-l\right]\left[\begin{array}{c}V_i\\V_j\\V_k\\V_l\\\hline M_i\\M_j\\M_k\\M_l\end{array}\right]+\left[\sum\eta_{i-l}\right]$$

Free-Fixed Beam



$$\left[\begin{array}{c}\delta_i\\0\\0\\0\\\hline\Theta_i\\0\\0\\0\end{array}\right]=\left[\varphi\Delta\sigma,\, i-l\right]\left[\begin{array}{c}0\\V_j\\V_k\\V_l\\\hline0\\M_j\\M_k\\M_l\end{array}\right]+\left[\sum\eta_{i-l}\right]$$

Hinged-Hinged Beam



$$\left[\begin{array}{c}0\\0\\0\\0\\\hline\Theta_i\\0\\0\\\Theta_l\end{array}\right]=\left[\varphi\Delta\sigma,\, i-l\right]\left[\begin{array}{c}V_i\\V_j\\V_k\\V_l\\\hline0\\M_j\\M_k\\0\end{array}\right]+\left[\sum\eta_{i-l}\right]$$

Stiffnesses

The end stiffnesses of a straight beam on elastic foundation is defined as the end reaction produced by a unit deformation or loads. Since the end reactions are forces and moments, the unit deformations are deflections and slopes, and the point of unit deformation is the near, or the far end, sixteen unit cause stiffnesses are required, for the formulation of a member stiffness matrix equation. Because of symmetry, and antisymmetry, only ten constants are necessary.

The derivation of the stiffness matrix follows from Eqs. (6, rows 1, and 2 of Table 4).

$$\begin{aligned} \bar{\Delta}_R &= \bar{l}_{\Delta,RL} + \bar{t}_{\Delta\Delta,RL}\bar{\Delta}_L + \bar{t}_{\Delta\sigma,RL}\bar{\sigma}_L, \\ \bar{\Delta}_L &= \bar{l}_{\Delta,LR} + \bar{t}_{\Delta\Delta,LR}\bar{\Delta}_R + \bar{t}_{\Delta\sigma,LR}\bar{\sigma}_R. \end{aligned} \tag{29}$$

As conventional in this case,

$$\sigma_{LR} = -\bar{\sigma}_L, \qquad \sigma_{RL} = \bar{\sigma}_R \tag{30}$$

and with these changes, the force-moment equations become:

$$\begin{aligned} \sigma_{LR} &= \bar{t}^{-1}_{\Delta\sigma,RL}\bar{t}_{\Delta\Delta,RL}\bar{\Delta}_L - \bar{t}^{-1}_{\Delta\sigma,RL}\bar{\Delta}_R + \bar{t}^{-1}_{\Delta\sigma,RL}\bar{l}_{\Delta,RL}, \\ \sigma_{RL} &= \bar{t}^{-1}_{\Delta\sigma,LR}\bar{\Delta}_L - \bar{t}^{-1}_{\Delta\sigma,LR}\bar{t}_{\Delta\Delta,LR}\bar{\Delta}_R - \bar{t}^{-1}_{\Delta\sigma,LR}\bar{l}_{\Delta,LR}. \end{aligned} \tag{31}$$

In these equations,

$$\bar{t}^{-1}_{\Delta\sigma,LR} = \frac{1}{(c^2-b\,d)}\begin{bmatrix} c & d \\ b & c \end{bmatrix}, \qquad \bar{t}^{-1}_{\Delta\sigma,RL} = \frac{1}{(c^2-b\,d)}\begin{bmatrix} c & -d \\ -b & c \end{bmatrix} \tag{32}$$

and $\bar{t}_{\Delta\Delta,RL}$, $\bar{t}_{\Delta\Delta,LR}$, $\bar{l}_{\Delta,LR}$, $\bar{l}_{\Delta,LR}$ are submatrices defined in Table 4.

The algebraical evaluation of Eqs. (31) is shown in Table 9, and recorded in rearranged form below.

$$\begin{bmatrix} V_{LR} \\ V_{RL} \\ \hline M_{LR} \\ M_{RL} \end{bmatrix} = \left[\begin{array}{cc|cc} T_{LL} & T_{LR} & S_{LL} & S_{LR} \\ T_{RL} & T_{RR} & S_{RL} & S_{RR} \\ \hline -S_{LL} & S_{LR} & K_{LL} & K_{LR} \\ S_{RL} & -S_{RR} & K_{RL} & K_{RR} \end{array}\right] \begin{bmatrix} \delta_L \\ \delta_R \\ \hline \Theta_L \\ \Theta_R \end{bmatrix} + \begin{bmatrix} FV_{LR} \\ FV_{RL} \\ \hline FM_{LR} \\ FM_{RL} \end{bmatrix}. \tag{33}$$

Eqs. (33) define analytically the end stress-resultants (forces and moments in the stiffness sign convention), and consequently are also the slope-deflection equations.

From the Table 9, the following identities are observed:

$$\begin{aligned} T_{LL} &= \quad T_{RR} = t, \\ S_{LL} &= -S_{RR} = s, \\ K_{LL} &= \quad K_{RR} = k. \end{aligned} \tag{34a} \qquad \begin{aligned} T_{LR} &= T_{RL} = c\,t, \\ -S_{LR} &= S_{RL} = c\,s, \\ K_{LR} &= K_{RL} = c\,k. \end{aligned} \tag{34b}$$

Constants t, s, k are the near, end stiffnesses, whereas $c\,t$, $c\,s$, $c\,k$ are the far, end stiffnesses (or carry-over stiffnesses). Constants $F\,V_{LR}$, $F\,V_{RL}$, $F\,M_{LR}$, $F\,M_{RL}$ are the fixed end stress resultants (fixed end forces and fixed end moments), caused by loads.

Eqs. (33), in observance of identities (34a, b), yield a new relationship of S-matrices.

$$\begin{bmatrix} V_{(LR)} \\ \hline M_{(LR)} \end{bmatrix} = \left[\begin{array}{c|c} T_{(LR)} & S_{(LR)} \\ \hline -S^*_{(LR)} & K_{(LR)} \end{array}\right] \begin{bmatrix} \delta_{(LR)} \\ \hline \Theta_{(LR)} \end{bmatrix} + \begin{bmatrix} FV_{(LR)} \\ \hline FM_{(LR)} \end{bmatrix}. \tag{35}$$

Table 9. Stiffnesses

(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
T_{LL}	$\frac{EJ}{l^3}\frac{\bar\lambda^3(ab+4cd)}{\beta}$	$\frac{EJ}{l^3}\frac{4\bar\lambda^3[\sinh(2\bar\lambda)+\sin(2\bar\lambda)]}{\gamma}$	t	T_{LR}	$\frac{-EJ}{l^3}\frac{\bar\lambda^3 b}{\beta}$	$\frac{EJ}{l^3}\frac{-8\bar\lambda^3[\cosh(\bar\lambda)\sin(\bar\lambda)+\sinh(\bar\lambda)\cos(\bar\lambda)]}{\gamma}$	ct
T_{RL}	$\frac{-EJ}{l^3}\frac{\bar\lambda^3 b}{\beta}$	$\frac{EJ}{l^3}\frac{-8\bar\lambda^3[\cosh(\bar\lambda)\sin(\bar\lambda)+\sinh(\bar\lambda)\cos(\bar\lambda)]}{\gamma}$	ct	T_{RR}	$\frac{EJ}{l^3}\frac{\bar\lambda^3(ab+4cd)}{\beta}$	$\frac{EJ}{l^3}\frac{4\bar\lambda^3[\sinh(2\bar\lambda)+\sin(2\bar\lambda)]}{\gamma}$	t
S_{LL}	$\frac{EJ}{l^2}\frac{\bar\lambda^2(b^2-ac)}{\beta}$	$\frac{EJ}{l^2}\frac{2\bar\lambda^2[\cosh(2\bar\lambda)-\cos(2\bar\lambda)]}{\gamma}$	s	S_{LR}	$\frac{EJ}{l^2}\frac{\bar\lambda^2 c}{\beta}$	$\frac{EJ}{l^2}\frac{8\bar\lambda^2\sinh(\bar\lambda)\sin(\bar\lambda)}{\gamma}$	$-cs$
S_{RL}	$\frac{-EJ}{l^2}\frac{\bar\lambda^2 c}{\beta}$	$\frac{EJ}{l^2}\frac{-8\bar\lambda^2\sinh(\bar\lambda)\sin(\bar\lambda)}{\gamma}$	cs	S_{RR}	$\frac{-EJ}{l^2}\frac{\bar\lambda^2(b^2-ac)}{\beta}$	$\frac{EJ}{l^2}\frac{-2\bar\lambda^2[\cosh(2\bar\lambda)-\cos(2\bar\lambda)]}{\gamma}$	$-s$
K_{LL}	$\frac{-EJ}{l}\frac{\bar\lambda(bc-ad)}{\beta}$	$\frac{EJ}{l}\frac{-2\bar\lambda[\sinh(2\bar\lambda)-\sin(2\bar\lambda)]}{\gamma}$	k	K_{LR}	$\frac{-EJ}{l}\frac{\bar\lambda d}{\beta}$	$\frac{EJ}{l}\frac{-4\bar\lambda[\cosh(\bar\lambda)\sin(\bar\lambda)-\sinh(\bar\lambda)\cos(\bar\lambda)]}{\gamma}$	ck
K_{RL}	$\frac{-EJ}{l}\frac{\bar\lambda d}{\beta}$	$\frac{EJ}{l}\frac{-4\bar\lambda[\cosh(\bar\lambda)\sin(\bar\lambda)-\sinh(\bar\lambda)\cos(\bar\lambda)]}{\gamma}$	ck	K_{RR}	$\frac{-EJ}{l}\frac{\bar\lambda(bc-ad)}{\beta}$	$\frac{EJ}{l}\frac{-2\bar\lambda[\sinh(2\bar\lambda)-\sin(2\bar\lambda)]}{\gamma}$	k
$F\,V_{LR}$	$\frac{-EJ}{l^3}\frac{\bar\lambda^3(b\bar L_{RL}-c\bar L'_{RL})}{\beta}$	$\frac{EJ}{l^3}\frac{-8\bar\lambda^3}{\gamma}\{[\cosh(\bar\lambda)\sin(\bar\lambda)+\sinh(\bar\lambda)\cos(\bar\lambda)]\bar L_{RL}-\sinh(\bar\lambda)\sin(\bar\lambda)\bar L'_{RL}\}$					
$F\,V_{RL}$	$\frac{-EJ}{l^3}\frac{\bar\lambda^3(b\bar L_{LR}+c\bar L'_{LR})}{\beta}$	$\frac{EJ}{l^3}\frac{-8\bar\lambda^3}{\gamma}\{[\cosh(\bar\lambda)\sin(\bar\lambda)+\sinh(\bar\lambda)\cos(\bar\lambda)]\bar L_{LR}+\sinh(\bar\lambda)\sin(\bar\lambda)\bar L'_{LR}\}$				$\bar\lambda=\lambda l$	
$F\,M_{LR}$	$\frac{EJ}{l^2}\frac{\bar\lambda^2(c\bar L_{RL}-d\bar L'_{RL})}{\beta}$	$\frac{EJ}{l^2}\frac{4\bar\lambda^2}{\gamma}\{2\sinh(\bar\lambda)\sin(\bar\lambda)\bar L_{RL}-[\cosh(\bar\lambda)\sin(\bar\lambda)-\sinh(\bar\lambda)\cos(\bar\lambda)]\bar L'_{RL}\}$				$\beta=c^2-bd$ $\gamma=16\beta=\cosh(2\bar\lambda)+\cos(2\bar\lambda)-2$	
$F\,M_{RL}$	$\frac{-EJ}{l^2}\frac{\bar\lambda^2(c\bar L_{LR}+d\bar L'_{LR})}{\beta}$	$\frac{EJ}{l}\frac{-4\bar\lambda^2}{\gamma}\{2\sinh(\bar\lambda)\sin(\bar\lambda)\bar L_{LR}-[\cosh(\bar\lambda)\sin(\bar\lambda)-\sinh(\bar\lambda)\cos(\bar\lambda)]\bar L'_{LR}\}$					

Stiffness Chain

Once the stiffness matrix (35) is available for a single segment, the analysis of multi-segment bars is accomplished by chain overlapping.

Considering again the bar $ijkl$ (Fig. 3), the static equilibrium at j (any station) requires that,

$$\begin{aligned} V_{ji} + V_{jk} &= 0, \\ M_{ji} + M_{jk} &= 0 \end{aligned} \tag{36}$$

with notation (33), and new equivalents:

$$\sum T_{jj};\ \sum S_{jj};\ \sum K_{jj};\ \sum F\,V_j;\ \sum F\,M_j$$

designating the sum of the respective, near, end stiffnesses and stiffness load functions at j, the equilibrium Eqs. (36) become typical joint, deflection-slope equations.

$$\begin{aligned} &\left\{\begin{matrix} T_{ji}\delta_i + \sum T_{jj}\delta_j + T_{jk}\delta_k \\ S_{ji}\Theta_i + \sum S_{jj}\Theta_j + S_{jk}\Theta_k \end{matrix}\right\} + \sum F V_j = 0, \\ &\left\{\begin{matrix} S_{ji}\delta_i - \sum S_{jj}\delta_j + S_{jk}\delta_k \\ K_{ji}\Theta_i + \sum K_{jj}\Theta_j + K_{jk}\Theta_k \end{matrix}\right\} + \sum F M_j = 0. \end{aligned} \tag{37}$$

Eqs. (37) have a general meaning, and are used as recurrence formulas.

With end conditions to be discussed later, the complete joint deflections-slope matrix takes the form shown in Table 10.

Table 10. Joint Deflection-Slope Matrix

$$\begin{bmatrix} V_{ij} \\ 0 \\ 0 \\ V_{lk} \\ \hline M_{ij} \\ 0 \\ 0 \\ M_{lk} \end{bmatrix} = \left[\begin{array}{cccc|cccc} T_{ii} & T_{ij} & & & S_{ii} & S_{ij} & & \\ T_{ji} & \sum T_{jj} & T_{jk} & & S_{ji} & \sum S_{jj} & S_{jk} & \\ & T_{kj} & \sum T_{kk} & T_{kl} & & S_{kj} & \sum S_{kk} & S_{kl} \\ & & T_{lk} & T_{ll} & & & S_{lk} & S_{ll} \\ \hline -S_{ii} & S_{ij} & & & K_{ii} & K_{ij} & & \\ S_{ji} & -\sum S_{jj} & S_{jk} & & K_{ji} & \sum K_{jj} & K_{jk} & \\ & S_{kj} & -\sum S_{kk} & S_{kl} & & K_{kj} & \sum K_{kk} & K_{kl} \\ & & S_{lk} & -S_{ll} & & & K_{lk} & K_{ll} \end{array}\right] \begin{bmatrix} \delta_i \\ \delta_j \\ \delta_k \\ \delta_l \\ \hline \Theta_i \\ \Theta_j \\ \Theta_k \\ \Theta_l \end{bmatrix} + \begin{bmatrix} F\,V_{ij} \\ \sum F\,V_j \\ \sum F\,V_k \\ F\,V_{lk} \\ \hline F\,M_{ij} \\ \sum F\,M_j \\ \sum F\,M_k \\ F\,M_{lk} \end{bmatrix}$$

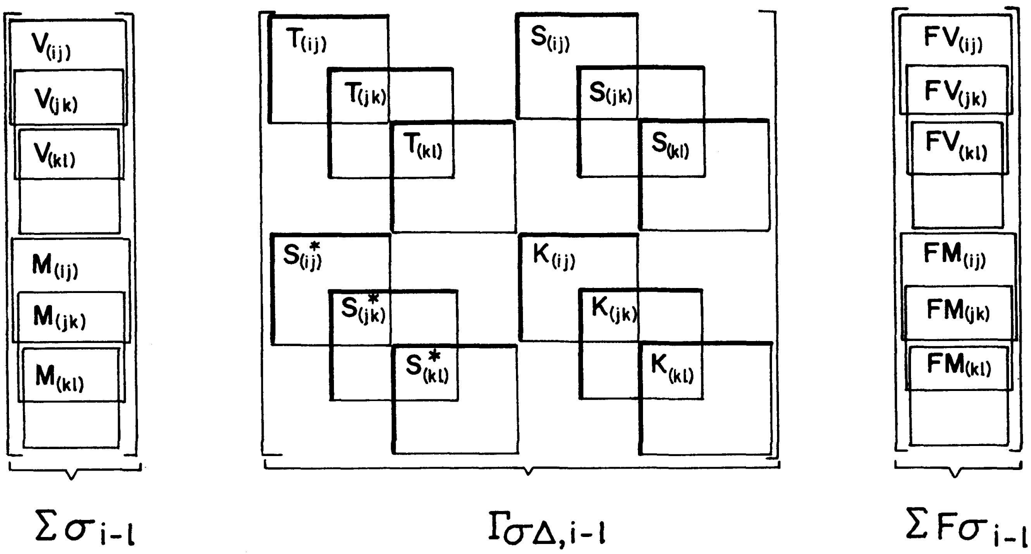
$$\begin{bmatrix} \sum V_{i-l} \\ \hline \sum M_{i-l} \end{bmatrix} = \left[\begin{array}{c|c} T_{i-l} & S_{i-l} \\ \hline -S^{*}_{i-l} & F_{i-l} \end{array}\right] \begin{bmatrix} \delta_{i-l} \\ \hline \Theta_{i-l} \end{bmatrix} + \begin{bmatrix} \sum F\,V_{i-l} \\ \hline \sum F\,M_{i-l} \end{bmatrix}$$

$$[\sum \sigma_{i-l}] = [\Gamma_{\sigma\Delta,\,i-l}]\,[\Delta_{i-l}] + [\sum F\sigma_{i-l}]$$

In symbolic form,

$$\sum \sigma_{i-l} = \Gamma_{\sigma\Delta,i-l}\Delta_{i-l} + \sum F\,\sigma_{i-l} = 0. \tag{38}$$

It is interesting to observe, that $\Gamma_{\sigma\Delta, i-l}$ is formed by four matrix chains, layed out diagonally, like a deck of cards, with overlapping corners.



Matrices $\sum \sigma_{i-l}$, $\sum F \sigma_{i-l}$ are formed by column matrix chains, layed out vertically like a deck of cards, with half length overlapping.

Matrix Eq. (38) may be written for any number of segments, and is designated as the static, stiffness chain.

Four typical cases solved symbolically by the transport chain method in Table 5, by the flexibility chain method in Table 8, are also symbolically solved by the stiffness chain method in Table 11.

Keywords

Beam; elastic foundation; matrix analysis; transport matrix; flexibility matrix; stiffness matrix; structural engineering.

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Summary

Three general solutions are given in matrix form for the analysis of beams on elastic foundation. From the transport matrix formed by the static parameters of a single bar, the transport chain, the flexibility chain, and the stiffness chain methods are developed and applied to the solution of particular cases. The study is restricted to coplanar systems, consisting of straight members, acted upon by transverse loads, and deforming elastically.

Résumé

On donne pour l'étude des poutres sur appui élastique trois solutions générales mises sous forme de matrices. A partir de la matrice de transport formée par les paramètres statiques d'une barre simple, les méthodes de la chaîne de transport, de la chaîne de fléxibilité et de la chaîne de rigidité sont développées et appliquées à la solution de cas particuliers. L'étude est restreinte aux systèmes coplanaires composés de pièces droites, solicités par des charges transversales et déformées élastiquement.

Zusammenfassung

Drei generelle Lösungen in genereller Matrixform sind für die Analysis von Balken auf elastischer Unterlage gegeben.

Ausgehend von der Übertragungsmatrix, gebildet von den statischen Parametern eines einzelnen Balkens wird die Übertragungs-, die Flexibilitäts- und die Steifigkeitskettenmethode entwickelt und auf Spezialfälle angewendet. Die Untersuchung ist beschränkt auf elastisch verformbare ebene, aus geraden Stäben gebildeten Systeme, die senkrecht zu ihrer Achse belastet werden.