Zeitschrift: IABSE publications = Mémoires AIPC = IVBH Abhandlungen

Band: 30 (1970)

Artikel: Tangent stiffness matrices for finite elements

Autor: Tezcan, Semih S. / Mahapatra, B.C. / Mathews, C.I.

DOI: https://doi.org/10.5169/seals-23586

Tangent Stiffness Matrices for Finite Elements

Matrices de rigidité tangentielle pour des éléments finis

Tangentiale Steifigkeitsmatrizen für endliche Elemente

SEMIH S. TEZCAN
Prof. of Civil Engineering, Robert College, Bebek, İstanbul, Turkey

B. C. MAHAPATRA
Research Engineer, Electronic Calculus Inc., New York, U.S.A.

C. I. MATHEWS
Post-Graduate Student, Civil Engineering Department, University of British, Columbia, Vancouver, Canada

Introduction

It is possible to take into account both the geometric and material nonlinearities in a structure by a suitable repetition of linear stiffness analysis. Basically, there are two distinct numerical methods for the nonlinear analysis of structures; (1) step by step application of incremental loads and (2) regular or modified Newton-Raphson's iterative solution under full loads. In either of these two numerical methods, it is assumed that during each solution cycle, the stiffness analysis proceeds along a straight line tangent to the curve characterizing the force-deflection relations of the structure. In order to achieve such a tangent solution, the stiffness matrix of each element should be modified to account for the accumulated stresses and the change in geometry. Once, the tangent stiffness matrices are available representing both the physical and geometric nonlinearities at any stage of deformed condition, the nonlinear analysis as well as the stability problems of a continuous medium may be performed by a repititious application of the direct linear stiffness method of analysis.

Nonlinearity was first introduced into the stiffness matrices by TURNER [1], et al. through the strain-displacement equations in connection with a truss element and a triangle in membrane. Using similar techniques, nonlinear stiffness matrices were obtained for a beam element in plane (2, 3, 4, 10, 12, 13, 14] and in space [5], for a triangular plate [2, 6, 10, 12], rectangular plate [7, 8, 12], axisymmetrical shell element [9], and a tetrahedron [10, 11].

Various valuable contributions have been made in connection with general formulation of both the geometric [15, 16] and material nonlinearities [17, 18, 19, 20]. As an alternative to solid finite elements, HRENNIKOFF lattice models were also used for large deflection analysis of thin plates [21]. ODEN [22] presented an excellent review and examined the applications of the finite element method to nonlinear problems in structural mechanics.

In this paper, it is intended to give a general matrix formulation for the derivation of tangent stiffness matrices for two and three dimensional finite elements. The approach is basically the same as followed by MARTIN [2], PRZEMIENIECKI [12], WISSMANN [16], ODEN [15, 17] and others. However, a systematic matrix manipulation scheme is presented for purposes of conveniently incorporating all of the higher order terms of the strain-displacement equations. The stiffness contribution of each energy term is formulated in a uniform fashion allowing the derivation of the complete tangent stiffness matrix for a wide range of one, two and three dimensional finite elements to be carried out by means of almost identical matrix operations. The final tangent stiffness matrix is obtained as the combination of a linear stiffness matrix plus seven or more different types of stiffness matrices which correspond to various higher order terms of the strain energy and are functions of stresses and displacements of the element at the stage of deformation concerned. This separate evaluation of the contribution of each higher order term, enables the analyst to include or discard any particular component of the tangent stiffness matrix, depending on the relative importance of the respective terms.

In fact, numerical experience on the use of tangent stiffness matrices of line elements [4, 13, 16] have indicated clearly, that the inclusion of some of the previously neglected higher order terms increase both the accuracy and the speed of convergence.

Although the formulation is general, due to space limitations, only the complete components of the tangent stiffness matrix of a triangle and of a tetrahedron are included at the end together with a discussion of the modified Newton-Raphson iteration scheme.

Total Strain Energy

Total strain energy U, by definition is

$$U = \tfrac{1}{2}\int_V \{\epsilon\}^T \{\sigma\} dV = \tfrac{1}{2}\int_V \{\epsilon\}^T [D] \{\epsilon\} dV, \tag{1}$$

in which, the material matrix D, relates the stresses to strains as

$$\{\sigma\} = [D]\{\epsilon\} \tag{2}$$

and is given by

$$D = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix}. \tag{3}$$

For an isotropic material,

$$D_{11} = D_{22} = \frac{E}{1-\nu^2}; \qquad D_{12} = \nu D_{11}; \qquad D_{33} = \frac{E}{2(1+\nu)}, \tag{4}$$

in which, E = Modulus of Elasticity, ν = Poisson's ratio.

The integration should be carried over the undeformed volume of the finite element as already proved in Ref. [16]. The strain vector ϵ, at any intermediate stage of deformations, may be expressed as the superposition of the instantaneous values of strains ϵ_i, and the additional strains ϵ_a which are developed due to the application of a new set of external loads [2]. Therefore,

$$\{\epsilon\} = \{\epsilon\}_i + \{\epsilon\}_a, \tag{5}$$

in which

$$\{\epsilon\}_a = \{\epsilon\}_0 + \{\epsilon\}_1 + \{\epsilon\}_3 + \{\epsilon\}_4 \tag{6}$$

or, in terms of the partial derivatives of the generic displacements u, v, and w with respect to the local coordinates x, y and z (24)

$$\underset{}{\{\epsilon\}_a} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \underset{\{\epsilon\}_0}{\begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y}+v_{,x} \end{Bmatrix}} + \underset{\{\epsilon\}_1}{\begin{Bmatrix} \frac{1}{2}u_{,x}^2 \\ \frac{1}{2}u_{,y}^2 \\ u_{,x}u_{,y} \end{Bmatrix}} + \underset{\{\epsilon\}_2}{\begin{Bmatrix} \frac{1}{2}v_{,x} \\ \frac{1}{2}v_{,y} \\ v_{,x}v_{,y} \end{Bmatrix}} + \underset{\{\epsilon\}_3}{\begin{Bmatrix} \frac{1}{2}w_{,x} \\ \frac{1}{2}w_{,y} \\ w_{,x}w_{,y} \end{Bmatrix}} + \underset{\{\epsilon\}_4}{\begin{Bmatrix} -z\,w_{,xx} \\ -z\,w_{,yy} \\ -2z\,w_{,xy} \end{Bmatrix}}. \tag{7a}$$

The middle plane of the two-dimensional finite element is assumed to coincide with the local xy-plane. In case of a three dimensional finite element with no rotational degrees of freedom at its nodes, the incremental strain vector ϵ_a, is given by

$$\{\epsilon\}_a = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} u_{,x} \\ v_{,y} \\ w_{,z} \\ u_{,y}+v_{,x} \\ u_{,z}+w_{,x} \\ v_{,z}+w_{,x} \end{Bmatrix} + \begin{Bmatrix} \frac{1}{2}u_{,x}^2 \\ \frac{1}{2}u_{,y}^2 \\ \frac{1}{2}u_{,z}^2 \\ u_{,x}u_{,y} \\ u_{,x}u_{,z} \\ u_{,y}u_{,z} \end{Bmatrix} + \begin{Bmatrix} \frac{1}{2}v_{,x}^2 \\ \frac{1}{2}v_{,y}^2 \\ \frac{1}{2}v_{,z}^2 \\ v_{,x}v_{,y} \\ v_{,x}v_{,z} \\ v_{,y}v_{,z} \end{Bmatrix} + \begin{Bmatrix} \frac{1}{2}w_{,x}^2 \\ \frac{1}{2}w_{,y}^2 \\ \frac{1}{2}w_{,z}^2 \\ w_{,x}w_{,y} \\ w_{,x}w_{,z} \\ w_{,y}w_{,z} \end{Bmatrix}. \tag{7b}$$

$$\{\epsilon\}_a = \{\epsilon\}_0 + \{\epsilon\}_1 + \{\epsilon\}_2 + \{\epsilon\}_3$$

Instantaneous strains ϵ_i, are considered to be present and numerically available before the application of a new set of external loads. For instance, the state of strain of the preceding cycle of analysis, thermal stresses, prestressing forces, yield stresses, lack of fit, etc., constitute the initial stress vector. It is interesting to note that the instantaneous strain vector ϵ_i, due to the loads of the preceding cycle is also calculated from Eq. (7) in the same way as the additional strain vector ϵ_a using all the nonlinear strain-displacement relations. There is one difference, however, that in the case of ϵ_i, the generic displacements u, v and w as well as their partial derivatives, are all numerically available, while in the case of ϵ_a, these displacements are variable functions of the nodal displacements. Therefore, from the view of load-deflection history, it is more appropriate to call ϵ_i, as the *"accumulated"* and ϵ_a, as the *"incremental"* strain vectors. Substituting these strain vectors in Eq. (1)

$$U = \int_V \{\epsilon_i + \epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4\}^T [D] \{\epsilon_i + \epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4\} \, dV \tag{8}$$

and after carrying out the products inside the integration, the total strain energy is expressed as the algebraic sum of thirty-six different energy components as follows:

$$U = \frac{1}{2}\int_V \left[\begin{array}{cccccc}
\epsilon_i^T D \epsilon_i \; + & \epsilon_i^T D \epsilon_0 \; + & \epsilon_i^T D \epsilon_1 \; + & \epsilon_i^T D \epsilon_2 \; + & \epsilon_i^T D \epsilon_3 & + \; \epsilon_i^T D \epsilon_4 \\
\text{Constant} & \text{First} & \text{Second} & \text{Second} & \text{Second} & \text{First} \\
\epsilon_0^T D \epsilon_i \; + & \epsilon_0^T D \epsilon_0 \; + & \epsilon_0^T D \epsilon_1 \; + & \epsilon_0^T D \epsilon_2 \; + & \epsilon_0^T D \epsilon_3 & + \; \epsilon_0^T D \epsilon_4 \\
\text{First} & \text{Second} & \text{Third} & \text{Third} & \text{Third} & \text{Second} \\
\epsilon_1^T D \epsilon_i \; + & \epsilon_1^T D \epsilon_0 \; + & \epsilon_1^T D \epsilon_1 \; + & \epsilon_1^T D \epsilon_2 \; + & \epsilon_1^T D \epsilon_3 & + \; \epsilon_1^T D \epsilon_4 \\
\text{Second} & \text{Third} & \text{Fourth} & \text{Fourth} & \text{Fourth} & \text{Third} \\
\epsilon_2^T D \epsilon_i \; + & \epsilon_2^T D \epsilon_0 \; + & \epsilon_2^T D \epsilon_1 \; + & \epsilon_2^T D \epsilon_2 \; + & \epsilon_2^T D \epsilon_3 & + \; \epsilon_2^T D \epsilon_4 \\
\text{Second} & \text{Third} & \text{Fourth} & \text{Fourth} & \text{Fourth} & \text{Third} \\
\epsilon_3^T D \epsilon_i \; + & \epsilon_3^T D \epsilon_0 \; + & \epsilon_3^T D \epsilon_1 \; + & \epsilon_3^T D \epsilon_2 \; + & \epsilon_3^T D \epsilon_3 & + \; \epsilon_3^T D \epsilon_4 \\
\text{Second} & \text{Third} & \text{Fourth} & \text{Fourth} & \text{Fourth} & \text{Third} \\
\epsilon_4^T D \epsilon_i \; + & \epsilon_4^T D \epsilon_0 \; + & \epsilon_4^T D \epsilon_1 \; + & \epsilon_4^T D \epsilon_2 \; + & \epsilon_4^T D \epsilon_3 & + \; \epsilon_4^T D \epsilon_4 \\
\text{First} & \text{Second} & \text{Third} & \text{Third} & \text{Third} & \text{Second}
\end{array} \right] dV. \tag{9}$$

Generic Displacement Functions

It is assumed that, for a finite element with n degrees of freedom, the displacements u, v and w of any particular point are expressed as a suitable polynomial of the local coordinates, xyz, involving n number of unknown coefficients $a_1, a_2, \ldots, a_n$ as

$$\underset{(3\times1)}{\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}} = \underset{(3\times n)}{[C]} \; \underset{(n\times1)}{\{a\}}, \tag{10}$$

in which, $[C]$ coordinate matrix containing the individual terms of the displacement polynomials in the form of certain powers of x, y and z; $\{a\}$ = column vector of unknown coefficients of the polynomial. When the actual coordinates of the element nodes are substituted inside the coordinate matrix C, the nodal displacements $d_1, d_2, \ldots, d_n$, which contain both the straight and rotational degrees of freedom, are obtained as

$$\{d\} = [A]\{a\}, \tag{11}$$

in which, $[A] = (n \times n)$ matrix containing the local coordinates of the element nodes; $\{d\}$ = column vector of nodal displacements. Since, the content of A matrix is numerically available, once the geometry of the element is decided, the unkown coefficients are obtainable from the nodal displacements as follows:

$$\underset{(n\times 1)}{\{a\}} = \underset{(n\times n)}{[B]} \; \underset{(n\times 1)}{\{d\}}, \tag{12}$$

where

$$\underset{(n\times n)}{[B]} = \underset{(n\times n)}{[A]^{-1}}. \tag{13}$$

Combining Eqs. (10) and (12), the generic displacements may be related to the nodal displacements by

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [C]\,[B]\{d\}. \tag{14}$$

Stiffness Matrix from Strain Energy

Since, the generic displacements u, v and w, in accordance with Eq. (14), are linear functions of the nodal displacements $d_1, d_2, \ldots, d_n$, it can be shown using Eq. (7) that the strain vectors $\{\epsilon\}_0$ and $\{\epsilon\}_4$, are also linear functions of the nodal displacements. However, the strain vectors $\{\epsilon\}_1$, $\{\epsilon\}_2$ and $\{\epsilon\}_3$ are quadratic functions of the nodal displacements, since these vectors contain the square of the partial derivatives of the generic displacements. Therefore, as already indicated in Eq. (9) the total strain energy U, contains not only constants as a result of instantaneous strains, but also a mixture of linear, quadratic, cubic and fourth order expressions in terms of the nodal displacements.

The stiffness influence coefficients k_{ij}, using Castigliano's Theorem [2], are obtained as the second partial derivative of the strain energy with respect to the nodal displacements, from

$$k_{ij} = \frac{\partial^2 U}{\partial d_i \, \partial d_j}. \tag{15}$$

This procedure requires rigidly, that U is expressed in terms of nodal displacements. Once this is done, the problem is merely taking the second partial derivatives of U. It should be realized, however, that the constants and the first order terms of U, will vanish after the second partial derivatives are taken, leaving behind only the second, third and fourth order terms to contribute to the stiffness. After the second partial derivative of U, no nodal displacements will exist in the second order expressions, therefore, they contribute to the regular *linear stiffness matrix*. All the third and fourth order terms contribute to the *nonlinear* components of the tangent stiffness matrix which are functions of intermediate displacements.

A summary of the strain energy components contributing to various types of stiffness matrices is given in Table 1. As already indicated in this table, for flat plate elements and also for three dimensional solid elements in which no rotational degrees of freedom are specified, type IV and IV stiffness matrices become zero. The general formulation for the systematic derivation of each of the six different types of stiffness matrices is briefly outlined below and later the respective stiffnes smatrix components are developed for a tetrahedron and a triangular element. Although all of the formulation is first given for a two-dimensional bending shell element, the modifications and translations necessary for a three dimensional solid finite element are also introduced later in the form of a table (Table 3).

Table 1. List of Contributing Strain Energy Components

Type	Order	Strain Energy Components		Resulting Stiffness Matrix	Remarks
I	2nd	U^{00}	$\int_V \epsilon_0^T D \epsilon_0$	K^{00}	Linear Stiffness Matrix
II	2nd	U^{i1}	$\frac{1}{2}\int_V \epsilon_i^T D \epsilon_1 + \frac{1}{2}\int_V \epsilon_1^T D \epsilon_i$	K^{i1}	Instantaneous Stress Matrix
		U^{i2}	$\frac{1}{2}\int_V \epsilon_i^T D \epsilon_2 + \frac{1}{2}\int_V \epsilon_2^T D \epsilon_i$	K^{i2}	
		U^{i3}	$\frac{1}{2}\int_V \epsilon_i^T D \epsilon_3 + \frac{1}{2}\int_V \epsilon_3^T D \epsilon_1$	K^{i3}	
III	2nd	U^{44}	$\int_V \epsilon_4^T D \epsilon_4$	K^{44}	Linear Stiffness Matrix (Bending)
IV	2nd	U^{04}	$\frac{1}{2}\int_V \epsilon_0^T D \epsilon_4 + \frac{1}{2}\int_V \epsilon_4^T D \epsilon_0$	K^{04}	Zero, if – no rotational degrees of freedom exist – the element is symmetrical about the xy-plane
V	3rd	U^{01}	$\frac{1}{2}\int_V \epsilon_0^T D \epsilon_1 + \frac{1}{2}\int_V \epsilon_1^T D \epsilon_0$	K^{01}	Displacement Stiffness Matrix as function of u and v
		U^{02}	$\frac{1}{2}\int_V \epsilon_0^T D \epsilon_2 + \frac{1}{2}\int_V \epsilon_2^T D \epsilon_0$	K^{02}	
		U^{03}	$\frac{1}{2}\int_V \epsilon_0^T D \epsilon_3 + \frac{1}{2}\int_V \epsilon_3^T D \epsilon_0$	K^{03}	Displacement Stiffness Matrix as function of w
VI	3rd	U^{41}	$\frac{1}{2}\int_V \epsilon_4^T D \epsilon_1 + \frac{1}{2}\int_V \epsilon_1^T D \epsilon_4$	K^{41}	Zero, if – no rotational degrees of freedom exist – the element is symmetrical about the xy-plane
		U^{42}	$\frac{1}{2}\int_V \epsilon_4^T D \epsilon_2 + \frac{1}{2}\int_V \epsilon_2^T D \epsilon_4$	K^{42}	
		U^{43}	$\frac{1}{2}\int_V \epsilon_4^T D \epsilon_3 + \frac{1}{2}\int_V \epsilon_3^T D \epsilon_4$	K^{43}	

Type I. Linear Stiffness Matrix

The respective strain energy component from Table 1, is

$$U^{00} = \tfrac{1}{2}\int_V \epsilon_0^T D\, \epsilon_0\, dV. \tag{16}$$

Differentiating, u, v and w, in accordance with Eq. (7), and rearranging the results in matrix form, the strain vector, $\{\epsilon\}_0$, becomes

$$\{\epsilon\}_0 = [G_0]\{a\}. \tag{17}$$

The size of the $[G_0]$ matrix, which contains a suitable combination of the x, y and z terms, is $(3\times n)$ for two dimensional, and $(6\times n)$ for three dimensional finite elements. Using Eqs. (17) and (12), Eq. (16) becomes

$$U^{00} = \tfrac{1}{2}\int_V \{d\}^T [B]^T [H^{00}][B]\{d\}\, dV \tag{18}$$

and after differentiating twice, in accordance with Eq. (15), the stiffness matrix K^{00} is obtained as

$$K^{00} = [B]^T [H^{00}][B], \tag{19}$$

in which

$$H^{00} = \int_V [G_0]^T [D][G_0]\, dV. \tag{20}$$

The K^{00} matrix corresponds to the linear stiffness matrix of the small deflection theory.

Type II. Instantaneous Strain Matrix

A typical component of this type, from Table 1, is

$$U^{i1} = \tfrac{1}{2}\int_V \{\epsilon\}_i^T [D]\{\epsilon\}_1 dV + \tfrac{1}{2}\int_V \{\epsilon\}_1^T [D]\{\epsilon\}_i dV, \tag{21}$$

in which, the instantaneous vector $\{\epsilon\}_i$, may be calculated numerically at the end of any intermediate cycle of analysis in terms of the numeric values of the nodal displacements and local coordinates.

With regard to the variation of strains within the element, there are two different cases as follows:

Case 1. Constant strain element: If the assumed displacement field corresponds to a constant state of strain within the element, that is, if the strain expressions are not functions of local coordinates, the strain energy U^{i1} may be written, using Eq. (2), as

$$U^{i1} = \tfrac{1}{2}\int_V \{\sigma\}_i^T \{\epsilon\}_1 dV + \tfrac{1}{2}\int_V \{\epsilon\}_1^T \{\sigma\}_i dV. \tag{22}$$

Taking advantage of the identity

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix}^T \begin{Bmatrix} \tfrac{1}{2} B_1^2 \\ \tfrac{1}{2} B_2^2 \\ B_1 B_2 \end{Bmatrix} = \tfrac{1}{2} \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix}^T \begin{bmatrix} A_1 & A_3 \\ A_3 & A_2 \end{bmatrix} \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix}, \tag{23}$$

the product of $\sigma_i^T \epsilon_1$ may be expressed as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^T \underset{(3\times 1)}{\begin{Bmatrix} \frac{1}{2} u_{,x}^2 \\ \frac{1}{2} u_{,y}^2 \\ u_{,x}\, u_{,y} \end{Bmatrix}} = \tfrac{1}{2} \underset{(2\times 1)}{\begin{Bmatrix} u_{,x} \\ u_{,y} \end{Bmatrix}}^T \overbrace{\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}}^{[\sigma]^i} \underset{(2\times 1)}{\begin{Bmatrix} u_{,x} \\ u_{,y} \end{Bmatrix}}, \tag{24}$$

in which the (3×1) column vector of quadratic terms are reduced to the product of two (2×1) column vectors of linear terms. The middle block $[\sigma]^i$, in Eq. (24) is the instantaneous stress tensor of constant magnitude. By taking the partial derivatives of the function for displacement u, as

$$\underset{(2\times 1)}{\begin{Bmatrix} u_{,x} \\ u_{,y} \end{Bmatrix}} = \underset{(2\times n)}{[G_1]} \underset{(n\times 1)}{\{a\}} \tag{25}$$

and substituting into Eq. (24), the strain energy U^{i1} becomes

$$U^{i1} = \tfrac{1}{2}\int_V \tfrac{1}{2}\{a\}^T [G_1]^T [\sigma]^i [G_1] \{a\}\, dV + \tfrac{1}{2}\int_V \tfrac{1}{2}\{a\} [G_1] [\sigma]^i [G_1]^T \{a\}^T dV. \tag{26}$$

Since, the middle block σ^i = stress tensor is symmetrical, the congruent transformation $G_1^T \sigma^i G_1$ is also symmetrical. Therefore, the two separate integrals of Eq. (26) are equal and may be combined as

$$U^{i1} = \tfrac{1}{2}\int_V \{a\}^T [G_1]^T [\sigma]^i [G_1] \{a\}\, dV. \tag{27}$$

After substituting Eq. (12) to eliminate $\{a\}$ and differentiating twice, the stiffness matrix K^{i1} is obtained as

$$K^{i1} = [B]^T [H^{i1}] [B], \tag{28}$$

in which

$$H^{i1} = \int_V [G_1]^T [\sigma]^i [G_1]\, dV. \tag{29}$$

By analogy, the stiffness matrices K^{i2} and K^{i3} are obtained from the same formula as given for K^{i1}, except H^{i1} is replaced by

$$H^{i2} = \int_V [G_2]^T [\sigma]^i [G_2]\, dV \tag{30}$$

and

$$H^{i3} = \int_V [G_3]^T [\sigma]^i [G_3]\, dV, \tag{31}$$

in which, G_2 and G_3 matrices are the same as G_1, except u of Eq. (25) is replaced by v and w, respectively.

Case 2. Variable strain element: If the strain vector is function of local coordinates, in order to be able to follow the same procedure as in Case 1, instead of replacing $D\epsilon_i$ by a constant stress vector σ, we shall introduce a fictitious vector s which will be in the same role as σ, except it will contain not only the local coordinates x, y and z but also the coefficients $a_1, a_2, \ldots, a_n$ of the displacement functions. The column vector $D\epsilon_i$, from Eq. (3), is

$$\{s\} = [D]\{\epsilon\}_i = \begin{Bmatrix} D_{11}\,\epsilon_x + D_{12}\,\epsilon_y \\ D_{12}\,\epsilon_x + D_{22}\,\epsilon_y \\ D_{33}\,\gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} s_1 \\ s_2 \\ s_3 \end{Bmatrix} \tag{32}$$

and regarding this vector as A vector in the identity of Eq. (23), the fictitious strain tensor S^i, by analogy, becomes

$$[S]^i = \begin{bmatrix} S_1 & S_3 \\ S_3 & S_2 \end{bmatrix} = \begin{bmatrix} (D_{11}\,\epsilon_x + D_{12}\,\epsilon_y) & D_{33}\,\gamma_{xy} \\ D_{33}\,\gamma_{xy} & (D_{12}\,\epsilon_x + D_{22}\,\epsilon_y) \end{bmatrix}_i, \tag{33}$$

in which, ϵ_x, ϵ_y and γ_{xy} are the total instantaneous strain components to be numerically calculated by means of Eq. (7), using the nodal displacements of the preceding cycle. The fictitious strain tensor S^i contains constants as well as the variable functions of the local coordinates. Then, the strain energy, by analogy to Eq. (27) is

$$U^{i1} = \tfrac{1}{2}\int_V \{a\}^T [G_1]^T [S]^i [G_1]\{a\}\,dV, \tag{34}$$

which yields, after differentiating twice, to

$$K^{i1} = [B]^T [H^{i1}]\,[B], \tag{35}$$

in which

$$H^{i1} = \int_V [G_1]^T [S]^i [G_1]\,dV. \tag{36}$$

Note that, K^{i2} and K^{i3} matrices are obtained from similar expressions, except G_1 is replaced by G_2 and G_3, which are the coefficient matrix of partial derivatives of v and w, respectively given by Eq. (25). The only difference between a constant and a variable strain element is thus reduced to introducing a variable fictitious strain tensor S^i in place of a constant stress tensor σ^i.

Type III. Linear Bending Stiffness Matrix

The respective strain energy component, from Table 1, is

$$U^{44} = \tfrac{1}{2}\int_V \epsilon_4^T D\,\epsilon_4\,dV. \tag{37}$$

Introducing

$$\{\epsilon\}_4 = -z \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x\,\partial y} \end{Bmatrix} = -z \underset{(3\times n)}{[G_4]}\ \underset{(n\times 1)}{\{a\}} \tag{38}$$

and eliminating $\{a\}$ by means of Eq. (12)

$$U^{44} = \tfrac{1}{2}\int_V \{d\}^T [B]^T [H^{44}]\,[B]\{d\}\,dV. \tag{39}$$

After differentiating twice, according to Eq. (15)

$$K^{44} = [B]^T [H^{44}] [B], \tag{40}$$

in which

$$H^{44} = \int_V z^2 [G_4]^T [D] [G_4] dV. \tag{41}$$

The matrix K^{44} contributes into the w-locations only, and it vanishes for finite elements with no rotational degrees of freedom.

Type IV and VI. Stiffness Matrix

In most practical finite elements, type IV and VI stiffness matrices are zero, because either there are no rotational degrees of freedom specified, i.e., $\epsilon_4 = 0$, or the middle surface of the element lies in the xy-plane, thus rendering the volume integral $\int z\, dV$ to zero. Therefore, for reasons of space limitations, the matrix formulation of these types are not discussed herein.

Type V. Displacement Stiffness Matrix

A typical strain energy component of this type, from Table 1, is

$$U^{01} = \tfrac{1}{2}\int_V \epsilon_0^T D\, \epsilon_1 dV + \tfrac{1}{2}\int_V \epsilon_1^T D\, \epsilon_0 dV. \tag{42}$$

Replacing the products of $D\epsilon_0$, by a fictitious single vector $\{f\}$

$$\{f\} = [D]\{\epsilon\}_0 = \underset{(3\times1)}{\begin{Bmatrix} D_{11} u_{,x} + D_{12} v_{,y} \\ D_{12} u_{,x} + D_{22} v_{,y} \\ D_{33}(u_{,y} + v_{,x}) \end{Bmatrix}} = \underset{(3\times1)}{\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}} \tag{43}$$

and using it in Eq. (42)

$$U^{01} = \tfrac{1}{2}\int_V \{f\}^T \{\epsilon\}_1 dV + \tfrac{1}{2}\int_V \{\epsilon\}_1^T \{f\} dV \tag{44}$$

and taking advantage of the identity of Eq. (23), the strain energy becomes

$$U^{01} = \tfrac{1}{2}\int_V \underset{(1\times2)}{\begin{Bmatrix} u_{,x} \\ u_{,y} \end{Bmatrix}^T} \underset{(2\times2)}{\begin{bmatrix} f_1 & f_3 \\ f_3 & f_2 \end{bmatrix}} \underset{(2\times1)}{\begin{Bmatrix} u_{,x} \\ u_{,y} \end{Bmatrix}} dV. \tag{45}$$

The individual terms f_1, f_2 and f_3 of the middle block of Eq. (45) are analogues to σ_x, σ_y and τ_{xy} of the instantaneous stress block of Eq. (27). It is important to note that this middle block, given by

$$F = \underset{(2\times2)}{\begin{bmatrix} f_1 & f_3 \\ f_3 & f_2 \end{bmatrix}} \tag{46}$$

contains the unknown displacement coefficients $\{a\}$, or by virtue of Eq. (12) the unknown nodal displacements $\{d\}$. Substituting Eqs. (25) and (46) in Eq. (45)

$$U^{01} = \tfrac{1}{2}\int_V \{d\}^T [B]^T [G_1]^T [F] [G_1] [B] \{d\} dV. \tag{47}$$

While differentiating U^{01} twice, in accordance with Eq. (15), a difficulty arises due to the presence of $\{a\}$ or $\{d\}$ terms inside the F matrix. Obeying the rules for the derivative of the product variables, however, it is possible to overcome this difficulty and obtain the correct form of the resulting stiffness matrix K^{01}. Depending on whether the F matrix is expressed in terms of displacement coefficients $a_1, a_2, \ldots, a_n$, or in terms of nodal displacements $d_1, d_2, \ldots, d_n$, the stiffness matrix is obtained in one of the two following ways:

Case 1. F matrix as function of displacement coefficients:

$$K^{01} = [B]^T([H^{01}]+[M_1]+[M_1^T])[B], \tag{48}$$

in which

$$H^{01} = \int_V [G_1]^T [F_a][G_1]\,dV, \tag{49a}$$

$$M_1 = \left[\left\{\frac{\partial H^{01}}{\partial a_1}\{a\}\right\}, \left\{\frac{\partial H^{01}}{\partial a_2}\{a\}\right\}, \ldots, \left\{\frac{\partial H^{01}}{\partial a_n}\{a\}\right\}\right]. \tag{49b}$$

As seen from Eq. (48), the stiffness matrix K^{01} requires the evaluation of three middle blocks H^{01}, M and M_1^T, which are functions of coefficients $\{a\}$. The supplementary blocks, M_1 and its transpose M_1^T, are obtained from the partial derivatives of H^{01} as shown in Eq. (49b). Any jth column of M_1 matrix, equals the partial derivative of H^{01} with respect to the coefficient a_j, multiplied by the column vector of $\{a\}$. This process of taking derivatives with respect to a's, rather than with respect to d's is especially convenient for finite elements for which the F matrix is available in terms of a's but not in terms of d's. (Note this significant difference in the triangle and tetrahedron elements discussed later).

Case 2. F matrix as function of nodal displacements: In this case, after the second partial derivative of Eq. (47), the stiffness matrix is obtained as

$$K^{01} = [B]^T[H^{01}][B]+[B]^T[M_1]+[M_1^T][B] \tag{50a}$$

or

$$K^{01} = [K^1]+\left[\frac{\partial K^1}{\partial d_j}\{d\}\right]+\left[\frac{\partial K^1}{\partial d_j}\{d\}\right]^T \quad (j = 1, \ldots, n), \tag{50b}$$

in which

$$K^1 = [B]^T[H^{01}][B], \quad H^{01} = \int_V [G_1]^T[F_d][G_1]\,dV, \tag{51a}$$

$$M_1 = \left[\left\{\frac{\partial H^{01}}{\partial d_1}[B]\{d\}\right\}, \left\{\frac{\partial H^{01}}{\partial d_2}[B]\{d\}\right\}, \ldots, \left\{\frac{\partial H^{01}}{\partial d_n}[B]\{d\}\right\}\right]. \tag{51b}$$

As seen from Eqs. (48) to (51), the stiffness matrix K^{01} of Type V is always dependent on the nodal displacements $\{d\}$, of the preceding cycle of analysis. Especially, it is this matrix component that reflects the nonlinearity of the element much. It would be therefore, more appropriate to call this component as the "*Displacement Stiffness Matrix*".

Following exactly the same procedure outlined above the other two displacement stiffness matrices K^{02} and K^{03} are obtained from Eqs. (48) and (50), respectively. Except, matrix H^{01} is replaced by

$$H^{02} = \int_V [G_2]^T [F] [G_2] dV \tag{51}$$

and

$$H^{03} = \int_V [G_3]^T [F] [G_3] dV, \tag{52}$$

in which, G_2 and G_3 matrices are prepared in accordance with Eq. (25), using derivatives of v and w, instead of u, respectively.

Total Tangent Stiffness Matrix

The total tangent stiffness matrix K, of a finite element is obtained by combining algebraically various component stiffness matrices as follows:

$$K = K^{00} + (K^{i1} + K^{i2} + K^{i3}) + K^{44} + (K^{01} + K^{02} + K^{03}) \tag{53}$$

or

$$K = [B]^T [H] [B], \tag{54}$$

in which, the combined middle block H, is

$$H = H^{00} + (H^{i1} + H^{i2} + H^{i3}) + H^{44} + (H^{01} + M_1 + M_1^T) + (H^{02} + M_2 + M_2^T) + [H^{03} + M_3 + M_3^T). \tag{55}$$

Various components of H matrix is summarized in Table 2.

Table 2. Components of Stiffness Matrix

Type	H Matrix		K Matrix
I	H^{00}	$\int_V [G_0]^T [D] [G_0] dV$	$K^{00} = B^T H^{00} B$
II	H^{i1}	$\int_V [G_1]^T [\sigma^i] [G_1] dV$	$K^{i1} = B^T H^{i1} B$
	H^{i2} H^{i3}	Same as H^{i1}, except use v and w instead of u, respectively in G_1	$K^{i2} = B^T H^{i2} B$ $K^{i3} = B^T H^{i3} B$
III	H^{44}	$\int_V z^2 [G_4]^T [D] [G_4] dV$	$K^{44} = B^T H^{44} B$
V	H^{01}	$\int_V [G_1]^T [F] [G_1] dV$	
	M_1	If, H^{01} is function of a_j's $\left[\frac{\partial H^{01}}{\partial a_j}\{a\}\right]$ $(j = 1, \ldots, n)$	$K^{01} = B^T H^{01} B + B^T M_1 B + B^T M_1^T B$
	M_1	If, H^{01} is function of d_j's $\left[\frac{\partial H^{01}}{\partial d_j}[B]\{d\}\right]$ $(j = 1, \ldots, n)$	$K^{01} = B^T H^{01} B + B^T M_1 + M_1^T B$
	H^{02}, H^{03} M_2, M_3	Same as above, except u is replaced by v and w, respectively	K^{02} and K^{03} same form as K^{01}

Modifications for Three Dimensional Elements

The above formulation for the derivation of tangent stiffness matrices is prepared for two-dimensional finite elements. In order to translate the results into the three-dimensional state, the contents of various key matrices are summarized in Table 3, for a three dimensional solid finite element with no rotational degrees of freedom specified at its nodes.

Table 3. Three Dimensional State

Material Matrix (Eq. 3)

Orthotropic material

$$D = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & 0 & 0 \\ D_{12} & D_{22} & D_{23} & D_{24} & 0 & 0 \\ D_{13} & D_{23} & D_{33} & 0 & 0 & 0 \\ D_{14} & D_{24} & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & D_{56} \\ 0 & 0 & 0 & 0 & D_{56} & D_{66} \end{bmatrix}$$

Isotropic material

$$D_{11} = D_{22} = D_{33} = \frac{1-\nu}{(1+\nu)(1-2\nu)} E$$

$$D_{12} = D_{13} = D_{23} = \frac{\nu}{(1+\nu)(1-2\nu)} E$$

$$D_{14} = D_{24} = D_{56} = 0$$

$$D_{44} = D_{55} = D_{66} = \frac{E}{2(1+\nu)}$$

Displacement Derivatives (Eq. 25)

$$\{\epsilon\}_1 = \begin{Bmatrix} u,_x \\ u,_y \\ u,_z \end{Bmatrix} = [G_1]\{a\}; \quad \{\epsilon\}_2 = \begin{Bmatrix} v,_x \\ v,_y \\ v,_z \end{Bmatrix} = [G_2]\{a\}; \quad \{\epsilon\}_3 = \begin{Bmatrix} w,_x \\ w,_y \\ w,_z \end{Bmatrix} = [G_3]\{a\}$$

Instantaneous Stress Tensor (Eq.. 24). Constant strain element

$$[D]\{\epsilon\}_i = [\sigma]^i = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{xz} & \sigma_z \end{bmatrix}$$

Instantaneous Strain Tensor (Eq. 33). Variable strain element

Strain vector (Eq. 32)

$$\{s\} = [D]\{\epsilon^i\} = \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix}_i = \begin{Bmatrix} D_{11}\epsilon_x + D_{12}\epsilon_y + D_{13}\epsilon_z + D_{14}\gamma_{xy} \\ D_{12}\epsilon_x + D_{22}\epsilon_y + D_{23}\epsilon_z + D_{24}\gamma_{xy} \\ D_{13}\epsilon_x + D_{23}\epsilon_y + D_{33}\epsilon_z \\ D_{14}\epsilon_x + D_{24}\epsilon_y + D_{44}\gamma_{xy} \\ D_{55}\gamma_{xz} + D_{56}\gamma_{yz} \\ D_{56}\gamma_{xz} + D_{66}\gamma_{yz} \end{Bmatrix}_i \; (6\times 1);$$

Strain tensor (Eq. 33)

$$[S]^i = \begin{bmatrix} S_1 & S_4 & S_5 \\ S_4 & S_2 & S_6 \\ S_5 & S_6 & S_3 \end{bmatrix} \; (3\times 3)$$

F Matrix (Eq. 46)

$$\{f\} = [D]\{\epsilon\}_9 = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix} = \begin{Bmatrix} D_{11}u,_x + D_{12}v,_y + D_{13}w,_z + D_{14}(u,_y + v,_x) \\ D_{12}u,_x + D_{22}v,_y + D_{23}w,_z + D_{24}(u,_y + v,_x) \\ D_{13}u,_x + D_{23}v,_y + D_{33}w,_z \\ D_{14}u,_x + D_{24}v,_y + D_{44}(u,_y + v,_x) \\ D_{55}(u,_z + w,_x) + D_{56}(v,_z + w,_x) \\ D_{56}(u,_z + w,_x) + D_{66}(v,_z + w,_x) \end{Bmatrix}$$

(6×1) (6×1)

$$[F] = \begin{bmatrix} f_1 & f_4 & f_5 \\ f_4 & f_2 & f_6 \\ f_5 & f_6 & f_3 \end{bmatrix} \; (3\times 3)$$

Note: F matrix may be expressed in terms of either dispalcement coefficients $a_1, a_2, \ldots, a_n$ or, nodal displacements $d_1, d_2, \ldots, d_n$

General Triangle

The local coordinate axes and the nine degrees of freedom of a general triangle are shown in Fig. 1. Corners are numbered counterclockwise. Assumed

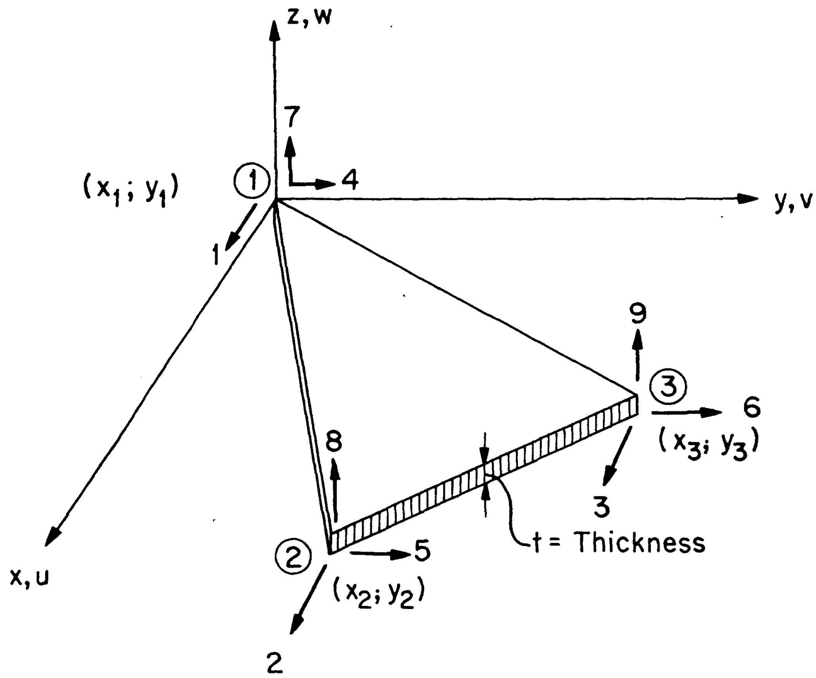

Fig. 1.

displacement functions are

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \underset{(3\times 3)}{\begin{bmatrix} 1 & x & y \\ 1 & x & y \\ 1 & x & y \end{bmatrix}} \{a\}. \tag{56}$$

Matrix A of Eq. 11 is

$$[A] = \underset{(9\times 9)}{\begin{bmatrix} [c] & 0 & 0 \\ 0 & [c] & 0 \\ 0 & 0 & [c] \end{bmatrix}}, \quad \text{in which } [c] = \underset{(3\times 3)}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}} \tag{57}$$

and after inverting A as in Eq. 13

$$[B] = [A]^{-1} = \underset{(9\times 9)}{\begin{bmatrix} c^{-1} & 0 & 0 \\ 0 & c^{-1} & 0 \\ 0 & 0 & c^{-1} \end{bmatrix}}, \quad \text{in which } [c]^{-1} = \frac{1}{2A} \underset{(3\times 3)}{\begin{bmatrix} 2A & 0 & 0 \\ y_{23} & y_3 & -y_2 \\ x_{32} & -x_3 & x_2 \end{bmatrix}}, \tag{58}$$

and

$$\begin{aligned} 2A &= x_2 y_3 - x_3 y_2, \\ x_{32} &= x_3 - x_2, \\ y_{23} &= y_2 - y_3. \end{aligned}$$

Type I. Linear Membrane Stiffness matrix: Taking derivatives of generic displacements, in accordance with Eq. (17) and subsequently employing G_0 matrix in Eq. (20), the linear membrane stiffness matrix K^{00} is obtained from Eq. (19) as given in Eq. (59). This matrix is the same as originally reported in Ref. [23].

$$
K_{00} = \begin{array}{c|c|c|c|c|c|c|c|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \text{ to } 9 \\
\hline
1 & \begin{matrix} D_{11}y_{23}^2 \\ D_{33}x_{32}^2 \end{matrix} & \begin{matrix} D_{11}y_3y_{23} \\ -D_{33}x_3x_{32} \end{matrix} & \begin{matrix} -D_{11}y_2y_{23} \\ D_{33}x_2x_{32} \end{matrix} & \begin{matrix} D_{33}x_{32}y_{23} \\ D_{12}x_{32}y_{23} \end{matrix} & \begin{matrix} D_{33}x_{32}y_3 \\ -D_{12}x_3y_{23} \end{matrix} & \begin{matrix} -D_{33}x_{32}y_2 \\ D_{12}x_2y_{23} \end{matrix} & \\
\hline
2 & \begin{matrix} D_{11}y_3y_{23} \\ -D_{33}x_3x_{32} \end{matrix} & \begin{matrix} D_{11}y_3^2 \\ D_{33}x_3^2 \end{matrix} & \begin{matrix} -D_{11}y_2y_3 \\ -D_{33}x_2x_3 \end{matrix} & \begin{matrix} -D_{33}x_3y_{23} \\ D_{12}x_{32}y_3 \end{matrix} & \begin{matrix} -D_{33}x_3y_3 \\ -D_{12}x_3y_3 \end{matrix} & \begin{matrix} D_{33}x_3y_2 \\ D_{12}x_2y_3 \end{matrix} & \\
\hline
3 & \begin{matrix} -D_{11}y_2y_{23} \\ D_{33}x_2x_{32} \end{matrix} & \begin{matrix} -D_{11}y_2y_3 \\ -D_{33}x_2x_3 \end{matrix} & \begin{matrix} D_{11}y_2^2 \\ D_{33}x_2^2 \end{matrix} & \begin{matrix} D_{33}x_2y_{23} \\ -D_{12}x_{32}y_2 \end{matrix} & \begin{matrix} D_{33}x_2y_3 \\ D_{12}x_3y_2 \end{matrix} & \begin{matrix} -D_{33}x_2y_2 \\ -D_{12}x_2y_2 \end{matrix} & \\
\hline
4 & \begin{matrix} D_{12}x_{32}y_{23} \\ D_{33}x_{32}y_{23} \end{matrix} & \begin{matrix} D_{12}x_{32}y_3 \\ -D_{33}x_3y_{23} \end{matrix} & \begin{matrix} -D_{12}x_{32}y_2 \\ D_{33}x_2y_{23} \end{matrix} & \begin{matrix} D_{33}y_{23}^2 \\ D_{22}x_{32}^2 \end{matrix} & \begin{matrix} D_{33}y_3y_{23} \\ -D_{22}x_3x_{32} \end{matrix} & \begin{matrix} -D_{33}y_2y_{23} \\ D_{22}x_2x_{32} \end{matrix} & 0 \\
\hline
5 & \begin{matrix} -D_{12}x_3y_{23} \\ D_{33}x_{32}y_3 \end{matrix} & \begin{matrix} -D_{12}x_3y_3 \\ -D_{33}x_3y_3 \end{matrix} & \begin{matrix} D_{12}x_3y_2 \\ D_{33}x_2y_3 \end{matrix} & \begin{matrix} D_{33}y_3y_{23} \\ -D_{22}x_3x_{32} \end{matrix} & \begin{matrix} D_{33}y_3^2 \\ D_{22}x_3^2 \end{matrix} & \begin{matrix} -D_{33}y_2y_3 \\ -D_{22}x_2x_3 \end{matrix} & \\
\hline
6 & \begin{matrix} D_{12}x_2y_{23} \\ -D_{33}x_{32}y_2 \end{matrix} & \begin{matrix} D_{12}x_2y_3 \\ D_{33}x_3y_2 \end{matrix} & \begin{matrix} -D_{12}x_2y_2 \\ -D_{33}x_2y_2 \end{matrix} & \begin{matrix} -D_{33}y_2y_{23} \\ D_{22}x_2x_{32} \end{matrix} & \begin{matrix} -D_{33}y_2y_3 \\ -D_{22}x_2x_3 \end{matrix} & \begin{matrix} D_{33}y_2^2 \\ D_{22}x_2^2 \end{matrix} & \\
\hline
7 \text{ to } 9 & & & & 0 & & & \\
\hline
\end{array} \; \frac{t}{4A} \tag{59}
$$

Type II. Instantaneous Strain Matrix: Partial derivatives of displacements in accordance with Eq. (25) give

$$
\underset{(2\times 9)}{G_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}, \tag{60}
$$

$$
G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{61}
$$

$$
G_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{62}
$$

Substituting these in Eqs. (29) to (31), the instantaneous strain matrices are obtained as follows:

$$
\underset{(9\times 9)}{K^{i1} = \begin{bmatrix} [i] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}; \quad \underset{(9\times 9)}{K^{i2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & [i] & 0 \\ 0 & 0 & 0 \end{bmatrix}}; \quad \underset{(9\times 9)}{K^{i3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & [i] \end{bmatrix}}, \tag{63}
$$

in which

$$[i] = \begin{bmatrix} \sigma_x y_{23}^2 + \sigma_y x_{32}^2 \\ 2\tau_{xy} x_{32} y_{23} & \sigma_x y_3 y_{23} - \sigma_y x_3 x_{32} \\ \tau_{xy}(y_3 x_{32} - x_3 y_{23}) & -\sigma_x y_2 y_{23} + \sigma_y x_2 x_{32} \\ \tau_{xy}(x_2 y_{23} - y_2 x_{32}) \\ \sigma_x y_3 y_{23} - \sigma_y x_3 x_{32} \\ \tau_{xy}(y_3 x_{32} - x_3 y_{23}) & \sigma_x y_3^2 + \sigma_y x_3^2 \\ -2\tau_{xy} x_3 y_3 & -\sigma_x y_2 y_3 - \sigma_y x_2 x_3 \\ \tau_{xy}(x_2 y_3 + y_2 x_3) \\ -\sigma_x y_2 y_{23} + \sigma_y x_2 x_{32} \\ \tau_{xy}(x_2 y_{23} - y_2 x_{32}) & -\sigma_x y_2 y_3 - \sigma_y x_2 x_3 \\ \tau_{xy}(x_2 y_3 + y_2 x_3) & \sigma_x y_2^2 + \sigma_y x_2^2 \\ -2\tau_{xy} x_2 y_2 \end{bmatrix}$$

$$[i] = \begin{bmatrix} \sigma_x y_{23}^2 + \sigma_y x_{32}^2 + 2\tau_{xy} x_{32} y_{23} & \sigma_x y_3 y_{23} - \sigma_y x_3 x_{32} + \tau_{xy}(y_3 x_{32} - x_3 y_{23}) & -\sigma_x y_2 y_{23} + \sigma_y x_2 x_{32} + \tau_{xy}(x_2 y_{23} - y_2 x_{32}) \\ \sigma_x y_3 y_{23} - \sigma_y x_3 x_{32} + \tau_{xy}(y_3 x_{32} - x_3 y_{23}) & \sigma_x y_3^2 + \sigma_y x_3^2 - 2\tau_{xy} x_3 y_3 & -\sigma_x y_2 y_3 - \sigma_y x_2 x_3 + \tau_{xy}(x_2 y_3 + y_2 x_3) \\ -\sigma_x y_2 y_{23} + \sigma_y x_2 x_{32} + \tau_{xy}(x_2 y_{23} - y_2 x_{32}) & -\sigma_x y_2 y_3 - \sigma_y x_2 x_3 + \tau_{xy}(x_2 y_3 + y_2 x_3) & \sigma_x y_2^2 + \sigma_y x_2^2 - 2\tau_{xy} x_2 y_2 \end{bmatrix} \frac{t}{4A}. \quad (64)$$

This matrix is the same as originally reported in Ref. [1].

Type III, IV and VI stiffness matrix components are all zero, since there are no rotational degrees of freedom.

Type V. Displacement Stiffness Matrix: If the partial derivatives of u and v are taken as in Eq. (43) and the displacement coefficients $\{a\}$, are replaced by nodal displacements $\{d\}$ by means of Eq. (12), the fictititous strain vector f becomes

$$\{f\} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \frac{1}{2A} \underset{(3\times 1)}{\begin{Bmatrix} D_{11}(y_{23} d_1 + y_3 d_2 - y_2 d_3) + D_{12}(x_{32} d_4 - x_3 d_5 + x_2 d_6) \\ D_{12}(y_{23} d_1 + y_3 d_2 - y_2 d_3) + D_{22}(x_{32} d_4 - x_3 d_5 + x_2 d_6) \\ D_{33}(x_{32} d_1 - x_3 d_2 + x_2 d_3) + D_{33}(y_{23} d_4 + y_3 d_5 - y_2 d_6) \end{Bmatrix}}. \quad (65)$$

By analogy to instantaneous strain matrix, it is seen that f_1, f_2 and f_3 terms of Eq. (46) are playing exactly the same role as σ_x, σ_y and τ_{xy} of Eq. (24). Therefore, the H matrices of this type, are in the same form as the H matrices of Type II.

From Eq. (50b)

$$K^{01} = [K^1] + \left[\frac{\partial K^1}{\partial d_j}\{d\}\right] + \left[\frac{\partial K^1}{\partial d_j}\{d\}\right]^T \quad (50\text{b})$$

and taking advantage of the analogy mentioned above, the first term of Eq. (50b), for each displacement component u, v and w, becomes

$$K^1 = \underset{(9\times 9)}{\begin{bmatrix} [h] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}; \qquad K^2 = \underset{(9\times 9)}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & [h] & 0 \\ 0 & 0 & 0 \end{bmatrix}}; \qquad K^3 = \underset{(9\times 9)}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & [h] \end{bmatrix}}, \quad (66)$$

in which, $[h]$ is identical to the (3×3) matrix $[i]$ given in Eq. (64), except σ_x, σ_y and τ_{xy} values are to be replaced by f_1, f_2 and f_3 of Eq. (65), respectively. In order to complete the stiffness matrix, partial derivatives of K^1 must be taken relative to $d_1, d_2, \ldots, d_n$ as indicated in Eq. (50b). Due to space limitations, the explicit contents of these derivatives are not presented herein. The total tangent stiffness matrix then becomes the combination of various components as follows:

$$K = K^{00} + K^{i1} + K^{i2} + K^{i3} + K^{1} + \left[\frac{\partial K^1}{\partial d_j}\{d\}\right] + \left[\frac{\partial K^1}{\partial d_j}\{d\}\right]^T + K^2 + \left[\frac{\partial K^2}{\partial d_j}\{d\}\right] + \left[\frac{\partial K^2}{\partial d_j}\{d\}\right]^T + K^3 + \left[\frac{\partial K^3}{\partial d_j}\{d\}\right] + \left[\frac{\partial K^3}{\partial d_j}\{d\}\right]^T. \quad (67)$$

Tetrahedron Element

A general tetrahedron element is shown in Fig. 2 with three degrees of freedom at each joint in the local coordinate axes, xyz. The displacement polynomials of Eq. (10), in terms of the xyz coordinates of a general point on the element, are assumed to be

$$\begin{aligned} u &= a_1 + a_2 x + a_3 y + a_4 z, \\ v &= a_5 + a_6 x + a_7 y + a_8 z, \\ w &= a_9 + a_{10} x + a_{11} y + a_{12} z. \end{aligned} \quad (68)$$

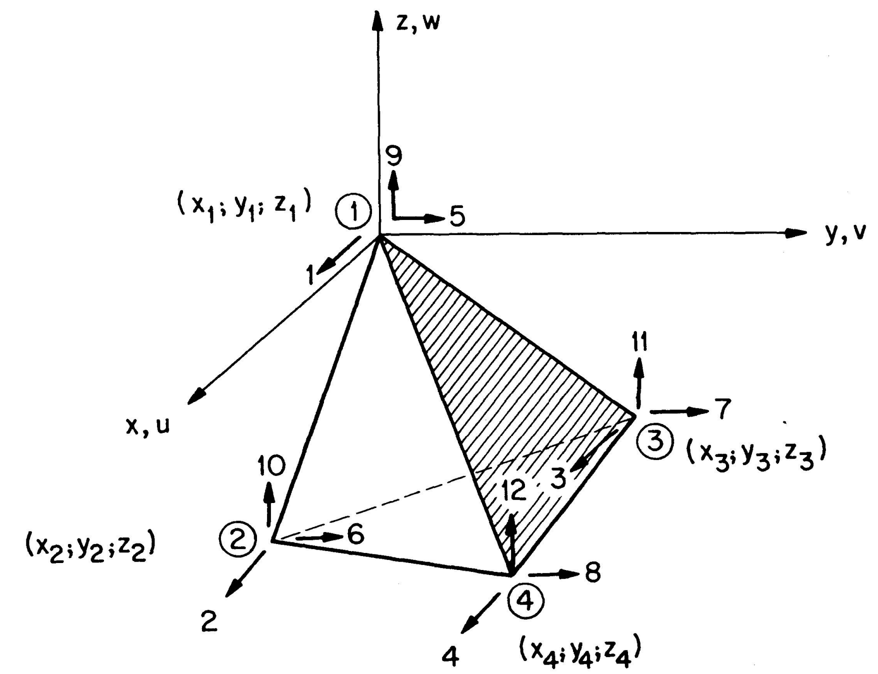

Fig. 2.

The matrix, B, relating the nodal displacements, $d_1, d_2, \ldots, d_{12}$ to the coefficients, $a_1, a_2, \ldots, a_{12}$, according to Eq. (12), is

$$[B] = \begin{bmatrix} [d]^{-1} & 0 & 0 \\ 0 & [d]^{-1} & 0 \\ 0 & 0 & [d]^{-1} \end{bmatrix}, \quad (69)$$

in which,
$$[d] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix}. \tag{70}$$

In order to develop a complete tangent stiffness matrix for the tetrahedron element, all components of the H matrix given in Table 2 should be evaluated. Note however, that the Types III, IV and VI matrices are all zero, since there are no rotations specified at the nodes. The rest of the H matrices are obtained as follows:

Type I. Linear Stiffness Matrix: Differentiating the generic displacements u, v and w with respect to x, y and z, in accordance with the ϵ_0 expression given in Eq. (7b), the content of the matrix G_0 of Eq. (17), is obtained as

$$G_0 = \underset{(6\times 12)}{\overset{\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}}}, \tag{71}$$

After pre- and post-multiplying D by G_0, H^{00} of Eq. (26) becomes

$$[H^{00}] = \frac{E\,V}{1+\nu} \overset{\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{matrix}}{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1-\nu}{1-2\nu} & 0 & 0 & 0 & 0 & \frac{\nu}{1-2\nu} & 0 & 0 & 0 & 0 & \frac{\nu}{1-2\nu} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\nu}{1-2\nu} & 0 & 0 & 0 & 0 & \frac{1-\nu}{1-2\nu} & 0 & 0 & 0 & 0 & \frac{\nu}{1-2\nu} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{\nu}{1-2\nu} & 0 & 0 & 0 & 0 & \frac{\nu}{1-2\nu} & 0 & 0 & 0 & 0 & \frac{1-\nu}{1-2\nu} \end{bmatrix}}, \tag{72}$$

in which, V = the volume of the tetrahedron element, which equals to one sixth of the determinant of the matrix d given in Eq. (70).

Type II. Instantaneous Stress Matrix: The partial derivatives of u, v and w displacements, in accordance with the expression of ϵ_1 given in Eq. (7b), yields the matrix G_1 of Eq. (25) as follows:

$$G_1 = \underset{(3\times 12)}{\left[\begin{array}{cccc|cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline 0 & 1 & 0 & 0 & & & & & & & & \\ 0 & 0 & 1 & 0 & & & & & 0 & & & \\ 0 & 0 & 0 & 1 & & & & & & & & \end{array}\right]}. \tag{73}$$

When the initial stress tensor σ^i of Eq. (24) is pre- and post-multiplied by G_1, in accordance with Eq. (29), the matrix H^{i1} is obtained as

$$[H^{i1}] = \underset{(12\times 12)}{\begin{bmatrix} [h] & & \\ & [0] & \\ & & [0] \end{bmatrix}}, \tag{74}$$

in which, using the initial stress tensor σ^i of Eq. (24), h matrix becomes

$$h = \underset{(4\times 4)}{\begin{array}{c|c|c|l} & 1 & \overbrace{2\text{ to }4} & \\ \hline & 0 & 0 & 1 \\ \hline & 0 & \sigma^i & 2\text{ to }4 \end{array}}. \tag{75}$$

As explained earlier in connection with Eqs. (30) and (31), the H^{i2} and H^{i3} matrices are obtained in the same manner as the H^{i1} matrix, except the displacement u, is replaced by v and w, respectively. Therefore,

$$H^{i2} = \begin{bmatrix} [0] & & \\ & [h] & \\ & & [0] \end{bmatrix}, \tag{76}$$

$$H^{i3} = \begin{bmatrix} [0] & & \\ & [0] & \\ & & [h] \end{bmatrix}. \tag{77}$$

Type V. Displacement Stiffness Matrix: In order to evaluate the stiffness matrix of Type V given by Eq. (48), at first, matrix H^{01} of Eq. (49a) should be determined. The matrix G_1 appearing inside matrix H^{01}, has been already evaluated in connection with Type II and given in Eq. (73). The middle block F, is obtained from Table 3 – Eq. (46) as

$$[F] = \begin{bmatrix} (1-\nu)a_2+\nu a_7 + \nu a_{12} & \frac{1-2\nu}{2}(a_3+a_6) & \frac{1-2\nu}{2}(a_{10}+a_4) \\ \frac{1-2\nu}{2}(a_3+a_6) & \nu a_2+(1-\nu)a_7 + \nu a_{12} & \frac{1-2\nu}{2}(a_8+a_{11}) \\ \frac{1-2\nu}{2}(a_{10}+a_4) & \frac{1-2\nu}{2}(a_8+a_{11}) & \nu a_2+\nu a_7 + (1-\nu)a_{12} \end{bmatrix}_{(3\times 3)} \frac{E\,V}{(1+\nu)(1-2\nu)}. \quad (78)$$

After pre- and post-multiplying F by G_1, in accordance with Eq. (49a) the matrix H^{01} is obtained, since G_1 is a partially unit matrix, as

$$H^{01} = \overset{\quad 1 \qquad 2\text{ to }4 \qquad 5\text{ to }12}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & [F] & 0 \\ 0 & 0 & 0 \end{bmatrix}} \begin{matrix} 1 \\ 2\text{ to }4 \\ 5\text{ to }12 \end{matrix}. \quad (79)$$

Similarly, exchanging the locations of u, with v and w, respectively, the matrices H^{02} and H^{03} are obtained as follows:

$$H^{02} = \overset{1\text{ to }5 \quad 6\text{ to }8 \quad 9\text{ to }12}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & [F] & 0 \\ 0 & 0 & 0 \end{bmatrix}} \begin{matrix} 1\text{ to }5 \\ 6\text{ to }8 \\ 9\text{ to }12 \end{matrix}, \quad (80)$$

$$H^{03} = \overset{1\text{ to }9 \quad 10\text{ to }12}{\begin{bmatrix} 0 & 0 \\ 0 & [F] \end{bmatrix}} \begin{matrix} 1\text{ to }9 \\ 10\text{ to }12 \end{matrix}. \quad (81)$$

As indicated in Eq. (49b), the jth column of matrix M^{01}, is obtained as the partial derivative of matrix H^{01}, with respect to the coefficients a_j $(j=1, \ldots, n)$. Therefore, from Eq. (49b) and (79)

$$M^{01} = \overset{1 \quad 2\text{ to }12}{\begin{bmatrix} 0 & 0 \\ 0 & [m] \\ 0 & 0 \end{bmatrix}} \begin{matrix} 1 \\ 2\text{ to }4 \\ 5\text{ to }12 \end{matrix}, \quad (82)$$

$$\text{in which} \quad [m]^T = \frac{E\,V}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu)\,a_2 & \nu\,a_3 & \nu\,a_4 \\ \frac{1-2\nu}{2}a_3 & \frac{1-2\nu}{2}a_2 & 0 \\ \frac{1-2\nu}{2}a_4 & 0 & \frac{1-2\nu}{2}a_2 \\ 0 & 0 & 0 \\ \frac{1-2\nu}{2}a_3 & \frac{1-2\nu}{2}a_2 & 0 \\ \nu\,a_2 & (1-\nu)\,a_3 & \nu\,a_4 \\ 0 & \frac{1-2\nu}{2}a_4 & \frac{1-2\nu}{2}a_3 \\ 0 & 0 & 0 \\ \frac{1-2\nu}{2}a_4 & 0 & \frac{1-2\nu}{2}a_2 \\ 0 & \frac{1-2\nu}{2}a_4 & \frac{1-2\nu}{2}a_3 \\ \nu\,a_2 & \nu\,a_3 & (1-\nu)\,a_4 \end{bmatrix}. \tag{83}$$

Differentiating H^{02} and H^{03} of Eqs. (80) and (81), in accordance with Eq. (49b), the matrices M^{02} and M^{03} are obtained as

$$M^{02} = \begin{array}{c} \begin{array}{cc} 1 & \overbrace{\text{2 to 12}} \end{array} \\ \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & [m'] \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} \text{1 to 5} \\ \text{6 to 8} \\ \text{9 to 12} \end{array} \end{array}, \tag{84}$$

$$M^{03} = \begin{array}{c} \begin{array}{cc} 1 & \overbrace{\text{2 to 12}} \end{array} \\ \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & [m''] \end{array}\right] \begin{array}{l} \text{1 to 9} \\ \text{10 to 12} \end{array} \end{array}, \tag{85}$$

in which, m' and m'' are the same as m, except the coefficients, a_2, a_3 and a_4 are to be replaced by a_6, a_7 and a_8 for m' and by a_{10}, a_{11} and a_{12} for m'', respectively. These coefficients are obtained, at the end of each cycle, from the nodal displacements, d_1, d_2, etc., by means of Eq. (12).

Finally, the total tangent stiffness matrix K of the tetrahedron is obtained in the computer, from

$$K = [B]^T [H] [B], \tag{86}$$

in which, B is given by Eq. (69) and

$$\begin{aligned} H = {} & H^{00} + H^{i1} + H^{i2} + H^{i3} + (H^{01} + M^{01} + M^{01\,T}) \\ & + (H^{02} + M^{02} + M^{02\,T}) + (H^{03} + M^{03} + M^{03\,T}). \end{aligned} \tag{87}$$

The linear component K^{00}, coincides exactly with the stiffness matrix given by PRZEMIENIECKI [14]. The rest of the components constitute the nonlinear part of the tangent stiffness matrix and are dependent on the instantaneous stresses and nodal displacements, which occur in the element at the end of previous cycle. Very few of these nonlinear stiffness components, in the author's knowledge, were reported until now.

Modified Newton-Raphson Iteration

For a given set of external loads, the objective of static nonlinear analysis is to determine the true values of the displacements and internal stress resultants. Since, the tangent stiffness matrix is dependent on the instantaneous strains and displacements, an iterative methods of solution is inevitable. Newton-Raphson method of successive cycles of linear analysis has been used in the solution of a variety of nonlinear structural problems with extremely satisfactory performance [3, 4, 5, 17, 20, 21]. The basic principals of this

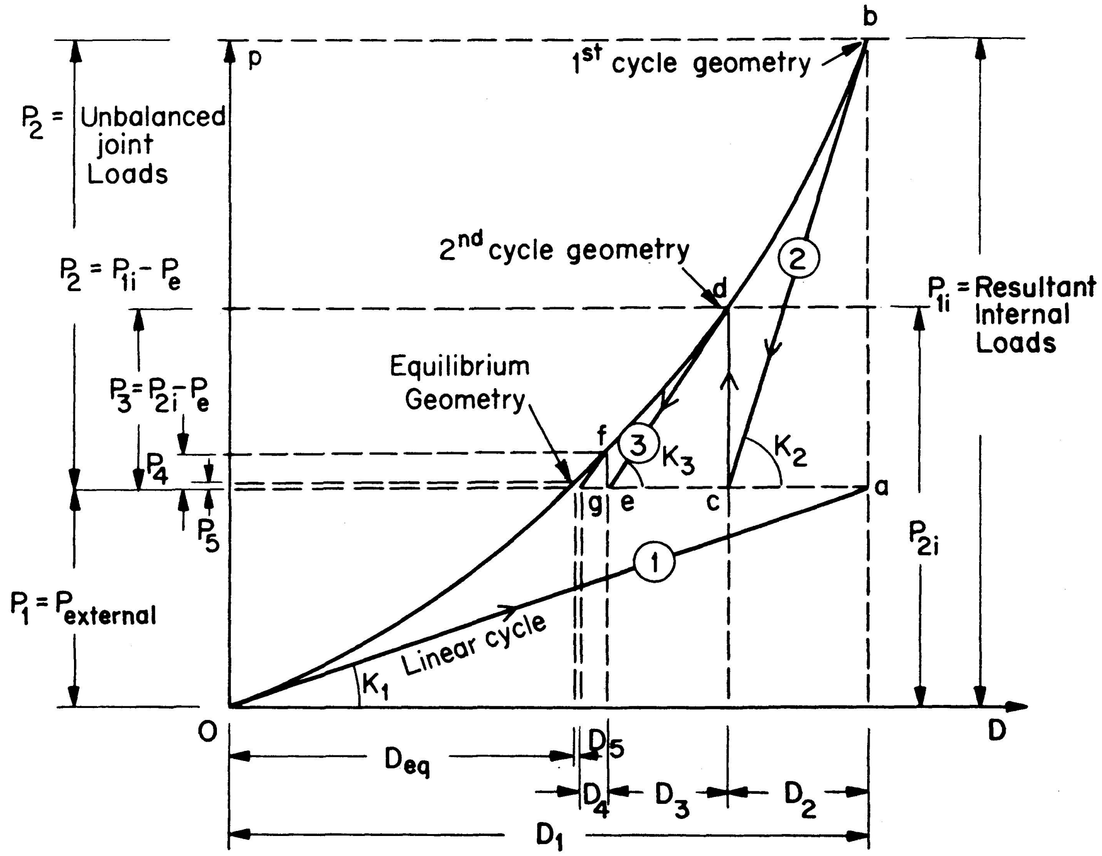

Fig. 3.

iterative method, as employed in the numeric examples of this presentation, are outlined graphically in Fig. 3.

At first, using the initial geometry and the external loads P_e, a linear stiffness analysis is performed corresponding to the straight line $\overline{oa}$. The slope of this line K_1, represents the linear stiffness matrix of the structure. Using, the nodal displacements D_1, of this first linear cycle of analysis, in the nonlinear expressions of stress-displacement relations (Eq. (7b)), the stresses as well as stress resultants are calculated at each node of every element. At a particular node of the system, in a particular direction of degree of freedom, the algebraic sum of all the calculated stress resultants must be equal to the external load given in that particular direction. Since, the nodal displacements obtained in the first cycle do not correspond to the true equilibrium geometry, the algebraic sum of internal stress resultants $P_{1,int}$ obtained using these first approximate displacements will not be equal to the given external loads. The difference between the internal and the external forces, that is

$$P_2 = P_{1,int} - P_{ext} \tag{88}$$

constitutes the new external loads to be used in the next second cycle of analysis.

In preparation for a second cycle, tangent stiffness matrices of each finite element are avaluated from Eq. (53), using all of the nonlinear components. Thus, the changes in the global co-ordinates of the nodes, as well as the presence of instantaneous strains and displacements, are all taken into account. A second set of nodal displacements D_2 are calculated, under the action of the unbalanced nodal forces P_2 following a purely linear analysis along the straight line $\overline{bc}$. The slope K_2 of this line represents the tangent stiffness matrix of this second cycle. When the nodal displacements of the first and second cycles are superimposed, the system comes closer to the actual equilibrium configuration. As a result, the difference between the external loads and internal stress resultants, calculated on the basis of the combined nodal displacements, is reduced. Denoting the internal stress resultants obtained at the end of the second cycle by $P_{2,int}$, the modified external loads P_3, of the subsequent third cycle becomes

$$P_3 = P_{2,int} - P_{ext}. \tag{89}$$

Using the strains and displacements obtained at the end of the second cycle, a new set of tangent stiffness matrices, represented by the slope K_3 of the straight line $\overline{de}$, are evaluated and under the application of the unbalanced nodal forces P_3, a third linear cycle of analysis is performed. The nodal displacements D_3 of the third cycle are superimposed on D_1 and D_2 and a new set of unbalanced nodal forces are calculated. The above iterative process is repeated until the maximum unbalanced nodal force in any direction becomes less than a tolerable value. The tangent stiffness matrices are successively altered after each cycle so as to include the latest global coordinates and the latest strain and displacements. Unbalanced nodal forces are continuously diminished and the algebraic sum of the nodal displacements at each cycle is taken to yield the final displacements D, of the true equilibrium configuration as

$$D = D_1 + D_2 + D_3 + \cdots + D_n . \tag{90}$$

Numerical Examples

For purposes of illustration of the use of tangent stiffness matrices in connection with the modified Newton-Raphson iteration scheme, as well as for assessment of relative importance of higher order terms, the following thin plate examples have been solved:

1. Fixed square plate – Uniformly loaded (Fig. 4, 5).
2. Fixed square plate – Centrally loaded (Fig. 6, 7).
3. Simply supported square plate with immovable edges – Uniformly Loaded (Fig. 8, 9).
4. Simply supported square plate with immovable edges – Centrally loaded (Fig. 10, 11).

Inveriably, due to four-way symmetry only one octant of the plate is analyzed dividing the plate into a 16×16 square mesh. The convergence criteria is taken to be the ratio of the maximum unbalanced nodal force, to the maximum stress resultant in that direction in the linear cycle. In most cases, this ratio is reduced to less than 1% within five to seven cycles. The nonlinear analysis of each particular plate has been performed for four different magnitudes of external loads. Although the results of the analyses are illustrated by continuous lines in all the diagrams (Fig. 4 to 9), the actual calculation points are indicated by circles and triangles. In order to evaluate the relative effects of the higher order terms of Eq. (53), the examples 3 and 4 have been solved for two cases:

Case 1. Tangent stiffness matrix, with types I, II and III, but excluding all three components of type V.

Case 2. Tangent stiffness matrix, including all components of type I, II, III and V.

The corresponding results have been shown separately in the diagrams, by means of putting triangular and circular signs, respectively, around the points of calculations. For a rectangular element lying in the local xy-plane, a series of 24×24 tangent stiffness matrices have been derived, using the general formulation presented herein. The assumed displacement functions are

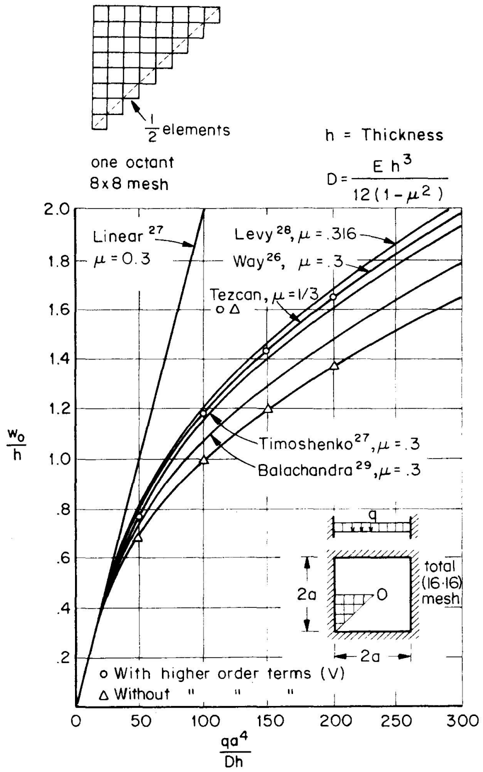

Fig. 4.

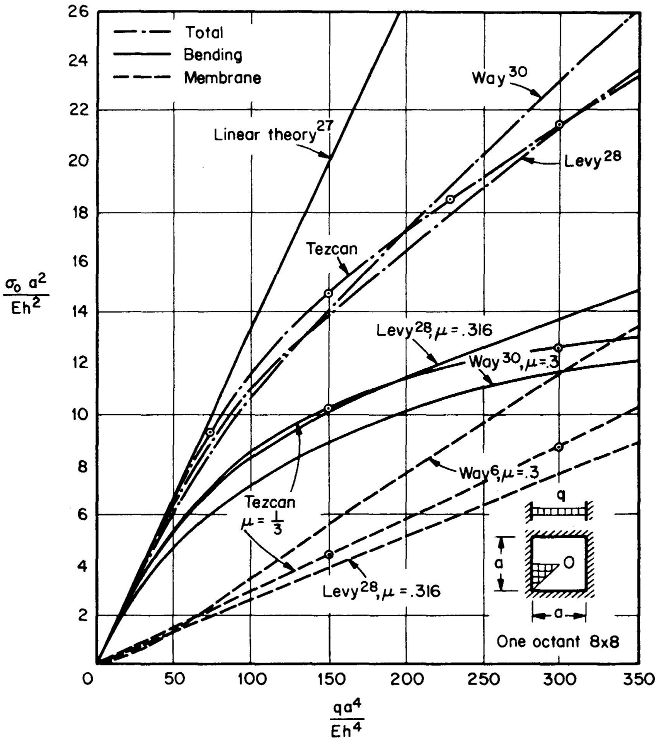

Fig. 5.

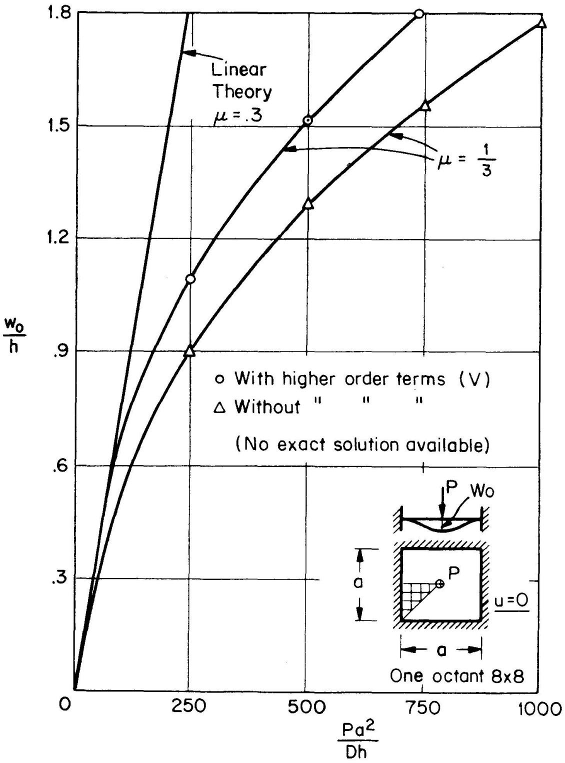

Fig. 6.

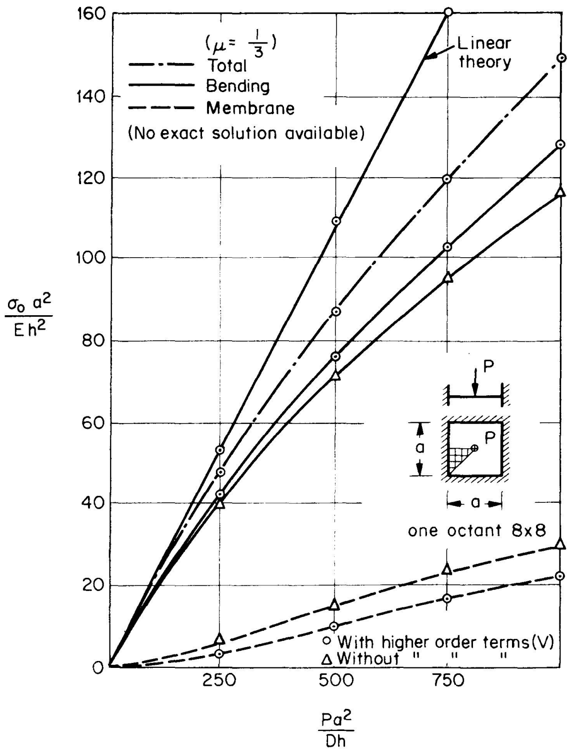

Fig. 7.

$$u = a_1 + a_2 x + a_3 y + a_4 x y,$$

$$v = a_5 + a_6 x + a_7 y + a_8 x y,$$

$$w = a_9 + a_{10} x + a_{11} y + a_{12} x^2 + a_{13} x y + a_{14} y^2 + a_{15} x^3 + a_{16} x^2 y + a_{17} x y^2$$
$$+ a_{18} y^3 + a_{19} x^3 y + a_{20} x^2 y^2 + a_{21} x y^3 + a_{22} x^3 y^2 + a_{23} x^2 y^3 + a_{23} x^3 y^3.$$

Due to space limitations the contents of the tangent stiffness matrix components of a rectangular element are not included in this paper. However, the derivation follows exactly the same procedure as employed for triangular and tetrahedron elements. After the first cycle, as a result of nodal displacements, originally square plates warp out of their planes and become spatial quadrilaterals. Consequently, the tangent stiffness matrices derived for rectangles, are no longer applicable to quadrilaterals in the subsequent cycles. However, in all the computations, although the changes in coordinates as well as the intermediate strains and displacements are duly taken into account, the change of shape from square to quadrilateral has been neglected.

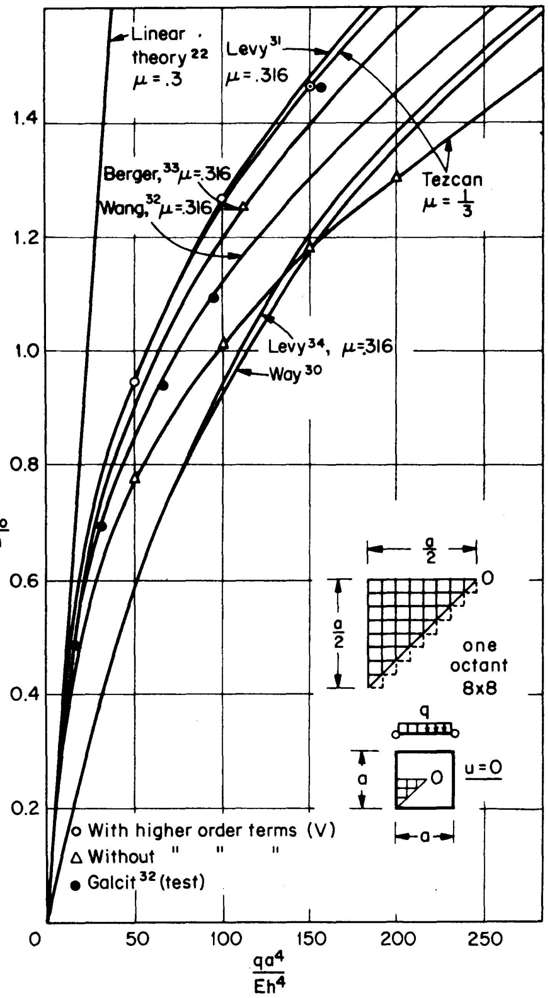

Fig. 8.

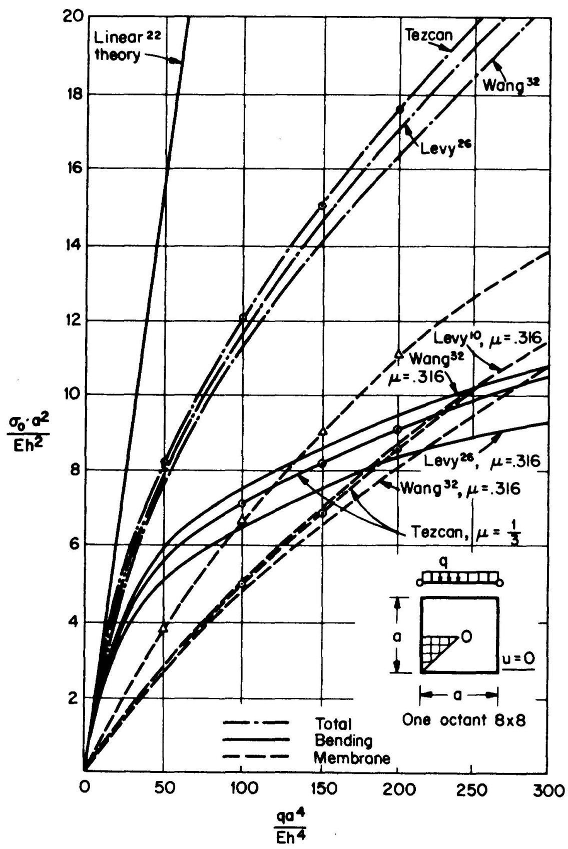

Fig. 9.

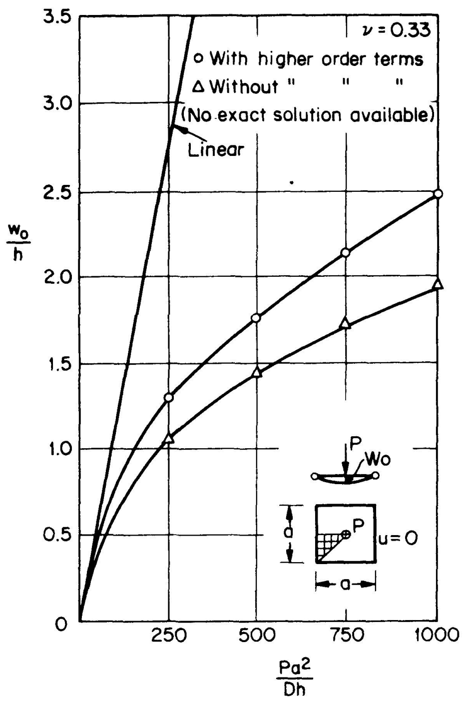

Fig. 10.

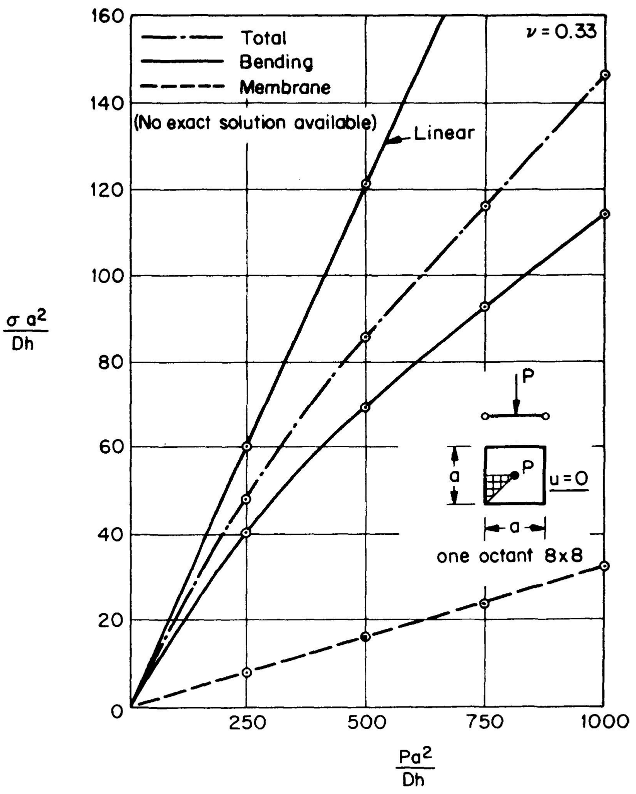

Fig. 11.

There are exact theory of elasticity solutions available in the literature for Examples 1 and 3, as already indicated in the respective diagrams. It is not possible, however, to compare the computer solutions of Examples 2 and 4 against any solution, since, to the knowledge of the author, no exact solution exists in the literature for centrally loaded square plates.

Conclusions

By using the strain energy approach, in conjunction with Castigliano's Theorem, a systematic method of derivation has been discussed for the development of tangent stiffness matrices for finite elements. The uniformity of formulations is expected to facilitate the inclusion of various high order terms of strain energy, which were ordinarily neglected due to complexities involved in the derivations.

As indicated in the numerical examples, when the higher order components of tangent stiffness matrices are included, a marked improvement is observed in the accuracy of the results as well as in the speed of convergence.

Although the numerical examples are for static analysis of nonlinear structures, the concept of tangent stiffness matrix is readily applicable to the stability problems in which the external loads are gradually increased until the displacements become excessively large [25].

Further studies are desirable, however, to investigate the relative effects of various higher order terms in more detail, and also to formulate different solution schemes for combined material and geometric nonlinearities. Tangent stiffness matrices for triangular finite elements with curvatures in two directions, would be a useful extension of the formulation presented.

Acknowledgement

The writers acknowledge gratefully, the financial support received in the early stages of this project from the National Research Council of Canada and in the later stages from the National Scientific and Technical Research Council (T.B.T.A.K.) of Turkey.

References

1. Turner, M. J., Dill, E. H., Martin, H. C., and Melosh, R. J., Large Deflections of Structures Subjected to Heating and External Loads, Journal of the Aerospace Sciences, Vol. 27, p. 97–102.
2. Martin, H. C., On Derivation of Stiffness Matrices for the Analysis of Large Deflection and Stability Problems, Proceedings of Conference on Matrix Methods in Structural Mechanics, Wright-Patterson Air Force Base, Ohio, U.S.A., AFFDL-TR-66-80, Dec. 1965, p. 647–716.
3. Saafan, A. S., Theoretical Analysis of Suspension Bridges, Journal of Structural Division, ASCE, Vol. 92, No. ST4, Aug. 1966, p. 1–11.
4. Tezcan, S. S., discussion of: Numerical Solution of Nonlinear Structures, by T. J. Poskitt, Proc. Paper 5362, Journal of Structural Division, ASCE, June 1968.
5. Tezcan, S. S. and Mahapatra, B. C., Tangent Stiffness Matrices for Space Frame Members, Journal of Structural Division, ASCE, Vol. 95, No. ST6, June, p. 1257–1270.
6. Lunder, C. A., Derivation of a Stiffness Matrix for a Right Triangular Plate in Bending Subjected to Initial Stresses, M.Sc., Thesis, Department of Aeronautics and Astronautics, Univ. of Washington, Seattle, January, 1962.
7. Kapur, K. K., and B. J. Hartz, Stability of Plates using Finite Elements, Journal of Engineering Mechanism Division, No. 92, EM2, ASCE, April, 1966, p. 177–195.
8. Gallagher, R. H., Gellatly, R. A., Padlog, J., and Mallet, R. A., A Discrete Element Procedure for Thin-Shell Instability Analysis, Journal Am. Institute of Aeronautics and Astronautics, Vol. 5, 1967, p. 138–145.
9. Wallace, C. D., Matrix Analysis of Axisymmetric Shells Under General Loading, M.Sc., Thesis, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, 1966.
10. Argyris, J. H., Continua and Discontinua, Proceedings of Conference on Matrix Methods in Structural Mechanics, Wright-Patterson Air Force Base, Ohio, U.S.A., AFFDL-TR-80, Dec. 1965, p. 11–189.
11. Argyris, J. H., Matrix Analysis of Three-Dimensional Elastic Media – Small and Large Displacements, Journal of Am. Institute of Aeronautics and Astronautics, Vol. 3, No. 1, Jan. 1965, p. 45–51.
12. Przemieniecki, J. S., Discrete Element Methods for Stability Analysis of Complex Structures, Journal of the Royal Aeronautical Society, Vol. 72, Dec. 1968, p. 1077–1086.

13. PURDY, D. M., and PRZEMIENIECKI, J. S., Influence of Higher-Order Terms in the Larger Deflection Analysis of Frameworks, Proceedings, Speciality Conference on Optimization and Nonlinear Problems, Am. Soc. of Civil Engineers, Chicago, April, 1968, p. 142–152.
14. PRZEMIENIECKI, J. S., Theory of Matrix Structural Analysis, McGraw-Hill Book Co., Inc., 1968.
15. ODEN, J. T., Calculation of Geometric Stiffness Matrices for Complex Structures, Journal of Am. Institute of Aeronautics and Astronautics, Vol. 4, p. 1480–1482, 1966.
16. WISSMANN, J. W., Nonlinear Structural Analysis; Tensor Formulation, Proceedings of the Conference on Matrix Methods, in Structural Mechanics, Wright-Patterson Air Force Base, Ohio, U.S.A., AFFDL-TR-66-80, Dec. 1965, p. 679–696.
17. ODEN, J. T. and SATO, T., Finite Strains and Displacements of Elastic Membranes by the Finite Element Method, International Journal of Solids and Structures, Vol. 3, p. 471–488, 1967.
18. ODEN, J. T., Numerical Formulation of Nonlinear Elasticity Problems, Journal of the Structural Division, Am. Society of Civil Engineers, Vol. 93, No. ST3, June 1967, p. 235–255.
19. MALLETT, H. R. and MARCAL, P. V., Finite Element Analysis of Nonlinear Structures, Journal of Structural Division, Am. Society of Civil Engineers, Vol. 94, No. ST 9, 1968, p. 2081–2105.
20. ODEN, J. T., Finite Plane Strain of Incompressible Elastic Solids by the Finite Element Method, The Aeronautical Quarterly, Vol. XIX, Aug. 1968, p. 254–264.
21. TEZCAN, S. S., Nonlinear Analysis of Thin Plates by Framework Method, Journal of Am. Institute of Aeronautics and Astronautics, Vol. 5, No. 10, Oct. 1967, p. 1890–1892.
22. ODEN, J. T., Finite Element Applications in Nonlinear Structural Analysis, Proceedings of the Symposium on Application of Finite Element Methods in Civil Engineering, Vanderbilt University, Nashville, Tennessee, Nov. 1969, p. 419–456.
23. TURNER, M. J., CLOUGH, R. W., MARTIN, H. C., and TOPP, L. J., Stiffness and Deflection Analysis of Complex Structures, Journal of Aerospace Sciences, Vol. 27, 1956, p. 97–102.
24. NOVOZHILOV, V. V., Foundations of the Nonlinear Theory of Elasticity, Graylock Press, Rochester, N.Y., 1953.
25. TEZCAN, S. S. and OVUNC, B., An Iteration Method for the Nonlinear Buckling of Framed Structures, Proceedings of the International Conference on Space Structures, Blackwell Scientific Publications, London, 1966.
26. LEVY, S., Bending of Rectangular Plates with Large Deflections, NACA, Technical Report, 737, 1942.
27. TIMOSHENKO, S., Theory of Plates and Shells, McGraw-Hill Book Co., Inc., 1940.
28. LEVY, S., Square Plate with Clamped Edges Under Normal Pressure Producing Large Deflections, NACA Technical Report 740, 1942.
29. BALACHANDRA, M. and GOPALACHARYULU, S., Large Deflections of Clamped Plate by Modified Fourier Series, Journal of Applied Mechanics, ASME, Vol. 32, Dec. 1965, p. 943.
30. WAY, S., Uniformly Loaded Clamped, Rectangular Plates with Large Deflections, Proc. Fifth International Congress of Applied Mechanics, John Wiley and Sons, Inc., 1939, p. 123–128.
31. LEVY, S., Square Plate with Clamped Edges Under Normal Pressure Producing Large Deflections, NACA Technical Report 847, 1942.
32. WANG, C. T., Nonlinear Large Deflection Boundary Value Problems of Rectangular Plates, NACA Technical Note, 1425, March 1948.

33. BERGER, H. M., A New Approach to the Analysis of Large Deflections of Plates, Journal of Applied Mechanics, Vol. 22, 1955, p. 465–472.
34. LEVY, S., Bending of Rectangular Plates with Large Deflections, NACA Technical Report 737, 1942.

Summary

A systematic method of derivation is presented for the tangent stiffness matrices of geometrically nonlinear two and three dimensional finite elements. At any deformed stage of the element, the strain energy is calculated by taking into account all of the nonlinear components of the strain-displacement equations. In addition to these nonlinear terms, the instantaneous values of the variable strains are also included in the strain energy.

Résumé

On présente une méthode systématique de dérivation pour les matrices de rigidité tangentielle d'éléments finis bidimensionnels ou tridimensionnels géometriquement non linéaires. Pour chaque stade de déformation de l'élément on calcule l'énergie de déformation en tenant compte de toutes les composantes non linéaires des équations déformation-déplacement. En plus, de ces termes non linéaires les valeurs instantanées des déformations variables sont aussi incluses dans l'énergie de déformation.

Zusammenfassung

Ein systematisches Herleitungsverfahren wird angegeben für die tangentialen Steifigkeitsmatrizen geometrisch nichtlinearer, zwei- und dreidimensionaler endlicher Elemente. Zu jedem Verformungszustand wird die Verzerrungsenergie unter Berücksichtigung aller nichtlinearer Glieder der Dehnungs-Verschiebungs-Gleichungen berechnet. Zu diesen nichtlinearen Gliedern werden auch die augenblicklichen Werte der variablen Dehnungen in die Verzerrungsenergie einbezogen.