Zeitschrift: IABSE publications = Mémoires AIPC = IVBH Abhandlungen

Band: 29 (1969)

Artikel: Application of the bending theory regarding hyperbolic paraboloid shells with straight edges : finite difference solutions

Autor: Stageboe, J.

DOI: https://doi.org/10.5169/seals-22925

Application of the Bending Theory Regarding Hyperbolic Paraboloid Shells with Straight Edges. Finite Difference Solutions

Application de la théorie de flexion aux coques en forme de paraboloïdes hyperboliques avec bords droits. Solutions de la différence finie

Anwendung der Biegetheorie auf hyperbolische Paraboloidschalen mit geraden Kanten. Endliche Differenzen-Lösung

J. STAGEBOE
Siv. Ing., Oslo

Introduction

Shells of cylindrical and positive gaussian curvature have been in use for many years. Practical interest in the hypar shell, however, occur only during the last twenty years. Investigations of the stresses in hypar shells appear to be receiving increasing attention from structural research engineers, both because of the economical uses of this type of shells and the striking architectural forms possible.

A great number of papers have been published on the membrane theory of the hyperbolic paraboloid shells [1, 2, 3]. The membrane theory gives the interesting answer that the shear force N_{xy} in the direction of the straight asymptotic lines is given without integration as function of the external load and without influence of the edge conditions. Thus for constant load in the plan area the shear force N_{xy} is constant all over the shell. The fact that for anticlastic shells the effect of disturbances is great in direction of the straight asymtotic lines shows that the membrane theory may be completely inadequate.

Shells of positive curvature, not too shallow, with no concentrated external loads have the important advantage that the membrane theory already gives a good approximation for the stress distribution in the shell. The bending stresses are confined to small zones along the boundaries. The calculation of the edge disturbances can for this type of shells be superimposed on the membrane state of stress.

For anticlastic shells, when the edge and the asymtotic lines coincide, the boundary disturbances may be completely different. The boundary effects can, in this case introduce stress forces which differ greatly from the membrane forces, and the bending stresses are not insignificant in comparison with membrane stresses. These boundary effects are no longer confined to a narrow zone along the edge.

For shells with negative curvature the differential equations for the membrane forces are of hyperbolic type, and in most cases the solution for the boundary conditions can not be prescribed in the usually way.

Hence it is necessary and essential to consider the complete bending theory for a proper design of hyperbolic paraboloid with straight edges.

Levy-type solutions are used [4] to determine boundary disturbances in hyperbolic paraboloid shells with straight edges, but a set of unrealistic boundary conditions is used, because it otherwise is impossible to satisfy all conditions by every individual term of an infinite Fourier series.

To obtain realistic boundary conditions the writer in this paper solves the three differential equations for the displacements by means of finite difference approximations. The boundary equations are established for boundary conditions which occur in practical design. The numerical calculations are restricted to shallow hyperbolic paraboloids square in plan for the following boundary conditions:

1. The shell is hinged to vertically supported edge members of constant and linear variable stiffness.
2. The shell is supported upon elastic edge members supported at the corners only.

The rise of the edge members are $^1/_5$ and $^1/_4$. To test the convergence the plan area of the shell is divided into 6×6, 8×8 and 12×12 meshes. The linear system of equations varies between 32 and 119 unknown and are solved with a high speed digital computer.

Notation

x_1, x_2	curviliniear coordinates
x, y, $z = z(x,y)$	cartesion coordinates
$\bar{r} = \bar{r}(x_1 x_2)$	Position vector of any points on the middle surface of the shell
$\frac{\partial \bar{r}}{\partial x_\alpha} = \bar{a}_\alpha$ [1])	Covariant base vectors of the surface
$a_{\alpha\beta} = \bar{a}_\alpha \bar{a}_\beta$, $a^{\alpha\beta} = \bar{a}^\alpha \bar{a}^\beta$	Symmetric metric surface tensors

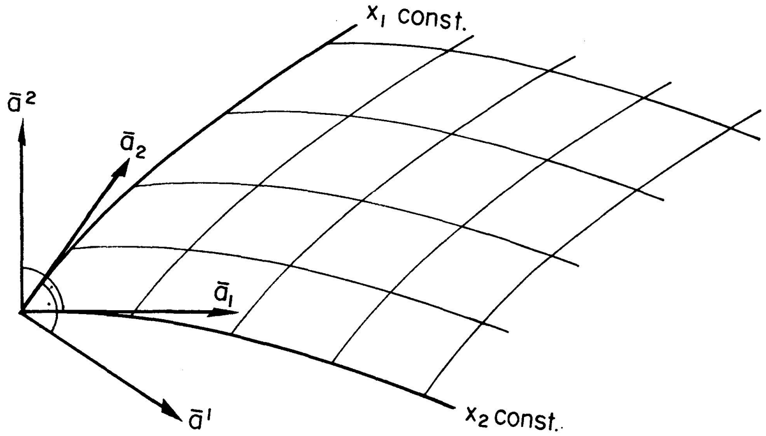


Fig. 1.

$$\left.\begin{aligned}\bar{a}^{\alpha} &= a^{\alpha\beta}\bar{a}_{\beta}\\ \bar{a}_{\alpha} &= a_{\alpha\beta}\bar{a}^{\beta}\end{aligned}\right\}$$ $\bar{a}^{\alpha}$ = Contravariant base vectors of the surface

$$a = a_{11}a_{22} - a_{12}^2$$

$$a = \frac{a_{22}}{a^{11}} = \frac{a_{11}}{a^{22}} = -\frac{a_{12}}{a^{12}}$$

$$\bar{a}_3 = \frac{\bar{a}_1 \times \bar{a}_2}{\sqrt{a}}$$ Unit normal vector

$$b_{\alpha\beta} = -\bar{a}_{\alpha}\frac{\partial \bar{a}_3}{\partial x_{\beta}} = \bar{a}_3\frac{\partial \bar{a}_{\alpha}}{\partial x_{\beta}}$$ Second fundamental form of the surface (symmetric tensor)

$$b = b_{11}b_{22} - b_{12}^2$$

$$\Gamma^{\gamma}_{\alpha\beta} = \tfrac{1}{2}a^{\rho\gamma}\left(\frac{\partial a_{\alpha\rho}}{\partial x_{\beta}} + \frac{\partial a_{\beta\rho}}{\partial x_{\alpha}} - \frac{\partial a_{\alpha\beta}}{\partial x_{\rho}}\right) = \bar{a}^{\gamma}\frac{\partial \bar{a}_{\alpha}}{\partial x_{\beta}}$$ Cristoffel symbol of the second kind

$$H^{\alpha\beta\rho\lambda} = \tfrac{1}{2}\left[a^{\alpha\lambda}a^{\beta\rho} + a^{\alpha\rho}a^{\beta\lambda} + \nu\left(\epsilon^{\alpha\rho}\epsilon^{\beta\lambda} + \epsilon^{\alpha\lambda}\epsilon^{\beta\rho}\right)\right]$$ Symmetric tensor

$$\epsilon^{12} = -\epsilon^{12} = \frac{1}{\sqrt{a}}$$

$$\epsilon^{11} = \epsilon^{22} = 0$$

$\sqrt{a_{\alpha\alpha}}\,p^{\alpha}, p^3$ External loads in curvilinear coordinates

X, Y, Z External loads in cartesian coordinates

$N_{(\alpha\beta)}$ Stress resultants directed along the covariant base vectors

$M_{(\alpha\beta)}$ Stress couples about the contravariant base vectors

$\hat{M}_{(\alpha\beta)}$ Stress couples about the covariant base vectors

$Q_{(\alpha)}$ Shearing forces in the direction of the normal vector

$$\left.\begin{aligned}&\sqrt{a_{\alpha\alpha}}\,v^{\alpha}\\ &\sqrt{a^{\alpha\alpha}}\,v_{\alpha}\\ &w\end{aligned}\right\}$$ Displacements in the direction of the covariant, contravarient and normal base vectors respectively

$$\left.\begin{aligned}\sqrt{a_{11}}\,v^1 &= u\\ \sqrt{a_{22}}\,v^2 &= v\\ \sqrt{a^{11}}\,v_1 &= \hat{u}\\ \sqrt{a^{22}}\,v_2 &= \hat{v}\end{aligned}\right\}$$ $v_{\alpha} = a_{\alpha\beta}v^{\beta}$

[1]) Greek indices range over the values 1,2.

$(\ )\vert_\alpha$	Covariant differentiation
$(\ )' = \dfrac{\partial(\ )}{\partial x}$	
$(\ )^{\cdot} = \dfrac{\partial(\ )}{\partial y}$	
h	Shell thickness
E	Young's modulus (E_c = edge member)
ν	Poisson's ratio
$D = \dfrac{E h}{1-\nu^2}$	
$B = \dfrac{E h^3}{12(1-\nu^2)}$	
l	Base length (square in plan)
φ	Slope of the edge members
$c = \dfrac{l}{2\,\mathrm{tg}\,\varphi}$	
F	Cross-sectional area of the edge member
J	Moment of inertia about the horizontal axis through the centroid c
e	Distance from the centroid of the edge member to the middle plane of the shell
$\alpha = \dfrac{F}{h^2}$	
$\beta = \dfrac{e}{h}$	
$\gamma = \dfrac{l}{h}$	
$\delta = \dfrac{F^2}{J}$	
$\rho = \dfrac{s}{l}$	(s mesh distance)
$\zeta = \dfrac{x}{l}$	
$\eta = \dfrac{y}{l}$	
$U = \dfrac{u}{l}$	
$V = \dfrac{v}{l}$	
$W = \dfrac{w}{10\,l}$	

General Shell Equations

The three equations for the displacements in general curvilinear coordinates for thin elastic shells may be written very compactly in tensor notation when using the same symbols as GREEN-ZERNA [5]:

$$\begin{aligned}
&\frac{D}{2} H^{\alpha\beta\rho\lambda}\,[v_\rho|_{\lambda\alpha} - v_\lambda|_{\rho\alpha} - 2\,(b_{\rho\lambda} w)|_\alpha] + p^\beta = 0\,,\\
&\frac{D}{2} H^{\alpha\beta\rho\lambda}\,(v_\rho|_\lambda + v_\lambda|_\rho - 2\,b_{\rho\lambda} w)\, b_{\alpha\beta} - B\,H^{\alpha\beta\rho\lambda}\, w|_{\rho\lambda\alpha\beta} + p^3 = 0\,.
\end{aligned} \tag{1}$$

To be suitable for practical applications the following approximations are introduced in (1): The equations refer to thin elastic shells. In the two first equations of equilibrium in the tangential directions of the curvilinear coordinates, the contribution of shearing forces are neglected. In the expressions for stress couples all members concerning the curvature of the shell are neglected.

These simplifications are for thin shells introduced by WLASSOW [6] and GREEN-ZERNA [5]. In the particular case of a circular cylindrical shell the equations (1) give the Donnel equation.

Neglecting terms in the covariant differentiation of the same order as the approximations done in the shell equations, they may be written in developed form:

$\beta = 1$:

$$\begin{aligned}
&H^{1111}\Big[\frac{\partial^2 v_1}{\partial x_1^2} - 3\,\Gamma^1_{11}\frac{\partial v_1}{\partial x_1} - 2\,\Gamma^2_{11}\frac{\partial v_2}{\partial x_1} - \Gamma^2_{11}\frac{\partial v_1}{\partial x_2}\\
&\quad - \Big(\frac{\partial b_{11}}{\partial x_1} - 2\,\Gamma^1_{11} b_{11} - 2\,\Gamma^2_{11} b_{12}\Big) w - b_{11}\frac{\partial w}{\partial x_1}\Big]\\
&\quad - H^{1122}\Big[\frac{\partial b_{22}}{\partial x_1} - 2\,\Gamma^1_{12} b_{12} + \Big(\frac{\partial w}{\partial x_1} - 2\,\Gamma^2_{12}\Big) b_{22}\Big] + H^{1112}\Big[\frac{\partial^2 v_2}{\partial x_1^2} + 2\,\frac{\partial^2 v_1}{\partial x_1\,\partial x_2}\\
&\quad - \Gamma^1_{11}\Big(\frac{\partial v_2}{\partial x_1} + 2\,\frac{\partial v_1}{\partial x_2}\Big) - 2\,\Gamma^2_{12}\Big(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\Big) - 3\,\Gamma^2_{11}\frac{\partial v_2}{\partial x_2} - 6\,\Gamma^1_{12}\frac{\partial v_1}{\partial x_1}\\
&\quad - 2\,\Gamma^2_{12}\frac{\partial v_2}{\partial x_1} - 3\Big(\frac{\partial b_{11}}{\partial x_2} - 2\,\Gamma^1_{12} b_{11} - 2\,\Gamma^2_{12} b_{12}\Big) w - b_{11}\frac{\partial w}{\partial x_2} - 2\,b_{12}\frac{\partial w}{\partial x_1}\Big]\\
&\quad + H^{1212}\Big[\frac{\partial^2 v_1}{\partial x_2^2} - (\Gamma^2_{22} + 2\,\Gamma^1_{12})\frac{\partial v_1}{\partial x_2} - \Gamma^1_{22}\frac{\partial v_1}{\partial x_1} - 2\,\Gamma^2_{12}\frac{\partial v_2}{\partial x_2}\\
&\quad - 2\Big(\frac{\partial b_{22}}{\partial x_1} - 2\,\Gamma^1_{12} b_{12} - 2\,\Gamma^2_{12} b_{22}\Big) w - 2\,b_{12}\frac{\partial w}{\partial x_2}\Big] + [H^{1212} + H^{1122}]\Big[\frac{\partial^2 v_2}{\partial x_1\,\partial x_2}\\
&\quad - (\Gamma^2_{22} + \Gamma^1_{12})\frac{\partial v_2}{\partial x_1} - \Gamma^1_{22}\frac{\partial v_1}{\partial x_1} - 2\,\Gamma^2_{12}\frac{\partial v_2}{\partial x_2} - \Gamma^1_{12}\frac{\partial v_1}{\partial x_2}\Big] + H^{1222}\Big[\frac{\partial^2 v_2}{\partial x_2^2} - 3\,\Gamma^2_{22}\frac{\partial v_2}{\partial x_2}\\
&\quad - 2\,\Gamma^1_{22}\frac{\partial v_1}{\partial x_2} - \Gamma^1_{22}\frac{\partial v_2}{\partial x_1} - \Big(\frac{\partial b_{22}}{\partial x_2} - 2\,\Gamma^1_{22} b_{12} - 2\,\Gamma^2_{22} b_{22}\Big) w - b_{22}\frac{\partial w}{\partial x_2}\Big] + \frac{p^1}{D} = 0
\end{aligned} \tag{2a}$$

and correspondingly for $\beta = 2$ (2b)

$$[H^{1111} b_{11} + 2 H^{1211} b_{12} + H^{2211} b_{22}] \left[\frac{\partial v_1}{\partial x_1} - b_{11} w\right]$$

$$+ [H^{1112} b_{11} + 2 H^{1212} b_{12} + H^{1222} b_{22}] \left[\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} - 2 b_{12} w\right]$$

$$+ [H^{1122} b_{11} + 2 H^{1222} b_{12} + H^{2222} b_{22}] \left[\frac{\partial v_2}{\partial x_2} - b_{22} w\right]$$

$$- \frac{B}{D} \Big\{ H^{1111} \left[\frac{\partial^4 w}{\partial x_1^4} - 6 \Gamma^1_{11} \frac{\partial^3 w}{\partial x_1^3} - 6 \Gamma^2_{11} \frac{\partial^3 w}{\partial x_1^2 \partial x_2}\right]$$

$$+ 4 H^{1112} \left[\frac{\partial^4 w}{\partial x_1^3 \partial x_2} - 3 (\Gamma^1_{11} + \Gamma^2_{12}) \frac{\partial^3 w}{\partial x_1^2 \partial x_2} - 3 \Gamma^2_{11} \frac{\partial^3 w}{\partial x_1 \partial x_2^2} - 3 \Gamma^1_{12} \frac{\partial^3 w}{\partial x_1^3}\right] \tag{2c}$$

$$+ [2 H^{1122} + 4 H^{1212}] \left[\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} - (4 \Gamma^1_{12} + \Gamma^2_{22}) \frac{\partial^3 w}{\partial x_1^2 \partial x_2} - (4 \Gamma^2_{12} + \Gamma^1_{11}) \frac{\partial^3 w}{\partial x_1 \partial x_2^2}\right.$$

$$\left. - \Gamma^2_{11} \frac{\partial^3 w}{\partial x_2^3} - \Gamma^1_{22} \frac{\partial^3 w}{\partial x_1^3}\right] + 4 H^{1222} \left[\frac{\partial^4 w}{\partial x_1 \partial x_2^3} - 3 (\Gamma^1_{12} + \Gamma^2_{22}) \frac{\partial^3 w}{\partial x_1 \partial x_2^2} - 3 \Gamma^2_{12} \frac{\partial^4 w}{\partial x_2^3}\right.$$

$$\left. - 3 \Gamma^1_{22} \frac{\partial^3 w}{\partial x_1^2 \partial x_2}\right] + H^{2222} \left[\frac{\partial^4 w}{\partial x_2^4} - 6 \Gamma^1_{22} \frac{\partial^3 w}{\partial x_1 \partial x_2^2} - 6 \Gamma^2_{22} \frac{\partial^3 w}{\partial x_2^3}\right] \Big\} + \frac{p^3}{D} = 0.$$

Where

$$v_1 = \frac{a_{11}}{\sqrt{a_{11}}} u + \frac{a_{12}}{\sqrt{a_{22}}} v, \qquad v_2 = \frac{a_{12}}{\sqrt{a_{11}}} u + \frac{a_{22}}{\sqrt{a_{22}}} v \tag{3}$$

and the tensor $H^{\alpha\beta\gamma\delta}$:

$$H^{1111} = (a^{11})^2, \quad H^{1112} = H^{1211} = a^{11} a^{12}, \quad H^{1122} = H^{2211} = (a^{12})^2 + \frac{\nu}{a},$$

$$H^{1222} = H^{2212} = a^{12} a^{22}, \quad H^{1212} = \frac{1}{2} \left[(a^{12})^2 + a^{11} a^{22} - \frac{\nu}{a}\right], \quad H^{2222} = (a^{22})^2. \tag{4}$$

Stress Resultants and Stress Couples

Stress resultants:

$$N_{(\alpha\beta)} = D \sqrt{\frac{a_{\beta\beta}}{a^{\alpha\alpha}}} H^{\alpha\beta\rho\lambda} (v_\rho|_\lambda - b_{\rho\lambda} w) \tag{5a}$$

or:
$$N_{(\alpha\beta)} = D\sqrt{\frac{a_{\beta\beta}}{a^{\alpha\alpha}}}\left[H^{\alpha\beta 11}\left(\frac{\partial v_1}{\partial x_1} - \Gamma^1_{11} v_1 - \Gamma^2_{11} v_2 - b_{11} w\right)\right.$$
$$+ H^{\alpha\beta 12}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} - 2\Gamma^1_{12} v_1 - 2\Gamma^2_{12} v_2 - 2 b_{12} w\right) \tag{5b}$$
$$\left. + H^{\alpha\beta 22}\left(\frac{\partial v_2}{\partial x_2} - \Gamma^1_{22} v_1 - \Gamma^2_{22} v_2 - b_{22} w\right)\right].$$

Stress couples:

$$\begin{aligned} M_{(11)} &= \quad m^{11}\sqrt{\frac{a_{11}}{a^{11}}}, \\ M_{(12)} &= -m^{12}\sqrt{a}. \\ M_{(21)} &= \quad m^{12}\sqrt{a}, \\ M_{(22)} &= -m^{22}\sqrt{\frac{a_{22}}{a^{22}}}. \end{aligned} \tag{6}$$

$$\begin{aligned} \hat{M}_{(11)} &= \sqrt{\frac{a\,a_{11}}{a^{11}}}\,(m^{11} a^{12} - m^{12} a^{11}), \\ \hat{M}_{(12)} &= a\,(m^{11} a^{22} - m^{12} a^{12}), \\ \hat{M}_{(21)} &= a\,(m^{12} a^{12} - m^{22} a^{11}), \\ \hat{M}_{(22)} &= \sqrt{\frac{a\,a_{22}}{a^{22}}}\,(m^{21} a^{12} - m^{22} a^{11}). \end{aligned} \tag{7}$$

$$m^{\alpha\beta} = -B\,H^{\alpha\beta\rho\lambda}\,w|_{\rho\lambda} \tag{8a}$$

or

$$m^{\alpha\beta} = -B\left[H^{\alpha\beta 11}\left(\frac{\partial^2 w}{\partial x_1^2} - \Gamma^1_{11}\frac{\partial w}{\partial x_1} - \Gamma^2_{11}\frac{\partial w}{\partial x_2}\right)\right. \tag{8b}$$
$$\left. + 2H^{\alpha\beta 12}\left(\frac{\partial^2 w}{\partial x_1\,\partial x_2} - \Gamma^1_{12}\frac{\partial w}{\partial x_1} - \Gamma^2_{12}\frac{\partial w}{\partial x_2}\right) + H^{\alpha\beta 22}\left(\frac{\partial^2 w}{\partial x_2^2} - \Gamma^2_{22}\frac{\partial w}{\partial x_2} - \Gamma^1_{22}\frac{\partial w}{\partial x_1}\right)\right].$$

Cartesian Coordinates

$$\begin{aligned} x &= x, \\ y &= y, \qquad \bar{r} = x\,\bar{i}_1 + y\,\bar{i}_2 + z(x,y)\,\bar{i}_3, \\ z &= z(x,y). \end{aligned}$$

The geometrical quantities for this coordinate-system can be compiled in the following table:

Geometrical quantities

	1	2	3
$\bar{a}_1$	1	0	z'
$\bar{a}_2$	0	1	$z^{\cdot}$
$a_{1\alpha}$	$1+(z')^2$	$z'z^{\cdot}$	
$a_{2\alpha}$	$z'z^{\cdot}$	$1+(z^{\cdot})^2$	
$a^{1\alpha}$	$\frac{1+(z^{\cdot})^2}{a}$	$-\frac{z'z^{\cdot}}{a}$	
$a^{2\alpha}$	$-\frac{z'z^{\cdot}}{a}$	$\frac{1+(z')^2}{a}$	
$\bar{a}_3$	$-\frac{z'}{\sqrt{a}}$	$-\frac{z^{\cdot}}{\sqrt{a}}$	$\frac{1}{\sqrt{a}}$
$\bar{a}_1'$	0	0	z''
$\bar{a}_1^{\cdot} = \bar{a}_2'$	0	0	$z'^{\cdot}$
$\bar{a}_2^{\cdot}$	0	0	$z^{\cdot\cdot}$
$b_{1\alpha}$	$\frac{z''}{\sqrt{a}}$	$\frac{z'^{\cdot}}{\sqrt{a}}$	
$b_{2\alpha}$	$\frac{z'^{\cdot}}{\sqrt{a}}$	$\frac{z^{\cdot\cdot}}{\sqrt{a}}$	
$\Gamma^1_{\alpha 1}$	$\frac{z'z''}{a}$	$\frac{z^{\cdot}z'^{\cdot}}{a}$	
$\Gamma^2_{\alpha 2}$	$\frac{z'z'^{\cdot}}{a}$	$\frac{z^{\cdot}z^{\cdot\cdot}}{a}$	
Γ^1_{22}		$\frac{z'z^{\cdot\cdot}}{a}$	
Γ^2_{11}	$\frac{z^{\cdot}z''}{a}$		
a	$1+(z')^2+(z^{\cdot})^2$		

(9)

$$p^1 = X a^{11} + Y a^{12} + Z \frac{z'}{a},$$

$$p^2 = Y a^{22} + X a^{12} + Z \frac{z^{\cdot}}{a},$$

$$p^3 = \frac{Z - X z' - Y z^{\cdot}}{\sqrt{a}}.$$

The Shell Equations for the Hyperbolic Paraboloid with Straight Boundaries

In this form the middle surface of the hyperbolic paraboloid shell is defined in cartesian coordinates by

$$z(x, y) = \frac{x\,y}{c} \quad \text{(Fig. 2)}.$$

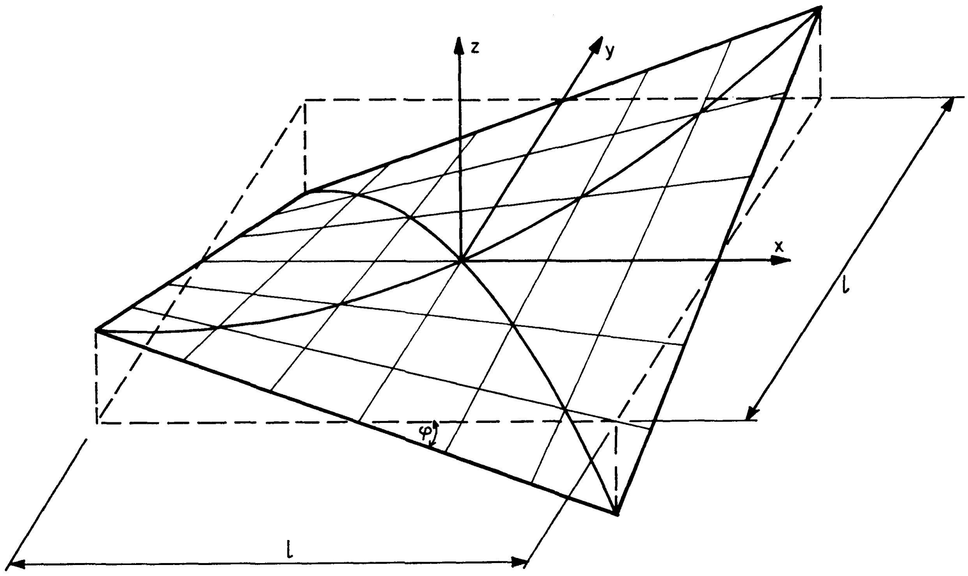


Fig. 2.

By means of the general equations (2a–c) and the table (9) we can easily establish the shell equations for the displacements:

Geometrial quantities from table (9):

$$a = 1 + \frac{y^2}{c^2} + \frac{x^2}{c^2}, \quad b_{11} = b_{22} = \Gamma^{\alpha}_{11} = \Gamma^{\alpha}_{22} = 0, \quad b_{12} = \frac{1}{c\sqrt{a}}, \quad \Gamma^1_{12} = \frac{y}{c^2\,a}, \quad \Gamma^2_{12} = \frac{x}{c^2\,a},$$

$$a_{11} = 1 + \frac{y^2}{c^2}, \qquad a_{12} = \frac{x\,y}{c^2}, \qquad a_{22} = 1 + \frac{x^2}{c^2},$$

$$a^{11} = \frac{1 + \frac{x^2}{c^2}}{a}, \qquad a^{12} = -\frac{x\,y}{a\,c^2}, \qquad a^{22} = \frac{1 + \frac{y^2}{c^2}}{a}$$

and the tensor $H^{\alpha\beta\gamma\delta}$ (4):

$$H^{1111} = \frac{1 + 2\left(\frac{x}{c}\right)^2 + \left(\frac{x}{c}\right)^4}{a^2}, \qquad H^{1122} = \frac{\frac{x^2 y^2}{c^4} + a\,\nu}{a^2}, \qquad H^{1112} = \frac{-\frac{x\,y}{c^2} - \frac{y\,x^3}{c^4}}{a^2},$$

$$H^{1222} = \frac{-\frac{x\,y}{c^2} + \frac{x\,y^3}{c^4}}{a^2}, \qquad H^{1212} = \frac{\frac{x^2 y^2}{c^4} + \frac{1}{2} a\,(1-\nu)}{a^2}, \qquad H^{2222} = \frac{1 + 2\left(\frac{y}{c}\right)^2 + \left(\frac{y}{c}\right)^4}{a^2}.$$

These quantities substituted into (2), neglecting 4 order terms and using (3) we have:

$$- \frac{4y}{c^3} w + (3-\nu)\left(\frac{xy}{c^2} u'^{\cdot} - \frac{y}{c^2} v'\right) - (1+\nu)\left(\frac{x}{c^2} u' - \frac{x}{c^2} v^{\cdot} - \frac{xy}{c^2} v^{\cdot\cdot} + v'^{\cdot}\right)$$

$$- \frac{4xy}{c^3} w' - \left(2 + \frac{x^2}{c^2} - \frac{y^2}{c^2}\right) u'' + (1-\nu)\left[\left(\frac{2}{c} - 2\frac{x^2}{c^3} - \frac{y^2}{c^3}\right) w^{\cdot} - \left(1 - \frac{x^2}{2c^2} + \frac{y^2}{2c^2}\right) u^{\cdot\cdot}\right] =$$

$$\frac{2}{D} p' a \left(1 - \frac{x^2}{2c^2}\right) = \frac{2}{D}\left[X\left(1 + \frac{x^2}{2c^2}\right) - Y\frac{xy}{c} + Z\frac{y}{c}\right] \tag{10}$$

and equally for the second equation.

The third equation:

$$(1+\nu)\frac{xy}{c^2}(u' + v^{\cdot}) + (1-\nu)\left[\left(\frac{2}{c} - \frac{3x^2}{c^3} - \frac{3y^2}{c^3}\right) w - \left(1 - \frac{x^2}{c^2} - \frac{y^2}{2c^2}\right) u^{\cdot}\right.$$

$$\left. - \left(1 - \frac{x^2}{2c^2} - \frac{y^2}{c^2}\right) v'\right] + \frac{B}{D} c \left[- \frac{8y}{c^2} w'''^{\cdot} - \frac{8x}{c^2} w'^{\cdot\cdot} - \frac{4xy}{c^2}(w'''^{\cdot} + w'^{\cdot\cdot\cdot})\right.$$

$$\left. + \left(1 + \frac{x^2}{2c^2} - \frac{3y^2}{2c^2}\right) w'''' + 2\left(1 - \frac{x^2}{2c^2} - \frac{y^2}{2c^2}\right) w''^{\cdot\cdot} + \left(1 - \frac{3x^2}{2c^2} + \frac{y^2}{2c^2}\right) w^{\cdot\cdot\cdot\cdot}\right] =$$

$$\frac{c}{D} p^3 \sqrt{a} = \frac{c}{D}\left[-X\frac{y}{c} - Y\frac{x}{c} + \left(1 - \frac{x^2}{2c^2} - \frac{y^2}{2c^2}\right) Z\right].$$

The difference compared with the work by BONGARD [7] is due to the simplifications in the third equation for the expressions for the stress couples and is insignificant. The two first equations are somewhat simpler than in [7] but of the same accuracy.

The same geometrical quantities substituted into (5) (7) (8) together with (3) gives the stress resultants and stress couples:

$$N_{(11)} = N_x =$$
$$D\left[\left(1 - \frac{y^2}{2c^2}\right) u' + \nu\left(1 - \frac{x^2}{2c^2}\right) v^{\cdot} - (1-\nu)\frac{xy}{c^2} u^{\cdot} + \frac{y}{c^2} v + \nu\frac{x}{c^2} u + 2\frac{xy}{c^3} w\right],$$

$$N_{(22)} = N_y =$$
$$D\left[\left(1 - \frac{x^2}{2c^2}\right) v^{\cdot} + \nu\left(1 - \frac{y^2}{2c^2}\right) u' - (1-\nu)\frac{xy}{c^2} v' + \frac{x}{c^2} u + \nu\frac{y}{c^2} v + 2\frac{xy}{c^3} w\right], \tag{11}$$

$$N_{(12)} = N_{xy} =$$
$$D\left\{\frac{(1-\nu)}{2}\left[\left(1 - \frac{x^2}{2c^2}\right) u^{\cdot} + \left(1 - \frac{y^2}{2c^2}\right) v' - 2\left(1 - \frac{x^2}{c^2} - \frac{y^2}{c^2}\right)\frac{w}{c}\right] - \frac{(1+\nu)}{2}\frac{xy}{c^2}(v^{\cdot} + u')\right\}.$$

$$-\hat{M}_{(12)} = M_x = B\left[\left(1 - \frac{y^2}{c^2}\right) w'' - (1+\nu)\frac{xy}{c^2} w'^{\cdot} + \left(1 - \frac{x^2}{c^2}\right)\nu w^{\cdot\cdot}\right],$$

$$\hat{M}_{(21)} = M_y = B\left[\left(1 - \frac{x^2}{c^2}\right) w^{\cdot\cdot} - (1+\nu)\frac{xy}{c^2} w'^{\cdot} + \left(1 - \frac{y^2}{c^2}\right)\nu w''\right], \tag{12}$$

$$\begin{aligned}\hat{M}_{(11)} = M_{xy} &= B(1-\nu)\left[-\frac{y}{c^2}w' - \frac{x}{c^2}w^{\cdot} + \left(1-\frac{x^2}{2c^2}-\frac{y^2}{2c^2}\right)w'^{\cdot} - \frac{xy}{c^2}w^{\cdot\cdot}\right],\\ \hat{M}_{(22)} = M_{yx} &= B(1-\nu)\left[\frac{y}{c^2}w' + \frac{x}{c^2}w^{\cdot} - \left(1-\frac{x^2}{2c^2}-\frac{y^2}{2c^2}\right)w'^{\cdot} + \frac{xy}{c^2}w''\right].\end{aligned} \tag{12}$$

When neglecting the quadratic terms in (10) (11) (12) we obtain the equations for the shallow hyperbolic paraboloid:

$$\begin{aligned} u'' + \frac{1}{2}(1-\nu)\,u^{\cdot\cdot} + \frac{1}{2}(1+\nu)\,v'^{\cdot} - \frac{1-\nu}{c}w^{\cdot} &= -\frac{X + Z\frac{y}{c}}{D},\\ v^{\cdot\cdot} + \frac{1}{2}(1-\nu)\,v'' + \frac{1}{2}(1+\nu)\,u'^{\cdot} - \frac{1-\nu}{c}w' &= -\frac{Y + Z\frac{y}{c}}{D},\\ (1-\nu)\left(u^{\cdot} + v' - \frac{2w}{c}\right) - \frac{h^2 c}{12}\Delta^2 w &= \frac{Xy + Yx - Zc}{D}. \end{aligned} \tag{13a–c}$$

Stress forces:

$$\begin{aligned} N_x &= D(u' + \nu v^{\cdot}),\\ N_y &= D(v^{\cdot} + \nu u'),\\ N_{xy} &= D\frac{(1-\nu)}{2}\left(u^{\cdot} + v' - 2\frac{w}{c}\right). \end{aligned} \tag{14}$$

Stress couples:

$$\begin{aligned} M_x &= B(w'' + \nu w^{\cdot\cdot}),\\ M_y &= B(w^{\cdot\cdot} + \nu w''),\\ M_{xy} &= B(1-\nu)\,w'^{\cdot}. \end{aligned} \tag{15}$$

Introducing non-dimensional coordinates and displacements we obtain the three basic equations (13a–c) in ordinary finite difference form (see Fig. 3):

$$\begin{array}{|ccc|}\hline & 4(1-\nu) & \\ 8- & \boxed{8(3-\nu)} & 8 \\ & 4(1-\nu) & \\ \hline\end{array}\, U + (1+\nu)\begin{array}{|ccc|}\hline -1 & . & 1\\ & \boxed{.} & \\ 1 & . & -1\\ \hline\end{array}\, V - 80(1-\nu)\rho\,\mathrm{tg}\,\varphi\begin{array}{|c|}\hline 1\\ \boxed{.}\\ -1\\ \hline\end{array}\, W =$$

$$\frac{-X - Z\eta\, 2\,\mathrm{tg}\,\varphi}{E}\, 8(1-\nu^2)\rho^2\gamma,$$

(16a–b)

$$\begin{array}{|ccc|}\hline & 8 & \\ 4(1-\nu)- & \boxed{8(3-\nu)} & 4(1-\nu) \\ & 8 & \\ \hline\end{array}\, V + (1+\nu)\begin{array}{|ccc|}\hline -1 & & 1\\ & \boxed{.} & \\ 1 & & -1\\ \hline\end{array}\, U$$

$$-80(1-\nu)\rho\,\mathrm{tg}\,\varphi\begin{array}{|ccc|}\hline -1 & \boxed{.} & 1\\ \hline\end{array}\, W = \frac{-Y - Z\zeta\, 2\,\mathrm{tg}\,\varphi}{E}\, 8(1-\nu^2)\rho^2\gamma,$$

$$\boxed{\begin{array}{c}1\\ \boxed{\cdot}\\ -1\end{array}}\,U+\boxed{-1\;\boxed{\cdot}\;1}\,V-\boxed{80\,\rho\,\mathrm{tg}\,\varphi}\,W$$

$$-\frac{10}{12\,(1-\nu)\,\rho^3\,\gamma^2\,\mathrm{tg}\,\varphi}\boxed{\begin{array}{ccccc} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & \boxed{20} & -8 & 1\\ & 2 & -8 & 2 & \\ & & 1 & & \end{array}}\,W=\frac{X\,\eta+Y\,\zeta-Z\frac{1}{2\,\mathrm{tg}\,\varphi}}{E}\,2\,(1+\nu)\,\rho\,\gamma. \qquad (16\,\mathrm{c})$$

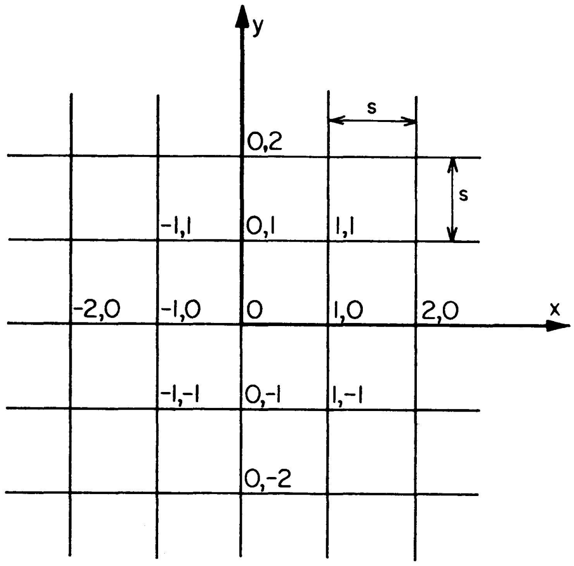


Fig. 3.

The two first equations (16a–b) are used at all points except the edge points. For these must be used equations with approximately the same accuracy.

Equally for the third equation (16c) for points near the edge. Similar finite difference equations may of course be established for the equations (10a–c).

1. The Cross Sectional Area of the Edge Member is Constant

Boundary Conditions

The edge member has bending rigidity in vertical direction. The torsional rigidity is neglected. Continuity and equilibrium conditions for shell and edge member (Fig. 4):

Equilibrium in x-direction:

$$N_x = D\,(u' + \nu\, v^{\cdot}) = 0\,, \qquad u' = -\nu\, v^{\cdot}. \qquad (17)$$

Equilibrium in y-direction (see Fig. 3):

$$N_{xy} = \overline{N}_c^{\cdot} = F\,E_c\,v_c^{\cdot\cdot\cdot}.$$

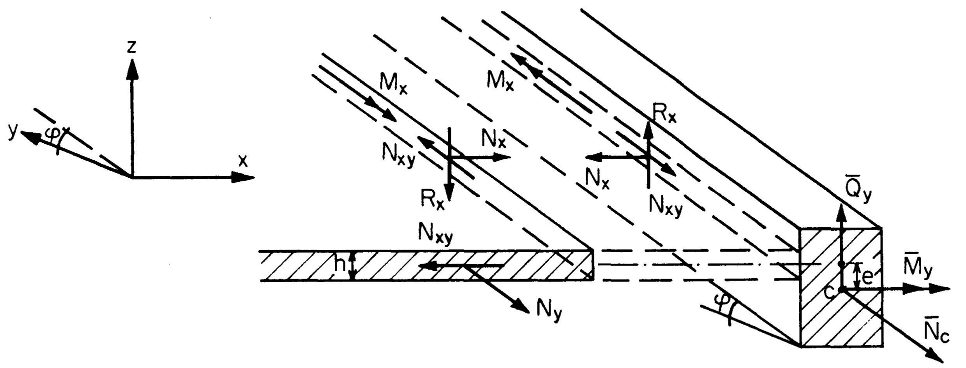


Fig. 4.

Same deformation shell edge member:

$$v = v_c - e\,w^{\cdot}.$$

We have thus the expression for the shear force:

$$N_{xy} = F\,E_c v^{\cdot\cdot} + e\,F\,E_c w^{\cdot\cdot\cdot}. \tag{18}$$

The normal force acting at the controid of the edge member:

$$\bar{N}_c = F\,E_c (v^{\cdot} + e\,w^{\cdot\cdot}) = \frac{F\,E_c}{h\,E}(N_y - \nu\,N_x) + \frac{12\,e\,F\,E_c}{h^3\,E}(M_y - \nu\,M_x). \tag{19}$$

From (18) and (14c) (or (11c) when not neglecting the quadratic terms) we find the expression:

$$v' = \frac{2\,F\,E_c}{h\,E}(1+\nu)\,v^{\cdot\cdot} - u^{\cdot} + \frac{2\,w}{c} + 2\,(1+\nu)\,\frac{e\,F\,E_c}{h\,E}\,w^{\cdot\cdot\cdot}. \tag{20}$$

Vertical equilibrium:

$$\bar{M}_y = \int (\bar{Q}_y - e\,N_{xy})\,dy,$$

$$\bar{M}_y^{\cdot\cdot} = \bar{Q}_y^{\cdot} - e\,N_{xy}^{\cdot} = R_x + g - e\,N_{xy}^{\cdot} = E_c\,J\,w^{\cdot\cdot\cdot\cdot}, \tag{21}$$

with $$R_x = B\,[w''' + (2-\nu)\,w'^{\cdot\cdot}]. \tag{22}$$

Expressions (18) and (22) substituted into (21) we obtain the following equation:

$$w''' + (2-\nu)\,w'^{\cdot\cdot} =$$

$$12\,(1-\nu^2)\frac{E_c}{E\,h^3}(J + e^2\,F)\,w^{\cdot\cdot\cdot\cdot} + 12\,(1-\nu^2)\frac{E_c e\,F}{E\,h^3}\,v^{\cdot\cdot\cdot} - 12\,(1-\nu^2)\,\frac{g}{E\,h^3}. \tag{23}$$

Before establishing the boundary equations we compile for later use various difference expressions. The differential expressions or quotients $L[f]$ are expanded in a Taylor-series with boundary conditions $M[f]$. Generally $L[f]$ and $M[f]$ are partial differential expressions in $f(x,y)$ and the expansion is to coincide with the differential expressions to as high an order as suitable [9]:

$$\sum L[f]_{jl} = \sum b_{rs} f_{rs} + M[f]_{ik} + \text{higher Taylor terms}. \tag{24}$$

In that way we obtain:

$$f_0' = \frac{1}{s}(-f_{-1}+f_0)+\frac{s}{2}f_0''+(s^2), \tag{25}$$

$$f_0'' = \frac{1}{2s^2}(-f_{-2}+8f_{-1}-7f_0)+\frac{3}{s}f_0'+(s^2), \tag{26}$$

$$f_0''' = \frac{1}{s^3}(f_{-3}-6f_{-2}+15f_{-1}-10f_0)+\frac{6}{s^2}f_0'+(s^2), \tag{27}$$

$$f_0''' = \frac{1}{3s^3}(-f_{-2}+9f_0-8f_1)+\frac{2}{s^2}f_1'+(s^2), \tag{28}$$

$$f_0''' = \frac{1}{11s^3}(-5f_{-2}+9f_{-1}-3f_0-f_1)+\frac{6}{11s}f_1''+(s^2), \tag{29}$$

$$f_0''' = \frac{1}{11s^3}(7f_{-3}-39f_{-2}+57f_{-1}-25f_0)+\frac{18}{11s}f_0''+(s^2), \tag{30}$$

$$f_0'''' = \frac{1}{10s^4}(-f_{-3}+16f_{-2}-54f_{-1}+64f_0-25f_1)+\frac{12}{10s^2}f_1''+(s^2), \tag{31}$$

$$f_0'''' = \frac{1}{5s^4}(-6f_{-4}+36f_{-3}-84f_{-2}+84f_{-1}-30f_0)+\frac{12}{5s^2}f_0''+(s^2), \tag{32}$$

$$f_0'''' = \frac{1}{6s^4}(-9f_{-4}+56f_{-3}-144f_{-2}+216f_{-1}-119f_0)+\frac{10}{s^3}f_0'+(s^2), \tag{33}$$

$$f_0'''' = \frac{1}{12s^4}(-3f_{-3}+32f_{-2}-108f_{-1}+192f_0-113f_1)+\frac{5}{s^3}(f')_1+(s^3). \tag{34}$$

$$21\,\Delta^2 f_0 s^4 = \left|\begin{array}{ccccc} & & & -4 & 25 \\ & -26 & 96 & -50 & -104 \\ -15 & 136 & -354 & 240 & \boxed{119} \\ & -26 & 96 & -50 & -104 \\ & & & -4 & 25 \end{array}\right| f + \nu \left|\begin{array}{ccccc} & & & 2 & -2 \\ & -8 & 36 & -80 & 52 \\ . & 16 & -72 & 156 & \boxed{-100} \\ & -8 & 36 & -80 & 52 \\ & & & 2 & -2 \end{array}\right| f$$

$$+24s^3[f'''+(2-\nu)f'^{\cdot\cdot}]_0+(s^6), \tag{35}$$

$$7(\Delta^2 f_0+\Delta^2 f_{-11})s^4 =$$

$$\left|\begin{array}{ccccc} 9 & -42 & 154 & \boxed{-219} & 82 \\ -13 & 76 & -344 & 552 & \boxed{-219} \\ & -6 & 136 & -344 & 154 \\ & & -6 & 76 & -42 \\ & & & -13 & 9 \end{array}\right| f + \nu \left|\begin{array}{ccccc} 1 & . & 14 & \boxed{-21} & 6 \\ -1 & -4 & -30 & 56 & \boxed{-21} \\ & 4 & 12 & -30 & 14 \\ & & 4 & -4 & . \\ & & & -1 & 1 \end{array}\right| f$$

$$+12s^3[f'''+(2-\nu)f'^{\cdot\cdot}]_0+12s^3[f^{\cdot\cdot\cdot}+(2-\nu)f''^{\cdot}]_{-11}+(s^6), \tag{36}$$

$$10\,\Delta^2 f_0 s^4 = \left|\begin{array}{ccccc} & & & 10 & \nu \\ & & 20 & -80 & 20-\nu 16 \\ -1 & 16 & -94 & \boxed{204} & -65+\nu 30 \\ & & 20 & -80 & 20-\nu 16 \\ & & & 10 & \nu \end{array}\right| f + 12\,(f'' + \nu f^{\cdot\cdot})_{10}\, s^2 + (s^6), \tag{37}$$

$$12\,\Delta^2 f_0 s^4 = \left|\begin{array}{ccccc} & & & 12 & \\ & & 24 & -96 & 24 \\ -3 & 32 & -156 & \boxed{360} & -161 \\ & & 24 & -96 & 24 \\ & & & 12 & \end{array}\right| f + 60\, f'_{10}\, s + (s^6), \tag{38}$$

$$10\,\Delta^2 f_0 s^4 = \left|\begin{array}{ccccc} & & 20 & 65 & 20 \\ -1 & 16 & -94 & \boxed{208} & 65 \\ & & 20 & -94 & 20 \\ & & & 16 & \\ & & & -1 & \end{array}\right| f + 12\,(f''_{10} + f^{\cdot\cdot}_{01})\, s^2 + (s^6), \tag{39}$$

$$12\,\Delta^2 f_0 s^4 = \left|\begin{array}{ccccc} & & 24 & -161 & 24 \\ -3 & 32 & -156 & \boxed{480} & -161 \\ & & 24 & -156 & 24 \\ & & & 32 & \\ & & & -3 & \end{array}\right| f + 60\,(f'_{10} + f^{\cdot}_{01})\, s + (s^6), \tag{40}$$

$$12\,\Delta^2 f_0 s^4 = \left|\begin{array}{ccccc} & & 24 & -78 & 24 \\ -3 & 32 & -156 & \boxed{364{,}8} & -161 \\ & & 24 & -112{,}8 & 24 \\ & & & 19{,}2 & \\ & & & -1{,}2 & \end{array}\right| f + 60\, f'_{10}\, s + 14{,}4\, f^{\cdot\cdot}_{01}\, s^2 + (s^6). \tag{41}$$

The difference expressions (34) (38) (40) are well suited for clamped boundary conditions.

For completeness we add the following difference expressions:

$$\Delta^2 f_0 s^4 = \left|\begin{array}{ccccc} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & \boxed{20} & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{array}\right| f + (s^6), \tag{42}$$

$$f_0^{\prime\prime\cdot\cdot}\, s^4 = \left|\begin{array}{ccc} 1 & -2 & 1 \\ -2 & \boxed{4} & -2 \\ 1 & -2 & 1 \end{array}\right| f + (s^6), \tag{43}$$

$$f_0'''' = \frac{1}{s^4}(f_{-2} - 4f_{-1} + 6f_0 - 4f_1 + f_2) + (s^2), \tag{44}$$

$$f_0''' = \frac{1}{2s^3}(-f_{-2} + 2f_{-1} - 2f_1 + f_2) + (s^2), \tag{45}$$

$$f_0''' = \frac{1}{2s^3}(-3f_{-1} + 10f_0 - 12f_1 + 6f_2 - f_3) + (s^2), \tag{46}$$

$$4f_0'^{\cdot\cdot}s^2 = \boxed{\begin{array}{ccc} -1 & & 1 \\ & \boxed{\cdot} & \\ 1 & & -1 \end{array}}\, f + (s^4), \tag{47}$$

$$4f_0'^{\cdot\cdot}s^2 = \boxed{\begin{array}{ccc} 1 & -4 & 3 \\ \cdot & \cdot & \boxed{\cdot} \\ -1 & 4 & -3 \end{array}}\, f + (s^4), \tag{48}$$

$$f_0'' = \frac{1}{s^2}(f_{-1} - 2f_0 + f_1) + (s^2), \tag{49}$$

$$f_0' = \frac{1}{2s}(f_1 - f_{-1}) + (s^2), \tag{50}$$

$$f_0' = \frac{1}{2s}(3f_0 - 4f_{-1} + f_{-2}) + (s^2). \tag{51}$$

Boundary Equations

The boundary condition (20) differentiated and substituted in the basic differential Eq. (13a), we obtain:

$$u'' - \nu u^{\cdot\cdot} + \frac{F E_c (1+\nu)^2}{h E} v^{\cdot\cdot\cdot} + \frac{e F E_c (1+\nu)^2}{h E} w^{\cdot\cdot\cdot\cdot} + \frac{2\nu}{c} w^{\cdot} = -\frac{X - Z\frac{y}{c}}{D}. \tag{52}$$

Introducing the difference expression (26) according to the boundary condition (17), and the finite differences (44) (45) (49) and (50) the boundary equation (52) may be written in difference form:

$$\boxed{\begin{array}{ccc} -1 & 8 & \boxed{-7} \end{array}}\, U + 2\nu \boxed{\begin{array}{c} -1 \\ \boxed{2} \\ -1 \end{array}}\, U + \frac{E_c \alpha (1+\nu^2)}{E \rho \gamma} \boxed{\begin{array}{c} 1 \\ -2 \\ \boxed{\cdot} \\ 2 \\ -1 \end{array}}\, V + 3\nu \boxed{\begin{array}{c} -1 \\ \boxed{\cdot} \\ 1 \end{array}}\, V$$

$$+ \frac{20 E_c \alpha \beta (1+\nu)^2}{E \rho^2 \gamma^2} \boxed{\begin{array}{c} 1 \\ -4 \\ \boxed{6} \\ -4 \\ 1 \end{array}}\, W + 40 \rho \operatorname{tg} \varphi \nu \boxed{\begin{array}{c} 1 \\ \boxed{\cdot} \\ -1 \end{array}}\, W = \frac{-X - Z\frac{y}{c}}{E} 2\rho^2 \gamma (1 - \nu^2). \tag{53a}$$

Repeating the procedure for the basic differential Eq. (13b) with the same boundary conditions (17) and (20) and the difference expressions (26) (45) (49) and (25), we obtain for hinged and clamped edge:

$$\boxed{-1\;\;8\;\;\boxed{-7}}\,V+\left[\frac{12\,E_c\,\alpha\,(1+\nu)}{E\,\rho\,\gamma}+\frac{4-2\,\nu\,(1+\nu)}{1-\nu}\right]\begin{array}{|c|}\hline 1\\ \boxed{-2}\\ 1\\ \hline\end{array}\,V-3\begin{array}{|c|}\hline 1\\ \boxed{\cdot}\\ -1\\ \hline\end{array}\,U$$

$$+\frac{60\,E_c\,\alpha\,\beta\,(1+\nu)}{E\,\rho^2\,\gamma^2}\begin{array}{|c|}\hline 1\\ -2\\ \boxed{\cdot}\\ 2\\ -1\\ \hline\end{array}\,W+\overbrace{80\,\rho\,\mathrm{tg}\,\varphi\,\boxed{1\;\;\boxed{2}}\,W+40\,\rho\,\mathrm{tg}\,\varphi\,\nu\begin{array}{|c|}\hline 1\\ \boxed{-2}\\ 1\\ \hline\end{array}\,W}^{\text{hinged}}=$$

$$\overbrace{+\,240\,\rho\,\mathrm{tg}\,\varphi\,\boxed{W}}^{\text{clamped}}\qquad\qquad (53\mathrm{b})$$

$$\frac{-Y-Z\frac{y}{c}}{E}\,4\,\rho^2\,\gamma\,(1+\nu)\,.$$

Substituting the boundary conditions (18) and (23), in the third basic Eq. (13c), and using the difference expressions (35) (44) (45) and (49), we find:

$$100{,}8\,(1-\nu^2)\,\mathrm{tg}\,\varphi\,\frac{E_c}{E}\,\alpha\,\rho^2\,\gamma\begin{array}{|c|}\hline 1\\ \boxed{-2}\\ 1\\ \hline\end{array}\,V+504\,(1-\nu^2)\,\mathrm{tg}\,\varphi\,\frac{E_c}{E}\,\alpha\,\beta\,\rho\begin{array}{|c|}\hline 1\\ -2\\ \boxed{\cdot}\\ 2\\ -1\\ \hline\end{array}\,W$$

$$-\begin{array}{|ccccc|}\hline & & & -4 & 25\\ & -26 & 96 & -50 & -104\\ -15 & 136 & -354 & 240 & \boxed{119}\\ & -26 & 96 & -50 & -104\\ & & & -4 & 25\\ \hline\end{array}\,W-\nu\begin{array}{|cccc|}\hline & & 2 & -2\\ -8 & 36 & -80 & 52\\ 16 & -72 & 156 & \boxed{-100}\\ -8 & 36 & -80 & 25\\ & & 2 & -2\\ \hline\end{array}\,W\qquad (53\,\mathrm{c})$$

$$-\,14{,}4\,(1-\nu^2)\,\frac{E_c}{E}\,\alpha\,\beta\begin{array}{|c|}\hline 1\\ -2\\ \boxed{\cdot}\\ 2\\ -1\\ \hline\end{array}\,V-288\,(1-\nu^2)\,\frac{E_c\,\alpha}{E\,\rho\,\gamma}\left(\frac{\alpha}{\delta}+\beta^2\right)\begin{array}{|c|}\hline 1\\ -4\\ \boxed{6}\\ -4\\ 1\\ \hline\end{array}=$$

$$\frac{(X\,\eta+Y\,\zeta)\,2\,\mathrm{tg}\,\varphi-Z}{E}\,25{,}2\,(1-\nu^2)\,\rho^4\,\gamma^3-28{,}8\,(1-\nu^2)\,\rho^3\,\gamma^3\,\frac{g}{a\,E}\,.$$

For points near the edge the third basic Eq. (13c) may be established with the difference expressions (37) and (39) for hinged edge, and (38) and (40) for clamped edge:

Hinged edge:

$$\begin{array}{|c|}\hline 1\\ \boxed{\cdot}\\ -1\\ \hline\end{array}\, U + \boxed{-1\;\boxed{\cdot}\;1}\, V - \boxed{80\,\rho\,\mathrm{tg}\,\varphi}\, W - \frac{1}{12\,(1-\nu)\,\rho^3\gamma^2\,\mathrm{tg}\,\varphi} \tag{54c}$$

$$\cdot \begin{array}{|ccccc|}\hline &&&10&\nu\\ &&20&-80&20-\nu 16\\ -1&16&-94&\boxed{204}&-65+\nu 30\\ &&20&-80&20-\nu 16\\ &&&10&\nu\\ \hline\end{array}\, W = \frac{X\eta + Y\zeta - Z\frac{1}{2\,\mathrm{tg}\,\varphi}}{E}\, 2\,(1+\nu)\,\rho\gamma .$$

Hinged edge, near the corner:

$$\begin{array}{|c|}\hline 1\\ \boxed{\cdot}\\ -1\\ \hline\end{array}\, U + \boxed{-1\;\boxed{\cdot}\;1}\, V - \boxed{80\,\rho\,\mathrm{tg}\,\varphi}\, W - \frac{1}{12\,(1-\nu)\,\rho^3\gamma^2\,\mathrm{tg}\,\varphi} \tag{55c}$$

$$\cdot \begin{array}{|ccccc|}\hline &&20&65&20\\ -1&16-&94&\boxed{208}&65\\ &&20&-94&20\\ &&&16&\\ &&&-1&\\ \hline\end{array}\, W = \frac{X\eta + Y\zeta - Z\frac{1}{2\,\mathrm{tg}\,\varphi}}{E}\, 2\,(1+\nu)\,\rho\gamma .$$

Clamped edge:

$$\begin{array}{|c|}\hline 1\\ \boxed{\cdot}\\ -1\\ \hline\end{array}\, U + \boxed{-1\;\boxed{\cdot}\;1}\, V - \boxed{80\,\rho\,\mathrm{tg}\,\varphi}\, W - \frac{10}{144\,(1-\nu)\,\rho^3\gamma^2\,\mathrm{tg}\,\varphi} \tag{56c}$$

$$\cdot \begin{array}{|ccccc|}\hline &&&12&\\ &&24&-96&24\\ -3&32&-156&\boxed{360}&-161\\ &&24&-96&24\\ &&&12&\\ \hline\end{array}\, W = \frac{X\eta + Y\zeta - Z\frac{1}{2\,\mathrm{tg}\,\varphi}}{E}\, 2\,(1+\nu)\,\rho\gamma .$$

Clamped edge, near the corner:

$$\left[\begin{array}{c}1\\ \boxed{\cdot}\\ -1\end{array}\right] U + \left[\begin{array}{ccc}-1 & \boxed{\cdot} & 1\end{array}\right] V - \left[\,80\,\rho\,\mathrm{tg}\,\varphi\,\right] W - \frac{10}{144\,(1-\nu)\,\rho^3\,\gamma^2\,\mathrm{tg}\,\varphi} \qquad (57\,\mathrm{c})$$

$$\cdot \left[\begin{array}{ccccc} & & 24 & -161 & 24\\ -3 & 32 & -156 & \boxed{480} & -161\\ & & 24 & -156 & 24\\ & & & 32 & \\ & & & -3 & \end{array}\right] W = \frac{X\,\eta + Y\,\zeta - Z\,\frac{1}{2\,\mathrm{tg}\,\varphi}}{E}\, 2\,(1+\nu)\,\rho\,\gamma .$$

We have now established the boundary equations except for the corners and points at the edge one mesh from the corners.

With the boundary condition (17) and according to symmetry the condition in the upper corner is:

$$u' = v^{\cdot} = 0 .$$

At points one mesh from the corners the difference expression $\left[\begin{array}{c}1\\ -2\\ \boxed{\cdot}\\ 2\\ -1\end{array}\right] V$ in Eq. (53a) is according to (18) replaced by $\frac{2}{3}\left[\begin{array}{c}-8\\ \boxed{9}\\ \cdot\\ -1\end{array}\right]^{v^{\cdot}=0} V$ at the upper corner and $\left[\begin{array}{c}-1\\ 6\\ -12\\ \boxed{10}\\ -3\end{array}\right] V$ at the lower corner. $\left[\begin{array}{c}1\\ -4\\ \boxed{6}\\ -4\\ 1\end{array}\right] W$ is replaced by $\frac{1}{10}\left[\begin{array}{c}-25\\ \boxed{64}\\ -54\\ 16\\ -1\end{array}\right]^{w^{\cdot\cdot}=0} W$ or $\frac{1}{12}\left[\begin{array}{c}-113\\ \boxed{192}\\ -108\\ 32\\ -3\end{array}\right]^{w^{\cdot}=0} W$ for respectively hinged and clamped corner. In Eq. (53b) the difference expression $\left[\begin{array}{c}1\\ -2\\ \boxed{\cdot}\\ 2\\ 1\end{array}\right] W$ is replaced by $\frac{2}{11}\left[\begin{array}{c}-1\\ \boxed{-3}\\ 9\\ -5\end{array}\right]^{w^{\cdot\cdot}=0} W$ or $\frac{2}{3}\left[\begin{array}{c}-8\\ \boxed{9}\\ \cdot\\ -1\end{array}\right]^{w^{\cdot}=0} W$ for hinged and clamped corner respectively.

For the third basic Eq. (13c) at points at the edge one mesh from the upper corner, we obtain with the difference expression (36) for hinged and clamped corner

$$67{,}2\,(1-\nu^2)\,\mathrm{tg}\,\varphi\,\frac{E_c}{E}\,\alpha\,\rho^2\gamma \left[\begin{array}{c} 1 \\ \boxed{-2} \\ 1 \end{array}\right] V + (1-\nu^2)\,\mathrm{tg}\,\varphi\,\frac{E_c}{E}\,\alpha\,\beta\,\rho \underbrace{\left[\begin{array}{c} -1 \\ \boxed{-3} \\ 9 \\ -5 \end{array}\right]_{w''=0} \frac{672}{11}\,W}_{\text{hinged}} \;\; \underbrace{\left[\begin{array}{c} -8 \\ \boxed{9} \\ \cdot \\ -1 \end{array}\right]_{w'=0} 224\,W}_{\text{clamped}}$$

$$- \left[\begin{array}{ccccc} 9 & -42 & 154 & \boxed{-219} & 82 \\ -13 & 76 & -344 & 552 & \boxed{-219} \\ & -6 & 136 & -344 & 154 \\ & & -6 & 76 & -42 \\ & & & -13 & 9 \end{array}\right] W - \nu \left[\begin{array}{ccccc} 1 & \cdot & 14 & \boxed{-21} & 6 \\ -1 & -4 & -30 & 56 & \boxed{-21} \\ & 4 & 12 & -30 & 14 \\ & & 4 & -4 & \cdot \\ & & & -1 & 1 \end{array}\right] W$$

$$-(1-\nu^2)\,\frac{E_c\,\alpha}{E\,\rho\,\gamma}\left(\frac{\alpha}{\delta}+\beta^2\right) \underbrace{\left[\begin{array}{c} -25 \\ \boxed{64} \\ -54 \\ 16 \\ -1 \end{array}\right]_{w''=0} 28{,}8\,W}_{\text{hinged}} \;\; \underbrace{\left[\begin{array}{c} -113 \\ \boxed{192} \\ -108 \\ 32 \\ -3 \end{array}\right]_{w'=0} 24\,W}_{\text{clamped}} \qquad (58\,\mathrm{c})$$

$$-9{,}6\,(1-\nu^2)\,\frac{E_c}{E_s}\,\alpha\,\beta \left[\begin{array}{c} -8 \\ \boxed{9} \\ \cdot \\ -1 \end{array}\right]_{v'=0} V = \frac{(X\,\eta + Y\,\zeta)\,2\,\mathrm{tg}\,\varphi - Z}{E}\,16{,}8\,(1-\nu^2)\,\rho^4\gamma^3$$

$$-\,28{,}8\,(1-\nu^2)\,\frac{\rho^3\,\gamma^3\,g}{l\,E}.$$

At the lower corner $\left[\begin{array}{c} -8 \\ \boxed{9} \\ \cdot \\ -1 \end{array}\right] V$ is replaced by $\frac{3}{2}\left[\begin{array}{c} -1 \\ 6 \\ -12 \\ \boxed{10} \\ -3 \end{array}\right] V.$

At the upper corner point we obtain according to symmetry for the two basic Eqs. (13a—b) when substituting the condition (20):

$$v^{\cdot\cdot} + \frac{F\,E_c\,(1+\nu)^2}{h\,E}\,v^{\cdot\cdot\cdot} + \frac{e\,F\,E_c\,(1+\nu)^2}{h\,E}\,w^{\cdot\cdot\cdot\cdot} + \frac{2\,\nu}{c}\,w^{\cdot} = \frac{-X - Z\frac{y}{c}}{D}.$$

With the condition $v^{\cdot}=0$ we use the difference expressions (26) (27) (32) and (33):

$$\left[\begin{array}{c}\boxed{-7}\\8\\-1\end{array}\right] v+\frac{2\,F\,E_c(1+\nu)^2}{s\,h\,E}\left[\begin{array}{c}\boxed{-10}\\15\\-6\\1\end{array}\right] v+\frac{F\,e\,E_c}{s^2\,h\,E}\underbrace{\left[\begin{array}{c}\boxed{-30}\\84\\-84\\36\\-6\end{array}\right]^{w^{\cdot\cdot}=0}\frac{2}{5}\,w}_{\text{hinged}}\;\underbrace{\left[\begin{array}{c}\boxed{-119}\\216\\-144\\56\\-9\end{array}\right]^{w^{\cdot}=0}\frac{w}{3}}_{\text{camped}}$$

$$+\frac{4\,\nu\,s}{c}\left[\begin{array}{c}\boxed{1}\\-1\end{array}\right]^{w^{\cdot\cdot}=0} w=\frac{-X-Z\frac{y}{c}}{D}\,2\,s^2. \tag{59 a–b}$$

At the lower corners the displacements u, v, w are supposed to be known, and in the upper corners the displacement w.

For $u=0$ and $w^{\cdot}=0$ along the boundary in y-direction the Eq. (53b) may be written:

$$\left[\begin{array}{ccc}-1 & 8 & \boxed{-7}\end{array}\right] v+\left[\frac{12\,E_c\,F\,(1+\nu)}{E\,s\,h}+\frac{4}{1-\nu}\right]\left[\begin{array}{c}1\\\boxed{-2}\\1\end{array}\right] v+\frac{6\,E_c\,e\,F\,(1+\nu)}{E\,s^2\,h}\left[\begin{array}{c}1\\-2\\\boxed{\cdot}\\2\\-1\end{array}\right] w$$

$$+\frac{12\,s}{c}\boxed{w}+\frac{2\,(1+\nu)}{1-\nu}\left[\begin{array}{ccc}1 & -4 & \cdot\\\cdot & \cdot & \boxed{\cdot}\\-1 & 4 & \cdot\end{array}\right] u=\frac{-Y-Z\frac{y}{c}}{D}\,\frac{4\,s^2}{1-\nu}. \tag{60 b}$$

The edge member is vertically supported when setting $w=0$ along the edge.

Putting $\operatorname{tg}\varphi=0\,(c=\infty)$ the equations can be applied to plate problems. The Eqs. (16a–c) (53c) (56c) (57c) (58c) and (60b) have been used in [8] on rectangular clamped plate on elastic beams with $e \geqq 0$. Compared with "Mehrstellen" expressions good agreement is ashieved.

2. The Cross Sectional Area of the Edge Member is Variable

Boundary Conditions

The edge member has no bending rigidity and is vertically supported. Continuity and equilibrium conditions:

Equilibrium and continuity along the boundary:

$$N_{xy}=\bar{N}_c^{\cdot}=E_c\,F\,(y)\,v^{\cdot\cdot}+E_c v^{\cdot}\,F\,(y)^{\cdot}, \tag{61}$$

where $F(y)$ is the cross sectional area of the edge member and only function of y from (14c) and (61) we obtain:

$$v' = \frac{2(1+\nu)E_c}{hE}[F v^{\cdot\cdot} + F^{\cdot} v^{\cdot}] - u^{\cdot}. \tag{62}$$

Boundary Equations

Differentiating the boundary condition (62) and substituting into the basic Eq. (13a), we find:

$$u'' - \nu u^{\cdot\cdot} + \frac{E_c(1+\nu)^2}{Eh}[F v^{\cdot\cdot\cdot} + 2F^{\cdot} v^{\cdot\cdot} + F^{\cdot\cdot} v^{\cdot}] = \frac{-X - Z\frac{y}{c}}{D}. \tag{63}$$

When $F(y)$ is given analytically we find with the same difference expressions as used in (53a–b).

$$\boxed{-1 \quad 8 \quad \boxed{-7}}\, u + 2\nu \boxed{\begin{array}{c} -1 \\ \boxed{2} \\ -1 \end{array}}\, u + \frac{E_c F(1+\nu)^2}{Esh} \boxed{\begin{array}{c} 1 \\ -2 \\ \boxed{\cdot} \\ 2 \\ -1 \end{array}}\, v + \frac{4E_c F^{\cdot}(1+\nu)^2}{Eh} \boxed{\begin{array}{c} 1 \\ \boxed{-2} \\ 1 \end{array}}\, v$$

$$+ \frac{E_c F^{\cdot\cdot} s(1+\nu)^2}{Eh} \boxed{\begin{array}{c} 1 \\ \boxed{\cdot} \\ -1 \end{array}}\, v = -X - Z\frac{y}{c}\frac{2s^2}{D}, \tag{64a}$$

$$\boxed{-1 \quad 8 \quad \boxed{-7}}\, v + \left[\frac{12E_c F(1+\nu)}{Esh} + \frac{4-2\nu(1+\nu)}{1-\nu}\right] \boxed{\begin{array}{c} 1 \\ \boxed{-2} \\ 1 \end{array}}\, v + \frac{6E_c F^{\cdot}(1+\nu)}{Eh} \boxed{\begin{array}{c} 1 \\ \boxed{\cdot} \\ -1 \end{array}}\, v$$

$$-3 \boxed{\begin{array}{c} 1 \\ \boxed{\cdot} \\ -1 \end{array}}\, u + \frac{4s}{c} \underset{\substack{=0 \text{ when} \\ \text{clamped}}}{\overset{\text{hinged}}{\boxed{1 \quad \boxed{\cdot}}}}\, w = -Y - Z\frac{y}{c}\frac{4s^2}{D(1-\nu)}. \tag{64b}$$

According to symmetry and differentiating and substituting Eq. (62) in the basic Eq. (13a), we obtain for the upper corner point:

$$v^{\cdot\cdot} + \frac{E_c(1+\nu)^2}{Eh}[F v^{\cdot\cdot\cdot} + 2F^{\cdot} v^{\cdot\cdot} + F^{\cdot\cdot} v^{\cdot}] = \frac{-X - Z\frac{y}{c}}{D} \tag{65}$$

and with the condition $v^{\cdot} = 0$ we find:

$$\left[1 + \frac{2E_c(1+\nu)^2 F^{\cdot}}{Eh}\right] \boxed{\begin{array}{c} \boxed{-7} \\ 8 \\ -1 \end{array}}\, v + \frac{2E_c(1+\nu)^2}{Esh} \boxed{\begin{array}{c} \boxed{-10} \\ 15 \\ -6 \\ 1 \end{array}}\, v = \left(-X - Z\frac{y}{c}\right)\frac{2s^2}{D}. \tag{66a–b}$$

In the special case when $F(y)$ is lineary variable:

$$F(y) = \frac{F_1 - F_2}{l} y + \frac{F_1 + F_2}{2}, \tag{67}$$

where F_1 and F_2 denote the cross sectional area at the upper and lower corner respectively.

$$F^{\cdot} = \frac{F_1 - F_2}{l}, \qquad F^{\cdot\cdot} = 0.$$

Stress Forces and Stress Couples

The stress forces and stress couples are computed from the main variables U, V, W according to (14) (15) (18) and (61). Applying simple differences we find for the stress forces:

$$N_x = \frac{E\,l}{2(1-\nu^2)\rho\gamma}\left[\boxed{\begin{array}{ccc} -1 & \boxed{.} & 1 \end{array}}\,U + \nu \boxed{\begin{array}{c} 1 \\ \boxed{.} \\ -1 \end{array}}\,V\right], \tag{68a}$$

$$N_y = \frac{E\,l}{2\rho\gamma}\boxed{\begin{array}{c} 1 \\ \boxed{.} \\ -1 \end{array}}\,V + \nu N_x, \tag{68b}$$

$$N_{xy} = \frac{E\,l}{4(1+\nu)\rho\gamma}\left[\boxed{\begin{array}{c} 1 \\ \boxed{.} \\ -1 \end{array}}\,U + \boxed{\begin{array}{ccc} -1 & \boxed{.} & 1 \end{array}}\,V - 80\rho\,\mathrm{tg}\,\varphi\,W\right]. \tag{68c}$$

Stress couples:

$$M_x = \frac{E\,l^2}{1{,}2(1-\nu^2)\rho^2\gamma^3}\left[\boxed{\begin{array}{ccc} 1 & \boxed{-2} & 1 \end{array}}\,W + \nu \boxed{\begin{array}{c} 1 \\ \boxed{-2} \\ 1 \end{array}}\,W\right], \tag{69a}$$

$$M_y = \frac{E\,l^2}{1{,}2\rho^2\gamma^3}\boxed{\begin{array}{c} 1 \\ \boxed{-2} \\ 1 \end{array}}\,W + \nu M_x, \tag{69b}$$

$$M_{xy} = \frac{E\,l^2}{4{,}8(1+\nu)\rho^2\gamma^3}\boxed{\begin{array}{ccc} -1 & . & 1 \\ . & \boxed{.} & . \\ 1 & . & -1 \end{array}}\,W \tag{69c}$$

and in corner points:

$$M_{xy}=\frac{E\,l^2}{2{,}4\,(1+\nu)\,\rho^2\,\gamma^3}\begin{array}{|ccc|}\hline 1 & -5 & \boxed{4} \\ -1 & 6 & -5 \\ & -1 & 1 \\ \hline\end{array}\,W. \tag{69e}$$

For clamped edge condition Eq. (37) is used together with (13c) and (14c): The negative clamping shell moment is:

$$M_{x\,edge}=$$

$$\frac{5}{3}\frac{s^2}{c}\boxed{N_{xy}}-\frac{5}{3}s^2\left(X\frac{y}{c}+Y\frac{x}{c}-Z\right)-\begin{array}{|ccccc|}\hline & & & 10 & \nu \\ & & 20 & -80 & 20-\nu\,16 \\ -1 & 16 & -94 & \boxed{204} & -65+\nu\,30 \\ & & 20 & -80 & 20-\nu\,16 \\ & & & 10 & \nu \\ \hline\end{array}\,\frac{w\,B}{12\,s^2}. \tag{69d}$$

Near the corner Eqs. (39) have to be used.

By using higher expressions

$$N_x=\frac{E\,l}{12\,(1+\nu^2)\,\rho\,\gamma}\left[\begin{array}{|ccccc|}\hline 1 & -8 & \boxed{\cdot} & 8 & -1 \\ \hline\end{array}\,U+\nu\begin{array}{|c|}\hline 1 \\ -8 \\ \boxed{\cdot} \\ 8 \\ -1 \\ \hline\end{array}\,V\right] \tag{70a}$$

and correspondingly for N_y and N_{xy}.

$$M_x=\frac{E\,l^2}{14{,}4\,(1-\nu^2)\,\rho^2\,\gamma^3}\left[\begin{array}{|ccccc|}\hline -1 & 16 & \boxed{-30} & 16 & -1 \\ \hline\end{array}\,W+\nu\begin{array}{|c|}\hline -1 \\ 16 \\ \boxed{-30} \\ 16 \\ -1 \\ \hline\end{array}\,W\right], \tag{71a}$$

$$M_{xy}=\frac{E\,l^2}{172{,}8\,(1+\nu)\,\rho^2\,\gamma^3}\begin{array}{|ccccc|}\hline 1 & -8 & \cdot & 8 & -1 \\ -8 & 64 & \cdot & -64 & 8 \\ \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot \\ 8 & -64 & \cdot & 64 & -8 \\ -1 & 8 & \cdot & -8 & 1 \\ \hline\end{array}\,W. \tag{71c}$$

The stresses in the centre of a mesh may be computed from the corner values by using simple differences

$$N_x=\frac{E\,l}{2\,(1-\nu^2)\,\rho\,\gamma}\left[\begin{array}{|cc|}\hline -1 & 1 \\ -1 & 1 \\ \hline\end{array}\,U+\nu\begin{array}{|cc|}\hline -1 & -1 \\ 1 & 1 \\ \hline\end{array}\,V\right] \tag{72a}$$

and correspondingly for N_y and N_{xy}

$$M_{xy} = \frac{E\,l^2}{1{,}2\,(1-\nu)\,\rho^2\gamma^3}\left[\begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array}\right] W. \tag{73c}$$

The shear force N_{xy} along the edge according to (18) by using simple differences:

$$N_{xy} = \frac{E_c\,l\,\alpha}{\rho^2\gamma^2}\left[\begin{array}{c} 1 \\ \boxed{-2} \\ 1 \end{array}\right] V + \frac{5\,E_c\,l\,\alpha\,\beta}{\rho^3\gamma^3}\left[\begin{array}{c} 1 \\ -2 \\ \boxed{\cdot} \\ 2 \\ -1 \end{array}\right] W \tag{74}$$

and by applying higher expressions:

$$N_{xy} = \frac{E_c\,l\,\alpha}{12\,\rho^2\gamma^2}\left[\begin{array}{c} -1 \\ 16 \\ \boxed{-30} \\ 16 \\ -1 \end{array}\right] V + \frac{5\,E_c\,l\,\alpha\,\beta}{4\,\rho^3\gamma^3}\left[\begin{array}{c} -1 \\ 8 \\ -13 \\ \boxed{\cdot} \\ 13 \\ -8 \\ 1 \end{array}\right] W. \tag{75}$$

One mesh from corner:

$$N_{xy} = \frac{E_c\,l\,\alpha}{\rho^2\gamma^2}\left[\begin{array}{c} 1 \\ \boxed{-2} \\ 1 \end{array}\right] V + \frac{10\,E_c\,l\,\alpha\,\beta}{\rho^3\gamma^3}\left[\begin{array}{c} -1 \\ \boxed{-3} \\ 9 \\ -5 \end{array}\right]^{w^{\cdot\cdot}=0} \frac{W}{11} \left[\begin{array}{c} -8 \\ \boxed{9} \\ \cdot \\ -1 \end{array}\right]^{w^{\cdot}=0} \frac{W}{3}. \tag{76}$$

Upper corner:

$$N_{xy} = \frac{E\,l}{6\,(1+\nu)\,\rho\,\gamma}\left[\begin{array}{c} \boxed{7} \\ -8 \\ 1 \end{array}\right] U. \tag{77}$$

Lower corner:

$$N_{xy} = \frac{E\,l}{2\,(1+\nu)\,\rho\,\gamma}\left[\begin{array}{c} -1 \\ 4 \\ \boxed{-3} \end{array}\right] U. \tag{78}$$

Variable cross sectional area of edge member. According to Eq. (61) with finite difference expression:

$$N_{xy} = \frac{E_c\,F}{s^2}\left[\begin{array}{c} 1 \\ \boxed{-2} \\ 1 \end{array}\right] v + \frac{E_c\,F^{\cdot}}{2\,s}\left[\begin{array}{c} 1 \\ \boxed{\cdot} \\ -1 \end{array}\right] v \tag{79}$$

and with higher expressions:

$$N_{xy} = \frac{E_c F}{12\, s^2} \begin{array}{|c|} \hline -1 \\ 16 \\ \boxed{-30} \\ 16 \\ -1 \\ \hline \end{array}\, v + \frac{E_c F^{\cdot}}{12\, s} \begin{array}{|c|} \hline 1 \\ -8 \\ \boxed{\cdot} \\ 8 \\ -1 \\ \hline \end{array}\, v. \qquad (80)$$

Corner points Eqs. (77) and (78).

The support reaction along the edge is according to Eqs. (35) (36) and (13c):

$$R_x = \frac{21}{6} \rho \operatorname{tg} \varphi \boxed{N_{xy}} - \frac{21}{24} s \left(X \frac{y}{c} + Y \frac{x}{c} - Z \right)$$

$$- \begin{array}{|ccccc|} \hline & & & -4 & 25 \\ & -26 & 96 & -50 & -104 \\ -15 & 136 & -354 & 240 & \boxed{119} \\ & -26 & 96 & -50 & -104 \\ & & & -4 & 25 \\ \hline \end{array} \frac{w\, B}{24\, s^3} \qquad (81)$$

$$- \nu \begin{array}{|ccccc|} \hline & & & 2 & -2 \\ & -8 & 36 & -80 & 52 \\ \cdot & 16 & -72 & 156 & \boxed{-100} \\ & -8 & 36 & -80 & 52 \\ & & & 2 & -2 \\ \hline \end{array} \frac{w\, B}{24\, s^3}$$

and correspondingly with Eqs. (36).

Numerical Examples

In the following figures the notation a is to be replaced by l (base length).

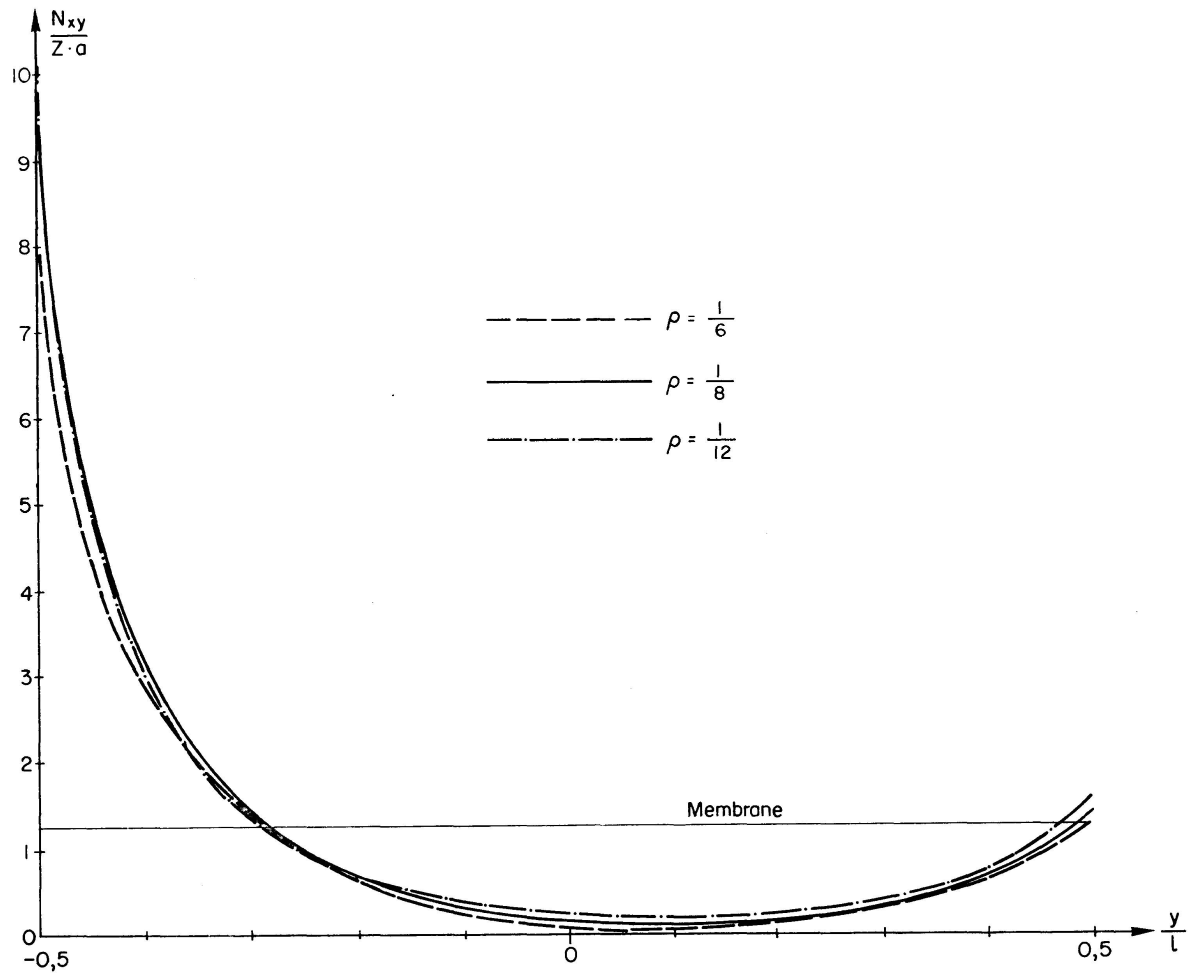
Example 1

Consider a H.P. shell over a square base, hinged to vertically supported edge members, with the following properties:

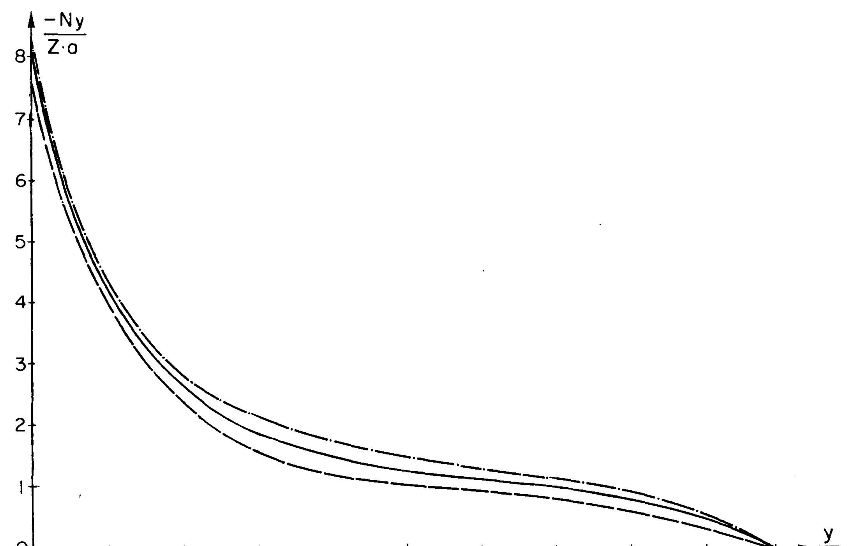
$$\operatorname{tg} \varphi = 1/5, \quad \alpha = F/h^2 = 50, \quad \gamma = l/h = 350, \quad \rho = s/l = 1/6, \quad \rho = 1/8, \quad \rho = 1/12.$$

All examples have the following condition at the lower corners: $u = -v = 0$. To test the convergence this example is computed with a mesh width $s = l/6$, $s = l/8$ and $s = l/12$.

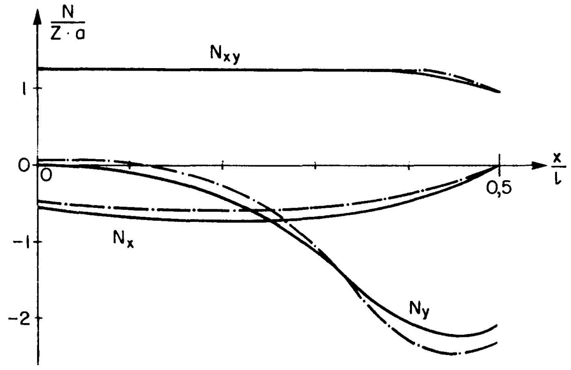
The results are as follows:



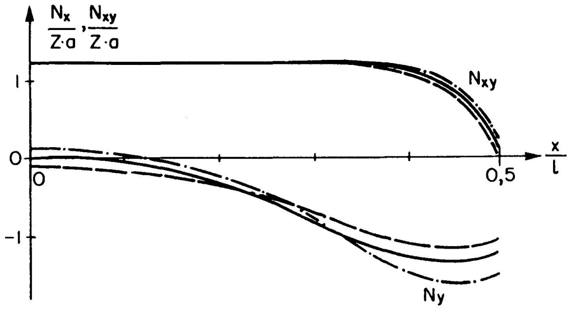
Shear force N_{xy} at the line $x = l/2$ (along the edge).



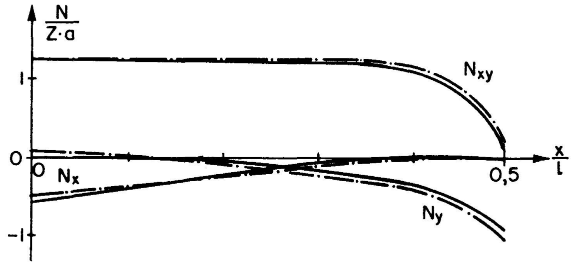
Stress forces N_y at the line $x = l/2$ (along the edge).



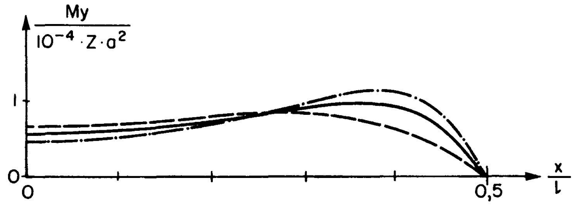
Stress forces N_x, N_y and N_{xy} at the line $y = -l/4$.



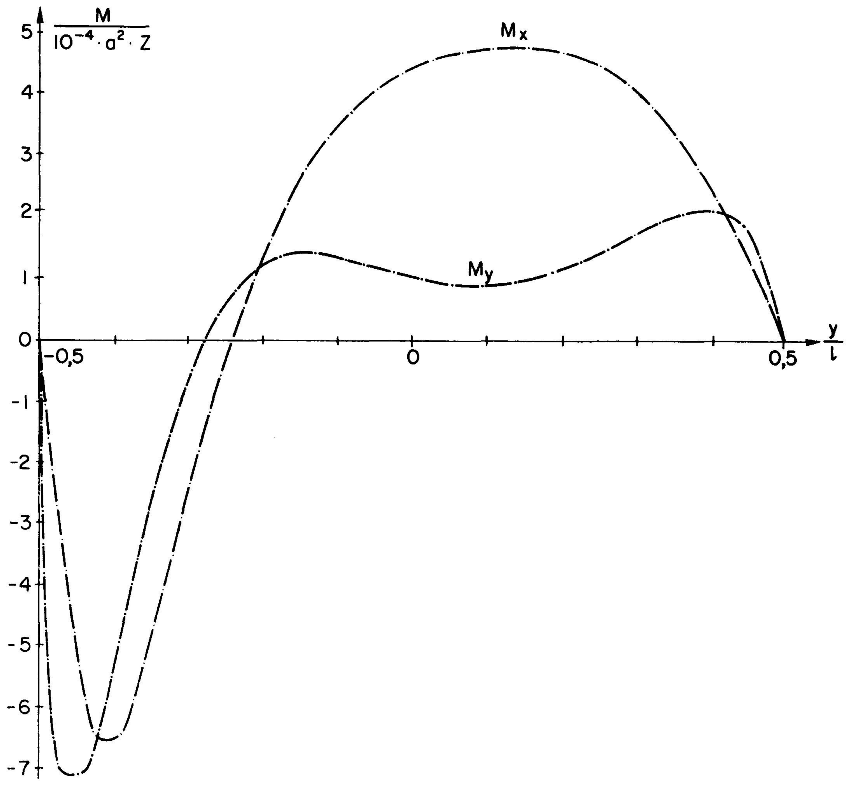
Stress forces N_{xy} and N_y at the line $y = 0$.



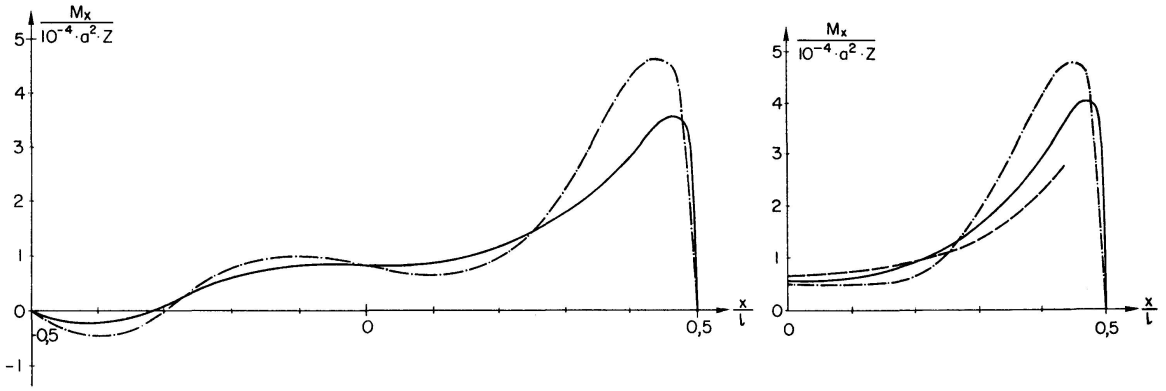
Stress forces N_x, N_y and N_{xy} at the line $y = l/4$.



Moments M_y at the line $y = 0$.



Moments M_x and M_y at the line $x = \frac{5}{12} l$.

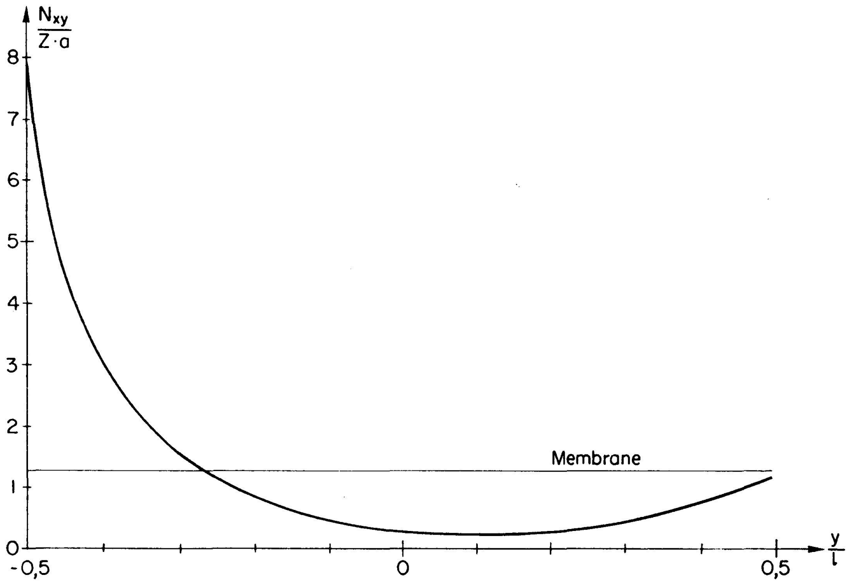


Moments M_x at the line $y = l/4$. Moments M_x at the line $y = 0$.

Example 2

The cross sectional area of the edge members is linear variable: Lower corner: $F_1/h^2 = 75$. Upper corner $F_2/h^2 = 25$.

The other properties the same as in example 1. Mesh width $s = l/8$.

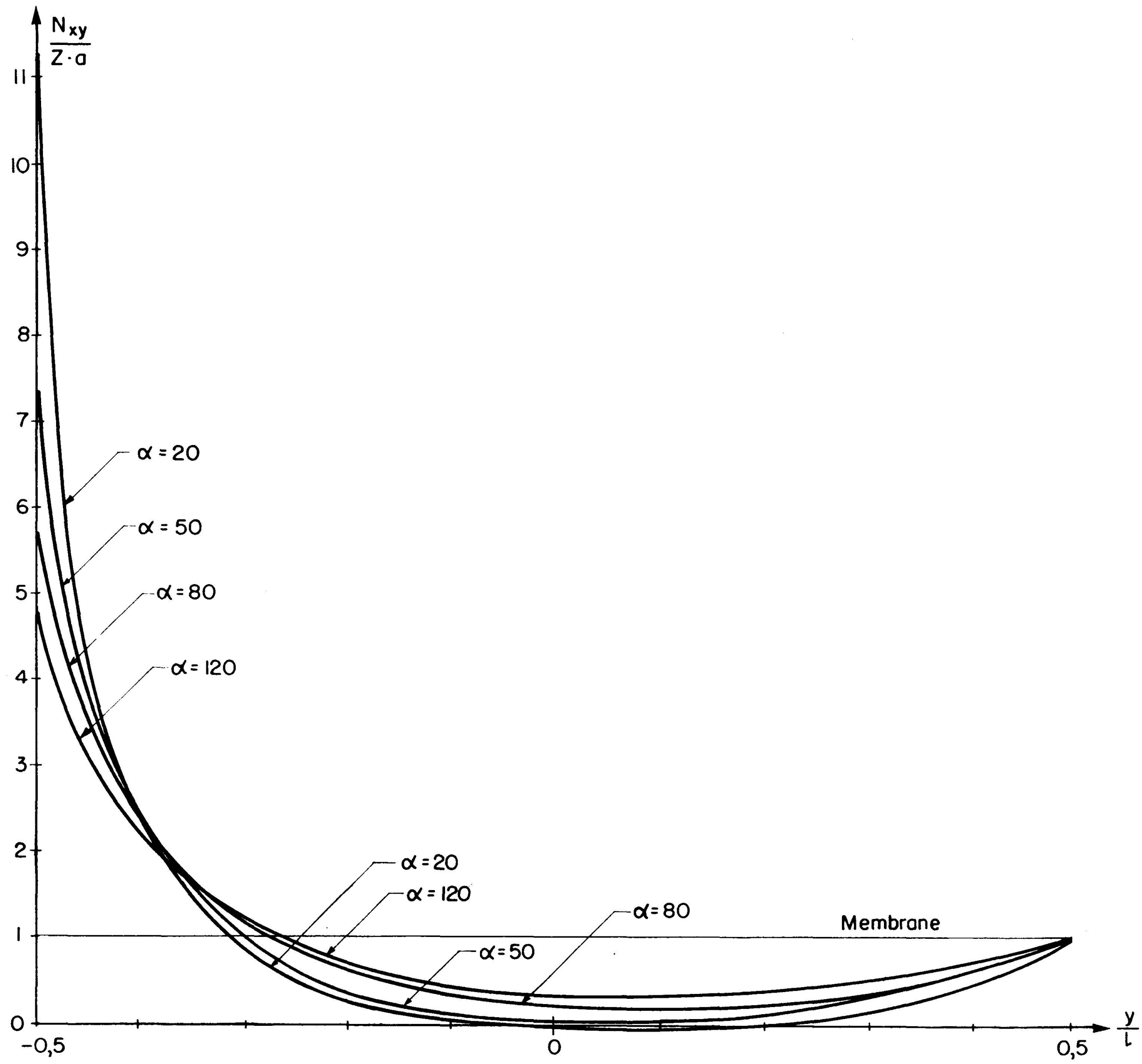


Shear forces at the line $x = l/2$ (along the edge).

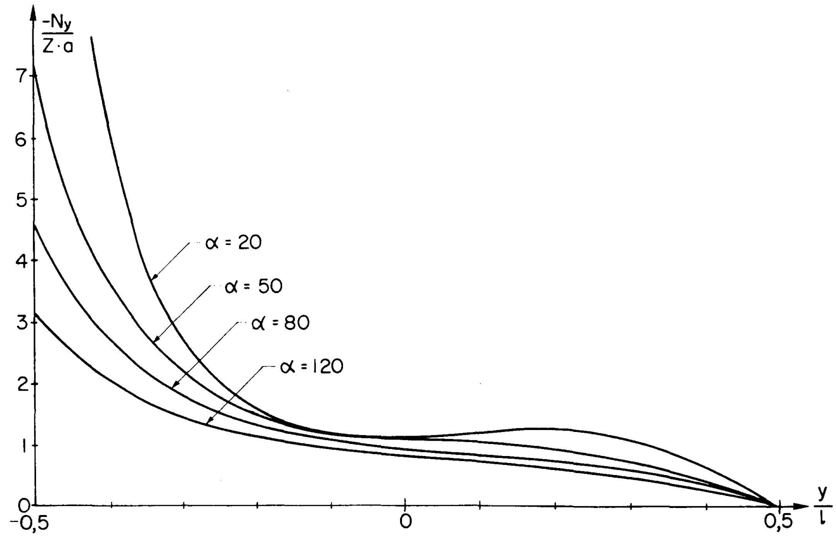
Example 3

The shell is hinged to vertically supported edge members with various stiffness. The following properties are used:

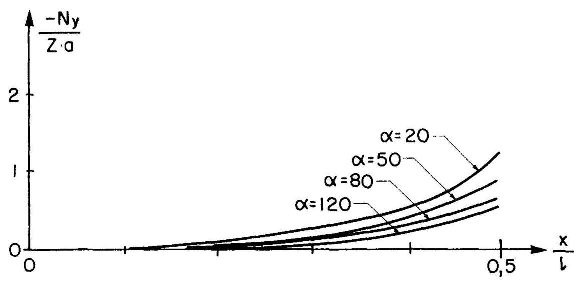
$$\operatorname{tg}\varphi = \frac{1}{4}, \qquad \varkappa = \frac{F}{h^2} = \begin{cases} 20 \\ 50 \\ 80 \\ 120 \end{cases}, \qquad \gamma = \frac{l}{h} = 400, \qquad \rho = \frac{s}{l} = \frac{1}{8}, \qquad E_c = E.$$



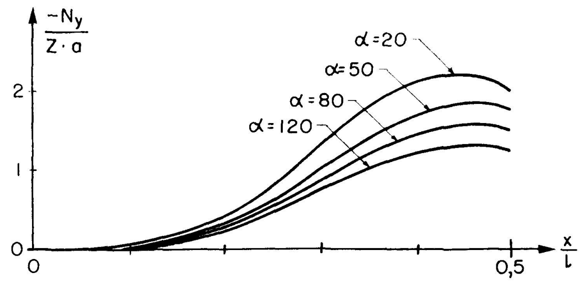
Shear forces N_{xy} at the line $x = l/2$ (along the edge).



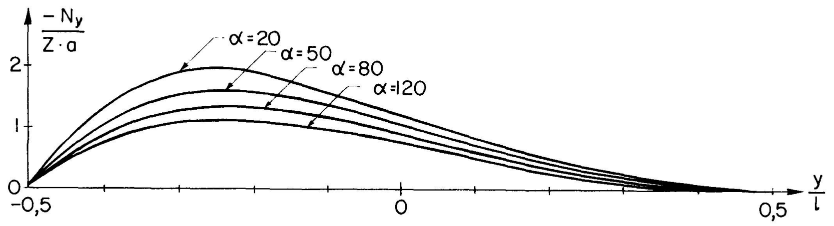
Stress force N_y at the line $x = l/2$ (along the edge).



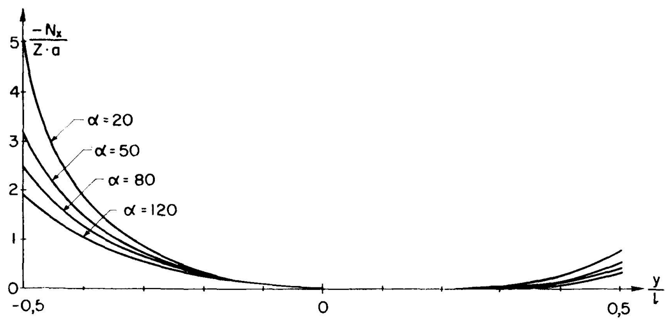
Stress force N_y at the line $y = l/4$.



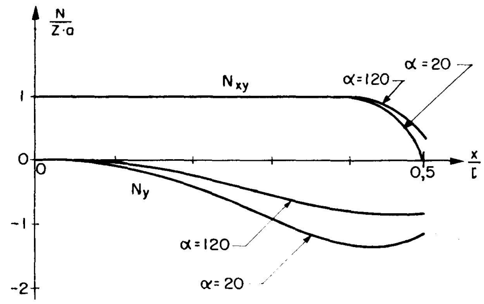
Stress force N_y at the line $y = -l/4$.



Stress force N_y at the line $x = \frac{3}{8} l$.



Stress force N_x at the line $x = 3/8\, l$.

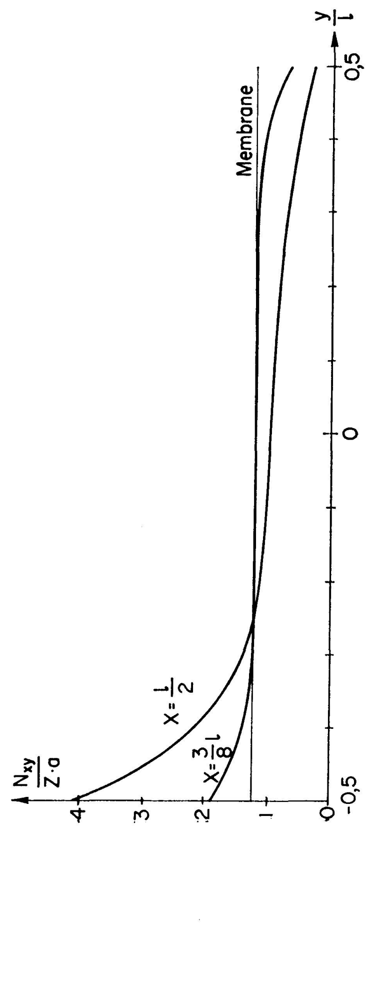


Stress forces N_y and N_{xy} at the line $y = 0$.

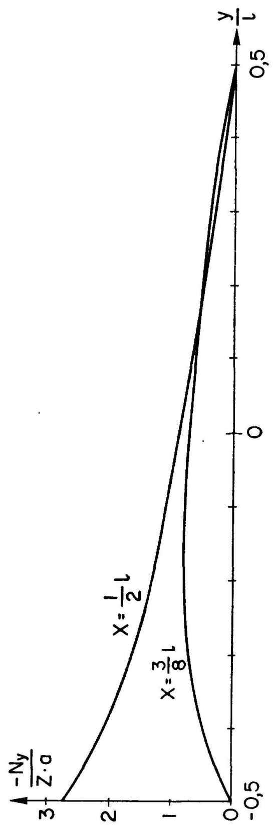
Example 4

The edge members have bending rigidity in the vertical direction and are only supported at the corners. The properties are:

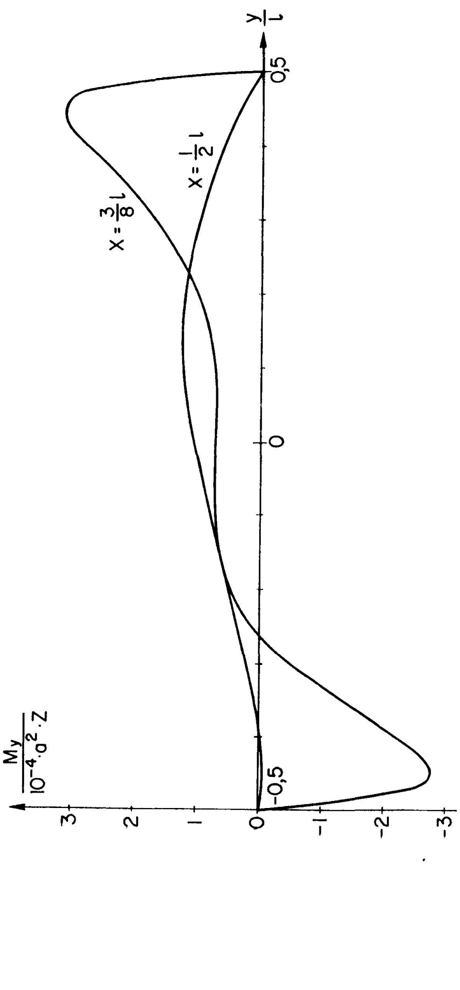
$$\operatorname{tg} \varphi = 1/5, \quad \alpha = 25, \quad \gamma = 220, \quad E_c/E = 4, \quad e = 0, \quad \nu = 0, \quad J = F^2/6.$$



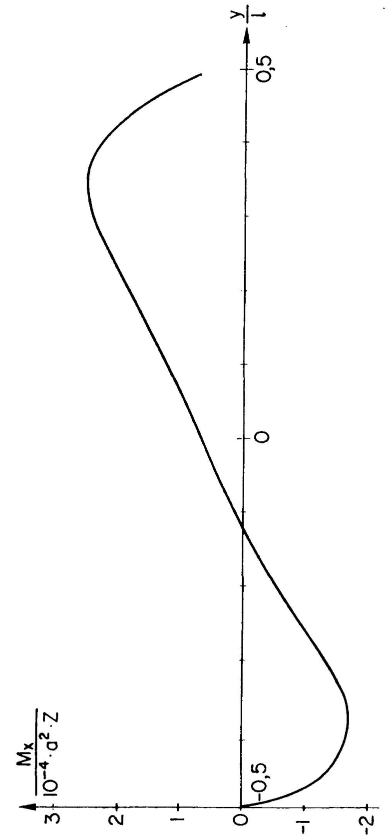
Shear forces N_{xy} at the lines $x = l/2$ (along the edge) and $x = \frac{3}{8}l$.



Stress forces N_y at the lines $x = \frac{1}{2}l$ and $x = \frac{3}{8}l$.



Moments M_y at the lines $x = \frac{1}{2}l$ (along the edge) and $x = \frac{3}{8}l$.



Moment M_x at the line $x = \frac{3}{8}l$.

Conclusions

From the calculations it is observed that the membrane analysis is in significant error, expecially in the region of the lower corners and along the boundary.

The stiffness of the edge member greatly influences the stress distribution and the displacements. In the interior of the shell, however, the stresses are in good agreement with the membrane theory.

By means of electronic computer, the entire process may be programmed from given data to end results.

Bibliography

1. Candela, F.: Structural Applications of Hyperbolic Paraboloidical Shells. Journ. Amer. Concrete Inst. 26 (1955), Title No. 51–20.
2. Aimond, F.: Etude statique des voiles minces en paraboloide hyperbolique. Mém. Assoc. Int. Ponts et Charpentes 4, 1–112 (1936).
3. Tester, K. G.: Berechnung der hyperbolischen Paraboloidschale. Ing.-Arch. 16, 39 (1947).
4. Apeland, K.: Analysis of bending stresses in translational shells. Proc. of the Colloquium on Simplified Calculation Methods, Brussels 1961. Published by North-Holland Pub. Co., Amsterdam 1962.
5. Green-Zerna: Theoretical Elasticity. Oxford 1954, Clarandon Press.
6. Wlassow, W. S.: Basic Differential Equations in General Theory of Elastic Shells. NACA, Techn. Mem. 1241, 1951. (Translated edition.)
7. Bongard, W.: Zur Theorie und Berechnung von Schalentragwerken in Form gleichseitiger hyperbolischer Paraboloide. Bautechnik-Archiv 15. Berlin 1959. Wilhelm Ernst & Sohn.
8. Stageboe, J.: Die orthotrope Platte als Pilzdecke und auf elastischen Trägern. (To be published.)
9. Collatz, L.: The numerical Treatment of Differential Equations. Berlin 1960. Springer-Verlag.

Summary

The present paper deals with the bending theory of hyperbolic paraboloid shells with straight edges. By applying tensor calculus the general shell equations for the displacements in arbitrary curvilinear coordinates are established. From the general equations, the equations for deep and shallow hyperbolic paraboloid shells are derived. The solutions are finally obtained by means of finite difference approximations. For shallow shells finite difference expressions are developed for various boundary conditions and the numerical calculations are accomplished for shell hinged to vertically supported edge member of constant and linear variable stiffness and shell supported upon elastic edge members only supported in the corners. To test the convergence the plan area is divided into 6×6, 8×8 and 12×12 meshes.

Résumé

Le présent travail traite de théorie de flexion appliquée aux coques en forme de paraboloïdes hyperboliques avec bords tendus. Les équations générales pour les coques sont établies, au moyen du calcul tensoriel appliqué, pour des déplacements en coordonnées curvilignes arbitraires. Des équations générales, on a dérivé les équations pour des coques paraboloïdes hyperboliques profondes et à courbure faible. Les solutions sont finalement obtenues à l'aide d'approximations par des différences finies. Pour les coques à courbure faible les expressions des différences finies sont développées pour des conditions limites variables et les calculs numériques sont effectués pour des coques reliées à des bords supportés verticalement dont la rigidité est constante ou linéairement variable et pour des coques reliées à des bords élastiques seulement appuyés dans les coins. Afin de tester la convergence, le plan est divisé en mailles 6×6, 8×8 et 12×12.

Zusammenfassung

Der vorliegende Bericht handelt von der Biegetheorie hyperbolischer Paraboloidschalen mit geraden Kanten. Mittels der Tensorrechnung wird die allgemeine Schalengleichung für die Verschiebungen in beliebigen, gekrümmten Koordinaten angegeben. Aus der allgemeinen Gleichung werden diejenigen für tiefe und flache Schalen hergeleitet. Die Lösungen wurden mit Hilfe der Endlichen-Differenzen-Näherung erhalten. Für flache Schalen sind Endliche-Differenzen-Ausdrücke für verschiedene Randbedingungen hergeleitet, und numerische Berechnungen sind für Schalen durchgeführt worden, die einerseits senkrecht unterstützte Randträger konstanter oder linear veränderlicher Steifigkeit und andererseits elastische Randträger, die nur in den Ecken aufgelegt sind, aufweisen. Um die Konvergenz zu prüfen, ist der Grundriß jeweils in 6×6, 8×8 und 12×12 Maschen aufgeteilt worden.