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Autor: Altmann, Wolf / Young, D.H.

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Stresses in Thin Conical Shells

Tensions dans les coques minces coniques

Spannungen in dünnen Kegeln

WOLF ALTMAN
Professor of Structural Engineering,
I.T.A., Sao Paulo, Brasil

D. H. YOUNG
Professor of Civil Engineering, Stanford
University, California

1. Introduction

This paper deals with the problem of stresses and deformations in thin conical shells subjected to axially symmetric edge loadings.

The shell material is assumed to be isotropic and to obey Hooke's law. In addition, the following assumptions are made:

1. The displacements are small in comparison with the thickness h of the shell and $h \ll R$, where R is a representative shell radius.

2. Straight fibers normal to the middle surface before deformation remain so after deformation and do not change their lengths.

3. Normal stresses acting on planes parallel to the middle surface may be neglected in comparison with the other stresses. The last two assumptions are due to Kirchhoff and they are equivalent to reducing the problem of deformation of the shell to that of deformation of its middle surface.

4. Finally, tangential displacements u and v will be neglected in comparison with normal displacements w in the formulae for change of curvature and twist of a shell element. This last assumption is justifiable if the bending stresses are of the same order of magnitude or less than the membrane stresses [1].

2. Basic Equations

Referring to Fig. 1, the cone is defined by its angle α and any point in the middle surface is located by the coordinates θ and s, where s replaces the usual coordinate φ for a shell of revolution since, in the case of a cone, $\varphi = \alpha$, is constant. The displacement of a point on the middle surface is defined by its orthogonal components u, v and w as shown.

The relations between middle surface strains ϵ_θ, ϵ_s, $\epsilon_{s\theta}$ and corresponding displacements u, v, w are given by:

$$\epsilon_s = \frac{\partial v}{\partial s}, \tag{1a}$$

$$\epsilon_\theta s \cos\alpha = \frac{\partial u}{\partial \theta} + v\cos\alpha + w \sin\alpha, \tag{1b}$$

$$2\,\epsilon_{s\theta} = \frac{\partial u}{\partial s} - \frac{u}{s} + \frac{1}{s\cos\alpha}\frac{\partial v}{\partial \theta}. \tag{1c}$$

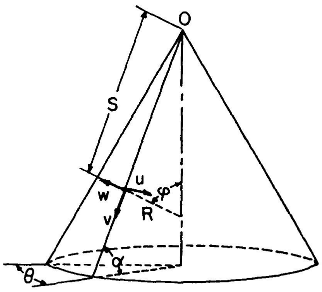

Fig. 1.

Neglecting u and v displacements in comparison with w displacements in the formulae for change in curvature κ_s, κ_θ and twist $\kappa_{s\theta}$, we can write

$$\kappa_s = -\frac{\partial^2 w}{\partial s^2}, \tag{1d}$$

$$\kappa_\theta = -\frac{1}{s^2\cos^2\alpha}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{s}\frac{\partial w}{\partial s}, \tag{1e}$$

$$\kappa_{s\theta} = -\frac{\partial}{\partial s}\left(\frac{1}{s\cos\alpha}\frac{\partial w}{\partial \theta}\right). \tag{1f}$$

As a result of the adoption of the Kirchhoff hypotheses and neglecting the small quantities of the same order of magnitude or less than h/R in comparison

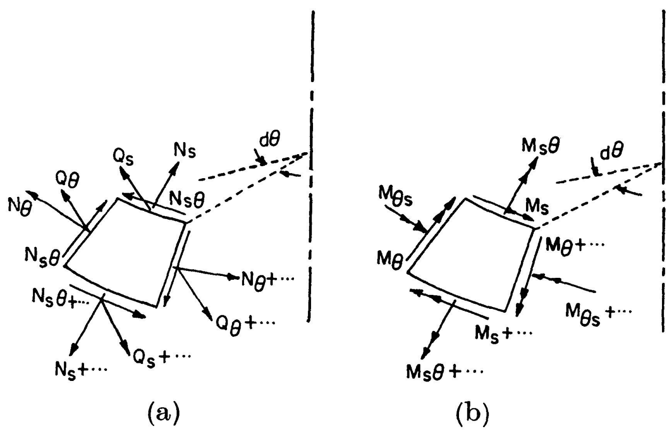

Fig. 2.

with unity, the force resultants N_s, N_θ, $N_{s\theta}$ (Fig. 2a) and moment resultants M_s, M_θ, $M_{s\theta}$ (Fig. 2b) can be written as follows:

$$N_s = \frac{E h}{1-\nu^2}(\epsilon_s + \nu\,\epsilon_\theta), \tag{2a}$$

$$N_\theta = \frac{E h}{1-\nu^2}(\epsilon_\theta + \nu\,\epsilon_s), \tag{2b}$$

$$N_{s\theta} = N_{\theta s} = \frac{E h}{1+\nu}\epsilon_{s\theta}, \tag{2c}$$

$$M_s = -D(\kappa_s + \nu\,\kappa_\theta), \tag{2d}$$

$$M_\theta = -D(\kappa_\theta + \nu\,\kappa_s), \tag{2e}$$

$$M_{s\theta} = M_{\theta s} = -D(1-\nu)\,\kappa_{s\theta}, \tag{2f}$$

where
$$D = \frac{E h^3}{12(1-\nu^2)}.$$

In these expressions h is the thickness of the shell, assumed to be constant, and E and ν are the elastic constants for the material.

From static equilibrium of the forces and moments acting on a small element defined by ds and $d\theta$ (Figs. 2a and 2b), we can write:

$$\frac{\partial(N_s s)}{\partial s} + \frac{1}{\cos\alpha}\frac{\partial N_{\theta s}}{\partial\theta} - N_\theta = 0, \tag{3a}$$

$$\frac{\partial(N_{s\theta} s)}{\partial s} + \frac{1}{\cos\alpha}\frac{\partial N_\theta}{\partial\theta} + N_{\theta s} - Q_\theta\tan\alpha = 0, \tag{3b}$$

$$\frac{\partial(Q_s s)}{\partial s} + \frac{1}{\cos\alpha}\frac{\partial Q_\theta}{\partial\theta} + N_\theta\tan\alpha = 0, \tag{3c}$$

$$\frac{\partial(M_s s)}{\partial s} + \frac{1}{\cos\alpha}\frac{\partial M_{s\theta}}{\partial\theta} - M_\theta - Q_s s = 0, \tag{3d}$$

$$\frac{\partial(M_{s\theta} s)}{\partial s} + \frac{1}{\cos\alpha}\frac{\partial M_\theta}{\partial\theta} + M_{s\theta} - Q_\theta s = 0, \tag{3e}$$

$$M_{s\theta} + (N_{s\theta} - N_{\theta s})\,s\cot\alpha = 0. \tag{3f}$$

With regard to these equations, two comments are in order. First, from (2c), we have $N_{s\theta} = N_{\theta s}$ and this result is incompatible with (3f); therefore, (3f) will simply be ignored. Secondly, we will take into account that $Q_\theta \ll \frac{\partial N_\theta}{\partial\theta}$ and thus neglect Q_θ in the tangential equilibrium Eq. (3b).

Substituting (1d–f) into (2d–f) we obtain

$$M_s = D\left[\frac{\partial^2 w}{\partial s^2} + \nu\left(\frac{\partial^2 w}{\partial\theta^2}\frac{1}{s^2\cos^2\alpha} + \frac{1}{s}\frac{\partial w}{\partial s}\right)\right], \tag{4a}$$

$$M_\theta = D\left[\left(\frac{\partial^2 w}{\partial\theta^2}\frac{1}{s^2\cos^2\alpha} + \frac{1}{s}\frac{\partial w}{\partial s}\right) + \nu\frac{\partial^2 w}{\partial s^2}\right], \tag{4b}$$

$$M_{s\theta} = D(1-\nu)\left[\frac{\partial^2 w}{\partial s\,\partial\theta}\frac{1}{s\cos\alpha} - \frac{\partial w}{\partial\theta}\frac{1}{s^2\cos\alpha}\right]. \tag{4c}$$

Addition of (4a) and (4b) gives

$$\overline{M} = \frac{M_s + M_\theta}{1+\nu} = D\,\nabla^2 w, \tag{5}$$

where
$$\nabla^2(\,) = \frac{\partial^2(\,)}{\partial s^2} + \frac{1}{s}\frac{\partial(\,)}{\partial s} + \frac{1}{s^2\cos^2\alpha}\frac{\partial^2(\,)}{\partial\theta^2}. \tag{5a}$$

Substituting (4a–c) into (3d–e), we have

$$Q_s = \frac{\partial\overline{M}}{\partial s}, \tag{6a}$$

$$Q_\theta = \frac{1}{s\cos\alpha}\frac{\partial\overline{M}}{\partial\theta} \tag{6b}$$

and when these are substituted into (3c), the result is

$$\nabla^2\overline{M} + \frac{N_\theta\tan\alpha}{s} = 0, \tag{7a}$$

or using (5)
$$D\,\nabla^2\nabla^2 w + N_\theta\frac{\tan\alpha}{s} = 0. \tag{7b}$$

The force resultants $N_s, N_\theta, N_{s\theta}$ can be expressed as functions of an auxiliary function Φ introduced by VLASOV [1] as follows:

$$N_s = -\frac{1}{s^2\cos^2\alpha}\frac{\partial^2\Phi}{\partial\theta^2} - \frac{1}{s}\frac{\partial\Phi}{\partial s}, \tag{8a}$$

$$N_\theta = -\frac{\partial^2\Phi}{\partial s^2}, \tag{8b}$$

$$N_{\theta s} = \frac{\partial}{\partial s}\left(\frac{1}{s\cos\alpha}\frac{\partial\Phi}{\partial\theta}\right). \tag{8c}$$

Substituting (8a–c) into the first two equations of equilibrium (3a–b), we can see that they are identically satisfied. Now, substituting (8b) into (7b) which actually is the third equation of equilibrium in terms of w, we obtain:

$$D\,\nabla^2\nabla^2 w - \frac{\tan\alpha}{s}\frac{\partial^2\Phi}{\partial s^2} = 0. \tag{9}$$

A second relation between the same functions w and Φ can be obtained by combining relations (1a–c) as follows:

$$\cos\alpha\frac{\partial}{\partial s}\left(s^2\frac{\partial\epsilon_\theta}{\partial s}\right) - 2\frac{\partial}{\partial s}\left(s\frac{\partial\epsilon_{s\theta}}{\partial\theta}\right) - s\cos\alpha\frac{\partial\epsilon_s}{\partial s} + \frac{1}{\cos\alpha}\frac{\partial^2\epsilon_s}{\partial\theta^2} = s\sin\alpha\frac{\partial^2 w}{\partial s^2}. \tag{10}$$

Solving (2a–c) for $\epsilon_\theta, \epsilon_s, \epsilon_{s\theta}$, substituting these into (10), and then eliminating the partial derivatives $\frac{\partial N_{s\theta}}{\partial\theta}, \frac{\partial^2 N_{s\theta}}{\partial\theta\,\partial s}$ from this result with the help of Eq. (3a), we have

$$s^2 \nabla^2 N_s + \frac{\partial^2 (s^2 N_s)}{\partial s^2}(1+\nu) + s^2 \frac{\partial^2 N_\theta}{\partial s^2} - \nu s \frac{\partial N_\theta}{\partial s} - \frac{\nu}{\cos^2\alpha} \frac{\partial^2 N_\theta}{\partial \theta^2}$$
$$-2(1+\nu) N_\theta = s \frac{\partial^2 w}{\partial s^2} \tan\alpha\, E h. \qquad (11)$$

Next, eliminating $\frac{\partial N_{\theta s}}{\partial \theta}$ and $\frac{\partial (N_{s\theta})}{\partial s}$ from (3a–b), we obtain

$$-\frac{\partial^2 (s^2 N_s)}{\partial s^2} + s \frac{\partial N_\theta}{\partial s} + \frac{1}{\cos^2\alpha} \frac{\partial^2 N_\theta}{\partial \theta^2} + 2 N_\theta = 0. \qquad (12)$$

Finally, multiplying (12) by $(1+\nu)$ and adding to (11), we have

$$\frac{1}{E h} \nabla^2 (N_s + N_\theta) = \frac{\tan\alpha}{s} \frac{\partial^2 w}{\partial s^2}, \qquad (13)$$

or using (8a—b)
$$\frac{1}{E h} \nabla^2 \nabla^2 \Phi + \frac{\tan\alpha}{s} \frac{\partial^2 w}{\partial s^2} = 0. \qquad (14)$$

Eqs. (9) and (14) are two simultaneous partial differential equations which describe the problem of a circular conical shell of constant wall thickness subjected to edge loads. They represent an eighth order system of equations, but we can combine them so as to obtain a fourth order partial differential equation in complex form. Indeed, (9) and (14) can be rewritten as follows:

$$\sqrt{\frac{h^2}{12(1-\nu^2)}} E h \nabla^2 \nabla^2 w - \sqrt{\frac{12(1-\nu^2)}{h^2}} \frac{\tan\alpha}{s} \frac{\partial^2 \Phi}{\partial s^2} = 0, \qquad (9a)$$

$$i \nabla^2 \nabla^2 \Phi + i \sqrt{\frac{12(1-\nu^2)}{h^2}} \sqrt{\frac{h^2}{12(1-\nu^2)}} E h \frac{\tan\alpha}{s} \frac{\partial^2 w}{\partial s^2} = 0, \qquad (14a)$$

where $i = \sqrt{-1}$. Adding, we have

$$\nabla^2 \nabla^2 \left(\sqrt{\frac{h^2}{12(1-\nu^2)}} E h w + i \Phi \right)$$
$$+ i \sqrt{\frac{12(1-\nu^2)}{h^2}} \frac{\tan\alpha}{s} \frac{\partial^2}{\partial s^2} \left(\sqrt{\frac{h^2}{12(1-\nu^2)}} E h w + i \Phi \right) = 0. \qquad (15)$$

Introducing the notations

$$c = \sqrt{\frac{12(1-\nu^2)}{h^2}}, \qquad (16a)$$

$$H = \frac{E h}{c}, \qquad (16b)$$

$$Z = H w + i \Phi \qquad (16c)$$

we have
$$\nabla^2 \nabla^2 Z + i c \frac{\tan\alpha}{s} \frac{\partial^2 Z}{\partial s^2} = 0. \qquad (17)$$

Eq. (17) is the basic differential equation governing the stresses and deformations of circular conical shells of constant thickness under edge loads.

3. Solution for the Axisymmetric Case

In order to have a separation of variables in (17), we will look for solutions of the form

$$Z = Z_j \cos j\,\theta , \tag{18}$$

where Z_j is a function of s only and j is an integer. Substitution of (18) into (17) leads to the ordinary differential equation

$$\begin{aligned} s^4 \frac{d^4 Z_j}{ds^4} + 2\,s^3 \frac{d^3 Z_j}{ds^3} + \left[s^2 \left(-1 - \frac{2\,j^2}{\cos^2\alpha} \right) + s^3\, i\, c \tan\alpha \right] \frac{d^2 Z_j}{ds^2} \\ + s \left(1 + \frac{2\,j^2}{\cos^2\alpha} \right) \frac{d\,Z_j}{ds} + \left(- \frac{4\,j^2}{\cos^2\alpha} + \frac{j^4}{\cos^4\alpha} \right) Z_j = 0, \end{aligned} \tag{19}$$

where

$$Z_j = H\,w_j + i\,\Phi_j .$$

For the axisymmetric case ($j=0$), Eq. (19) reduces to

$$s^4 \frac{d^4 Z}{ds^4} + 2\,s^3 \frac{d^3 Z}{ds^3} + (-s^2 + s^3\, i\, c \tan\alpha) \frac{d^2 Z}{ds^2} + s \frac{dZ}{ds} = 0 . \tag{19a}$$

We can see that $s=0$ is a regular singular point of the differential Eq. (19a). Using Frobenius' method, we assume a solution of the form

$$Z = s^{\mu} (A_0 + A_1 s + A_2 s^2 + \cdots), \tag{20}$$

where

$$A_n = a_n + i\,b_n . \tag{20a}$$

The complex coefficients A_n and the exponent μ are constants to be determined.

Substituting the assumed solution (20) into the differential Eq. (19a), and arranging the result according to ascending powers of s, we see that the coefficients of all powers of s must vanish independently in order to have the differential equation satisfied. The vanishing of the coefficient of the lowest power of s gives the indicial equation

$$\mu^2 (\mu^2 - 4\,\mu + 4) = 0 \tag{21}$$

whose roots are

$$\mu_1 = \mu_2 = 2; \qquad \mu_3 = \mu_4 = 0 . \tag{21a}$$

The vanishing of the coefficient of the nth power of s leads to the following recurrence relation:

$$A_n \{(\mu + n)^2 [\mu + (n-2)]\} = -[\mu + (n-1)]\, i\, c \tan\alpha\, A_{n-1} . \tag{22}$$

The four exponents of the differential equation together with the recurrence relation will provide four independent power series solutions if the exponents are different and no pair of them differs by an integer. Since in our case

$\mu_1 = \mu_2 = 2$ and $\mu_3 = \mu_4 = 0$, the recurrence relation (22) will give only two independent solutions, but two other independent solutions can be obtained as follows [2]:

$$Z_3(s) = \left.\frac{\partial Z(\mu, s)}{\partial \mu}\right|_{\mu=0} \tag{23}$$

and

$$Z_4(s) = \left.\frac{\partial Z(\mu, s)}{\partial \mu}\right|_{\mu=2}, \tag{24}$$

where $Z(\mu, s)$ is given by (20) and the coefficients A_n are expressed in terms of A_0 and μ by the recurrence formula (22).

Following the above procedure, after separating real and imaginary parts, we can write the general solution in the form:

$$\begin{aligned} Hw = {} & c_1 \sum_{n=0,2}^{\infty} a_n(2)\, s^{n+2} - c_2 \sum_{n=1,3}^{\infty} b_n(2)\, s^{n+2} \\ & + d_1 \Big[\ln s \sum_{n=0,2}^{\infty} a_n(2)\, s^{n+2} + \sum_{n=2,4}^{\infty} a'_n(2)\, s^{n+2}\Big] \\ & - d_2 \Big[\ln s \sum_{n=1,3}^{\infty} b_n(2)\, s^{n+2} + \sum_{n=1,3}^{\infty} b'_n(2)\, s^{n+2}\Big] \\ & + e_1 \sum_{n=0,2}^{\infty} a_n(0)\, s^{n} - e_2 \sum_{n=1,3}^{\infty} b_n(0)\, s^{n} \\ & + f_1 \Big[\ln s \sum_{n=0,2}^{\infty} a_n(0)\, s^{n} + \sum_{n=2,4}^{\infty} a'_n(0)\, s^{n}\Big] \\ & - f_2 \Big[\ln s \sum_{n=1,3}^{\infty} b_n(0)\, s^{n} + \sum_{n=1,3}^{\infty} b'_n(0)\, s^{n}\Big] \end{aligned} \tag{25}$$

and

$$\begin{aligned} \Phi = {} & c_2 \sum_{n=0,2}^{\infty} a_n(2)\, s^{n+2} + c_1 \sum_{n=1,3}^{\infty} b_n(2)\, s^{n+2} \\ & + d_2 \Big[\ln s \sum_{n=0,2}^{\infty} a_n(2)\, s^{n+2} + \sum_{n=2,4}^{\infty} a'_n(2)\, s^{n+2}\Big] \\ & + d_1 \Big[\ln s \sum_{n=1,3}^{\infty} b_n(2)\, s^{n+2} + \sum_{n=1,3}^{\infty} b'_n(2)\, s^{n+2}\Big] \\ & + e_2 \sum_{n=0,2}^{\infty} a_n(0)\, s^{n} + e_1 \sum_{n=1,3}^{\infty} b_n(0)\, s^{n} \\ & + f_2 \Big[\ln s \sum_{n=0,2}^{\infty} a_n(0)\, s^{n} + \sum_{n=2,4}^{\infty} a'_n(0)\, s_{n}\Big] \\ & + f_1 \Big[\ln s \sum_{n=1,3}^{\infty} b_n(0)\, s^{n} + \sum_{n=1,3}^{\infty} b'_n(0)\, s^{n}\Big], \end{aligned} \tag{26}$$

where $c_1, c_2, d_1, d_2, e_1, e_2, f_1, f_2$ are arbitrary constants and the coefficients of the powers of s are given in Table 1.

We can observe that for $\alpha = 0$, we have $a_0(0) = a_0(2) = 1$, and all the other coefficients are zero. Therefore, Eq. (25) takes the form

$$Hw = c_1 s^2 + d_1 s^2 \ln s + e_1 + f_1 \ln s, \tag{27}$$

Table 1

$\mu = 2$	$\mu = 0$
$a_0(2) = 1$	$a_0(0) = 1$
$a_2(2) = -\frac{3c^2\tan^2\alpha}{4^2\cdot 3^2}$	$a_2(0) = \frac{c^2\tan^2\alpha}{2^2\cdot 1^2}$
$a_4(2) = \frac{5c^4\tan^4\alpha}{6^2\cdot 5^2\cdot 4^2\cdot 3^2}$	$a_4(0) = -\frac{3c^4\tan^4\alpha}{4^2\cdot 3^2\cdot 2^2\cdot 1^2}$
$\vdots$	$\vdots$
$a_n(2) = (-1)^{n/2}\frac{[(n-1)+2]\,c^n\tan^n\alpha}{[(n+2)(n+1)\ldots 4\cdot 3]^2\cdot 1}$	$a_n(0) = (-1)^{n/2}\frac{(n-1)\,c^n\tan^n\alpha}{[n(n-1)\ldots 2\cdot 1]^2(-1)}$
$b_1(2) = -\frac{2c\tan\alpha}{3^2}$	$b_1(0) = 0$
$b_3(2) = \frac{4c^3\tan^3\alpha}{5^2\cdot 4^2\cdot 3^2}$	$b_3(0) = -\frac{2c^3\tan^3\alpha}{3^2\cdot 2^2\cdot 1^2}$
$b_5(2) = -\frac{6c^5\tan^5\alpha}{7^2\cdot 6^2\cdot 5^2\cdot 4^2\cdot 3^2}$	$b_5(0) = \frac{4c^5\tan^5\alpha}{5^2\cdot 4^2\cdot 3^2\cdot 2^2\cdot 1^2}$
$\vdots$	$\vdots$
$b_n(2) = (-1)^{(n+1)/2} \times \frac{[(n-1)+2]\,c^n\tan^n\alpha}{[(n+2)(n+1)\ldots 4\cdot 3]^2\cdot 1}$	$b_n(0) = (-1)^{(n+1)/2} \times \frac{(n-1)\,c^n\tan^n\alpha}{[n(n-1)\ldots 2\cdot 1]^2(-1)}$
$a'_0(2) = 0$	$a'_0(0) = 0$
$a'_2(2) = a_2(2)\left[\frac{1}{3} - 2\left(\frac{1}{3}+\frac{1}{4}\right) - 1\right]$	$a'_2(0) = a_2(0)\left[1 - 2\left(1+\frac{1}{2}\right) + 1\right]$
$a'_4(2) = a_4(2)\left[\frac{1}{5} - 2\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right) - 1\right]$	$a'_4(0) = a_4(0)\left[\frac{1}{3} - 2\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) + 1\right]$
$\vdots$	$\vdots$
$a'_n(2) = a_n(2) \times \left[\frac{1}{(n-1)+2} - 2\sum_{k=1}^{n}\frac{1}{k+2} - 1\right]$	$a'_n(0) = a_n(0) \times \left[\frac{1}{n-1} - 2\sum_{k=1}^{n}\frac{1}{k} + 1\right]$
$b'_1(2) = b_1(2)\left[\frac{1}{2} - 2\left(\frac{1}{3}\right) - 1\right]$	$b'_1(0) = c\tan\alpha$
$b'_3(2) = b_3(2)\left[\frac{1}{4} - 2\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right) - 1\right]$	$b'_3(0) = b_3(0)\left[\frac{1}{2} - 2\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right) + 1\right]$
$b'_5(2) = b_5(2) \times \left[\frac{1}{6} - 2\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right) - 1\right]$	$b'_5(0) = b_5(0) \times \left[\frac{1}{4} - 2\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right) + 1\right]$
$\vdots$	$\vdots$
$b'_n(2) = b_n(2) \times \left[\frac{1}{(n-1)+2} - 2\sum_{k=1}^{n}\frac{1}{k+2} - 1\right]$	$b'_n(0) = b_n(0) \times \left[\frac{1}{n-1} - 2\sum_{k=1}^{n}\frac{1}{k} + 1\right]$

which is the known plate solution.

The series solutions (25) and (26) can be written in terms of Thomson functions as follows [3]:

$$\begin{aligned} H w = {} & A_1\left(\operatorname{ber} x-\frac{x}{2}\operatorname{ber}' x\right)+A_2\left(\operatorname{bei} x-\frac{x}{2}\operatorname{bei}' x\right) \\ & +B_1\left(\operatorname{ker} x-\frac{x}{2}\operatorname{ker}' x\right)+B_2\left(\operatorname{kei} x-\frac{x}{2}\operatorname{kei}' x\right)+E_1 \\ & +\bar{d}_1[\ln s]-\bar{d}_2[b_1'(0)\, s] \end{aligned} \tag{28}$$

and

$$\begin{aligned} \Phi = {} & A_2\left(\operatorname{ber} x-\frac{x}{2}\operatorname{ber}' x\right)-A_1\left(\operatorname{bei} x-\frac{x}{2}\operatorname{bei}' x\right) \\ & +B_2\left(\operatorname{ker} x-\frac{x}{2}\operatorname{ker}' x\right)-B_1\left(\operatorname{kei} x-\frac{x}{2}\operatorname{kei}' x\right)+E_2 \\ & +\bar{d}_2[\ln s]+\bar{d}_1[b_1'(0)\, s], \end{aligned} \tag{29}$$

where $x = 2\sqrt{s c \tan\alpha}$.

Before proceeding with any applications of these equations, it can be shown that the constant $\bar{d}_2 = 0$. For this purpose, we rederive Eq. (14) specifically for the axisymmetric case.

Eqs. (1a–b) can be written as

$$\epsilon_s = \frac{dv}{ds}, \qquad \epsilon_\theta s = v + w\tan\alpha,$$

which combine to give

$$\frac{d(\epsilon_\theta s)}{ds} - \epsilon_s = \tan\alpha\frac{dw}{ds}, \tag{30}$$

wherein

$$\epsilon_\theta = \frac{1}{Eh}(N_\theta - \nu N_s); \qquad \epsilon_s = \frac{1}{Eh}(N_s - \nu N_\theta). \tag{31}$$

After substitution of (31) into (30), we have

$$\frac{1}{Eh}\left(\frac{d(sN_\theta)}{ds} - N_s\right) = \tan\alpha\frac{dw}{ds}. \tag{32}$$

Eq. (32) can also be written in terms of Φ as follows:

$$\frac{1}{Eh}\left(s\frac{d^3\Phi}{ds^3} + \frac{d^2\Phi}{ds^2} - \frac{1}{s}\frac{d\Phi}{ds}\right) = -\tan\alpha\frac{dw}{ds}$$

or

$$s\frac{d}{ds}\left[\frac{1}{s}\frac{d}{ds}\left(s\frac{d\Phi}{ds}\right)\right] = -Eh\tan\alpha\frac{dw}{ds}. \tag{33}$$

Eq. (33) (axisymmetric case) corresponds to Eq. (14) (general case).

Alternatively, we can particularize Eq. (14) to the axisymmetric case by dropping all derivatives with respect to θ. In this way it can be written as

$$\frac{d}{ds}\left\{s\frac{d}{ds}\left[\frac{1}{s}\frac{d}{ds}\left(s\frac{d\Phi}{ds}\right)\right]\right\} = -Eh\tan\alpha\frac{d^2 w}{ds^2}. \tag{34}$$

Integrating (34), we obtain

$$s\frac{d}{ds}\left[\frac{1}{s}\frac{d}{ds}\left(s\frac{d\Phi}{ds}\right)\right] = -E\,h\tan\alpha\frac{dw}{ds}+c\,. \tag{35}$$

Since expressions (35) and (33) have to be identical, it follows that the constant c must be zero. Then substituting the solutions (28) and (29) into Eq. (33) and performing the indicated differentiations, it will be found that $\bar{d}_2$ must be zero in order to have the equation satisfied.

4. Stress Resultants

For the axisymmetric case, the stress resultants (8), (4), (6) can be written in the form

$$N_s = -\frac{1}{s}\frac{d\Phi}{ds}, \tag{36a}$$

$$N_\theta = -\frac{d^2\Phi}{ds^2}, \tag{36b}$$

$$M_s = \frac{E\,h^3}{12\,(1-\nu^2)}\left[\frac{d^2 w}{ds^2}+\frac{\nu}{s}\frac{dw}{ds}\right], \tag{36c}$$

$$M_\theta = \frac{E\,h^3}{12\,(1-\nu^2)}\left[\frac{1}{s}\frac{dw}{ds}+\nu\frac{d^2 w}{ds^2}\right], \tag{36d}$$

$$Q_s = \frac{d\bar{M}}{ds}, \tag{36e}$$

where $\bar{M}$ is given by (5).

Also, the rotation of an element of a meridian, during deformation, will be

$$\chi = \frac{dw}{ds}. \tag{36f}$$

We can show that part of the solutions corresponding to (28) and (29) or more precisely, the A- and B-terms of these solutions, when substituted into (36a–f) will give exactly the same stress resultants and rotation as shown in ([4], p. 373). In fact, carrying out the above substitution, we have

$$\begin{aligned} N_s = -\frac{\cot\alpha}{s}\Big[&\bar{A}_1\left(\operatorname{ber}x-\frac{2}{x}\operatorname{bei}'x\right)+\bar{A}_2\left(\operatorname{bei}x+\frac{2}{x}\operatorname{ber}'x\right)\\ &+\bar{B}_1\left(\operatorname{ker}x-\frac{2}{x}\operatorname{kei}'x\right)+\bar{B}_2\left(\operatorname{kei}x+\frac{2}{x}\operatorname{ker}'x\right)\Big], \end{aligned} \tag{37a}$$

$$N_\theta = -\frac{\cot\alpha}{2s}\left[\bar{A}_1\left(x\,\mathrm{ber}'x - 2\,\mathrm{ber}\,x + \frac{4}{x}\mathrm{bei}'x\right) + \bar{A}_2\left(x\,\mathrm{bei}'x - 2\,\mathrm{bei}\,x - \frac{4}{x}\mathrm{ber}'x\right) + \bar{B}_1\left(x\,\mathrm{ker}'x - 2\,\mathrm{ker}\,x + \frac{4}{x}\mathrm{kei}'x\right) + \bar{B}_2\left(x\,\mathrm{kei}'x - 2\,\mathrm{kei}\,x - \frac{4}{x}\mathrm{ker}'x\right)\right], \tag{37b}$$

$$M_s = \frac{2}{x^2}\left\{\bar{A}_1\left[x\,\mathrm{bei}'x - 2(1-\nu)\left(\mathrm{bei}\,x + \frac{2}{x}\mathrm{ber}'x\right)\right] - \bar{A}_2\left[x\,\mathrm{ber}'x - 2(1-\nu)\left(\mathrm{ber}\,x - \frac{2}{x}\mathrm{bei}'x\right)\right] + \bar{B}_1\left[x\,\mathrm{kei}'x - 2(1-\nu)\left(\mathrm{kei}\,x + \frac{2}{x}\mathrm{ker}'x\right)\right] - \bar{B}_2\left[x\,\mathrm{ker}'x - 2(1-\nu)\left(\mathrm{ker}\,x - \frac{2}{x}\mathrm{kei}'x\right)\right]\right\}, \tag{37c}$$

$$M_\theta = \frac{2}{x^2}\left\{\bar{A}_1\left[\nu x\,\mathrm{bei}'x + 2(1-\nu)\left(\mathrm{bei}\,x + \frac{2}{x}\mathrm{ber}'x\right)\right] - \bar{A}_2\left[\nu x\,\mathrm{ber}'x + 2(1-\nu)\left(\mathrm{ber}\,x - \frac{2}{x}\mathrm{bei}'x\right)\right] + \bar{B}_1\left[\nu x\,\mathrm{kei}'x + 2(1-\nu)\left(\mathrm{kei}\,x + \frac{2}{x}\mathrm{ker}'x\right)\right] - \bar{B}_2\left[\nu x\,\mathrm{ker}'x + 2(1-\nu)\left(\mathrm{ker}\,x - \frac{2}{x}\mathrm{kei}'x\right)\right]\right\}, \tag{37d}$$

$$Q_s = -N_s\tan\alpha, \tag{37e}$$

$$\chi = \frac{\sqrt{12(1-\nu^2)}}{E h^2}\cot\alpha\left[\bar{A}_1\left(\mathrm{bei}\,x + \frac{2}{x}\mathrm{ber}'x\right) - \bar{A}_2\left(\mathrm{ber}\,x - \frac{2}{x}\mathrm{bei}'x\right) + \bar{B}_1\left(\mathrm{kei}\,x + \frac{2}{x}\mathrm{ker}'x\right) - \bar{B}_2\left(\mathrm{ker}\,x - \frac{2}{x}\mathrm{kei}'x\right)\right], \tag{37f}$$

where $\bar{A}_1 = c\tan^2\alpha A_1;\quad \bar{A}_2 = c\tan^2\alpha A_2;$

$\bar{B}_1 = c\tan^2\alpha B_1;\quad \bar{B}_2 = c\tan^2\alpha B_2.$

In these expressions the $\bar{A}$-terms are regular functions, and describe stresses produced by edge loads in a complete cone (Fig. 3a). The $\bar{B}$-terms have a singularity at $x=0$ and describe stresses caused by loads applied to the upper edge of a truncated cone (Fig. 3b).

Eq. (37e) can be interpreted as a condition of equilibrium for the part of the shell above any parallel circle. Indeed, multiplying (37e) by $2\pi s\cos^2\alpha$, we obtain

$$(N_s \sin\alpha + Q_s \cos\alpha)\, 2\pi s \cos\alpha = 0. \tag{38}$$

This equation expresses the fact that the vertical resultant of all forces transmitted through a parallel circle must be zero. In summary, we can see that Eq. (38) is a consequence of neglecting the $\bar{d}$-terms in Eqs. (28) and (29) and therefore characterizes conical shells with edge loading as shown in Figs. 3a, b. The remainig part of solutions (28) and (29), i. e., the $\bar{d}$-terms, have a singularity at $x=0$ and do not descibe the same kind of stress resultants as those due to the above edge loadings.

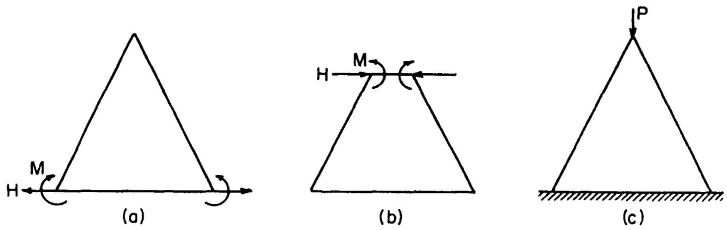

Fig. 3.

5. Solution for a Finite Conical Shell with a Concentrated Load at the Apex

For this case (Fig. 3c), the following conditions can be used in order to determine the constants*) B_1, B_2, $\bar{d}_1$, $\bar{d}_2$ in Eqs. (28) and (29):

a) At $s = 0$; $\chi = 0$,

b) $\lim\limits_{s\to 0} (N_s \sin\alpha + Q_s \cos\alpha)\, 2\pi s \cos\alpha = -P$.

Eqs. (28) and (29) can be written in the form

$$\begin{aligned} c\tan^2\alpha\, H w = {} & \bar{A}_1\left(\operatorname{ber} x - \frac{x}{2}\operatorname{ber}' x\right) + \bar{A}_2\left(\operatorname{bei} x - \frac{x}{2}\operatorname{bei}' x\right) \\ & + \bar{B}_1\left(\operatorname{ker} x - \frac{x}{2}\operatorname{ker}' x\right) + \bar{B}_2\left(\operatorname{kei} x - \frac{x}{2}\operatorname{kei}' x\right) \\ & + \bar{D}_1\left[2\ln x - \ln(4c\tan\alpha)\right] \end{aligned} \tag{39}$$

and

$$\begin{aligned} c\tan^2\alpha\, \Phi = {} & \bar{A}_2\left(\operatorname{ber} x - \frac{x}{2}\operatorname{ber}' x\right) - \bar{A}_1\left(\operatorname{bei} x - \frac{x}{2}\operatorname{bei}' x\right) \\ & + \bar{B}_2\left(\operatorname{ker} x - \frac{x}{2}\operatorname{ker}' x\right) - \bar{B}_1\left(\operatorname{kei} x - \frac{x}{2}\operatorname{kei}' x\right) + \bar{D}_1\left(\frac{x^2}{4}\right). \end{aligned} \tag{40}$$

In these expressions we have omitted the E_1 and E_2 solutions. For χ and N_s, we have

*) It has already been shown that $\bar{d}_2 = 0$, see p. 10.

$$\chi = \frac{\sqrt{12(1-\nu^2)}}{E h^2} \cot\alpha \left\{ \bar{A}_1 \left(\operatorname{bei} x + \frac{2}{x} \operatorname{ber}' x \right) - \bar{A}_2 \left(\operatorname{ber} x - \frac{2}{x} \operatorname{bei}' x \right) \right.$$
$$\left. + \bar{B}_1 \left(\operatorname{kei} x + \frac{2}{x} \operatorname{ker}' x \right) - \bar{B}_2 \left(\operatorname{ker} x - \frac{2}{x} \operatorname{kei}' x \right) + \bar{D}_1 \left(\frac{4}{x^2} \right) \right\}, \tag{41}$$

$$N_s = -\frac{\cot\alpha}{s} \left\{ \bar{A}_1 \left(\operatorname{ber} x - \frac{2}{x} \operatorname{bei}' x \right) + \bar{A}_2 \left(\operatorname{bei} x + \frac{2}{x} \operatorname{ber}' x \right) \right.$$
$$\left. + \bar{B}_1 \left(\operatorname{ker} x - \frac{2}{x} \operatorname{kei}' x \right) + \bar{B}_2 \left(\operatorname{kei} x + \frac{2}{x} \operatorname{ker}' x \right) + \bar{D}_1 \right\}. \tag{42}$$

Now, substituting the series expansion of Thomson functions [5] into condition a), we find

$$\bar{B}_1 = 2\bar{D}_1 \quad \text{and} \quad \bar{B}_2 = 0. \tag{43}$$

Before we will make use of condition b), we want to show that $\lim_{s\to 0} N_s s = 0$. In fact,

$$\lim_{s\to 0} N_s s = -\lim \left[\bar{A}_1 \left(\operatorname{ber} x - \frac{2}{x} \operatorname{bei}' x \right) + \bar{A}_2 \left(\operatorname{bei} x + \frac{2}{x} \operatorname{ber}' x \right) \right.$$
$$\left. + \bar{B}_1 \left(\operatorname{ker} x - \frac{2}{x} \operatorname{kei}' x \right) + \bar{D}_1 \right] \cot\alpha, \tag{44}$$
$$= -[\bar{B}_1(-\tfrac{1}{2}) + \bar{D}_1] \cot\alpha = 0.$$

Condition b) can now be written in the form

$$\lim_{s\to 0} (Q_s s)\, 2\pi \cos^2\alpha = -P, \tag{45}$$

where

$$Q_s = \frac{1}{s} \left[\bar{A}_1 \left(\operatorname{ber} x - \frac{2}{x} \operatorname{bei}' x \right) + \bar{A}_2 \left(\operatorname{bei} x + \frac{2}{x} \operatorname{ber}' x \right) + \bar{B}_1 \left(\operatorname{ker} x - \frac{2}{x} \operatorname{kei}' x \right) \right]. \tag{46}$$

Therefore,

$$\lim_{s\to 0} \left[\bar{A}_1 \left(\operatorname{ber} x - \frac{2}{x} \operatorname{bei}' x \right) + \bar{A}_2 \left(\operatorname{bei} x + \frac{2}{x} \operatorname{ber}' x \right) \right.$$
$$\left. + \bar{B}_1 \left(\operatorname{ker} x - \frac{2}{x} \operatorname{kei}' x \right) \right] 2\pi \cos^2\alpha \tag{47}$$
$$= \bar{B}_1 (-\tfrac{1}{2})\, 2\pi \cos^2\alpha = -P,$$

or

$$\bar{B}_1 = \frac{P}{\pi \cos^2\alpha} \tag{48}$$

and then from (43) we have

$$\bar{D}_1 = \frac{P}{2\pi \cos^2\alpha}. \tag{49}$$

The final results of the bending analysis for this case can now be written as follows:

$$w = \frac{P}{E\,h\,\pi \sin^2\alpha}\left[\ker x - \frac{x}{2}\ker' x + \frac{1}{2}\ln s\right] + \frac{1}{\tan^2\alpha\,E\,h}\left[\bar{A}_1\left(\operatorname{ber} x - \frac{x}{2}\operatorname{ber}' x\right) + \bar{A}_2\left(\operatorname{bei} x - \frac{x}{2}\operatorname{bei}' x\right)\right], \tag{50a}$$

$$\chi = \frac{\sqrt{12(1-\nu^2)}}{E\,h^2}\cot\alpha\left\{\frac{P}{\pi\cos^2\alpha}\left(\operatorname{kei} x + \frac{2}{x}\ker' x + \frac{2}{x^2}\right) + \bar{A}_1\left(\operatorname{bei} x + \frac{2}{x}\operatorname{ber}' x\right) - \bar{A}_2\left(\operatorname{ber} x - \frac{2}{x}\operatorname{bei}' x\right)\right\}, \tag{50b}$$

$$N_s = -\frac{\cot\alpha}{s}\left\{\frac{P}{\pi\cos^2\alpha}\left(\ker x - \frac{2}{x}\operatorname{kei}' x + \frac{1}{2}\right) + \bar{A}_1\left(\operatorname{ber} x - \frac{2}{x}\operatorname{bei}' x\right) + \bar{A}_2\left(\operatorname{bei} x + \frac{2}{x}\operatorname{ber}' x\right)\right\}, \tag{50c}$$

$$N_\theta = -\frac{\cot\alpha}{2s}\left\{\frac{P}{\pi\cos^2\alpha}\left(x\ker' x - 2\ker x + \frac{4}{x}\operatorname{kei}' x\right) + \bar{A}_1\left(x\operatorname{ber}' x - 2\operatorname{ber} x + \frac{4}{x}\operatorname{bei}' x\right) + \bar{A}_2\left(x\operatorname{bei}' x - 2\operatorname{bei} x - \frac{4}{x}\operatorname{ber}' x\right)\right\}, \tag{50d}$$

$$M_s = \frac{2}{x^2}\left\{\frac{P}{\pi\cos^2\alpha}\left[x\operatorname{kei}' x - 2(1-\nu)\left(\operatorname{kei} x + \frac{2}{x}\ker' x\right) - \frac{4}{x^2}(1-\nu)\right] + \bar{A}_1\left[x\operatorname{bei}' x - 2(1-\nu)\left(\operatorname{bei} x + \frac{2}{x}\operatorname{ber}' x\right)\right] - \bar{A}_2\left[x\operatorname{ber}' x - 2(1-\nu)\left(\operatorname{ber} x - \frac{2}{x}\operatorname{bei}' x\right)\right]\right\}, \tag{50e}$$

$$M_\theta = \frac{2}{x^2}\left\{\frac{P}{\pi\cos^2\alpha}\left[\nu x\operatorname{kei}' x + 2(1-\nu)\left(\operatorname{kei} x + \frac{2}{x}\ker' x\right) + \frac{4}{x^2}(1-\nu)\right] + \bar{A}_1\left[\nu x\operatorname{bei}' x + 2(1-\nu)\left(\operatorname{bei} x + \frac{2}{x}\operatorname{ber}' x\right)\right] - \bar{A}_2\left[\nu x\operatorname{ber}' x + 2(1-\nu)\left(\operatorname{ber} x - \frac{2}{x}\operatorname{bei}' x\right)\right]\right\}, \tag{50f}$$

$$Q_s = \frac{1}{s}\left\{\frac{P}{\pi\cos^2\alpha}\left(\ker x - \frac{2}{x}\operatorname{kei}' x\right) + \bar{A}_1\left(\operatorname{ber} x - \frac{2}{x}\operatorname{bei}' x\right) + \bar{A}_2\left(\operatorname{bei} x + \frac{2}{x}\operatorname{ber}' x\right)\right\}. \tag{50g}$$

The constants $\bar{A}_1$ and $\bar{A}_2$ are determined from given edge conditions for each particular problem.

We want to observe that the membrane solution for a concentrated load at the apex, namely

$$N_s = -\frac{P}{2\,\pi\, s \sin\alpha\cos\alpha}, \qquad N_\theta = 0,$$

is contained in solutions (50c, d).

Some of the stress resultants represented by Eqs. (50) have extremely large magnitudes in the vicinity of the apex. For example, the graphical representation of Q_s and M_s is shown in Fig. 4.

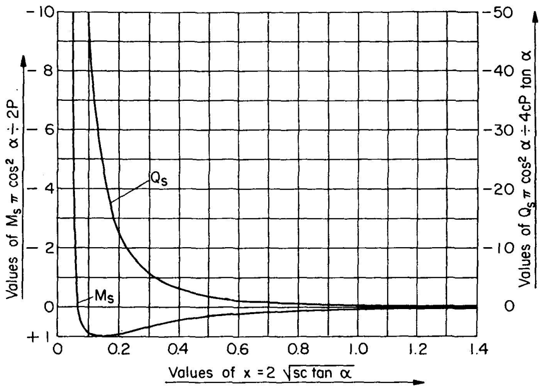

Fig. 4.

6. References

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Summary

The differential equations governing the bending behavior of a thin conical shell subjected to edge loading represent an eighth order system. By the introduction of a complex variable, this system of equations is reduced to a fourth order differential equation in complex form. For axisymmetric loading, this fourth order equation is solved by using the method of Frobenius. Subsequently, the series solution, so obtained, is transformed into a closed form solution in terms of Thompson functions. As an example, a complete set of expressions for all stress resultants is given for the case of a cone subjected to a concentrated load at the apex. It is believed that this solution, including both bending and membrane stresses in the cone, has not previously been shown.

Résumé

Les équations différentielles régissant la flexion d'une coque mince conique soumise à une force agissant sur son bord représentent un système du huitième ordre. En introduisant une variable complexe, ce système est réduit en une équation différentielle complexe du quatrième ordre. Pour des charges symétriques par rapport à un axe, cette équation du quatrième ordre est résolue par la méthode de Frobenius. Ensuite, la solution en séries ainsi obtenue est transformée en une solution de forme fermée en termes de fonction de Thompson. Comme exemple, on a donné l'ensemble des expressions pour toutes les tensions dans le cas du cône soumis à une force concentrée en son sommet. Nous croyons que cette solution englobant à la fois les tensions de flexion et les tensions de membrane n'a pas encore été trouvée dans le cas du cône.

Zusammenfassung

Die Differentialgleichungen, welche das Biegeverhalten dünner Kegel unter Randlasten wiedergeben, führen zu solchen achter Ordnung. Wenn man eine komplexe Unabhängige einführt, läßt sich dieses System auf eine komplexe Form vierter Ordnung abmindern. Sind die Lasten achsialsymmetrisch, führt das Verfahren Frobenius' zur Lösung der Differentialgleichung vierter Ordnung. Hernach kann die so erhaltene Reihenlösung in eine geschlossene Form mit Gliedern der Thompson-Funktion übergeführt werden. Als Beispiel wird ein vollständiger Satz für alle Spannungen angegeben, falls auf der Spitze des Kegels eine Einzellast wirkt. Wir glauben, daß diese Lösung, einschließend Biege- und Membranspannungen im Kegel, bis jetzt noch nie gezeigt ward.