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The Finite Element Method in Application to Plane Stress

La méthode des éléments finis appliquée à des contraintes bi-dimensionnelles

Das Verfahren der Endlichen Elemente in der Anwendung auf ebene Spannungszustände

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General

The differential equations of the theory of elasticity governing the conditions of plane stress are based on statics, continuity of the material and its elasticity, in accordance with the constants E and μ. The assumption of continuity implies that the intermolecular spacing is an infinitesimal of a higher order compared to the dimensions dx and dy of the element analyzed.

With clear realization of these assumptions of the rigorous theory consider a model of a plate made of polygonal cells of repeating pattern, joined to each other at the nodes and possessing such properties as to make the nodes in the model and the plate move identically in conditions of any arbitrary uniform stress. If furthermore the size of mesh is visualized as an infinitesimal of a higher order than dx and dy, the equations of elasticity describing the action of the prototype should be equally applicable to the model which thus becomes in effect a simplified representation of the molecular structure of the plate. This reasoning is a demonstration of the proposition, sometimes questioned, that the finite element solution involving proper cells, would converge to the true values on infinite reduction of the size of mesh, provided of course that the rounding off errors of the computer solution are negligible.

Types of Finite Element Cells

Two kinds of cells are used in the finite element solution of plane stress problems: the framework cells, made of elastic bars[1] [2] and the no-bar cells[3] [4]. The former represent true elastic structures which may be actually constructed and sometimes even experimented with physically. The no-bar cells on the other hand are mathematical abstractions suitable for calculation but unsusceptible to physical reproduction.

The nature of the no-bar cells may be explained in the following manner. Imagine a plate of wide extent subjected to a simple stress condition such as uniform unidimensional strain ϵ_x or shearless bending in X direction. Isolate from this plate a polygonal area of the shape of the assumed finite element, holding it in equilibrium by the appropriate peripheral stresses. Assume now that the edge stresses are replaced by proper statically equivalent corner forces without affecting by this operation the corner displacements. This transforms the given piece of plate into the finite element proper which is in effect a body, whose corners move through the same distances as the same points in the plate when acted upon by corner forces statically equivalent to the edge stresses. The finite element must behave in such manner under several simple stress conditions. By combining these conditions in proper proportions it is possible to effect the separate x or y displacements of the corners of the cell and through that to find the stiffness matrix of the element relating its corner forces to the displacements.

There are certain ambiguities associated with the transformation of the edge stresses into the corner forces, for illustration of which it is necessary to consider in detail a specific example — a cell in the form of an equilateral trapezoid (Fig. 1).

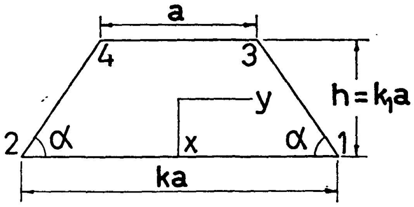

Fig. 1.

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[1]) A. HRENNIKOFF: "Solution of Problems of Elasticity by the Framework Method". Journal of Applied Mechanics, ASME, New York, Vol. 63, December 1941.

[2]) A. HRENNIKOFF: "Framework Method and its Technique for Solving Plane Stress Problems". Publications of International Association for Bridge and Structural Engineering, Zurich, Switzerland, Vol. 9, 1949.

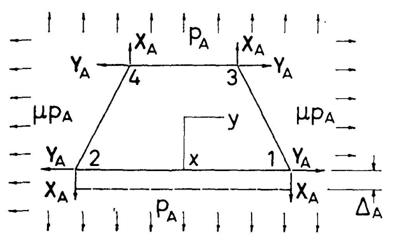
[3]) M. J. TURNER, R. W. CLOUGH, H. C. MARTIN and L. J. TOPP: "Stiffness and Deflection Analysis of Complex Structures". Journal of the Aeronautical Sciences, Vol. 23, No. 9, September 1956.

[4]) R. W. CLOUGH: "The Finite Element Method in Plane Stress Analysis". Proceedings of ASCE 2nd Conference on Electronic Computation, Pittsburgh, September, 1960.

Statics Type Stiffness Matrix of Equilateral Trapezoid

A quadrilateral cell has eight degrees of freedom with regard to u and v displacements of its four corners along the x and y axes. Single corner displacements may be effected by a combination of three rigid body movements in the plane of the cell and five basic stress conditions which may be assumed as follows:

A. Unilateral unit strain in x direction, ϵ_x (Fig. 2).



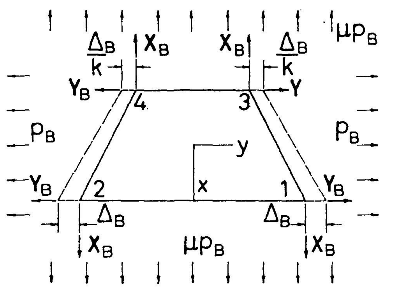
$$X_A = \frac{(1+k)\,t\,E\,\Delta_A}{4\,k_1\,(1-\mu^2)}$$

$$Y_A = \frac{\mu\,t\,E\,\Delta_A}{2\,(1-\mu^2)}$$

$$\Delta_A = \frac{p_A\,(1-\mu^2)\,k_1\,a}{t\,E}$$

Fig. 2.

B. Unilatera unit strain in y direction, ϵ_y (Fig. 3).



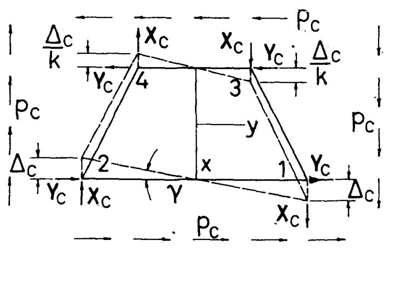
$$X_B = \frac{\mu\,(k+1)\,t\,E\,\Delta_B}{2\,k\,(1-\mu^2)}$$

$$Y_B = \frac{k_1\,t\,E\,\Delta_B}{k\,(1-\mu^2)}$$

$$\Delta_B = \frac{p_B\,(1-\mu^2)\,k\,a}{2\,t\,E}$$

Fig. 3.

C. Unit shear strain γ_{xy} (Fig. 4).



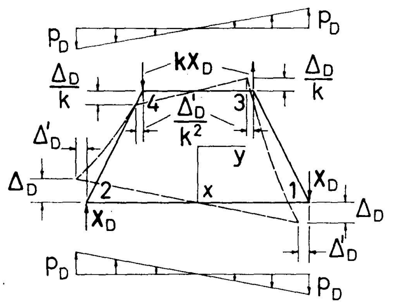
$$X_C = \frac{k_1\,t\,E\,\Delta_C}{2\,k\,(1+\mu)}$$

$$Y_C = \frac{(k+1)\,t\,E\,\Delta_C}{4\,k\,(1+\mu)}$$

$$\Delta_C = \frac{p_C\,(1+\mu)\,k\,a}{t\,E}$$

Fig. 4.

D. Shearless bending with stresses in x direction (Fig. 5).



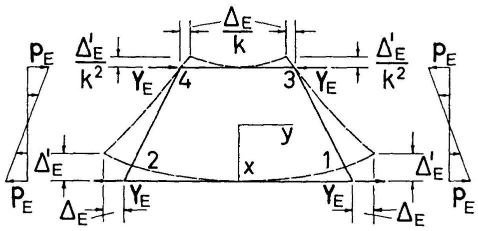
$$X_D = \frac{(k+1)\,t\,E\,\Delta_D}{6\,k\,k_1}$$

$$\Delta_D = \frac{p_D\,k_1\,a}{2\,t\,E}$$

$$\Delta'_D = \frac{\mu\,p_D\,k\,a}{4\,t\,E}$$

Fig. 5.

E. Shearless bending with stresses in y direction (Fig. 6).



$$Y_E = \frac{k_1\,t\,E\,\Delta_E}{3\,k}$$

$$\Delta_E = \frac{p_E\,k\,a}{2\,t\,E}$$

$$\Delta'_E = \frac{p_E\,k^2\,a}{4\,k_1\,t\,E}$$

Fig. 6.

Other suitable conditions may be used in place of D and E but the three uniform conditions A, B and C, or their equivalents, the uniform stresses, σ_x, σ_y and τ_{xy}, are compulsory.

The simplest way to determine the statically equivalent corner forces in the cell is to transfer its edge stresses to the adjacent corners by the law of the lever. This operation satisfies statics and at the same time carries the stresses only to the corners situated in the immediate vicinity of the edges in question. A weakness of this procedure will be pointed out later. The corner forces found by this method in all five stress conditions considered here are stated in Figs. 2—6.

The combination of the conditions A, C and D with $\Delta_A = 1/2\,u_1$, $\Delta_C = 1/4\,u_1$, and $\Delta_D = 1/4\,u_1$ results in a vertical displacement u_1 of the corner 1, with zero vertical displacements of the three other corners. However the condition D causes some unequal horizontal movements of the corners which must be cancelled by a horizontal rigid body movement and a horizontal displacement of the top edge to the left of the bottom edge by a shear condition C through a shear angle $\gamma = \frac{\mu\,(k^2-1)}{8\,k\,k_1^2\,a}\,u_1$. The corresponding corner forces are proportional to the ones shown in Fig. 4. The summation of the four sets of corner forces gives the stiffness matrix coefficients corresponding to the vertical movement u_1 of the corner 1.

The horizontal displacement v_1 of the same corner may be accomplished by the combination of the conditions B and E with $\Delta_A = \Delta_E = 1/4\,v_1$ and C, involving a horizontal displacement $\Delta_C = 1/2\, v_1$ of the bottom edge to the right of the top edge, with the shear strain $\frac{v_1}{2\,k_1\,a_1}$. Since the condition E moves the corners of the top edge down towards the bottom edge this displacement must be corrected by addition of the condition A with $\Delta_A = \frac{(k^2-1)\,v_1}{8\,k\,k_1}$. The combination of these four conditions gives another set of stiffness matrix coefficients.

The corner forces produced by the displacements of the node 2 may be found by symmetry with the node 1, and the ones brought about by the movement of the top corners — by replacement of the parameters k and k_1 with $1/k$ and k_1/k respectively, in the corresponding expressions of forces caused by the bottom corner movements.

The force-displacement relation of the trapezoidal cell is given by Eq. (1)

$$\begin{Bmatrix} P_1^x \\ P_1^y \\ P_2^x \\ P_2^y \\ P_3^x \\ P_3^y \\ P_4^x \\ P_4^y \end{Bmatrix} = \begin{bmatrix} X_1^{u1} & X_1^{v1} & X_1^{u2} = X_2^{u1} & X_1^{v2} = -X_2^{v1} & X_1^{u3} & X_1^{v3} & X_1^{u4} = X_2^{u3} & X_1^{v4} = -X_2^{v3} \\ Y_1^{u1} & Y_1^{v1} & Y_1^{u2} = -Y_2^{u1} & Y_1^{v2} = Y_2^{v1} & Y_1^{u3} & Y_1^{v3} & Y_1^{u4} = -Y_2^{u3} & Y_1^{v4} = Y_2^{v3} \\ X_2^{u1} & X_2^{v1} & X_2^{u2} = X_1^{u1} & X_2^{v2} = -X_1^{v1} & X_2^{u3} & X_2^{v3} & X_2^{u4} = X_1^{u3} & X_2^{v4} = -X_1^{v3} \\ Y_2^{u1} & Y_2^{v1} & Y_2^{u2} = -Y_1^{u1} & Y_1^{v2} = Y_1^{v1} & Y_2^{u3} & Y_2^{v3} & Y_2^{u4} = -Y_1^{u3} & Y_2^{v4} = Y_1^{v3} \\ X_3^{u1} & X_3^{v1} & X_3^{u2} = X_4^{u1} & X_3^{v2} = -X_4^{v1} & X_3^{u3} & X_3^{v3} & X_3^{u4} = X_4^{u3} & X_3^{v4} = -X_4^{v3} \\ Y_3^{u1} & Y_3^{v1} & Y_3^{u2} = -Y_4^{u1} & Y_3^{v2} = Y_4^{v1} & Y_3^{u3} & Y_3^{v3} & Y_3^{u4} = -Y_4^{u3} & Y_3^{v4} = Y_4^{v3} \\ X_4^{u1} & X_4^{v1} & X_4^{u2} = X_3^{u1} & X_4^{v2} = -X_3^{v1} & X_4^{u3} & X_4^{v3} & X_4^{u4} = X_3^{u3} & X_4^{v4} = -X_3^{v3} \\ Y_4^{u1} & Y_4^{v1} & Y_4^{u2} = -Y_3^{u1} & Y_4^{v2} = Y_3^{v1} & Y_4^{u3} & Y_4^{v3} & Y_4^{u4} = -Y_3^{u3} & Y_4^{v4} = Y_3^{v3} \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}. \tag{1}$$

and the explicit expressions of the stiffness coefficients X and Y are presented in Table 1. The system adopted in their nomenclature involves one digit subscripts and two digit superscripts. The former indicate the node at which the force in question is applied, and the latter — the corner moved and the kind of the unit movement $u = 1$ or $v = 1$ creating the nodal force.

An important characteristic of this stiffness matrix is the absence of symmetry about the principal diagonal. Thus for example the coefficients Y_1^{u1} and X_1^{v1} are not equal. The asymmetry of the matrix leads to results violating the Betti's reciprocal theorem thus evincing a theoretical deficiency of the matrix.

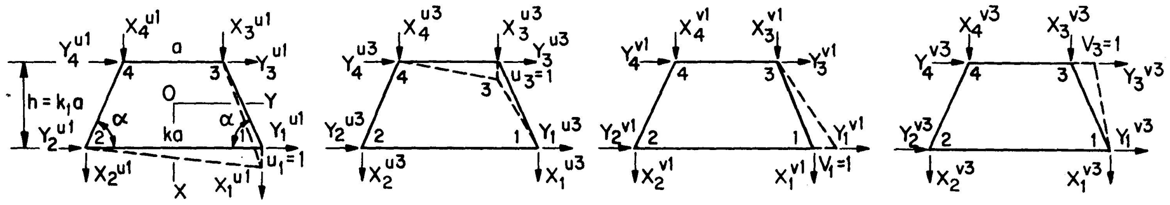
If the trapezoidal cell is transformed into a rectangle by making the parameter k equal to unity the matrix becomes symmetrical. Apparently its lack of symmetry reflects the asymmetry of the cell itself about the horizontal axis.

Energy Type Stiffness Matrix of Equilateral Trapezoid

Most of the authors employ a different stiffness matrix which is derived from the energy considerations[5]). Its commonly used implicit expression in

[5]) R. H. GALLACHER: "A Correlation Study of Methods of Matrix Structural Analysis". A Pergammon Press Book, the MacMillan Company, New York.

Table 1. No-bar Trapezoidal Cell Statics Type Stiffness Matrix (Unsymmetrical)



t = plate thickness $\qquad L = \dfrac{tE}{4(1-\mu^2)}$

$$X_1^{u1} = \left\{\frac{(1-\mu^2+3k)(k+1)}{6kk_1} + \frac{(1-\mu)[4k_1^2+\mu(k^2-1)]}{8kk_1}\right\}L, \qquad Y_1^{u1} = \left\{\frac{(k+1)+\mu(3k-1)}{4k} + \frac{\mu(1-\mu)(k^3+k^2-k-1)}{16kk_1^2}\right\}L.$$

$$X_2^{u1} = \left\{\frac{(-1+\mu^2+3k)(k+1)}{6kk_1} - \frac{(1-\mu)[4k_1^2+\mu(k^2-1)]}{8kk_1}\right\}L, \qquad Y_2^{u1} = \left\{\frac{(k+1)-\mu(5k+1)}{4k} + \frac{\mu(1-\mu)(k^3+k^2-k-1)}{16kk_1^2}\right\}L.$$

$$X_3^{u1} = \left\{-\frac{(4-\mu^2)(k+1)}{6k_1} + \frac{(1-\mu)[4k_1^2+\mu(k^2-1)]}{8kk_1}\right\}L, \qquad Y_3^{u1} = \left\{-\frac{(k+1)-\mu(5k+1)}{4k} - \frac{\mu(1-\mu)(k^3+k^2-k-1)}{16kk_1^2}\right\}L.$$

$$X_4^{u1} = \left\{-\frac{(2+\mu^2)(k+1)}{6k_1} - \frac{(1-\mu)[4k_1^2+\mu(k^2-1)]}{8kk_1}\right\}L, \qquad Y_4^{u1} = \left\{-\frac{(k+1)+\mu(3k+1)}{4k} - \frac{\mu(1-\mu)(k^3+k^2-k-1)}{16kk_1^2}\right\}L.$$

$$X_1^{u3} = \left\{-\frac{(4-\mu^2)(k+1)}{6k_1} + \frac{(1-\mu)[4k_1^2-\mu(k^2-1)]}{8k_1}\right\}L, \qquad Y_1^{u3} = \left\{\frac{(k+1)-\mu(k+5)}{4} - \frac{\mu(1-\mu)(k^3+k^2-k-1)}{16k_1^2}\right\}L.$$

$$X_2^{u3} = \left\{-\frac{(2+\mu^2)(k+1)}{6k_1} - \frac{(1-\mu)[4k_1^2-\mu(k^2-1)]}{8k_1}\right\}L, \qquad Y_2^{u3} = \left\{\frac{(k+1)+\mu(3-k)}{4} + \frac{\mu(1-\mu)(k^3+k^2-k-1)}{16k_1^2}\right\}L.$$

$$X_3^{u3} = \left\{\frac{[k(1-\mu^2)+3](k+1)}{6k_1} + \frac{(1-\mu)[4k_1^2-\mu(k^2-1)]}{8k_1}\right\}L, \qquad Y_3^{u3} = \left\{-\frac{(k+1)+\mu(3-k)}{4} + \frac{\mu(1-\mu)(k^3+k^2-k-1)}{16k_1^2}\right\}L.$$

$$X_4^{u3} = \left\{\frac{[-k(1-\mu^2)+3](k+1)}{6k_1} - \frac{(1-\mu)[4k_1^2-\mu(k^2-1)]}{8k_1}\right\}L, \qquad Y_4^{u3} = \left\{-\frac{(k+1)-\mu(k+5)}{4} + \frac{\mu(1-\mu)(k^3+k^2-k-1)}{16k_1^2}\right\}L.$$

$$X_1^{v1} = \frac{1}{2}\left\{\frac{k+\mu}{k} + \frac{k^3+k^2-k-1}{4kk_1^2}\right\}L, \qquad Y_1^{v1} = \left\{\frac{(4-\mu^2)k_1}{3k} + \frac{(k-\mu)(k+1)}{4kk_1}\right\}L.$$

$$X_2^{v1} = \frac{1}{2}\left\{\frac{-k+\mu(2k+1)}{k} + \frac{k^3+k^2-k-1}{4kk_1^2}\right\}L, \qquad Y_2^{v1} = \left\{-\frac{(4-\mu^2)k_1}{3k} + \frac{k(k+1)-\mu(2k^2+k-1)}{4kk_1}\right\}L.$$

$$X_3^{v1} = \frac{1}{2}\left\{\frac{k-\mu(2k+1)}{k} - \frac{k^3+k^2-k-1}{4kk_1^2}\right\}L, \qquad Y_3^{v1} = \left\{\frac{(2+\mu^2)k_1}{3k} - \frac{k(k+1)-\mu(2k^2+k-1)}{4kk_1}\right\}L.$$

$$X_4^{v1} = -\frac{1}{2}\left\{\frac{k+\mu}{k} + \frac{k^3+k^2-k-1}{4kk_1^2}\right\}L, \qquad Y_4^{v1} = -\left\{\frac{(2+\mu^2)k_1}{3k} + \frac{(k-\mu)(k+1)}{4kk_1}\right\}L.$$

$$X_1^{v3} = \frac{1}{2}\left\{-1+\mu(k+2) - \frac{k^3+k^2-k-1}{4k_1^2}\right\}L, \qquad Y_1^{v3} = \left\{\frac{(2+\mu^2)k_1}{3} - \frac{(k+1)+\mu(k^2-k-2)}{4k_1}\right\}L.$$

$$X_2^{v3} = \frac{1}{2}\left\{1+\mu k - \frac{k^3+k^2-k-1}{4k_1^2}\right\}L, \qquad Y_2^{v3} = \left\{-\frac{(2+\mu^2)k_1}{3} - \frac{(k+1)(1-\mu k)}{4k_1}\right\}L.$$

$$X_3^{v3} = \frac{1}{2}\left\{-1-\mu k + \frac{k^3+k^2-k-1}{4k_1^2}\right\}L, \qquad Y_3^{v3} = \left\{\frac{(4-\mu^2)k_1}{3} + \frac{(k+1)(1-\mu k)}{4k_1}\right\}L.$$

$$X_4^{v3} = \frac{1}{2}\left\{1-\mu(k+2) + \frac{k^3+k^2-k-1}{4k_1^2}\right\}L, \qquad Y_4^{v3} = \left\{-\frac{(4-\mu^2)k_1}{3} + \frac{(k+1)+\mu(k^2-k-2)}{4k_1}\right\}L.$$

the form of a product of several component matrices is convenient for computer work, but is apt to conceal some of its peculiarities and defects. This energy type matrix will be presented here explicitly, and it will be correlated with the matrix found by statics.

As explained above the stress condition of a cell corresponding to a unit displacement of one of its nodes may be described by combination of several stress conditions A to E. This makes it possible to express the stresses and displacements in the plate in terms of the particular node displacement. When all nodes are moved simultaneously, the stresses and displacements in the cell, including the values along the periphery, become linear functions of eight corner displacements u_1, v_1, u_2, etc.

Imagine eight nodal forces P_1^x, P_1^y, P_2^x, etc. statically equivalent to the edge stresses $\sigma_\perp$ (normal) and τ (tangential) and apply them in reversed directions, as the equilibrants of the edge stresses. The application of these corner forces does not affect the peripheral stresses $\sigma_\perp$ and τ. Give one of the corners, such as #1, a displacement du_1 and equate to zero the virtual work of the system corresponding to this displacement. As is well known, the work of deformation of a plate may be expressed either as an area integral or a line integral taken along the periphery, and the latter version will be used here. Of the eight corner forces only P_1^x does the virtual work.

Calling the virtual displacements developed on the edges of the cell $\lambda_\perp$-normal and λ_{11}-tangential, and the thickness of the plate t, the work equation may be stated in the following form:

$$P_1^x\, du_1 = t \int_{l\triangle} \sigma_\perp\, dl \frac{\partial \lambda_\perp}{\partial u_1} du_1 + t \int_{l\triangle} \tau\, dl \frac{\partial \lambda_{11}}{\partial u_1} du_1. \tag{2}$$

The increment du_1 may be cancelled and the partial derivatives may be viewed as the edge displacements produced by the unit movement of the corner $u_1 = 1$

$$\frac{\partial \lambda_\perp}{\partial u_1} = \lambda_\perp^{u_1=1} \quad \text{and} \quad \frac{\partial \lambda_{11}}{\partial u_1} = \lambda_{11}^{u_1=1}.$$

Then
$$P_1^x = t \int_{l\triangle} \sigma_\perp \lambda_\perp^{u_1=1}\, dl + t \int_{l\triangle} \tau \lambda_{11}^{u_1=1}\, dl. \tag{3}$$

This equation defines the terms in the first row of the stiffness matrix. Its individual terms are expressed thus:

$$X_1^{u1} = t \int_{l\triangle} (\sigma_\perp^{u1} \lambda_\perp^{u1} + \tau^{u1} \lambda_{11}^{u1})\, dl, \tag{4}$$

$$X_1^{v1} = t \int_{l\triangle} (\sigma_\perp^{v1} \lambda_\perp^{u1} + \tau^{v1} \lambda_{11}^{u1})\, dl \quad \text{etc.} \tag{5}$$

The terms in the other rows of the matrix are found in a similar manner, for example:

$$Y_1^{u1} = t \int_{l\triangle} (\sigma_\perp^{u1} \lambda_\perp^{v1} + \tau^{u1}\, {}_{11}^{v1})\, dl. \tag{6}$$

The significance of the subscript and the superscript symbols in X and Y was explained before. The edge stresses and displacements under the integral signs are produced by the unit displacements of one of the nodes. The stresses $\sigma_{\perp}$ and τ correspond to unit displacement of the corner indicated by the superscript in X or Y, while the displacements $\lambda_{\perp}$ and λ_{11} are produced by the unit displacements stated in their superscript, which matches the subscript of the term X or Y.

For simplicity of calculation the line integrals along the sloping sides of the cell 1—3 and 2—4 may be expressed through the X and Y components of the stresses and displacements rather than through the normal and tangential components, in accordance with the relation

$$\int (\sigma_{\perp} \lambda_{\perp} + \tau \lambda_{11})\, dl = \int \sigma_x \lambda_x \, dy + \int \sigma_y \lambda_y \, dx + \int \tau_{xy} \lambda_y \, dy + \int \tau_{xy} \lambda_x \, dx .$$

Care must be taken in using proper signs of stresses and displacements in these expressions.

Although the expressions X and Y (Eqs. 4, 5, 6) are extensively used (sometimes in a modified form) they are incorrect, because they imply that the virtual work of deformation of the material within the cell subjected to a simultaneous action of the edge stresses and the opposing corner forces is zero. This supposition is erroneous. Even though the corner forces and the edge stresses are balanced the internal stresses within the cell are still extant and there is no reason for their virtual work to be zero should the nodes be displaced.

The edge stresses brought about by any two single corner displacements, such as $u_1 = 1$ or $v_1 = 1$, are of course mutually balanced by themselves without the addition of the corner forces and so the magnitudes of the work done by the edge stresses of one of these conditions on the edge deformations of the other must be equal. The two sides of this work equation are the integral expressions of the matrix terms X_1^{v1} and Y_1^{u1} in Eqs. (5) and (6). This signifies the equality of these as well as of other symmetrically situated terms in the stiffness matrix. The symmetry of the energy type matrix is an advantage over the unsymmetrical matrix (Table 1) since it simplifies the computer work. The energy type matrix involves however some inconsistency to be pointed out presently.

Comparison of the Matrices

A comparison of the two types of stiffness matrix discussed here would clarify some of their peculiarities. In the stiffness matrix found by statics the edge stresses are assembled into the corner forces at the two adjacent corners by the law of the lever, while in the energy type matrix the same edge stresses are multiplied by the edge displacements produced by the movement of the corner and then added up. For a specific comparison take the term X_1^{v1} (Eq. (5)).

The edge stresses $\sigma_{\perp}^{v1}$ and τ^{v1} in its expression are produced by the displacement $v_1 = 1$, and the edge displacements $\lambda_{\perp}^{u1}$ and λ_{11}^{u1} — by the displacement $u_1 = 1$. The diagrams of $\lambda_{\perp}^{u1}$ and λ_{11}^{u1} may be viewed as the influence lines by whose ordinates the stresses $\sigma_{\perp}$ and τ created by any unit corner movement ($v_1 = 1$ in this case) must be multiplied and then summed up in order to form the force X_1 produced by the given movement ($u_1 = 1$ in this case). Of the eight corner displacements seven are equal to zero and only one, $u_1 = 1$, is distinct from zero.

Taking as an example a rectangular cell imagine as a possibility that the edge displacements corresponding to $u_1 = 1$ are all linear (Fig. 7). This means

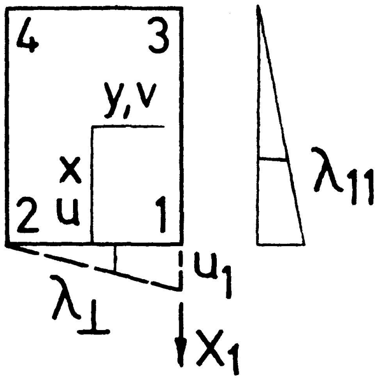


Fig. 7.

that of all edge displacements only $\lambda_{\perp}$ on the edge 1—2 and λ_{11} on the edge 1—3 are distinct from zero. Employment of the influence lines of the type of $\lambda_{\perp}$ and λ_{11} in Fig. 7 for calculation of X_1 is obviously equivalent to the application of the law of the lever. Under the stated conditions the terms X_1 found by the two methods should then be identical.

The linearity of the edge displacements was however used in this reasoning only as a tentative supposition, true for the uniform strain conditions A, B, and C but false for the flexural conditions D and E. Thus the condition D required for the displacement of $u_1 = 1$, results in parabolic normal displacements on the edges 1—3 and 2—4 directed inward on one edge and outward on the other. Parabolically distributed tangential displacements are present also on the edges 1—2 and 3—4. It so happens however that in all five basic stress conditions A to E the edge stresses on the opposite sides of the rectangle are equal, which results in mutual cancellation of the energy terms corresponding to the non-linear displacements. This makes the statics and the energy matrices for a rectangular cell identical, as well as symmetrical about the principal diagonal.

The situation is however different in the case of a trapezoidal cell. When such a cell is subjected to bending condition D its side edges become curved, and under the condition E its all four edges are curved. The diagrams of the edge displacements thus become non-linear, making the matrix terms, determined by the two methods, different. It is significant that a single corner displacement, such as $u_1 = 1$, produces displacements on the non-adjacent edges 2—4 and 4—3, as well as on the adjacent ones 1—2 and 1—3. This means that the terms X_1 and Y_1 are contributed to partly by the stresses on the non-adjacent edges. The presence of such contributions appears logically unsound and must be viewed as a defect of the energy matrix.

Correlation of the Matrices

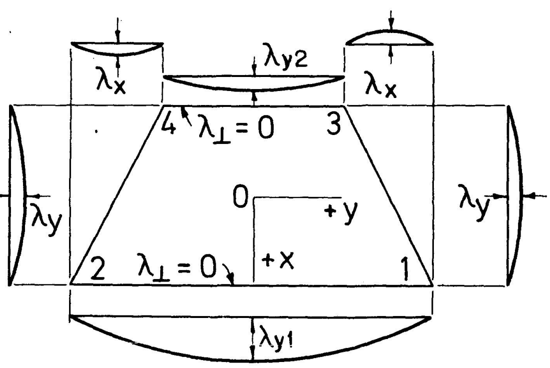
The two matrices may be correlated by finding the complementary terms whose addition to the terms of the statics type matrix transforms them into the terms of the energy matrix.

It was pointed out that integration of the edge stresses over the linear parts of the edge displacements, such as the ones in Fig. 7, produces the terms of the stiffness matrix found by statics. Then the complementary terms may be found by integration of the products of the edge stresses over the non-linear parts of the edge displacements with zero values at the corners. The nonlinear displacements are produced exclusively by the flexural conditions D and E, and their shape always conforms to the second order parabola with the ordinates measured from the chords passing through the ends. The ordinates of these parabolas are fully described by the midlength values.

Influence Lines of Complementary Terms X_1

The edge displacements considered here correspond to the corner movement $u_1 = 1$ (Fig. 8). Only the condition D is effective in producing the non-linear components of the edge displacements.

$$\lambda_x = \frac{k-1}{8k} u_1$$



$$\lambda_y = \left[\frac{k_1}{8k} + \frac{\mu (k-1)^2}{32 k k_1}\right] u_1$$

$$\lambda_{y1} = \frac{\mu k}{8 k_1} u_1$$

$$\lambda_{y2} = \frac{\mu}{8 k k_1} u_1$$

Fig. 8.

Edge 1—2. The normal displacements are linear. The nodes 1 and 2 move to the left (Fig. 5) the amount $\frac{\mu k}{2 k_1} \Delta_D = \frac{\mu k}{8 k_1} u_1$, while the mid-point remains at rest. The relative tangential displacement of the mid-point in relation to the ends is $\lambda_{y1} = + \frac{\mu k}{8 k_1} u_1$.

Edge 3—4. Similarly, the mid-point displacements are $\lambda_{\perp}=0$, $\lambda_{y\,2} = +\frac{\mu}{8\,k\,k_1}u_1$.

Edge 1—3. The vertical displacements of the corners 1 and 3 are $+1/4\,u_1$ and $-\frac{u_1}{4\,k}$ respectively, and of the mid-point — zero. Then the displacement of the mid-point in relation to the ends is up $\lambda_x=\frac{k-1}{8\,k}u_1$.

The horizontal displacements are produced by two effects: the curvature and the μ effect.

By the elementary flexure formula, $\frac{1}{R}=\frac{M}{EI}=\frac{2\,\sigma}{E\,h}=\frac{u_1}{4\,k\,k_1\,a^2}$ and the horizontal corner displacement to the left with reference to the mid-point is $\frac{(h/2)^2}{2\,R}=\frac{k_1}{8\,k}u_1$.

The horizontal displacements to the left of the corners 1 and 3 and of the mid-point produced by μ effect are respectively $\frac{\mu\,k\,u_1}{8\,k_1}$, $\frac{\mu\,u_1}{8\,k\,k_1}$ and $\frac{\mu\,(k+1)^2\,u_1}{8\,k\,k_1}$. This makes the relative displacement of the mid- point in relation to the straight line through the ends

$$\frac{1}{2}\left(\frac{\mu\,k\,u_1}{8\,k_1}+\frac{\mu\,u_1}{8\,k\,k_1}\right)-\frac{\mu\,(k+1)^2\,u_1}{32\,k\,k_1}=\frac{\mu\,(k-1)^2}{32\,k\,k_1}.$$

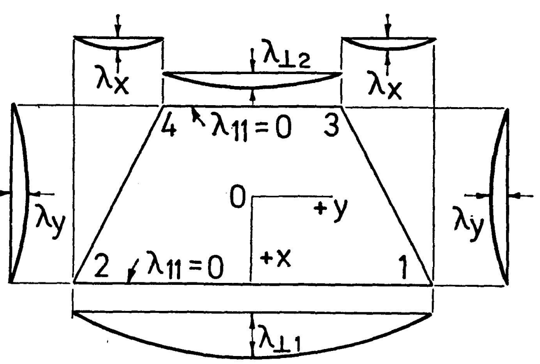
The total horizontal displacement of the mid-point of the side 1—3 is

$$\lambda_y = +\frac{k_1}{8\,k}u_1+\frac{\mu\,(k-1)^2}{32\,k\,k_1}u_1.$$

Edge 2—4. The displacements are equal in magnitude and opposite in sign compared to the ones on the edge 1—3.

The edge displacements determined here are shown in Fig. 8. If the quantity u_1 is made unity the curves become the influence lines of the complementary terms X_1 and X_2 of the energy stiffness matrices. They are valid for X_1 and X_2 produced by unit displacements u or v of any of the four corners. These influence lines are antisymmetrical about the vertical axis X.

$$\lambda_{\perp 1} = \frac{k}{8\,k_1}u_1 \qquad \lambda_{\perp 2} = \frac{v_1}{8\,k\,k_2}$$



$$\lambda_x = \left[\frac{\mu\,k_1}{8\,k}+\frac{(k-1)^2}{32\,k\,k_1}\right]v_1 \qquad \lambda_y = \frac{k-1}{8\,k}v_1$$

Fig. 9.

The influence lines of the terms Y_1 and $-Y_2$ corresponding to the displacement $v_1 = 1$ are shown in Fig. 9. They are brought about by the flexural condition E with $\Delta_E = 1/4\, v_1$. These lines are symmetrical about the vertical axis.

Complementary Term $X_1^{v_1}$. Energy Type Matrix Terms

The use of the influence lines of Figs. 8 and 9 is illustrated on the example of the complementary factor X_1^{v1} whose combination with the factor determined by statics produces the term X_1^{v1} of the energy stiffness matrix.

The displacement $v_1 = 1$ which creates the edge stresses used in calculation of X_1^{v1}, is made up of the stress conditions A, B, C and E of which only the stresses of the condition C are capable of making the products with the edge displacements of Fig. 8 distinct from zero, since the stresses of the condition C and the displacements are both antisymmetrical about the X axis. On the other hand the stresses of the condition A, B and E are symmetrical about the X axis, and the integral of their products with the ordinates of Fig. 8 is zero.

The edge stresses of the condition C are constant along all edges and they must be multiplied by the appropriate length of the edge and the mean displacement equal to 2/3 of the maximum ordinate of the parabolic influence line.

The unit shear force in condition C corresponding to $u_1 = 1$ is $\frac{tE}{4(1+\mu)\, k a_1}$. Then the values of the complementary terms of X_1^{v1} are as follows (Fig. 8).

Edge 1—2: $$\frac{E t}{4(1+\mu)\, k_1 a} \frac{2}{3} \frac{\mu k}{8 k_1} k a = \frac{\mu k^2 E t}{48 (1+\mu)\, k_1^2}.$$

Edge 3—4: $$-\frac{E t}{4(1+\mu)\, k_1 a} \frac{2}{3} \frac{\mu}{8 k k_1} a = -\frac{\mu E t}{48 (1+\mu)\, k k_1^2}.$$

Edges 1—3 and 2—4 are replaced with stepped lines made up of infinitesimal horizontal and vertical steps.

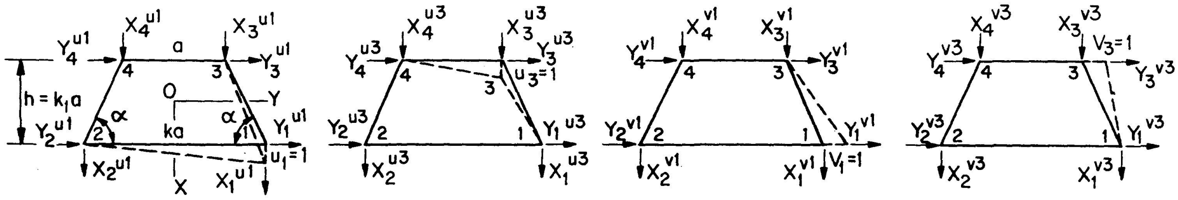
$$-\frac{2 E t}{4(1+\mu)\, k_1 a}\left\{\frac{2}{3} \frac{(k-1)}{8 k} k_1 a + \frac{2}{3}\left[\frac{k_1}{8 k} + \frac{\mu (k-1)^2}{48 k k_1}\right] \frac{(k-1)\, a}{2}\right\}.$$

Adding these up. The complement to

$$X_1^{v1} = \left[\frac{\mu (k+1)(k^2-1)}{64 k k_1^2} - \frac{k-1}{16 k}\right] \frac{E t}{(1+\mu)}.$$

The procedure described here allows determination of all complementary terms. Combination of the complementary terms with the statics type matrix terms (Table 1) results in the energy type stiffness matrix whose terms are presented in Table 2. As was pointed out earlier this matrix is symmetrical about its principal diagonal.

Table 2. No-bar Trapezoidal Cell Energy Type Stiffness Matrix (Symmetrical)



$$t = \text{plate thickness} \qquad L = \frac{tE}{4(1-\mu^2)}$$

$$X_1^{u1} = \left\{\frac{k+1}{2k_1} + \frac{(1-\mu^2)(k+1)(k^2+1)}{12k^2k_1} + \frac{(1-\mu)(k+1)k_1}{4k^2} + \frac{\mu(1-\mu)(k^2-1)(k+1)[8k_1^2+\mu(k^2-1)]}{64k^2k_1^3}\right\}L,$$

$$Y_1^{u1} = \left\{\frac{(\mu+1)(k+1)}{4k} + \frac{[2+\mu(1-\mu)](k^2-1)(k+1)}{16kk_1^2}\right\}L,$$

$$X_2^{u1} = \left\{\frac{k+1}{2k_1} - \frac{(1-\mu)(k+1)k_1}{4k^2} - \frac{(1-\mu^2)(k+1)(k^2+1)}{12k^2k_1} - \frac{\mu(1-\mu)(k^2-1)(k+1)[8k_1^2+\mu(k^2-1)]}{64k^2k_1^3}\right\}L,$$

$$Y_2^{u1} = \left\{\frac{(1-3\mu)(k+1)}{4k} - \frac{[2-\mu(1-\mu)](k^2-1)(k+1)}{16kk_1^2}\right\}L,$$

$$X_3^{u1} = \left\{-\frac{k+1}{2k_1} + \frac{(1-\mu)(k+1)k_1}{4k} - \frac{(1-\mu^2)(k+1)(k^2+1)}{12kk_1} - \frac{\mu^2(1-\mu)(k^2-1)^2(k+1)}{64kk_1^3}\right\}L,$$

$$Y_3^{u1} = \left\{-\frac{1-3\mu}{4} + \frac{\mu k}{2} - \frac{1-\mu}{4k} - \frac{[2k+\mu(1-\mu)](k^2-1)(k+1)}{16kk_1^2}\right\}L,$$

$$X_4^{u1} = \left\{-\frac{k+1}{2k_1} - \frac{(1-\mu)(k+1)k_1}{4k} + \frac{(1-\mu^2)(k+1)(k^2+1)}{12kk_1} + \frac{\mu^2(1-\mu)(k^2-1)^2(k+1)}{64kk_1^3}\right\}L,$$

$$Y_4^{u1} = \left\{-\frac{1+\mu}{4} - \frac{\mu k}{2} - \frac{1-\mu}{4k} + \frac{[2k-\mu(1-\mu)](k^2-1)(k+1)}{16kk_1^2}\right\}L,$$

$$Y_1^{v1} = \left\{\frac{3(2-\mu^2)kk_1+(2+\mu^2)k_1}{6k^2} + \frac{(k^2-\mu)(k+1)}{4k^2k_1} + \frac{(k^2-1)^2(k+1)}{32k^2k_1^3}\right\}L,$$

$$X_2^{v1} = -Y_2^{u1},$$

$$Y_2^{v1} = \left\{ -\frac{3(2-\mu^2)kk_1+(2+\mu^2)k_1}{6k^2} + \frac{[(1-2\mu)k^2+\mu](k+1)}{4k^2k_1} - \frac{(k^2-1)^2(k+1)}{32k^2k_1^3} \right\} L,$$

$$X_3^{v1} = \left\{ \frac{k+1}{4} - \frac{\mu(3+k)}{4} - \frac{\mu}{2k} - \frac{[2+\mu(1-\mu)k](k^2-1)(k+1)}{16kk_1^2} \right\} L,$$

$$Y_3^{v1} = \left\{ \frac{(2+\mu^2)(k+1)k_1}{6k} - \frac{(1-\mu)(k+1)}{4k_1} - \frac{(k^2-1)^2(k+1)}{32kk_1^3} \right\} L,$$

$$X_4^{v1} = \left\{ \frac{\mu(k-1)}{4} - \frac{\mu}{2k} - \frac{k+1}{4} - \frac{[2-\mu(1-\mu)k](k^2-1)(k+1)}{16kk_1^2} \right\} L,$$

$$Y_4^{v1} = \left\{ -\frac{(2+\mu^2)(k+1)k_1}{6k} - \frac{(1-\mu)(k+1)}{4k_1} + \frac{(k^2-1)^2(k+1)}{32kk_1^3} \right\} L,$$

$$X_2^{u2} = X_2^{u1}, \qquad Y_2^{u2} = -Y_1^{u1}, \qquad X_3^{u2} = X_4^{u1}, \qquad X_4^{u2} = X_3^{u1}, \qquad Y_4^{u2} = -Y_3^{u1},$$

$$Y_3^{u2} = \left\{ \frac{k+1}{4k} + \frac{\mu k}{2} + \frac{\mu(k-1)}{4k} - \frac{[2k-\mu(1-\mu)](k^2-1)(k+)}{16kk_1^2} \right\} L,$$

$$Y_2^{v2} = Y_1^{v1}, \qquad X_3^{v2} = -X_4^{v1}, \qquad Y_3^{v2} = Y_4^{v1},$$

$$X_4^{v2} = \left\{ -\frac{1-3\mu}{4} + \frac{\mu}{2k} - \frac{(1-\mu)k}{4} + \frac{[2+\mu(1-\mu)k](k^2-1)(k+1)}{16kk_1^2} \right\} L,$$

$$Y_4^{v2} = \left\{ \frac{(2+\mu^2)(k+1)k_1}{6k} - \frac{(1-\mu)(k+1)}{4k_1} - \frac{(k^2-1)^2(k+1)}{32kk_1^3} \right\} L,$$

$$X_3^{u3} = \left\{ \frac{k+1}{2k_1} + \frac{(1-\mu^2)(k+1)(k^2+1)}{12k_1} + \frac{(1-\mu)(k+1)k_1}{4} - \frac{\mu(1-\mu)(k^2-1)(k+1)[8k_1^2-\mu(k^2-1)]}{64k_1^3} \right\} L,$$

$$Y_3^{u3} = \left\{ \frac{[2+\mu(1-\mu)](k^2-1)(k+1)}{16k_1^2} - \frac{(1+\mu)(k+1)}{4} \right\} L,$$

$$X_4^{u3} = \left\{ \frac{k+1}{2k_1} - \frac{(1-\mu)(k+1)k_1}{4} - \frac{(1-\mu^2)(k+1)(k^2+1)}{12k_1} + \frac{\mu(1-\mu)(k^2-1)(k+1)[8k_1^2-\mu(k^2-1)]}{64k_1^3} \right\} L,$$

$$Y_4^{u3} = \left\{ -\frac{(1-3\mu)(k+1)}{4} - \frac{[2-\mu(1-\mu)](k^2-1)(k+1)}{16k_1^2} \right\} L,$$

$$Y_3^{v3} = \left\{ \frac{3(2-\mu^2)k_1+(2+\mu^2)kk_1}{6} + \frac{(1-\mu k^2)(k+1)}{4k_1} + \frac{(k^2-1)^2(k+1)}{32k_1^3} \right\} L,$$

$$Y_4^{v3} = \left\{ -\frac{3(2-\mu^2)k_1+(2+\mu^2)kk_1}{6} + \frac{(k+1)(1-2\mu+\mu k^2)}{4k_1} - \frac{(k^2-1)^2(k+1)}{32k_1^3} \right\} L,$$

$$X_4^{v3} = -Y_4^{u3}, \qquad X_4^{u4} = X_3^{u3}, \qquad Y_4^{u4} = -Y_3^{u3}, \qquad Y_4^{v4} = Y_3^{v3}.$$

Intercell Continuity

The continuity of displacements of the neighbouring cells in the finite element model is preserved at the nodes, but not necessarily along the intercell boundaris and there is a difference of opinion with regard to the significance of this discontinuity. Some authors maintain firmly that the displacements must be continuous across the boundaries if the solution of the model is to converge to the true solution of the structure. This view has apparently been prompted by the use of the Rayleigh-Ritz principle for the derivation of the energy stiffness matrix.

For this purpose a grid of reference points is established over the plate under investigation with the interspaces between the nodes having the shape of the finite elements. The values of the displacements of these points, converging to the exact values on reduction of mesh, may be determined by the Rayleigh-Ritz principle, whose application results in the same energy type matrix as the one found above. The necessary condition for the applicability of the principle is continuity of the displacements along the internodal lines.

It follows from this discussion that the displacement continuity along the intercell boundaries is a *sufficient* condition for the validity of the finite element method, but not the *necessary* condition. Irrespective of the edge continuity the finite element method is valid because of the identity of the differential equations of elasticity in application, on the one hand, to the solid plate and on the other — to the finite element model of proper deformability with infinitesimal mesh size.

It may be pointed out that the edge continuity has no meaning in application to the bar cells, which by their very nature are joined only at the nodes.

Triangular no-bar cells preserve edge continuity under all conditions, while the rectangular do not. Yet the precision of the results obtained with the rectangular cells has been found invariably much better than with the triangular ones of comparable size.

Appraisal of Imperfections of Stiffness Matrices

The existence of two different stiffness matrices of unsymmetrical no-bar finite elements calls for their comparison. No fault can be detected in the derivation of the statics type matrix, yet its asymmetry violates the basic structural principle of reciprocity. The inconsistency must be charged against the inexact nature of the method of no-bar finite element.

The energy type matrix may appear more attractive in view of its symmetry, but the error committed in its derivation in neglecting a part of the virtual work speaks against it. Furthermore the inclusion of certain edge stresses into the corner forces on the opposite side of the cell is contrary to common sense.

The defects discussed here do not apply to a rectangular no-bar cell or to a cell in the form of a triangle of any kind. It may be observed that the matrix of a quadrilateral framework cell is free from the faults of its no-bar counterpart because unlike the latter it is an actual structure and as such it is subject to the law of reciprocity.

The practical significance of the defects of the unsymmetrical no-bar cells should not, however, be exaggerated. On reduction of the size of cells the state of stress in the neighbourhood of the individual units approaches uniformity, diminishing by that the effect of the flexure conditions D and E, responsible for the inconsistencies of both types of the no-bar matrix. This means that the errors induced by the defects of the matrices tend to disappear on reduction of the size of mesh.

Comparison of the models made of bar and no-bar cells is worthy of comment. The stiffness matrix of the former is based on the uniform stress conditions, and of the latter — on the uniform conditions assisted by shearless bending. At first glance one might expect better results with the no-bar cells in view of their seemingly better conformity to the actual state of stress in the prototype. The present author himself at one time held this view[6]). However, the actual results have not confirmed the expectation, apart from some cases (like that of an end loaded wide cantilever beam), favouring the no-bar cells. It appears that the non-uniform components of the actual stresses in the prototype are normally not of the nature of shearless bending of conditions D and E, and for this reason they are not described any better by the matrix of the no-bar cell than by the one of the framework cell.

Conclusions

1. On infinite reduction of the size of the mesh the finite element solution of a plane stress problem converges to the true solution, provided the rounding off errors of the computer are negligible. This is true with regard to both the bar cells and the no-bar cells of a proper pattern.

2. Two types of stiffness matrix are available for the unsymmetrical no-bar cells, the statics type and the energy type. They are mutually related by the "complementary" terms which may be found by the use of influence lines.

3. Both types of the no-bar stiffness matrix contain some theoretical defects reflecting the inexact nature of the method utilizing cells of finite size. The framework cells are free from comparable defects.

4. The stiffness matrix of the framework cell and the energy type matrix of the no-bar cell are symmetrical about their principal diagonals, and so they

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[6]) A. Hrennikoff and S. Tezcan: "Analysis of Cylindrical Shells by the Finite Element Method". International Conference on Shell Structures, Leningrad, U.S.S.R., 1966.

have an advantage in the computer work over the statics matrix of the no-bar cell, which does not possess the symmetry.

5. The continuity of the displacements on the intercell boundaries is not compulsory for the validity of the method.

6. The precision of the results obtained with the three matrices is not greatly different for the same shape and size of cells.

Appendix. Notation

a	small base of a trapezoidal cell
E	modulus of elasticity
h	height of a trapezoidal cell
k	ratio of bases of a trapezoid
k_1	ratio of height to base of a trapezoid
l	distance along boundary of cell
t	plate thickness
u, v	displacements along x and y axes, nodal displacements
x, y	coordinates, coordinate axes
X, Y	nodal forces, terms of stiffness matrix
α	base angle in a trapezoidal cell
ϵ	normal strain
γ	shearing strain
σ, $\sigma_{\perp}$	normal stress, normal stress on cell boundary
τ	shearing stress, shearing stress on cell boundary
$\lambda_{\perp}$, λ_{11}	displacements on cell boundary perpendicular and parallel to it
λ_x, λ_y	displacements on cell boundary parallel to x and y axes
μ	Poisson's ratio
Δ	nodal displacement

Summary

The purpose of this paper is clarification of some aspects of the Finite Element method which so far have not been studied sufficiently closely.

Two kinds of cells used in the analysis of plane stress by the Finite Element method, framework cells and the no-bar cells, are examined in detail. Two distinct types of stiffness matrix associated with the latter are presented in explicit form for a cell having the shape of an equilateral trapezoid. Some inconsistencies inherent in these matrices are pointed out and their effect on the results is discussed.

Résumé

Ce papier a pour but de mettre plus de clarté dans quelques aspects de la méthode des éléments finis, qui n'ont pas encore été étudiés à fond jusqu'à ce jour.

Deux éléments utilisés par la méthode des éléments finis dans l'analyse de contraintes bi-dimensionnelles, sont examinés en détail: l'élément de barre et l'élément de plaque. Deux types distincts de matrice associée avec ces éléments sont présentés explicitement pour un élément ayant la forme d'un trapèze équilatéral. L'attention est tirée sur quelques irrégularités inhérentes à ces matrices, et leur effet sur le résultat est discuté.

Zusammenfassung

Zweck dieser Schrift ist, verschiedene Auffassungen des Endlichen-Elementen-Verfahrens zu klären, derart wie es bis jetzt in seinem Umfange noch nicht geschehen ist.

Zwei Zellenarten, die in der Analyse der ebenen Spannungen vom Endlichen-Elementen-Verfahren angewandt werden, nämlich Stabwerkszelle und Scheibenelement, werden genau untersucht. Zwei verschiedene Steifigkeitsmatrizen, mit letzteren verbunden, werden explizit für ein Trapezelement aufgeführt. Auf einige innewohnenden Unstetigkeiten dieser Matrizen wird hingewiesen und der Einfluß auf das Ergebnis untersucht.

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