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Autor: Zaslavsky, Aron

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Beams on Immovable Supports

Poutres sur appuis fixes

Träger auf unbeweglichen Auflagern

ARON ZASLAVSKY
Associate Professor of Civil Engineering, Technion, Israel Institute of Technology, Haifa, Israel

Introduction

Beams are usually represented as borne by one immovable support, the others being assumed as movable. This assumption is an idealisation, since even a support on rollers is restrained by considerable friction.

In this paper, simple beams are elastically analysed under the opposite extreme limiting assumption: that of perfectly immovable supports. (Actually either the supports may yield and/or stresses become so high as to render their transmission impossible.) A rigorous solution (though neglecting shear forces and, of course, stress concentrations near concentrated forces) has been given by WASZCZYSZYN and ZYCZKOWSKI [1] for a beam loaded by a concentrated force at midspan, the mathematics being rather involved and not very suitable for engineering purposes. A simpler and sufficiently accurate analysis has been given by TIMOSHENKO for cylindrically-bent plates [2]. In both works, however, the analysis is based on the assumption that the supports act at the level of the neutral axis. In beams supported at their bottom surface this assumption is misleading, and therefore the main part of the paper deals with this latter case.

Longitudinal Changes

Fig. 1 shows the deflected shape of a simply-supported beam in bending. The reduction Δn of the span at the level of the *axis* equals:

$$\Delta n = \int_0^l (ds - dx) \simeq \frac{1}{2} \int_0^l \left(\frac{dy}{dx}\right)^2 dx. \tag{1}$$

Substituting $\frac{dy}{dx}$ from the equation of the elastic deflection line for the corresponding loading, Eq. (1) may be written in the form:

$$\Delta n = \beta \frac{f^2}{l}, \tag{2}$$

where the coefficient β varies only little for the most important cases, as shown in Table: $\beta = 2.4 - 2.78$.

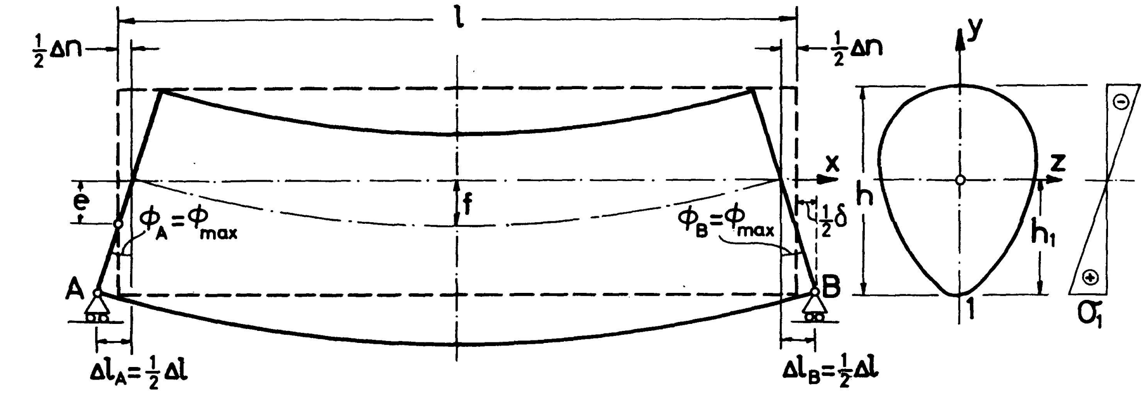

Fig. 1.

Table

		$\Delta n = \beta \frac{f^2}{l}$	$\Delta l = \gamma f \frac{h}{l}$
1	$l/2$, P, f, l	$2.4 \frac{f^2}{l}$	$3.0\, f \frac{h}{l}$
2	$\bar{q}$, $q = \bar{q} \sin \pi x/l$	$2.47 \frac{f^2}{l}$	$\pi f \frac{h}{l}$
3	q	$2.51 \frac{f^2}{l}$	$3.2\, f \frac{h}{l}$
4	M, M	$2.67 \frac{f^2}{l}$	$4.0\, f \frac{h}{l}$

The extension Δl of the *bottom fibre* equals:

$$\Delta l = \int_0^l \epsilon_1 dx = \int_0^l \frac{\sigma_1}{E} dx = \int_0^l \frac{M h_1}{EI} dx = h_1 \int_0^l \frac{M}{EI} dx. \tag{3}$$

The last integral represents the reduced moment area and equals the sum of the slopes $\phi_A + \phi_B$ multiplied by h_1, which result may be directly written down from examination of Fig. 1:

$$\Delta l = h_1 (\phi_A + \phi_B). \tag{4}$$

In the case of symmetrical loading $\phi_A = \phi_B = \phi_{max} = (dy/dx)_{max}$:

$$\Delta l = 2 h_1 \phi_{max}. \tag{5a}$$

And if, in addition, the section also has two axes of symmetry:

$$\Delta l = h\,\phi_{max}. \tag{5}$$

Eq. (5) may be given the form:

$$\Delta l = \gamma h \frac{f}{l}, \tag{6}$$

where the coefficient γ (varying from 3.0 to 4.0) is shown in the Table. While Δl is proportional to the loading, Δn is not.

The displacement δ of the movable support of a simply (at the bottom) supported beam equals

$$\delta = \Delta l - \Delta n = f(\gamma h - \beta f)/l. \tag{6a}$$

Criterion for the Horizontal Reactions

If the beam is supported at the level of its axis by immovable supports, horizontal reactions H will appear, always producing a *tensile* axial force since the span tends to shorten. However, if the immovable supports act at the bottom, the reactions H will produce compression whenever $\Delta l_0 > \Delta n_0$ or, by Eqs. (2) and (6), whenever

$$f_0 < \frac{\gamma}{\beta} h \tag{7}$$

or approximately when

$$f_0 < 1.25\,h, \tag{7,a}$$

which is the case in ordinary practice.

The subscript 0 has been added to indicate that Δn_0, Δl_0 and f_0 apply to the ordinary simply-supported beam (with one movable support).

In the case $f_0 = \frac{\gamma}{\beta} h$, no horizontal reactions are involved, but will appear in the course of the loading history (and change sign in the case $f_0 > \frac{\gamma}{\beta} h$; see Fig. 6).

By adjusting the level of the supports the reactions H may be regulated. For instance, in order to obtain $H = 0$ the supports should be raised to the level e (Fig. 1):

$$e = \frac{\frac{1}{2} n_0}{\phi_{0,max}} = \frac{\beta f_0}{2\gamma} \tag{8}$$

e depending on the magnitude and type of loading.

For case 1 (Table) we have:

$$e = 0.4 f_0 = \frac{P l^3}{120\,EI}. \tag{8a}$$

If the supports act at the bottom then $e = \frac{1}{2} h$ and the load producing $H = 0$ will be found by (8) from

$$f_0 = \frac{\gamma h}{\beta}; \tag{8b}$$

for case 1 this leads to

$$P_{H=0} = \frac{60\,E\,I\,h}{l^3}. \tag{8c}$$

Loads smaller or greater than $P_{H=0}$ produce horizontal reactions H.

Beams Supported at Axis Level (Fig. 2)

The analysis involves the two unknowns H and $f = f_0 - f_H$, where f_0 is the deflection produced by the loads in an ordinary simply-supported beam and

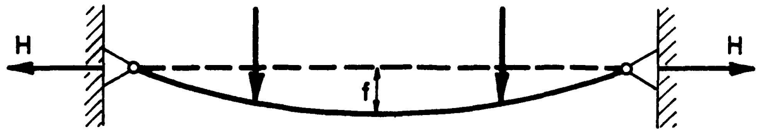

Fig. 2.

f_H the deflection produced by the reactions H. Assuming approximately a sinusoidal deflection curve we have

$$f_H = \frac{H\,f\,l^2}{\pi^2 E\,I}$$

and

$$f = f_0 - f_H = f_0 - \frac{H\,f\,l^2}{\pi^2 E\,I},$$

$$f = \frac{f_0}{1 + f_H/f} = \frac{f_0}{1 + \frac{H\,l^2}{\pi^2\,EI}} = \frac{f_0}{1+\alpha}, \tag{9}$$

where $\alpha = \frac{H\,l^2}{\pi^2\,EI}$ and the denominator $(1+\alpha)$ denotes a reduction factor with regard to f_0.

The reactions H also produce, in addition to f_H, the elongation $\frac{H\,l}{E\,A}$, so that the condition for the immovability of the supports is:

$$\Delta\, n_M - \frac{H\,l}{E\,A} = 0. \tag{10}$$

$\Delta\, n_M$ being due to the bending moments only.

Or, by Eqs. (2) and (9):

$$H = \frac{\beta\,EA\,f^2}{l^2} = \frac{\beta\,EA}{l^2}\,\frac{f_0^2}{\left(1 + \frac{H\,l^2}{\pi^2\,EI}\right)^2} = \frac{\beta\,EA}{l^2}\,\frac{f_0^2}{(1+\alpha)^2}, \tag{11}$$

where $\beta = \pi^2/4 = 2.47$ (see Table, case 2).

From Eq. (11) the reaction H will be found, and from Eq. (9) — also f. In many practical cases $\alpha \ll 1$ and Eq. (11) reduces to

$$H \cong \frac{2.4\,EA\,f_0^2}{l^2}. \tag{12}$$

Examples. Consider first the INP 20 ($I = 2140\,\mathrm{cm}^4$; $Z = 214\,\mathrm{cm}^3$) beam of Fig. 3 (dead weight neglected). In this beam both $\sigma_{max} = \sigma_{adm} = \pm 1400\,\mathrm{kg/cm}^2$, and $f_0 = l/400 = 1.125$ cm are utilised when one support is movable. When both supports are immovable, however, we obtain from Eqs. (11) and (9):

$H = 1050$ kg ($\sigma_H = H/A = 32\,\mathrm{kg/cm}^2$) and $f = f_0/1.005$. We see that here the influence of H is very small. (A reduction in temperature by 1.5° would produce the same H.)

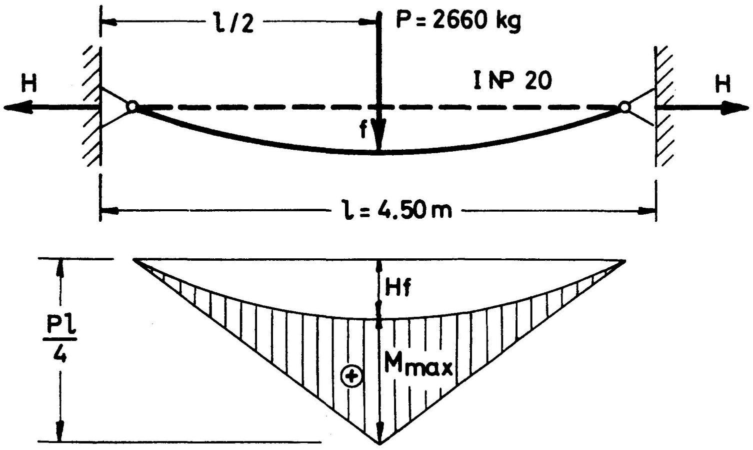

Fig. 3.

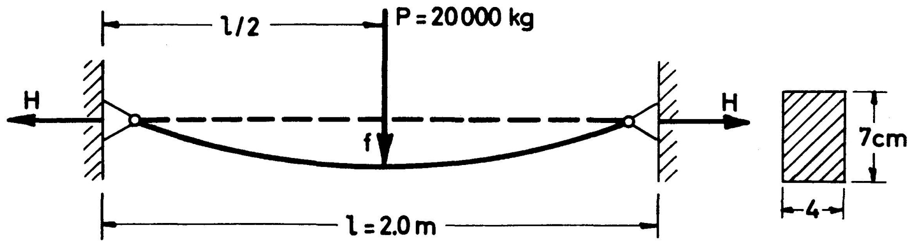

Fig. 4.

Even when the beam is twice as long ($l = 9.0$ m) and loaded by $P = \frac{1}{2} \times 2660 = 1330$ kg (i. e. $\sigma_{max} = \sigma_{adm}$; $f_0 = l/200 = 4.5$ cm) the horizontal reaction is not considerable:

$$H \cong 4 \times 1050 = 4200\ \mathrm{kg}; \qquad f \cong f_0/1.20 = 4.4\ \mathrm{cm}.$$

The maximum tensile normal stress at midspan equals:

$$\begin{array}{lll} \frac{1}{4} Pl/Z = & & 1400 \\ -Hf/Z = & -4200 \times 4.4/214 = & -\ \ 86 \\ +H/A = & 4200/33.5 = & +\ 125 \\ \hline & & 1439\ \mathrm{kg/cm}^2 \end{array}$$

As an opposite example, showing a very large H, consider the rectangular steel beam of Fig. 4 ($A = 28\,\mathrm{cm}^2$; $I = 114\,\mathrm{cm}^4$; $Z = 32.7\,\mathrm{cm}^3$; $E = 2.1 \times 10^6\,\mathrm{kg/cm}^2$) for which $f_0 = 13.88$ cm. This example was given by WASZCZYSZYN and ZYCZKOWSKI [1], who obtained $H = 97{,}508$ kg and $f = 5.34$ cm.

Our calculation by Eqs. (11) and (9), using $\beta = 2.47$, yields: $H \simeq 98{,}500$ kg and $f \simeq f_0/2.65 = 5.24$ cm, which is in good agreement with the rigorous results. It may be noted that this example does not represent usual practical cases of steel beams since the deflection is of the order $l/40$ and the maximum normal stress equals:

$$
\begin{array}{lll}
\frac{1}{4} P l/Z & = \frac{1}{4} 2000 \times 200/32.7 \simeq & +30{,}600 \\
-H f/Z & = 98{,}500 \times 5.24/32.7 \simeq & -15{,}800 \\
+H/A & = 98{,}500/28 \simeq & +\ \ 3{,}520 \\
\hline
 & & 18{,}320 \text{ kg/cm}^2
\end{array}
$$

Beams Supported at the Bottom Surface

Fig. 5 shows a beam on immovable supports. The section is assumed to have two axes of symmetry and a concentrated load P acts at midspan (but the following analysis is equally applicable for different loading conditions and sections).

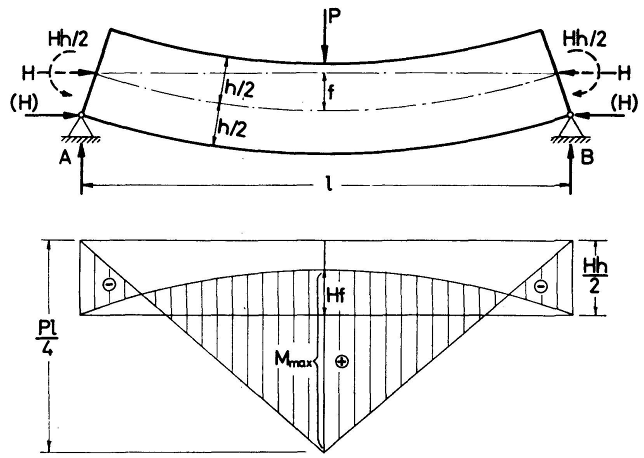

Fig. 5.

Assuming compressive horizontal reactions H, they may be equivalently represented by bending moments $H h/2$ (producing f'_H and $\Delta l'_{H,M}$) and axial forces H causing both bending moments with maximum $H f$ (producing f''_H and $\Delta l''_{H,M}$) and a contraction $\frac{H l}{E A}$.

We have therefore the condition:

$$\Delta l_M - \frac{H l}{E A} - \Delta n_M = 0, \tag{13}$$

where Δl_M and Δn_M are due to the combined action of all the bending moments. Δn_M may be represented by Eq. (2) using again $\beta = 2.47$; Δl_M may be represented by Eq. (5):

$$\Delta l_M = \Delta l_0 - (\Delta l'_{H,M} - \Delta l''_{H,M}) = h \left[\phi_{0,max} - \left(\frac{H h l}{4 E I} - \frac{H f l}{\pi E I}\right)\right]. \tag{14}$$

Eq. (13) then reads:

$$h\left[\phi_{0,max} - \frac{H\,l}{E\,I}\left(\frac{h}{4} - \frac{f}{\pi}\right)\right] - \frac{H\,l}{EA} - \frac{\beta f^2}{l} = 0 \tag{15a}$$

or:

$$\frac{H}{E\,I} = \frac{\phi_{0,max} - \frac{\beta}{h}\frac{f^2}{l}}{\frac{l\,I}{h\,A} + \frac{h\,l}{4} - \frac{l\,f}{\pi}}. \tag{15b}$$

In our case $\phi_{0,max} = \frac{P\,l^2}{16\,E\,I}$ and for $H=0$ Eq. (15b) reduces to (8c) since then $f=f_0$.

On the other hand the deflection equals:

$$f = f_0 - f_H = f_0 - (f'_H - f''_H) = f_0 - \frac{H\,l^2\,h}{16\,E\,I} + \frac{H\,l^2\,f}{\pi^2\,E\,I},$$

$$f = \frac{f_0 - \frac{H\,l^2\,h}{16\,E\,I}}{1 - \frac{H\,l^2}{\pi^2\,E\,I}} = \frac{f_0 - 0.62\,\alpha\,h}{1-\alpha}. \tag{16}$$

In our case $f_0 = P\,l^3/(48\,E\,I)$.

When H approaches Euler's buckling load ($\alpha \to 1$), f becomes very large, as it is always the case in compressed beam-columns. (It is assumed that buckling is precluded perpendicularly to the plane of the Figure, and also that local buckling does not occur.) From Eqs. (15) and (16) H and f may be calculated. Eq. (15b) reduces to simple proportionality for small values of f when $\Delta\,n$ and $\Delta\,l''_{H,M}$ (i.e. $H\,f$) may be neglected. In the case of concentrated force P at midspan, this yields (Z-section modulus):

$$H \cong \frac{P\,l}{4\,h + 8\,Z/A} \tag{17}$$

and in the case of a uniformly distributed load $Q = q\,l$:

$$H \cong \frac{Q\,l}{6\,h + 12\,Z/A}. \tag{18}$$

Eqs. (17) and (18) may be also derived by Castigliano's theorem (U-elastic energy): $\partial\,U/\partial\,H = 0$.

If in addition the beam is subjected to a uniform *temperature* $+\,t$ (α_t – thermal coefficient) Eq. (13) becomes:

$$\Delta\,l_M - \frac{H\,l}{EA} - \Delta\,n_M + \alpha_t\,l\,t = 0 \tag{13a}$$

and Eq. (15b):

$$\frac{H}{E\,I} = \frac{\phi_{0,max} - \frac{\beta}{h}\frac{f^2}{l} + \frac{\alpha_t\,l\,t}{h}}{\frac{l\,I}{h\,A} + \frac{h\,l}{4} - \frac{l\,f}{\pi}} \tag{15c}$$

and Eq. (17):

$$H \cong \frac{Pl + 16\,E\,I\,\alpha_t t/h}{4h + 8\,Z/A}. \tag{18a}$$

(Likewise a small movement of the supports may be accounted for.)

Examples. Referring to the beam of Fig. 3, supported at the bottom, we find from Eq. (15) and (16):

$$H \cong 9300\text{ kg} \quad \text{and} \quad f \cong 0.65\text{ cm} < 1.125\text{ cm} = f_0.$$

The linear Eq. (17) yields $H \cong 9150$ kg (only 1.6% less). The maximum normal (compressive) stress at midspan equals:

$$\begin{array}{rlr}
\tfrac{1}{4}Pl/Z & = & 1400 \\
-\tfrac{1}{2}Hh/Z & = 9300 \times 10/214 \;= & -\;435 \\
Hf/Z & = 9300 \times 0.65/214 = & 28 \\
H/A & = 9300/33.5 \;= & \underline{276} \\
 & & 1271\text{ kg/cm}^2 < 1400
\end{array}$$

Taking $l = 9.0$ m and $P = 1330$ kg we obtain from Eq. (17) the same $H = 9150$ kg but more accurately from (15) and (16): $H = 9500$ kg; $f = 2.85$ cm < 4.4 cm. The maximum stress remains below the admissible value (1367 kg/cm²) and the loading history for this beam would show that up to $P = 3000$ kg there is no considerable deviation from proportionality.

Now consider the beam of Fig. 4, with supports at the bottom. Although, for the given load $P = 20{,}000$ kg, this is a hypothetical case, it may be of interest to analyse its loading history.

Using Eq. (7) as a criterion for H, we expect it to be *tension:*

$$1.25\,h = 8.75\text{ cm} < f_0 = 13.88\text{ cm}.$$

From Eq. (15) and (16) we obtain $H \cong -38{,}500$ kg and $f \cong 10.2$ cm, the minus sign denoting tension. The maximum normal (tensile) stress at midspan equals:

$$\begin{array}{rlr}
\tfrac{1}{4}Pl/Z & = & +30{,}600 \\
+\tfrac{1}{2}Hh/Z & = \;\;38{,}500 \times 3.5/32.7 \;= & +\;\;4{,}120 \\
-Hf/Z & = -38{,}500 \times 10.2/32.7 = & -12{,}000 \\
+H/A & = +38{,}500/28 \;= & \underline{+\;\;1{,}375} \\
 & & +24{,}100\text{ kg/cm}^2
\end{array}$$

The load $P_{H=0}$, for which $H = 0$, may be readily found by Eq. (8c):

$$P_{H=0} = 60\,E\,I\,h/l^3 = 60 \times 2.1 \times 10^6 \times 114.3 \times 20/200^3 \cong 12{,}600\text{ kg}.$$

The loading history is shown in Fig. 6. It is seen that H changes from compression (+) to tension (−) passing through a maximum and through zero at $P_{H=0}$, after which it continues to increase quickly while the deflection f and the stresses are flattening off. Both tensile (σ_{max}) and compressive (σ_{min}) normal stresses at midspan are shown, their mutual ratio changing in the course of history; also $10\,\delta$ is indicated (Eq. (6, a)).

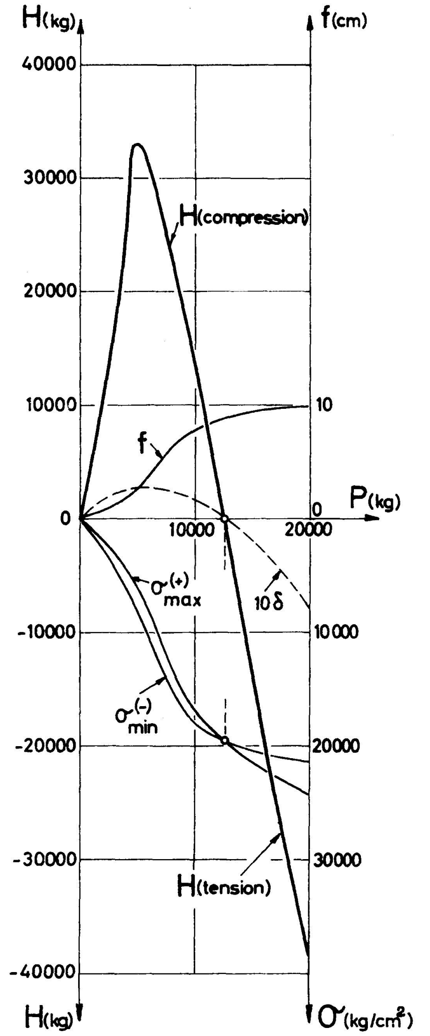

Fig. 6.

Using a simply supported rectangular rubber beam, the movement δ of its roller-support under increasing load P followed Eq. (6, a): first the movement was outward and later inward.

From Fig. 6 we also see that $H/P \simeq 5$, i. e. in the case of a *movable* support its movement would start from the very beginning and would be resisted by $H = \frac{1}{2} \mu P$, μ being the friction coefficient.

It should be noted that the analysis of the loading history in cases like this (non-linear, non-monotonic relationship) is indispensable for a complete picture. Also, the question of the safety factor k arises. Although we are dealing with elastic analysis, the admissible load P_{adm} should be defined here as $P_{adm} = P_Y/k$, where P_Y is the load producing the first yield in the material.

It has, of course, been assumed that the supports and connections are capable of transmitting the large forces, that the material is still in the proportionality range, and that transverse buckling is precluded, the Euler load being $H_{cr} = 18{,}000$ kg, while in the plane of loading it equals $H_{cr} = 59{,}000$ kg. (Should buckling occur, the beam would be relieved of part of the reaction H.)

Conclusions

An approximate "second-order" elastic analysis leading to simple algebraic equations is presented for simple beams supported at their bottom surface under the limiting assumption of perfectly immovable supports. In beams of high-strength material the axial force produced by the horizontal reactions may change from compression to tension under increased loading and deflections. By changing the level of the supports the horizontal reactions may be regulated.

References

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Summary

Simple steel beams supported at their bottom surface are analysed for the limiting case of perfectly immovable supports. It is shown that in most practical cases the horizontal reactions cause compression, but tension is also possible. The case of supports acting at the level of the neutral axis is also reviewed. The analysis uses approximations permitting very simple calculations. Numerical examples are given.

Résumé

L'auteur étudie des poutres simples métalliques dont les appuis, situés à la hauteur de la membrure inférieure, sont considérés comme parfaitement fixes. Pour ce cas limite, les réactions horizontales mettent presque toujours la poutre en compression, mais une traction est aussi possible. On examine également le cas d'appuis agissant à la hauteur de l'axe neutre. L'auteur utilise une méthode approximative, permettant des calculs simples. Il donne des applications numériques.

Zusammenfassung

Auf dem unteren Flansch aufgelagerte einfache Stahlträger werden für den Grenzfall unbeweglicher Auflagerung untersucht. Dabei zeigt sich, daß in den meisten praktischen Fällen die horizontalen Auflagerkräfte Druckspannungen im Träger erzeugen, wobei aber auch Zugspannungen möglich sind. Der Sonderfall von Auflagerung auf der Höhe der Neutralachse wird ebenfalls untersucht. Die theoretischen Näherungen erlauben eine einfache Berechnung, wie anhand numerischer Beispiele gezeigt wird.