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Analysis of Plane and Space Frameworks with Curved Members

Calcul des structures bi- et tridimensionnelles comportant des éléments courbes

Berechnung ebener und räumlicher Rahmentragwerke mit gekrümmten Elementen

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1. Introduction

The stiffness method of analysis in conjunction with high speed digital computers has proved to be the most efficient tool in structural engineering. With the stiffness matrices of typical individual members established, the method of analysis is the same for a great variety of structures such as plane trusses, plane frames, plane grids, space trusses or space frames. Although there is abundance of literature for the stiffness matrices of straight members [1, 2, 3], there is not sufficient material available for the stiffness matrices of curved members.

Usually, the curved members in a structure are replaced by a series of straight members. The disadvantages of this replacement are the great increase in the number of degrees of freedom of the structure and the approximations involved in the analysis. For instance, as will be demonstrated later, an analysis of a semi-circular arch, to an acceptable degree of accuracy, would require at least more than twenty straight members. This would mean that a spherical dome with one hundred circular parts would involve some two thousand straight members. This increase in the number of members may cause a serious problem in regards to the limited core memory capacities of the computers.

In the following presentation, stiffness matrices are developed for the circular members of space frames, plane frames and plane grids. At first, the stiffness matrix of a space member is determined relative to the radial, tangential and transverse axes of the member. Then, through successive orthogonal transformations, the member stiffness matrix is transformed to the common coord nate system. The use of common coordinates in the stiffness matrices of individual members is imperative for generation of the main stiffness matrix of the structure by direct combination of appropriate matrix elements of the members. The stiffness matrices presented for the circular curved members are very general and they may be used even for straight members by equating the radius of curvature to infinity and the central angle to zero in the results.

2. Coordinate Axes and Sign Convention

Circular curved members with doubly symmetric cross sections will be considered. The stress resultants and deformations in the following discussion of each member will be referred to a right-hand orthogonal coordinate system xyz. These "Member Axes" are different for different members. The tangential axis directed from the i end towards the j end is taken as the y-axis, while the transversal and radial directions, which are also the principal inertia axes of the cross section, are taken as the x- and z-axes respectively, as shown in Fig. 1. For consistency, the principal inertia axis within the plane of curvature

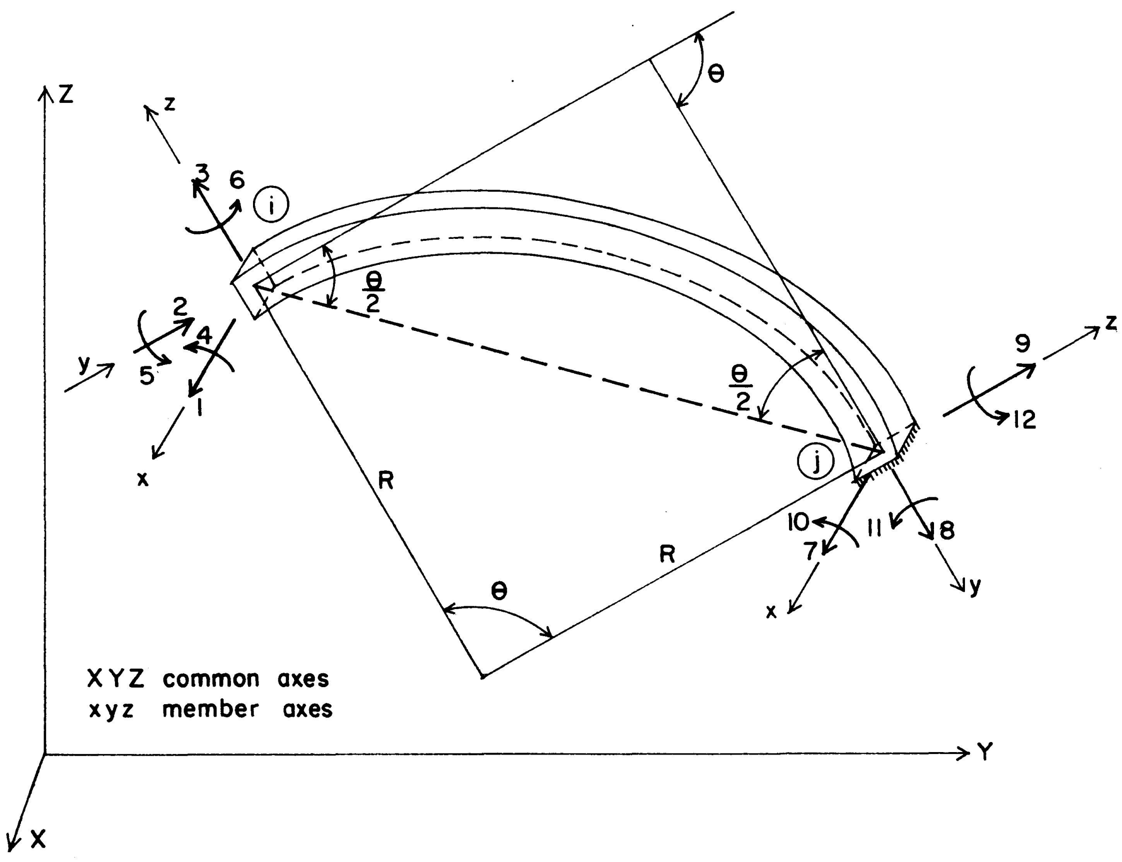


Fig. 1. Coordinate systems and numbering of deformations.

is taken as the member's z-axis. The positive sense of the z-axis is so determined that it always makes an angle smaller than 90° with the common Z-axis.

The joint deformations and external loads on the structure are expressed relative to a global coordinate system XYZ, called the "Common Axes". Positive directions of rotations and moments are determined in accordance with the right hand screw rule.

3. Stiffness Matrix of a Space Member Relative to the Member Axes

Flexibility influence coefficients. The flexibility influence coefficients of a curved member may be obtained from the unit load theorem in the following manner:

$$f_{km} = \int_s \left[\left(\frac{M_k M_m}{E I_x}\right)_{yz} + \left(\frac{M_k M_m}{E I_z}\right)_{xy} + \left(\frac{N_k N_m}{EA}\right)_y + k'\left(\frac{V_k V_m}{GA}\right)_x + k'\left(\frac{V_k V_m}{GA}\right)_z + \left(\frac{T_k T_m}{GJ}\right)_{xz}\right] ds, \tag{1}$$

where f_{km} is the deformation in the "k"th direction due to a unit load in the "m"th direction. Unit loads are applied non-concurrently first in the "k"th and then in the "m"th direction at the free i end of the member, following which the algebraic expressions of the bending moment M, axial force N, shear V and torque T, are evaluated.

These are substituted into Eq. (1) for determination of the flexibility coefficients f_{km}. After repetition of this procedure for each of the six degrees of freedom at the end i, the following flexibility matrix f_i is obtained:

$$[f]_i = \begin{bmatrix} f_{11} & 0 & 0 & 0 & f_{51} & f_{61} \\ 0 & f_{22} & f_{32} & f_{42} & 0 & 0 \\ 0 & f_{32} & f_{33} & f_{43} & 0 & 0 \\ 0 & f_{42} & f_{43} & f_{44} & 0 & 0 \\ f_{51} & 0 & 0 & 0 & f_{55} & f_{65} \\ f_{61} & 0 & 0 & 0 & f_{65} & f_{66} \end{bmatrix}, \tag{2}$$

in which the individual flexibility coefficients f_{km} are

$$\begin{aligned} f_{11} &= R^3 a/E I_z + R^3(2b-a)/GJ + k' R\theta/GA, \\ f_{51} &= R^2 a/E I_z - R^2(\sin\theta - c)/GJ, \\ f_{61} &= R^2 e/E I_z + R^2(d-e)/GJ, \\ f_{22} &= R^3(2b-a)/E I_x + Rc/EA + k' Ra/GA, \\ f_{32} &= R^3(d-e)/E I_x - Re/EA + k' e/GA, \end{aligned} \tag{2a}$$

$$\begin{aligned} f_{42} &= -R^2 b/E I_x; \quad f_{43} = -R^2 d/E I_x; \quad f_{44} = R\theta/E I_x, \\ f_{33} &= R^3 a/E I_x + R a/E A + k' R c/G A, \\ f_{55} &= R a/E I_z + R c/G J; \quad f_{65} = R e/E I_z - R e/G J, \\ f_{66} &= R c/E I_z + R a/G J. \end{aligned} \tag{2a}$$

The trigonometric terms a, b, c, d and e in these expressions are

$$\begin{aligned} a &= (\theta - \tfrac{1}{2}\sin 2\theta)/2; \qquad c = (\theta + \tfrac{1}{2}\sin 2\theta)/2; \\ b &= \theta - \sin\theta; \qquad d = 1 - \cos\theta; \qquad e = \tfrac{1}{2}\sin^2\theta. \end{aligned} \tag{2b}$$

The stress resultants $\{p\}_i$ at the i end may then be related to the deformations $\{\delta\}_i$ by means of the inverse of the flexibility matrix as

$$\{p\}_i = [f]_i^{-1}\{\delta\}_i. \tag{3}$$

Static Equilibrium Matrix. Considering the static equilibrium of the member, the six stress resultants $\{P\}_j$ at the j end may be expressed in terms of those at the i end as follows:

$$\{P\}_j = [S]\{P\}_i \tag{4}$$

in which S is the equilibrium matrix given by

$$[S] = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\cos\theta & \sin\theta & 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta & 0 & 0 & 0 \\ 0 & R - R\cos\theta & R\sin\theta & -1 & 0 & 0 \\ R - R\cos\theta & 0 & 0 & 0 & -\cos\theta & \sin\theta \\ -R\sin\theta & 0 & 0 & 0 & -\sin\theta & -\cos\theta \end{bmatrix}. \tag{5}$$

It is possible to derive from energy considerations, that a relation similar to Eq. (4) exists between the deformations of the i and j ends of the member as follows:

$$\{\delta\}_i = -[S]^T\{\delta\}_j. \tag{6}$$

Stiffness Matrix relative to the Member Axes. By making use of Eqs. (3), (4) and (6), the stiffness matrix k_{xyz} of a curved space member relative to the member axes may be obtained from the following relation:

$$\begin{Bmatrix} \{P\}_i \\ \{P\}_j \end{Bmatrix} = \left[\begin{array}{c|c} [f]_i^{-1} & [f]_i^{-1}[S]^T \\ \hline [S][f]_i^{-1} & [S][f]_i^{-1}[S]^T \end{array}\right] \begin{Bmatrix} \{\delta\}_i \\ \{\delta\}_j \end{Bmatrix} \quad (12 \text{ by } 12) \tag{7}$$

or

$$\{P\} = [k]_{xyz}\{\delta\} \quad (12 \text{ by } 12), \tag{7a}$$

in which:

$$[k]_{xyz} = \begin{bmatrix} k_{11} & \cdot & \cdot & \cdot & k_{51} & k_{61} & -k_{11} & \cdot & \cdot & \cdot & -k_{51} & k_{61} \\ \cdot & k_{22} & k_{32} & k_{42} & \cdot & \cdot & \cdot & k_{82} & -k_{83} & k_{84} & \cdot & \cdot \\ \cdot & k_{32} & k_{33} & k_{43} & \cdot & \cdot & \cdot & k_{83} & k_{93} & -k_{94} & \cdot & \cdot \\ \cdot & k_{42} & k_{43} & k_{44} & \cdot & \cdot & \cdot & k_{84} & k_{94} & k_{10,4} & \cdot & \cdot \\ k_{51} & \cdot & \cdot & \cdot & k_{55} & k_{65} & -k_{51} & \cdot & \cdot & \cdot & k_{11,5} & -k_{11,6} \\ k_{61} & \cdot & \cdot & \cdot & k_{65} & k_{66} & -k_{61} & \cdot & \cdot & \cdot & k_{11,6} & k_{12,6} \\ -k_{11} & \cdot & \cdot & \cdot & -k_{51} & -k_{61} & k_{11} & \cdot & \cdot & \cdot & k_{51} & -k_{61} \\ \cdot & k_{82} & k_{83} & k_{84} & \cdot & \cdot & \cdot & k_{22} & -k_{32} & k_{42} & \cdot & \cdot \\ \cdot & -k_{83} & k_{93} & k_{94} & \cdot & \cdot & \cdot & -k_{32} & k_{33} & -k_{43} & \cdot & \cdot \\ \cdot & k_{84} & -k_{94} & k_{10,4} & \cdot & \cdot & \cdot & k_{42} & -k_{43} & k_{44} & \cdot & \cdot \\ -k_{51} & \cdot & \cdot & \cdot & k_{11,5} & k_{11,6} & k_{51} & \cdot & \cdot & \cdot & k_{55} & -k_{65} \\ k_{61} & \cdot & \cdot & \cdot & -k_{11,6} & k_{12,6} & -k_{61} & \cdot & \cdot & \cdot & -k_{65} & k_{66} \end{bmatrix} \tag{8}$$

Space Frame Member

and, the individual stiffness coefficients are

$$\begin{aligned}
k_{11} &= (f_{55} f_{66} - f_{65}^2)/W, & k_{51} &= (f_{61} f_{65} - f_{51} f_{66})/W, \\
k_{61} &= (f_{51} f_{65} - f_{55} f_{61})/W, & k_{22} &= (f_{33} f_{44} - f_{43}^2)/U, \\
k_{32} &= (f_{42} f_{43} - f_{32} f_{44})/U, & k_{42} &= (f_{32} f_{43} - f_{33} f_{42})/U, \\
k_{82} &= -k_{22} \cos\theta + k_{32} \sin\theta, & k_{83} &= -k_{32} \cos\theta + k_{33} \sin\theta, \\
k_{84} &= -k_{42} \cos\theta + k_{43} \sin\theta, & k_{33} &= (f_{22} f_{44} - f_{42}^2)/U, \\
k_{43} &= (f_{32} f_{42} - f_{22} f_{43})/U, & k_{93} &= -k_{32} \sin\theta - k_{33} \cos\theta, \\
k_{94} &= -k_{42} \sin\theta - k_{43} \cos\theta, & k_{44} &= (f_{22} f_{33} - f_{32}^2)/U,
\end{aligned}$$

$$k_{10,4} = k_{42} R(1-\cos\theta) + k_{43} R \sin\theta - k_{44}, \tag{8a}$$

$$\begin{aligned}
k_{55} &= (f_{11} f_{66} - f_{61}^2)/W, & k_{65} &= (f_{51} f_{61} - f_{11} f_{65})/W, \\
k_{66} &= (f_{11} f_{55} - f_{51}^2)/W,
\end{aligned}$$

$$\begin{aligned}
k_{11,5} &= k_{51} R(1-\cos\theta) - k_{55} \cos\theta + k_{65} \sin\theta, \\
k_{11,6} &= k_{61} R(1-\cos\theta) - k_{65} \cos\theta + k_{66} \sin\theta, \\
k_{12,6} &= -k_{61} R \sin\theta - k_{65} \sin\theta - k_{66} \cos\theta, \\
W &= [(f_{11} f_{55} - f_{51}^2)(f_{11} f_{66} - f_{61}^2) - (f_{11} f_{65} - f_{51} f_{61})^2]/f_{11}, \\
U &= [(f_{22} f_{33} - f_{32}^2)(f_{22} f_{44} - f_{42}^2) - (f_{22} f_{43} - f_{32} f_{42})^2]/f_{22}.
\end{aligned}$$

4. Transformation from the Member Axes to the Common Axes

Ultimately, all the member stiffness matrices must be reduced to the common axes so that the main stiffness of the system can be generated by direct superposition of the stiffness matrices of individual members. Normally, the member axes $x\,y\,z$ may be directed in any manner. The orthogonal transformation of the member axes $x\,y\,z$, from such a general state of inclination to the common coordinate system can be conveniently achieved by the following three successive transformation operations.

Step 1. The member axes $x y z$ are first rotated through an angle β about the straight line connecting the points i and j, until the member's x-axis becomes horizontal, or the yz-plane becomes vertical. After this transformation, the axes are referred to as x_0, y_0, z_0, as shown in Fig. 2. The transforma-

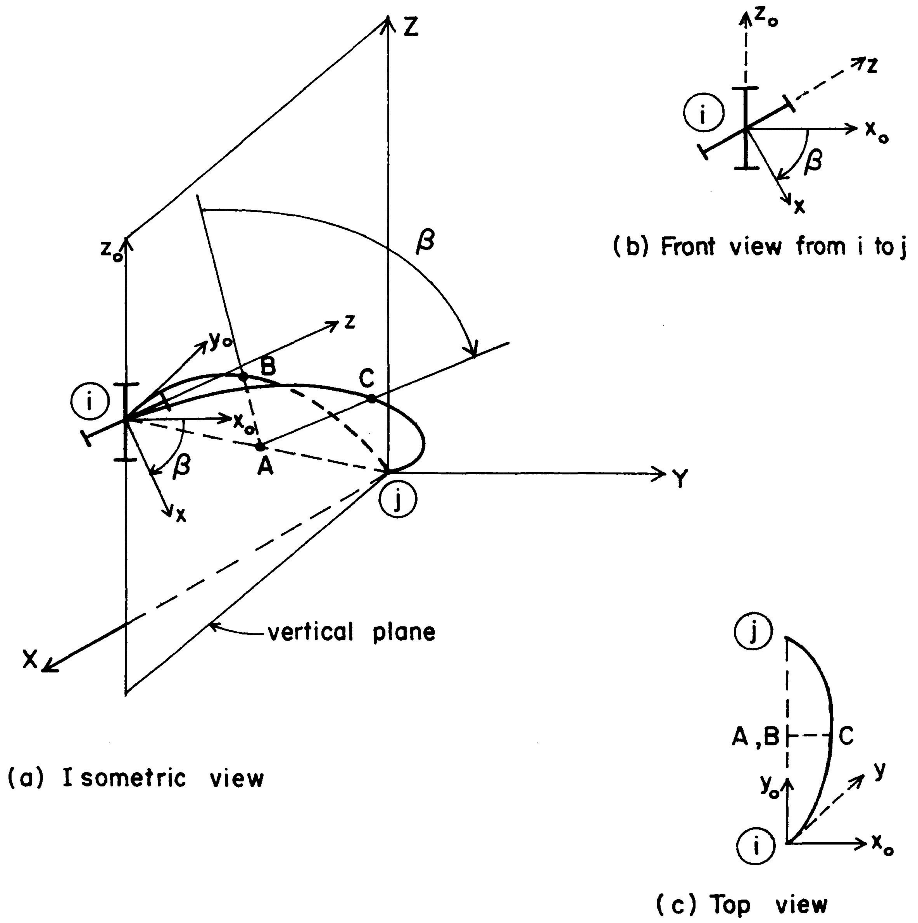


Fig. 2. Transformation from a general plane to the vertical plane.

tion equation corresponding to such rotation is

$$\{P\}_{xyz} = [t]_1 \{P\}_{x_0 y_0 z_0} \tag{9}$$

in which,

$$[t]_1 = \begin{bmatrix} \cos\beta & -\sin\beta\sin\frac{\theta}{2} & -\sin\beta\cos\frac{\theta}{2} \\ \sin\beta\sin\frac{\theta}{2} & \left(1-2\sin^2\frac{\theta}{2}\sin^2\frac{\beta}{2}\right) & -\frac{1}{2}(1-\cos\beta)\sin\theta \\ \sin\beta\cos\frac{\theta}{2} & -\frac{1}{2}(1-\cos\beta)\sin\theta & \left(1-2\cos^2\frac{\theta}{2}\sin^2\frac{\beta}{2}\right) \end{bmatrix}. \tag{9a}$$

The angle β is positive if it is measured clockwise from the positive direction of the horizontal x_0-axis to the positive direction of the actual member's x-axis, when viewed looking straight in the direction from i to j.

Step 2. Imagine a rectangular coordinate system $x_s y_s z_s$ in which the axis y_s co-incides with the straight line between the points i and j and the axis x_s is horizontal, which makes the plane $y_s z_s$ vertical or, in other words, perpendicular to the xy-plane. The axis x_s of this system is co-incidental with the axis x_0. Now rotate the previously rotated coordinate system $x_0 y_0 z_0$ of the curved member about x_0 through an angle $\theta/2$, until the axes y_0 and z_0 coincide with the auxiliary axes y_s and z_s respectively. This rotation may be expressed in its effect on the stress resultants $\{P\}$ by the equation:

$$\{P\}_{x_0 y_0 z_0} = [t]_2 \{P\}_{x_s y_s z_s}, \tag{10}$$

where

$$[t]_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ 0 & -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}. \tag{10a}$$

Step 3. The auxiliary axes $x_s y_s z_s$ are transformed to the common axes $X\,Y\,Z$ by means of the following orthogonal transformation [3]:

$$\{P\}_{x_s y_s z_s} = [t]_3 \{P\}_{XYZ} \quad \text{and} \quad [t]_3 = \begin{bmatrix} m_y/Q & -l_y/Q & 0 \\ l_y & m_y & n_y \\ -l_y n_y/Q & -m_y n_y/Q & Q \end{bmatrix}, \tag{11}$$

in which l_y, m_y, n_y, are the direction cosines of the straight line connecting the points i and j. These direction cosines can be readily obtained from the member end coordinates as $l_y = (X_j - X_i)/L$, $m_y = (Y_j - Y_i)/L$, $n_y = (Z_j - Z_i)/L$, and $Q^2 = 1 - n_y^2$.

The total transformation achieved through Steps 1, 2 and 3 may be combined in a single expression as follows:

$$\{P\}_{xyz} = [t]_i \{P\}_{XYZ} \tag{12}$$

or

$$[t]_i = [t]_1 [t]_2 [t]_3, \tag{12a}$$

in which

$$\{t\}_i = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \quad \text{(3 by 3)} \tag{13}$$

and
$$
\begin{aligned}
t_{11} &= \frac{m_y}{Q}\cos\beta + \frac{l_y n_y}{Q}\sin\beta,\\
t_{21} &= \frac{m_y}{Q}\sin\frac{\theta}{2}\sin\beta + l_y\left(\cos\frac{\theta}{2} - \frac{n_y}{Q}\sin\frac{\theta}{2}\cos\beta\right),\\
t_{31} &= \frac{m_y}{Q}\cos\frac{\theta}{2}\sin\beta - l_y\left(\sin\frac{\theta}{2} + \frac{n_y}{Q}\cos\frac{\theta}{2}\cos\beta\right),\\
t_{12} &= -\frac{l_y}{Q}\cos\beta + \frac{m_y n_y}{Q}\sin\beta,\\
t_{22} &= -\frac{l_y}{Q}\sin\frac{\theta}{2}\sin\beta + m_y\left(\cos\frac{\theta}{2} - \frac{n_y}{Q}\sin\frac{\theta}{2}\cos\beta\right),\\
t_{32} &= -\frac{l_y}{Q}\cos\frac{\theta}{2}\sin\beta - m_y\left(\sin\frac{\theta}{2} + \frac{n_y}{Q}\cos\frac{\theta}{2}\cos\beta\right),\\
t_{13} &= -Q\sin\beta,\\
t_{23} &= n_y\cos\frac{\theta}{2} + Q\sin\frac{\theta}{2}\cos\beta,\\
t_{33} &= Q\cos\frac{\theta}{2}\cos\beta - n_y\sin\frac{\theta}{2}.
\end{aligned}
\tag{13a}
$$

At each end of a space frame member there are three forces and three moments, i.e., altogether six vectors. Therefore, the transformation matrix for a space member, including all twelve stress resultants at both ends, is

$$[T] = \begin{bmatrix} [T]_i & \\ & [T]_j \end{bmatrix}, \tag{14}$$

where
$$[T]_i = \begin{bmatrix} [t]_i & \\ & [t]_i \end{bmatrix} \quad\text{and}\quad [T]_j = \begin{bmatrix} [t]_j & \\ & [t]_j \end{bmatrix}. \tag{14a}$$

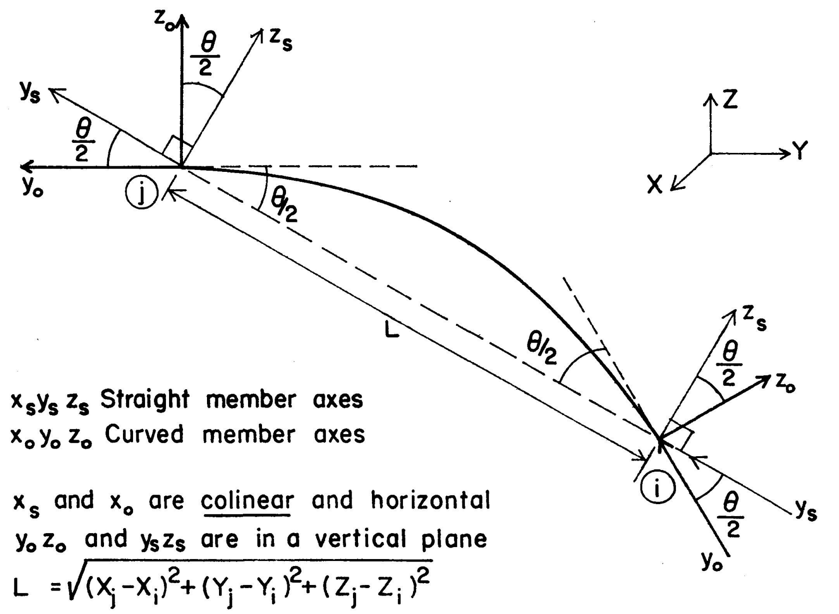


Fig. 3. Transformation from straight member axes to curved member axes.

The transformation matrix t_j for the j end is identical with t_i of the i end, except that $-(\theta/2)$ should be used in t_j instead of $(\theta/2)$.

Note that, in all the above derivations it is assumed that the positive sense of the member z-axis is directed away from the center of curvature. If, however, the curvature of the member were opposite to that which is shown in Fig. 3, the member z-axis would be directed towards the center of curvature so as to satisfy the previous assumption that the z-axis should always make an angle smaller than 90° with the Z-axis. In such a case the numerical values of θ and R in Eqs. (5) and (14) should be used as $-\theta$ and $-R$, in order to account for this direction change in the curvature.

5. Common Axes Stiffness Matrix of a Space Member

Once the transformation matrices T_i and T_j are evaluated from Eq. (14a), the stiffness matrix relative to the member axes $[k]_{xyz}$ should be reduced to the common axes by means of the following standard transformation formula [4]:

$$[k]_{XYZ} = [T]^T [k]_{xyz} [T]. \tag{15}$$

6. Stiffness Matrix of a Plane Frame Member

The positive directions of the end deformations and stress resultants of a plane frame member are shown in Fig. 4. The member is assumed to lie in the YZ-plane and the member x-axis is always taken to be directed parallel to the common X-axis. By selecting the appropriate rows and columns from Eq. (8), in accordance with the numbering system given in Fig. 4, the stiffness

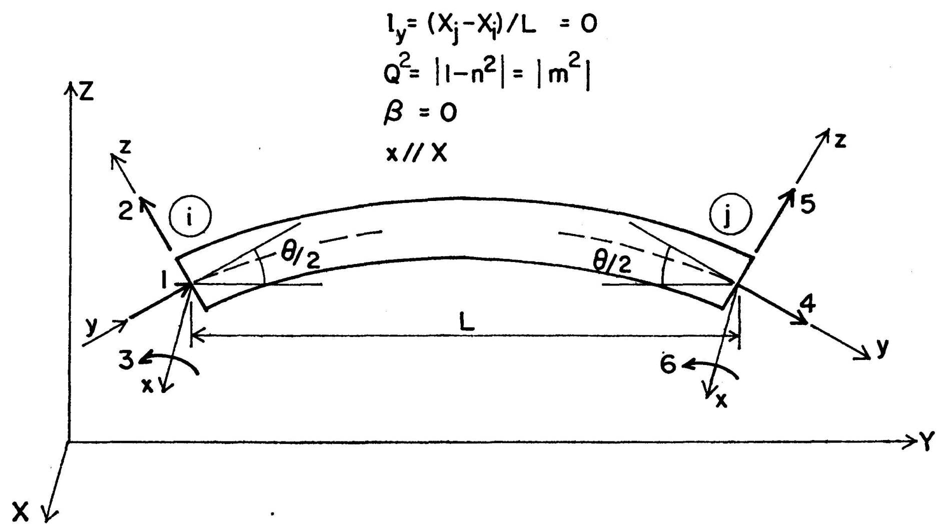


Fig. 4. Curved plane frame member.

matrix of a plane frame member, relative to the member axes, is obtained as

$$\begin{array}{l}[k]_{xyz} = \\ \textit{Plane} \\ \textit{Frame} \\ \textit{Member}\end{array} \begin{bmatrix} k_{22} & k_{32} & k_{42} & k_{82} & -k_{83} & k_{84} \\ k_{32} & k_{33} & k_{43} & k_{83} & k_{93} & -k_{94} \\ k_{42} & k_{43} & k_{44} & k_{84} & k_{94} & k_{10,4} \\ k_{82} & k_{83} & k_{84} & k_{22} & -k_{32} & k_{42} \\ -k_{83} & k_{93} & k_{94} & -k_{32} & k_{33} & -k_{43} \\ k_{84} & -k_{94} & k_{10,4} & k_{42} & -k_{43} & k_{44} \end{bmatrix}. \tag{16}$$

The direction cosine l_y of the member's centerline is zero because the X-coordinates are zero. Therefore, $Q = m$. For plane frame members the angle β is always zero because the principal z-axis lies always in the vertical YZ-plane. Substituting $l_y = \beta = 0$ and $Q = m$ in Eq. (14), and considering that there are only two forces and one moment at the i end of the member, the transformation matrix T_i of Eq. (14) for a plane frame member becomes

$$[T]_i = \begin{bmatrix} m_y \cos\frac{\theta}{2} - n_y \sin\frac{\theta}{2} & n_y \cos\frac{\theta}{2} + m_y \sin\frac{\theta}{2} & 0 \\ -m_y \sin\frac{\theta}{2} - n_y \cos\frac{\theta}{2} & -n_y \sin\frac{\theta}{2} + m_y \cos\frac{\theta}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{17}$$

The common axes stiffness matrix k_{XYZ} of a curved plane frame member is then obtained from Eq. (15) by substituting the values of k_{xyz} and T_i from Eq. (16) and (17), respectively.

7. Stiffness Matrix of a Plane Grid Member

The positive directions of the end deformations and stress resultants of a plane grid member are shown in Fig. 5. The grid member is assumed to lie in the XY-plane and the member x-axis is always vertical, i.e., parallel to the common Z-axis. If appropriate rows and columns are selected from Eq. (8), in accordance with the numbering system shown in Fig. 5, the stiffness matrix of a plane grid member relative to the member axes is obtained as

$$\begin{array}{l}[k]_{xyz} = \\ \textit{Plane} \\ \textit{Grid} \\ \textit{Member}\end{array} \begin{bmatrix} k_{11} & k_{51} & k_{61} & -k_{11} & -k_{51} & k_{61} \\ k_{51} & k_{55} & k_{65} & -k_{51} & k_{11,5} & -k_{11,6} \\ k_{61} & k_{65} & k_{66} & -k_{61} & k_{11,6} & k_{12,6} \\ -k_{11} & -k_{51} & -k_{61} & k_{11} & k_{51} & -k_{61} \\ -k_{51} & k_{11,5} & k_{11,6} & k_{51} & k_{55} & -k_{65} \\ k_{61} & -k_{11,6} & k_{12,6} & -k_{61} & -k_{65} & k_{66} \end{bmatrix}. \tag{18}$$

The direction cosine n_y of the member's centerline is zero, since the member's y-axis lies always in the XY-plane. Therefore, $Q = 1$. Because the curvature of the member is in the XY-plane, the z-axis is always horizontal making

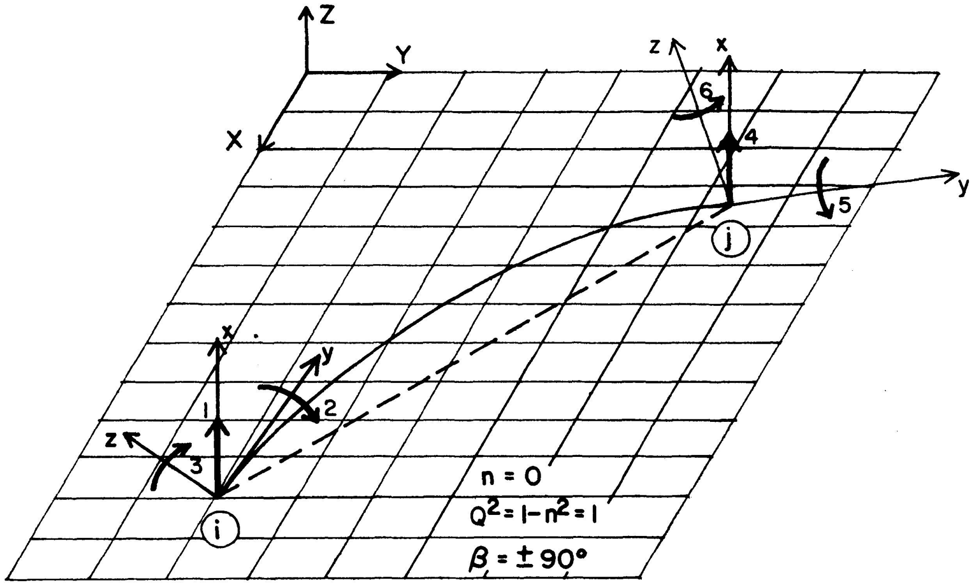


Fig. 5. Curved plane grid member.

$\beta = 90°$. Substituting $n_y = 0$, $\beta = 90°$ and $Q = 1$ in Eq. (14), and taking into account that there is only one force and two moments at the i end of the member, the combined stiffness matrix T_i of Eq. (14) becomes

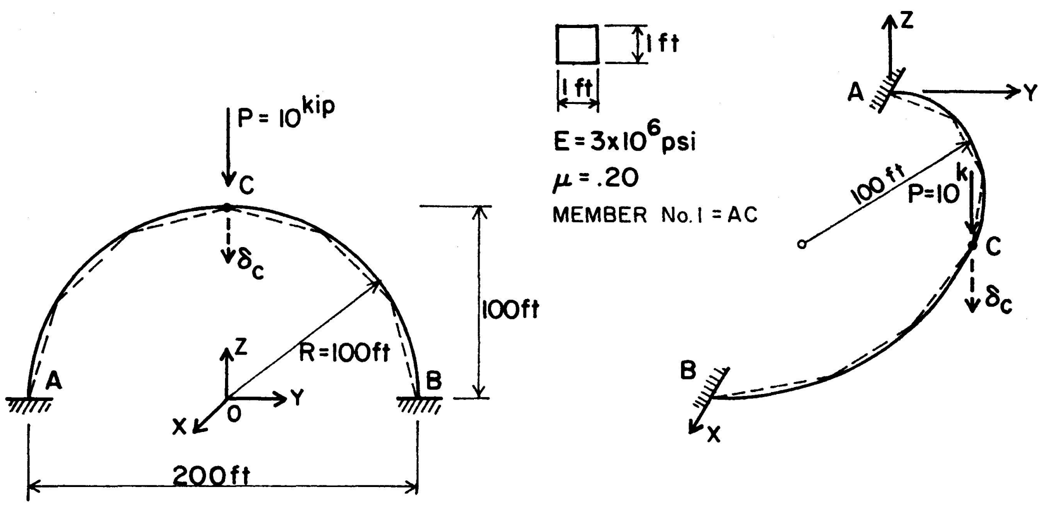
$$[T]_i = \begin{bmatrix} -\mathrm{Sin}\,\beta & 0 & 0 \\ 0 & \left(m_y \sin\frac{\theta}{2}\sin\beta + l_y\cos\frac{\theta}{2}\right) & \left(-l_y \sin\frac{\theta}{2}\sin\beta + m_y\cos\frac{\theta}{2}\right) \\ 0 & \left(m_y \cos\frac{\theta}{2}\sin\beta - l_y\sin\frac{\theta}{2}\right) & \left(-l_y \cos\frac{\theta}{2}\sin\beta - m_y\sin\frac{\theta}{2}\right) \end{bmatrix}. \quad (19)$$

If the common axes stiffness matrix k_{XYZ} is required, Eqs. (18) and (19) are substituted in the standard transformation formula of Eq. (15).

8. Numerical Examples

Example 1. The semi-circular fixed-ended arch shown in Fig. 6 was analyzed both as a plane frame and as a grid, taking into account all the axial and shear deformations. The arch was first considered as composed of two circular members and then was replaced by a number of straight members. For a varying number of divisions, the comparative results are summarized in Table 1.

Example 2. The sperical dome structure supported on four columns as shown in Fig. 7, was analyzed by considering the individual members first as curved then as straight. Some of the comparative results are summarized in Table 2.



(a) Arch as a frame (b) Arch as a grid

Fig. 6. Semi-circular arch as a frame and a grid.

Table 1. Comparative Results for the Semi-Circular Arch

$k' = 1.5$		Analyses with straight members					Analysis with curved members (Exact)
		Number of divisions					
Member No. 1		2	4	6	10	20	
As a plane frame	$10^4\, \delta_C$ (ft.)	3.3	33.3	35.3	36.2	36.7	*37.3*
	M_A (k–ft.)	0.25	–7.56	–9.33	–10.26	–10.61	*–10.71*
	M_C	0.25	12.19	13.95	14.83	15.21	*15.35*
As a plane grid	$10^4\, \delta_C$	910	732	727	728	729	*731*
	Torque T_A	14.56	–.02	–5.69	–10.68	–14.29	*–18.17*
	M_C	14.56	–27.04	–29.66	–31.05	–31.63	*–31.83*

Table 2. Comparative Results for the Spherical Dome

$k' = 1.5$	Bending moments in the yz-plane (kip-ft.)							Vertical deflection
Location	M_{CE}	M_{EC}	M_{GF}	M_{CG}	M_{GC}	M_{GA}	M_{AC}	Joint A
Case A	295.6	187.1	74.5	–71.8	–67.6	10.6	15.3	0.06 ft.
Case B	145.2	145.6	38.4	–110.7	20.5	–31.0	317.8	0.45 ft.
Members between joints were considered as straight in the Case A, and as curved in the Case B								

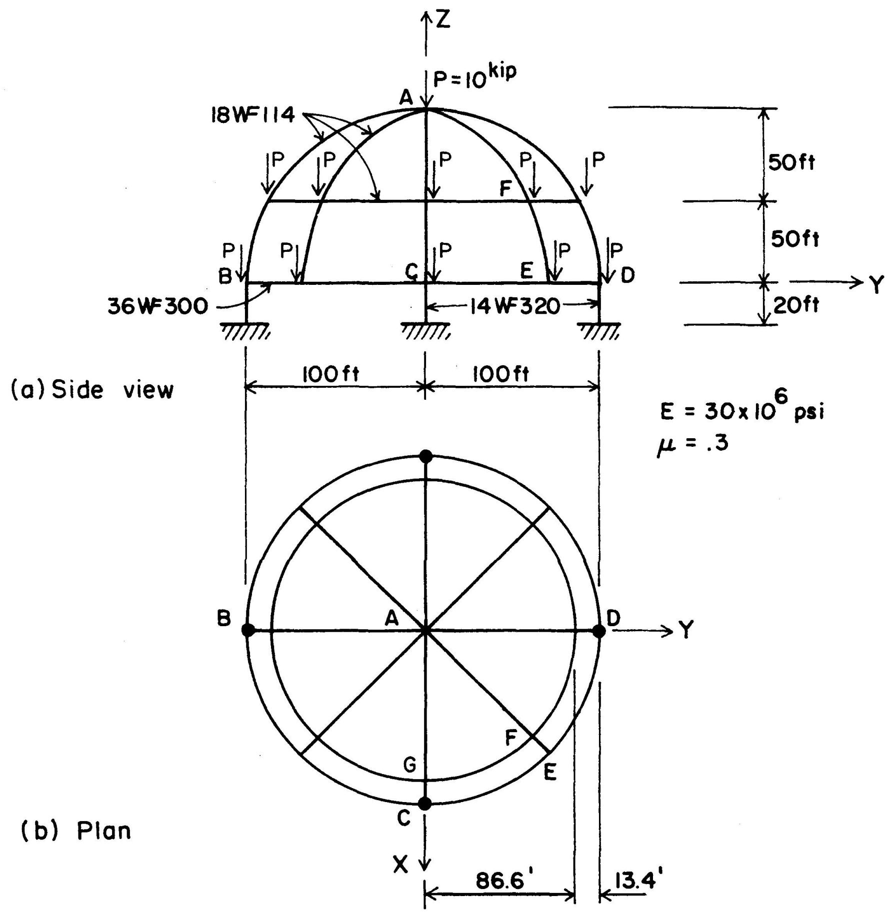


Fig. 7. Spherical dome supported on four columns.

References

1. Gere, J. M., and Weaver, W., "Analysis of Framed Structures". D. Van Nostrand Co., Inc., New York, 1965.
2. Asplund, O., "Matrix Method of Structures". Prentice-Hall Inc., New Jersey, U.S.A., 1965.
3. Tezcan, S. S., "Computer Analysis of Plane and Space Structures". Journ. of Structural division Am. Soc. of Civil Eng., ST 2, No. 4780, April, 1966, pp. 143—173.
4. Argyris, J. H., and Kelsey, S., "Energy Theorems and Structural Analysis". Butterworths, London, 1960.

Summary

The general stiffness matrices of circular members are presented for plane frames, plane grids, and space frames. First, the flexibility matrix of the unsupported end of the member is determined from the unit load theorem. Then, utilizing the conditions of static equilibrium and making use of the inverse of the flexibility matrix, the stiffness matrix is obtained with relation

to the member axes. After three successive orthogonal transformations, the stiffness matrix is transformed from the member axes to the common axes of the system.

When curved members are idealized as a series of straight members two disadvantages are in evidence. First, the results become approximate, and second, the problem requires ten to twenty times more work in order to achieve reasonable accuracy. On the other hand, with the availability of stiffness matrices for curved members, these two disadvantages disappear.

Résumé

Les auteurs présentent les matrices de rigidité générales des éléments circulaires, pour des portiques plans, des réseaux de poutres plans ou des ossatures tridimensionnelles. On détermine d'abord, en utilisant le théorème des forces unitaires, la matrice de souplesse de l'extrémité libre de l'élément. En utilisant les conditions d'équilibre et en inversant la matrice de souplesse, on obtient la matrice de rigidité rapportée aux axes de l'élément. Par trois transformations orthogonales successives, on rapporte la matrice de rigidité aux axes principaux du système.

Lorsque l'on assimile les éléments courbes à une série d'éléments droits, on rencontre les deux désavantages suivants. Premièrement, il s'agit d'une solution approximative. Deuxièmement, pour obtenir une précision raisonnable, la durée des calculs est à multiplier par dix ou vingt. Lorsque l'on dispose des matrices pour les éléments courbes, ces inconvénients disparaissent.

Zusammenfassung

Die allgemeinen Steifigkeitsmatrizen für kreisförmig gekrümmte Elemente werden angeschrieben für ebene Rahmen, ebene Trägerroste und für räumliche Rahmen. Zuerst wird die Verformungsmatrix des freien Endes des Elementes aus dem Einheitslasttheorem bestimmt. Anschließend, unter Benützung der statischen Gleichgewichtsbedingungen und der Umkehrmatrix der Verformungsmatrix, wird die Steifigkeitsmatrix bezogen auf die Achsen des Elementes hergeleitet. Nach drei sukzessiven orthogonalen Transformationen wird die Steifigkeitsmatrix von den Achsen der Elemente auf die Hauptachsen des Systems umgeformt.

Wenn gekrümmte Elemente als eine Folge gerader Elemente idealisiert werden, sind zwei Nachteile augenscheinlich. Erstens sind die Ergebnisse Näherungslösungen und zweitens verlangt das Problem das Zehn- bis Zwanzigfache an Zeit, um eine vernünftige Genauigkeit zu erreichen. Mit der Einführung der Steifigkeitsmatrix für gekrümmte Elemente treten diese beiden Nachteile nicht mehr auf.