A Contribution to the Bending Theory of Elliptic Paraboloid shells

Contribution à la théorie de la fléxion des voiles minces en forme de paraboloïde elliptique

Beitrag zur Biegetheorie der elliptischen Paraboloidschalen

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1. Introduction

a) In recent years shells of double curvature have been of great interest to engineers and architects. Although the hyperbolic paraboloid is the most popular type, due to its attractive form and simple construction, investigations have also been carried out in the field of paraboloid shells of positive Gaussian curvature. These shells, also very attractive from an architectural point of view, have the important advantage that the bending stresses are confined to narrow zones along the boundaries and are very small. Thus the membrane theory provides already a good approximation for the stress condition of the surface. For proper design, it is nevertheless necessary to calculate bending stresses due to edge conditions incompatible with the membrane theory. For the calculation of shear forces N_{xy} at the corners it is also necessary to take bending stresses into consideration [1]).

b) Numerous authors have dealt with the analysis of EP-shells, but very few consider bending stresses. A general bending theory for shallow shells was first presented by K. MARGUERRE [2] in 1938. In 1944 V. Z. VLASOV [3] published his basic theory, and based upon this S. A. AMBARTSUMYAN [4] presented a solution for shallow shells, rectangular in plan and simply supported along the edges in 1947. This solution, of the Navier-type using double trigonometric series is, however, of little practical importance as the series are very slowly convergent.

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[1]) The shear forces N_{xy} are singular at the corners. (See for instance TIMOSHENKO [1], p. 464.)

An approximate solution to the bending theory was given by K. HRUBAN [5] in 1953. From the general theory of VLASOV he obtained, by approximation, a fourth order differential equation identical to the one for cylindrical containers. A better approximation was obtained by W. ZERNA [6] also in 1953. Here the solution again leads to the fourth order differential equation for cylindrical containers, only in this case the coefficient k is no more a constant. In 1957 V. Z. VLASOV [7] proposed a Levy-type solution where two opposite edges are simply supported and the two remaining edges may have arbitrary boundary conditions. A solution for the case of clamped boundaries is given by W. A. NASH and P. L. SHENG [8] also in 1957.

For the special case where $k_1 = k_2 = 1/R$ (sphere!) B. B. DIKOVICH [9] presented a very useful contribution in 1960. The shells, having the base proportions 1 : 1 or 1 : 2, are assumed to be simply supported along the edges. With these limitations DIKOVICH established graphs for the distribution of the stress resultants depending on one parameter only.

A very extensive study of paraboloid shells with elastic edge members has been carried out by H. C. SHAH [10] (1960). The solution is obtained by using double trigonometric series, for which the convergence is so hopeless that a practicable application seems almost impossible.

Levy-type solutions, similar to the one treated by V. Z. VLASOV [7], are presented by A. L. BOUMA [11] (1959), K. APELAND [12] (1961) and I. DOGANOFF [13] (1961). DOGANOFF gives also a simplified solution for simply supported edges suitable for pre-dimensioning. K. APELAND and E. P. POPOV [14] have established tables for paraboloidal shells of positive and negative Gaussian curvature, similar to those for circular cylindrical shells compiled by D. RUDIGER and J. URBAN.

c) The papers mentioned above have two features in common [2]):

1. The solutions consist of infinite series.
2. The solutions satisfy the basic equations exactly and satisfy the boundary conditions approximately.

In most cases the shell is assumed to be simply supported on shear diaphragms perpendicular to the shell surface, at least on two opposite edges. The boundary conditions were therefore:

$$v = 0, \qquad w = 0, \qquad N_1 = 0, \qquad M_1 = 0. \tag{a}$$

In practice the diaphragms are, however, always made vertical and the more realistic boundary conditions should therefore be

$$v = 0, \quad w \cos\varphi_0 - u \sin\varphi_0 = 0, \quad N_1 \cos\varphi_0 + Q_1 \sin\varphi_0 = 0, \quad M_1 = 0, \tag{b}$$

as was mentioned by BOUMA [11]. It is readily seen that $\sin\varphi_0$ is not negligible;

[2]) This does not apply to the approximate solutions given by Prof. HRUBAN [5] and by Prof. ZERNA [6].

a square spherical shell with a total rise to span ratio of 1/5 has the slope $\varphi_0 = 22.5°$ at the edges. Thus $\cos\varphi_0 = 0.92$ and $\sin\varphi_0 = 0{,}38$. How the boundary conditions (b) can be considered will be shown in § 6.

d) In this paper the writer proposes another way in which the solutions satisfy the boundary conditions exactly but the basic equations only approximately. To obtain such a solution a variational technique has been used in connection with the matrix progression method and numerial computation with the aid of digital computers [3]).

The procedure is applied to shallow elliptic paraboloids, rectangular in plan, for the following boundary conditions:

1. Edges rigidly clamped.
2. Edges simply supported on shear diaphragms perpendicular to the shell surface (Navier, equation (a)).
3. Edges simply supported on vertical shear diaphragms (equation (b)) [4]).

subjected to uniformly distributed loads.

2. Geometry

The equation for a surface in the form of an elliptic paraboloid is given by (see Fig. 1):

$$z = -\left[h_1\left(\frac{x}{a}\right)^2 + h_2\left(\frac{y}{b}\right)^2\right]. \tag{1}$$

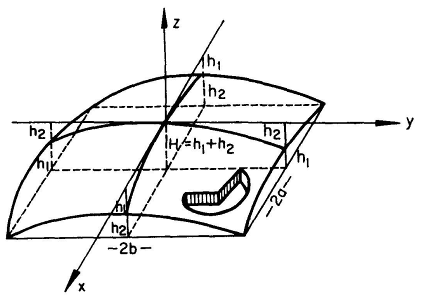

Fig. 1.

The principal curvatures are thus

$$k_1 = -\frac{\partial^2 Z}{\partial x^2} = +\frac{2h_1}{a^2}, \quad k_2 = -\frac{\partial^2 Z}{\partial y^2} = +\frac{2h_2}{b^2}, \quad k_{12} = \frac{\partial^2 Z}{\partial x\,\partial y} = 0. \tag{2}$$

Introducing the slopes of the surface at the edges

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[3]) The technique was first introduced by S. M. K. CHETTY [15] for hyperbolic paraboloid shells. The mathematical foundation for variational methods may be found in [16] and their applications for shell problems are discussed by H. TOTTENHAM [17] and CHETTY [15]. The matrix progression method, first introduced for analysis of shells by H. TOTTENHAM [18], has been elementarily by the writer [19].

[4]) In this case the shell must be square in plan and $k_1 = k_2$.

$$\alpha = -\frac{2\,h_1}{a} \quad \text{and} \quad \beta = -\frac{2\,h_2}{b} \tag{3}$$

the expressions for the principal curvature take the forms

$$k_1 = -\frac{\alpha}{a}, \qquad k_2 = -\frac{\beta}{b}, \qquad k_{12} = 0. \tag{2a}$$

3. Basic Equations

a) From the shallow shell theory of V. Z. VLASOV [20] the following equilibrium equations are obtained:

$$\begin{aligned}
L_1(u,v,w) &= u'' + \tfrac{1}{2}(1-\nu)\,u^{\cdot\cdot} + \tfrac{1}{2}(1+\nu)\,v'^{\cdot} + (k_1+\nu\,k_2)\,w' = -\frac{1-\nu^2}{E\,t}X,\\
L_2(u,v,w) &= v^{\cdot\cdot} + \tfrac{1}{2}(1-\nu)\,v'' + \tfrac{1}{2}(1+\nu)\,u'^{\cdot} + (k_2+\nu\,k_1)\,w^{\cdot} = -\frac{1-\nu^2}{E\,t}Y,\\
L_3(u,v,w) &= (k_1+\nu\,k_2)\,u' + (k_2+\nu\,k_1)\,v^{\cdot} + (k_1^2 + 2\,\nu\,k_1\,k_2 + k_2^2)\,w\\
&\quad + \frac{t^2}{12}\nabla^4 w = +\frac{1-\nu^2}{E\,t}Z.
\end{aligned} \tag{4}$$

Here u, v and w are the displacements, X, Y and Z the external loads, positive in the positive direction of the co-ordinate axes. L_i denotes a linear differential operator.

The stress resultants, as functions of the displacements, are given by:

$$\begin{aligned}
N_1 &= +\frac{E\,t}{1-\nu^2}[u' + \nu\,v^{\cdot} + (k_1+\nu\,k_2)\,w],\\
N_2 &= +\frac{E\,t}{1-\nu^2}[v^{\cdot} + \nu\,u' + (k_2+\nu\,k_1)\,w],\\
S &= +\frac{E\,t}{2\,(1+\nu)}[u^{\cdot} + v'],\\
M_1 &= +\frac{E\,t^3}{12\,(1-\nu^2)}[w'' + \nu\,w^{\cdot\cdot}],\\
M_2 &= +\frac{E\,t^3}{12\,(1-\nu^2)}[w^{\cdot\cdot} + \nu\,w''],\\
M_{12} &= -\frac{E\,t^3}{12\,(1+\nu)}w'^{\cdot},\\
Q_1 &= -\frac{E\,t^3}{12\,(1-\nu^2)}[w''' + w'^{\cdot\cdot}],\\
Q_2 &= -\frac{E\,t^3}{12\,(1-\nu^2)}[w^{\cdot\cdot\cdot} + w''^{\cdot}].
\end{aligned} \tag{5}$$

The sign conventions for the stress resultants are as shown in Fig. 2.

For convenience the non-dimensional co-ordinates

$$\bar{x} = \frac{x}{a} \quad \text{and} \quad \bar{y} = \frac{y}{b} \tag{6a}$$

and the dimensionless unknown displacements

$$\bar{u} = \frac{u}{a}, \qquad \bar{v} = \frac{v}{b}, \qquad \bar{w} = \frac{w}{a} \tag{6b}$$

are introduced.

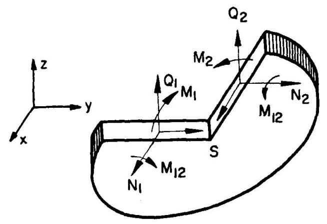

Fig. 2.

Introducing the expressions (2a) for the curvatures and the non-dimensional co-ordinates and displacements (6) in the basic Eqs. (4) and (5), we obtain the non-dimensional basic equations:

$$\begin{aligned}
L_1(u,v,w) &= u'' + \tfrac{1}{2}(1-\nu)\,r^2 u^{\cdot\cdot} + \tfrac{1}{2}(1+\nu)\,v'^{\cdot} - (\alpha+\nu r\beta)\,w' = -(1-\nu^2)\frac{a}{t}\frac{X}{E},\\
L_2(u,v,w) &= v^{\cdot\cdot} + \tfrac{1}{2}(1-\nu)\frac{1}{r^2}v'' + \tfrac{1}{2}(+\nu)\,u'^{\cdot} - (r\beta+\nu\alpha)\,w^{\cdot} = -(1-\nu^2)\frac{b}{t}\frac{Y}{E},\\
L_3(u,v,w) &= -(\alpha+\nu r\beta)\,u' - (r\beta+\nu\alpha)\,v^{\cdot} + (\alpha^2+2\nu r\alpha\beta+r^2\beta^2)\,w\\
&\quad + \frac{1}{\gamma}(w'''' + 2r^2 w''^{\cdot\cdot} + r^4 w^{\cdot\cdot\cdot\cdot}) = +(1-\nu^2)\frac{a}{t}\frac{Z}{E}.
\end{aligned} \tag{7}$$

$$\begin{aligned}
N_1 &= +\frac{E\,t}{1-\nu^2}[u' + \nu v^{\cdot} - (\alpha+\nu r\beta)\,w],\\
N_2 &= +\frac{E\,t}{1-\nu^2}[v^{\cdot} + \nu u' - (r\beta+\nu\alpha)\,w],\\
S &= +\frac{E\,t}{2(1+\nu)}\left[r u^{\cdot} + \frac{1}{r}v'\right],\\
M_1 &= +\frac{E\,t^3}{12(1-\nu^2)}\frac{1}{a}[w'' + \nu r^2 w^{\cdot\cdot}],\\
M_2 &= +\frac{E\,t^3}{12(1-\nu^2)}\frac{1}{a}[r^2 w^{\cdot\cdot} + \nu w''],\\
M_{12} &= -\frac{E\,t^3}{12(1+\nu)}\frac{1}{b}w'^{\cdot},\\
Q_1 &= -\frac{E\,t^3}{12(1-\nu^2)}\frac{1}{a^2}[w''' + r^2 w'^{\cdot\cdot}],\\
Q_2 &= -\frac{E\,t^3}{12(1-\nu^2)}\frac{1}{a^2}[r^3 w^{\cdot\cdot\cdot} + r\,w''^{\cdot}],
\end{aligned} \tag{8}$$

where $\alpha = -\dfrac{2h_1}{a}, \qquad \beta = -\dfrac{2h_2}{b}, \qquad \gamma = 12\dfrac{a^2}{t^2}$ and $r = \dfrac{a}{b}.$ (9)

In Eqs. (7) and (8) the bars introduced in (6) have been omitted for convenience. The three partial differential equations for the three non-dimensional quantities u, v and w have now to be solved in order to obtain the stress resultants (8) at any point $P(x, y)$ of the surface.

b) Assuming a solution in the form

$$u = c_1 f_1(x) g_1(y), \qquad v = c_2 f_2(x) g_2(y), \qquad w = c_3 f_3(x) g_3(y), \tag{10}$$

where $f_1(x)$ and $g_i(y)$ are functions of one variable only and c_i are unknown constants, we like to determine c_i such that the energy of the system is a minimum, i.e. $\frac{\partial E}{\partial c_i} = 0$. This provides three equations for the three unknowns c_i[5]:

$$\iint_S [L_i(u, v, w) - Q_i] f_i g_i \, dx \, dy = 0 \qquad (i = 1, 2 \text{ and } 3), \tag{11}$$

where L_i is the differential operator and Q_i denotes the loading term in (7). The equation (11) is the well-known Galerkin equation.

c) Assuming a solution in the form

$$u = f_1(x) g_1(y), \qquad v = f_2(x) g_2(y), \qquad w = f_3(x) g_3(y), \tag{12}$$

where g_i are known, we obtain three equations for the three unknowns $f_i(x)$ putting $\frac{\partial E}{\partial f_i(x)} = 0$. These are

$$\int_S [L_i(u, v, w) - Q_i] g_i \, dx \, dy = 0. \tag{13}$$

4. Solution for Rigidly Clamped Edges

a) Boundary conditions

At the edges $x = \pm 1$ and $y = \pm 1$ there shall be no movement and the rotation of the normal must vanish. Thus

$$\text{At } x = \pm 1 \quad u = 0, \; v = 0, \; w = 0, \; w' = 0. \tag{14}$$

$$\text{At } y = \pm 1 \quad u = 0, \; v = 0, \; w = 0, \; w^{\cdot} = 0. \tag{15}$$

b) Kantorovich method. (Reduction to ordinary differential equations.)

The solution is sought in the form

$$u = f_1(x) g_1(y), \qquad v = f_2(x) g_2(y), \qquad w = f_3(x) g_3(y), \tag{16}$$

where $f_i(x)$ and $g_i(y)$ are functions of only one variable, x or y respectively.

As a first approximation we assume the distributions $g_i(y)$ in the y-direction, such that the boundary conditions at the edges $y = \pm 1$ are satisfied.

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[5]) The functions f_i and g_i are here assumed to be known.

Under uniformly distributed loads it is seen that

$$g_1(y) = \text{even function}, \quad g_2(y) = \text{odd function}, \quad g_3(y) = \text{even function}. \tag{17}$$

We assume thus polynomials of the simplest possible forms:

$$u = f_1(x)(1-y^2), \quad v = f_2(x)(y-y^3), \quad w = f_3(x)(1-y^2)^2. \tag{18}$$

Differentiating and substituting in the basic Eqs. (7), we obtain three ordinary differential equations with variable coefficients for the three unknown functions f_i:

$$f_1''(1-y^2) + \tfrac{1}{2}(1-\nu)r^2(-2f_1) + \tfrac{1}{2}(1+\nu)f_2'(1-3y^2) - (\alpha+\nu r\beta)f_3'(1-y^2)^2 =$$
$$-(1-\nu^2)\frac{a}{t}\frac{X}{E},$$

$$-6f_2 y + \tfrac{1}{2}(1-\nu)\frac{1}{r^2}f_2''(y-y^3) + \tfrac{1}{2}(1+\nu)f_1'(-2y) - (r\beta+\nu\alpha)f_3(-4)(1-y^2) =$$
$$-(1-\nu^2)\frac{b}{t}\frac{Y}{E},$$

$$-(\alpha+\nu r\beta)f_1'(1-y^2) - (r\beta+\nu\alpha)f_2(1-3y^2) + (\alpha^2+2\nu r\alpha\beta+r^2\beta^2)f_3(1-y^2)^2$$
$$+\frac{1}{\gamma}[f_3''''(1-y^2)^2 + 2r^2 f_3''(-4)(1-3y^2) + r^4 24 f_3] = +(1-\nu^2)\frac{a}{t}\frac{Z}{E}.$$

The error made by using the assumed functions (18) will now be minimized using Eq. (13). Integrating over the area bounded by $y = \pm 1$ we thus obtain minimum potential energy and in the same time the equations will be transformed to equations with constant coefficients. These are:

$$\begin{aligned}
&1.0667 f_1'' - 1.3333(1-\nu)r^2 f_1 + 0.2667(1+\nu)f_2' - 0.9143(\alpha+\nu r\beta)f_3' = 0,\\
&\frac{1}{r^2}0.0762(1-\nu)f_2'' - 1.6 f_2 - 0.2667(1+\nu)f_1' + 0.6095(r\beta+\nu\alpha)f_3 = 0,\\
&-0.9173(\alpha+\nu r\beta)f_1' - 0.6095(r\beta+\nu\alpha)f_2 + 0.8127(\alpha^2+2\nu r a\beta+r^2\beta^2)f_3\\
&\qquad + \frac{1}{\gamma}[0.8127 f_3'''' - 4.8768 r^2 f_3'' + 25.6 r^4 f_3] = +1.0667(1-\nu^2)\frac{a}{t}\frac{Z}{E},
\end{aligned} \tag{20}$$

if we assume normal loading only, i. e.

$$X = Y = 0 \quad \text{and} \quad Z = \text{constant}.$$

c) *Tottenham's method for solving the equations*

For the solution of the Eqs. (20), it is convenient to use the matrix progression method due to Tottenham [14, 19]. With this method the equations are integrated directly without using the laborious traditional way in finding the roots of the auxiliary and the particular integral.

Introducing the new functions:

$$f_4 = f_1', \quad f_5 = f_2', \quad f_6 = f_3', \quad f_7 = f_3'' = f_6', \quad f_8 = f_3''' = f_6'' = f_7', \tag{21}$$

we have from (20) and (21) the following eight first order differential equations, with constant coefficients, for the eight unknown functions $f_1, f_2, \ldots f_8$:

$$
\begin{aligned}
f_1' &= f_4, \\
f_2' &= f_5, \\
f_3' &= f_6, \\
f_4' &= f_1'' = +1.25\,(1-\nu)\,r^2 f_1 - 0.25\,(1+\nu)\,f_5 - 0.8572\,(\alpha+\nu r \beta)\,f_6, \\
f_5' &= f_2'' = +21.0\frac{r^2}{1-\nu} f_2 + 3.5\frac{1+\nu}{1-\nu} r^2 f_4 + 8.0\frac{(r\beta+\nu\alpha)}{1-\nu} r^2 f_3, \\
f_6' &= f_7, \\
f_7' &= f_8, \\
f_8' &= +0.75\,\gamma\,(r\beta+\nu\alpha)\,f_2 - [\gamma\,(\alpha^2+2\nu r\alpha\beta+r^2\beta^2)+31.5\,r^4]\,f_3 \\
&\quad + 1.125\,(\alpha+\nu r\beta)\,f_4 + 6.0\,r^2 f_7 + 1.3125\,(1-\nu^2)\frac{a}{t}\frac{Z}{E}\gamma
\end{aligned}
\tag{22}
$$

or in matrix form:

$$
\frac{d}{dx}\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix} = \begin{bmatrix}
\cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\
a_{41} & \cdot & \cdot & \cdot & a_{45} & a_{46} & \cdot & \cdot \\
\cdot & a_{52} & a_{53} & a_{54} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\
\cdot & a_{82} & a_{83} & a_{84} & \cdot & \cdot & a_{87} & \cdot
\end{bmatrix}
\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix} + \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_8 \end{bmatrix},
$$

which may be written as

$$
\frac{d}{dx} Z = A\,Z + B. \tag{22}
$$

Here Z is a column matrix containing the unknown functions $f_1, f_2 \ldots f_8$, A is a 8×8 square matrix and B a column matrix with the coefficients

$$
\begin{aligned}
a_{14} &= +1, \\
a_{25} &= +1, \\
a_{36} &= +1, \\
a_{41} &= +1.25\,(1-\nu)\,r^2, \\
a_{45} &= -0.25\,(1+\nu), \\
a_{46} &= -0.8572\,(\alpha+\nu r\beta), \\
a_{52} &= +21\frac{r^2}{1-\nu}, \\
a_{53} &= +3.5\frac{1+\nu}{1-\nu} r^2,
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
a_{54} &= +8.0 \frac{r\beta + \nu\alpha}{1-\nu} r^2, \\
a_{67} &= +1, \\
a_{78} &= +1, \\
a_{82} &= +0.75\gamma(r\beta + \nu\alpha), \\
a_{83} &= -\gamma(\alpha^2 + 2\nu r\alpha\beta + r^2\beta^2) - 31.5 r^4, \\
a_{84} &= +1.125(\alpha + \nu r\beta)\gamma, \\
a_{87} &= +6.0 r^2, \\
b_8 &= +1.3125(1-\nu^2)\frac{a}{t}\frac{Z}{E}\gamma
\end{aligned}
\tag{22}
$$

the remaining coefficients being zero.

The Eq. (21) is the same as the one for a simple beam [6]) and the solution will readily be found to be

$$Z(x) = e^{Ax} Z_0 - [I - e^{Ax}] A^{-1} B,$$

where Z_0 are the values of $Z(x)$ at $x=0$ and I is the unit matrix.

Introducing

$$G(x) = e^{Ax} = I + \frac{Ax}{1!} + \frac{A^2x^2}{2!} + \cdots = I + Ax\left[I + \frac{Ax}{2}\left(I + \frac{Ax}{3}(I + \cdots)\right)\right] \quad ^{7)}$$

and $$\tilde{Z}(x) = -[I - e^{Ax}] A^{-1} B$$

we have thus the matrix progression equation [8])

$$\boxed{Z(x) = G(x) Z_0 + \tilde{Z}(x)} \tag{23}$$

In (23) $Z(x)$ are the unknown functions f_i

$G(x) = e^{Ax}$ is called the distribution matrix,

Z_0 are the eight unknown values of $Z(x)$ for $x=0$,

$\tilde{Z}(x) = -[I - e^{Ax}] A^{-1} B$ is called the loading solution matrix.

Having solved the equations, it remains to determine Z_0 according to the boundary conditions.

d) Determination of Z_0 from the boundary conditions

If the shell is symmetrically loaded about the x and y axes, the stressed state of the shell must also be symmetric about the x and y axes. Therefore

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[6]) See (19).

[7]) In this quickly convergent matrix series 12 terms will be more than sufficient for the accuracy required.

[8]) See for example (19).

it will be seen that u is an odd function of x,
v is an even function of x,
w is an even function of x.

We thus have the following four conditions at $x=0$:

$$u=f_1=0, \quad v'=f_5=0, \quad w'=f_6=0, \quad w'''=f_8=0. \tag{24}$$

We have hence only four unknowns in Z_0 and we write

$$Z_0 = \begin{bmatrix} \cdot \\ f_2 \\ f_3 \\ f_4 \\ \cdot \\ \cdot \\ f_7 \\ \cdot \end{bmatrix}_{x=0} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} f_2 \\ f_3 \\ f_4 \\ f_7 \end{bmatrix}_{x=0} = k_1 \bar{Z}_0, \tag{25}$$

where $\bar{Z}_0$ contains the four unknowns $f_2(0), f_3(0), f_4(0)$ and $f_7(0)$.

At $x=+1$ we have, according to (14), (25) and (23)

$$Z(x=1) = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ f_4(1) \\ f_5(1) \\ \cdot \\ f_7(1) \\ f_8(1) \end{bmatrix} = G(1.0)\, k_1 \bar{Z}_0 + \tilde{Z}(1.0). \tag{26}$$

The first, second, third and sixth equations provide four equations for the four unknowns $\bar{Z}_0$. Using the "Isolation Matrix"

$$k_2 = \begin{bmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \end{bmatrix}, \tag{27}$$

we have thus $\quad k_2 Z(1.0) = k_2 G(1.0)\, k_1 \bar{Z}_0 + k_2 \tilde{Z}(1.0) = 0$

and hence $\quad \bar{Z}_0 = -[k_2 G(1.0)\, k_1]^{-1} k_2 \tilde{Z}(1.0).$

From (25) the unknown functions Z_0 are thus

$$\boxed{Z_0 = -k_1 [k_2 G(1.0)\, k_1]^{-1} k_2 \tilde{Z}(1.0).} \tag{28}$$

All matrices on the R. H. S. of Eq. (28) are known and with (23) the unknown functions may readily be calculated with the aid of a digital computer.

It is interesting to notice that also Eq. (28) coincides with the one for a simple beam [9]).

e) Improvements to the solution

1. Having found the functions f_i from

$$Z(x) = G(x)\, Z_0 + \tilde{Z}(x) \tag{23}$$

we can compute the displacements from (18) and the stress resultants from (8). It may be expected that the distribution in the x-direction will be reasonably good; the distribution in the y-direction, however, may be very erroneous. In order to obtain a better approximation we assume firstly that the shell is almost square in plan (say $\frac{1}{2} b < a < 2b$). In this case it may be assumed that the stress distributions in the two directions will have the same pattern if the load is uniformly distributed over the surface. Therefore $g_1(y)$ will be equal to $f_2(x)$, $g_2(y)$ equal to $f_1(x)$ and $g_3(y)$ equal to $f_3(x)$, apart from some constant factors. We have thus as a second approximation

$$u = \bar{c}_1 f_1(x) f_2(y), \qquad v = \bar{c}_2 f_2(x) f_1(y), \qquad w = \bar{c}_3 f_3(x) f_3(y). \tag{29}$$

The constants $\bar{c}_i$ may now be determined from the known conditions at the lines $x=0$ and $y=0$. Comparing the values of u and w given by (29) and (18) at these lines we find

$$\text{from (18)}\quad \begin{aligned} u(x,0) &= f_1(x), \\ w(0,0) &= f_3(x), \end{aligned} \qquad \text{from (29)}\quad \begin{aligned} u(x,0) &= \bar{c}_1 f_1(x) f_2(0), \\ w(0,0) &= \bar{c}_3 f_3(0) f_3(0), \end{aligned}$$

it is readily seen that

$$\bar{c}_1 = \frac{1}{f_2(0)} \quad \text{and} \quad \bar{c}_3 = \frac{1}{f_3(0)}.$$

Due to symmetry we can also put

$$\bar{c}_2 = \bar{c}_1 = \frac{1}{f_2(0)}$$

and we have the second approximation

$$u = \frac{1}{f_2(0)} f_1(x) f_2(y), \qquad v = \frac{1}{f_2(0)} f_2(x) f_1(y), \qquad w = \frac{1}{f_3(0)} f_3(x) f_3(y). \tag{30}$$

2. By using the system (30) instead of (18), the energy will no longer be a minimum. Assuming

$$u = c_1 \frac{1}{f_2(0)} f_1(x) f_2(y), \quad v = c_2 \frac{1}{f_2(0)} f_2(x) f_1(y), \quad w = c_3 \frac{1}{f_3(0)} f_3(x) f_3(y) \tag{31}$$

---

[9]) See for instance (19).

and minimizing the energy w.r.t. c_i, i.e. $\frac{\partial E}{\partial c_i}=0$, we get the Galerkin Eq. (11):

$$\int_{-1}^{+1}\int_{-1}^{+1}[L_i(f_k)-Q_i]\,f_i(x)\,g_i(y)\,dx\,dy = 0. \tag{11}$$

Here L_i denotes the L.H.S. of the basic differential Eq. (7) and Q_i the loading terms of same.

Substituting (31) in (7) we find

$$[L_1-Q_1] = c_1 f_4' f_2 + \tfrac{1}{2}(1-\nu)\,r^2 c_1 f_1 f_5' + \tfrac{1}{2}(1+\nu)\,c_2 f_5 f_4 - (\alpha+\nu r\beta)\,c_3 f_6 f_3 \frac{f_2(0)}{f_3(0)},$$

$$[L_2-Q_2] = c_2 f_2 f_4' + \tfrac{1}{2}(1-\nu)\frac{1}{r^2} c_2 f_5' f_1 + \tfrac{1}{2}(1+\nu)\,c_1 f_4 f_5 - (r\beta+\nu\alpha)\,c_3 f_3 f_6 \frac{f_2(0)}{f_3(0)},$$

$$\begin{aligned}[L_3-Q_3] = &-(\alpha+\nu r\beta)\,c_1 f_4 f_2 - (r\beta+\nu\alpha)\,c_2 f_2 f_4 + (\alpha^2+2\nu r\alpha\beta+r^2\beta^2)\,c_3 f_3 f_3 \frac{f_2(0)}{f_3(0)}\\ &+\frac{1}{\gamma} c_3\,[f_8' f_3 + 2r^2 f_7 f_7 + r^4 f_3 f_8']\,\frac{f_2(0)}{f_3(0)} - (1-\nu^2)\,f_2(0)\frac{a}{t}\frac{Z}{E},\end{aligned}$$

where, for example, $f_4'(x)\,f_2(y)$ is abbreviated to $f_4' f_2$ etc. In these expressions all functions $f_1 \ldots f_8$ are known from the first approximation, and the functions f_4', f_5' and f_8' can be expressed as functions of $f_1 \ldots f_8$ through the Eq. (22).

We are now able to construct the integrands $[L_i(f_k)-Q_i]\,f_i(x)\,g_i(y)$ in the Galerkin equation. In matrix form these may be expressed as

$$\begin{bmatrix} r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\\ r_{31} & r_{32} & r_{33}\end{bmatrix}\begin{bmatrix} c_1\\ c_2\\ c_3\end{bmatrix} - \begin{bmatrix} \cdot\\ \cdot\\ q\end{bmatrix} \tag{32}$$

or

$$R\,C - Q, \tag{32}$$

where

$$\begin{aligned}
r_{11} &= (f_4' f_1)(f_2 f_2) + \tfrac{1}{2}(1-\nu)\,r^2 (f_1 f_1)(f_5' f_2),\\
r_{12} &= \tfrac{1}{2}(1+\nu)(f_5 f_1)(f_4 f_2),\\
r_{13} &= -(\alpha+\nu r\beta)\frac{f_2(0)}{f_3(0)}(f_6 f_1)(f_3 f_2),\\
r_{21} &= (f_2 f_2)(f_4' f_1) + \tfrac{1}{2}(1-\nu)\frac{1}{r^2}(f_5' f_2)(f_1 f_1),\\
r_{22} &= \tfrac{1}{2}(1+\nu)(f_4 f_2)(f_5 f_1),\\
r_{23} &= -(r\beta+\nu\alpha)\frac{f_2(0)}{f_3(0)}(f_3 f_2)(f_6 f_1),\\
r_{31} &= -(\alpha+\nu r\beta)(f_4 f_3)(f_2 f_3),\\
r_{32} &= -(r\beta+\nu\alpha)(f_2 f_3)(f_4 f_3),\\
r_{33} &= (\alpha^2+2\nu r\alpha\beta+r^2\beta^2)\frac{f_2(0)}{f_3(0)}(f_3 f_3)(f_3 f_3)\\
&\quad+\frac{1}{\gamma}\frac{f_2(0)}{f_3(0)}[(f_8' f_3)(f_3 f_3) + 2r^2 (f_7 f_3)(f_7 f_3) + r^4 (f_3 f_3)(f_8' f_3)]
\end{aligned}$$

and

$$q = +(1-\nu^2)\,f_2(0)\frac{a}{t}\frac{Z}{E}(f_3)(f_3). \tag{32}$$

In the expressions (32) the abbreviations $(f_4' f_1)(f_2 f_2)$ have been used for $f_4'(x) f_1(x) f_2(y) f_2(y)$, etc.

Using (32), the Galerkin Eq. (11) may now be written as

$$\int_{-1}^{+1}\int_{-1}^{+1} R c\, dx\, dy - \int_{-1}^{+1}\int_{-1}^{+1} Q\, dx\, dy = 0,$$

whence we have
$$c = +\left[\int_{-1}^{+1}\int_{-1}^{+1} R\, dx\, dy\right]^{-1} \int_{-1}^{+1}\int_{-1}^{+1} Q\, dx\, dy. \tag{33}$$

In (33) c will be a (3×1) matrix, providing the three wanted constants c_1, c_2 and c_3. With (33) and (31) we have thus established a third approximation.

3. *Note on computation.* The formulae (23) and (28) may easily be programmed for a digital computer, thus providing numerical values for the functions $f_1 \ldots f_8$ for, say, 11 points $(x = 0, 0.1, 0.2 \ldots 1.0)$. These values may be stored in the machine as a (11×8) matrix. Denoting this matrix as $(F, 11\times 8)$, we have for instance f_4 as a column matrix denoted by $(f_4, 11\times 1)$. The integrations in expression (33) can now be performed numerically with, for example, Simpson's formula with step $h = 0.1$ (here). The coefficients of this formula are to be considered as a diagonal matrix. This is, if we omit the common factor $\frac{0.1}{3}$,

$$(S, 11/) = \mathrm{Diag}\,(1\ 4\ 2\ 4\ 2\ 4\ 2\ 4\ 2\ 4\ 1).$$

The integration of, for instance, r_{12} in (31) is thus as follows:

$$r_{12}\, dx\, dy = +\tfrac{1}{2}(1+\nu)(f_5, 1\times 11)(S, 11/)(f_1, 11\times 1)(f_4, 1\times 11)(S, 11/)(f_2, 11\times 1)$$

where $(f_i, 1\times 11)$ is simply the transpose of $(f_i, 11\times 1)$.

5. Edges Simply Supported on Diaphragms Perpendicular to the Surfase

The procedure for this case will be exactly as for the clamped edges described in § 4. By examination of the solution established in the previous paragraph, it will be seen that the only differences between the two cases are the coefficients of the matrices A, B and k_2 (Eqs. (22) and (27)). Therefore the same computer programme may be used for the two cases provided the appropriate data matrices are used.

a) Boundary conditions

According to Eq. (a) we have at $x = \pm 1$

$$v = 0,$$
$$w = 0,$$
$$N_1 = \frac{E t}{1-\nu^2}[u' + \nu v^{\cdot} - (\alpha + \nu r \beta) w] = 0,$$
$$M_1 = \frac{E t^3}{12(1-\nu^2)} \frac{1}{a}[w'' + \nu r^2 w^{\cdot\cdot}] = 0.$$

Assuming that the shear diaphragms are rigid in their plane, we may put

$$v^{\cdot} = w^{\cdot\cdot} = 0$$

and the boundary conditions take the form

At $x = \pm 1$

$$v = 0, \quad w = 0, \quad u' = 0, \quad w'' = 0 \quad (\text{and} \quad v^{\cdot} = w^{\cdot\cdot} = 0). \tag{34}$$

At $y = \pm 1$, similarly

$$u = 0, \quad w = 0, \quad v^{\cdot} = 0, \quad w^{\cdot\cdot} = 0 \quad (\text{and} \quad u' = w'' = 0). \tag{35}$$

b) Solution

1. Similarly to (19) we assume

$$u = f_1(x)(1-y^2), \qquad v = f_2(x)\tfrac{1}{2}(3y-y^3), \qquad w = f_3(x)(1-1.2y^2+0.2y^4). \tag{36}$$

These functions satisfy the boundary conditions at $y = \pm 1$, as may easily be checked. Substituting these expressions in the basic differential Eqs. (7) and integrating over the area bounded by $y = \pm 1$, using (15), we obtain:

$$+0.8127 f_1'' - 1.2191(1-\nu) r^2 f_1 + 0.6857(1+\nu) f_2' - 0.8940(\alpha+\nu r\beta) f_3' = 0,$$

$$+0.4857(1-\nu)\frac{1}{r^2} f_2'' - 2.4 f_2 - 0.6857(1+\nu) f_1' + 1.5543(r\beta+\nu\alpha) f_3 = 0,$$

$$-0.8940(\alpha+\nu r\beta) f_1' - 1.5543(r\beta+\nu\alpha) f_2 + 1.0077(\alpha^2+2\nu r\alpha\beta+r^2\beta^2) f_3$$
$$+\frac{1}{\gamma}[1.0077 f_3'''' - 4.9738 r^2 f_3'' + 6.1440 r^4 f_3] = 1.28(1-\nu^2)\frac{a}{t}\frac{Z}{E}.$$

The coefficients of the A and B matrix are thus

$$\begin{aligned}
a_{14} &= +1, \\
a_{25} &= +1, \\
a_{36} &= +1, \\
a_{41} &= +1.5(1-\nu) r^2, \\
a_{45} &= -0.8437(1+\nu), \\
a_{46} &= +1.1(\alpha+\nu r\beta), \\
a_{52} &= +4.9416\frac{r^2}{1-\nu}, \\
a_{53} &= -3.2(r\beta+\nu\alpha)\frac{r^2}{1-\nu}, \\
a_{54} &= +1.4119\frac{1+\nu}{1-\nu} r^2, \\
a_{67} &= +1, \\
a_{78} &= +1,
\end{aligned} \tag{37}$$

$$\begin{aligned}
a_{82} &= +1.5425\,(r\beta+\nu\alpha)\gamma,\\
a_{83} &= -(\alpha^2+2\nu r\alpha\beta+r^2\beta^2)\gamma-6.0973\,r^4,\\
a_{84} &= +0.8872\,(\alpha+\nu r\beta)\gamma,\\
a_{87} &= +4.9360\,r^2,\\
b_8 &= +1.2703\,(1-\nu^2)\gamma\frac{a}{t}\frac{Z}{E}.
\end{aligned}\tag{37}$$

2. The first solution is as before

$$\boxed{Z(x) = G(x)\,Z_0+\tilde{Z}(x)}\,,\tag{23}$$

where

$$\underline{Z_0 = -k_1\,[k_2\,G(1.0)\,k_1]^{-1}\,k_2\tilde{Z}(1.0)}\,.\tag{28}$$

Here

$$G(x) = e^{Ax},$$

$$Z(x) = -(I-e^{Ax})\,A^{-1}B$$

and

$$k_1 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot\\ +1 & \cdot & \cdot & \cdot\\ \cdot & +1 & \cdot & \cdot\\ \cdot & \cdot & +1 & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & +1\\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}\tag{25}$$

as for the clamped case. The k_2 matrix takes now the form

$$k_2 = \begin{bmatrix} \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \end{bmatrix}.\tag{38}$$

3. The improvements to this first approximation may be obtained as described in § 4e.

6. Edges Simply Supported on Vertical Diaphragms

a) Boundary conditions

In this case we use the boundary conditions (b) which may be written in the form:

At $x = \pm 1$

$$\begin{aligned}
v &= 0,\\
w-u\,\mathrm{tg}\,\varphi_0 &= 0,\\
\frac{E\,t^3}{12\,(1-\nu^2)}\,\frac{1}{a}\,[w''+\nu r^2 w^{\cdot\cdot}] &= 0,
\end{aligned}$$

$$-\frac{E\,t^3}{12\,(1-\nu^2)}\,\frac{1}{a^2}\,[w'''+r^2 w'^{\cdot\cdot}]\,\mathrm{tg}\,\varphi_0+\frac{E\,t}{1-\nu^2}\,[u'+\nu v^{\cdot}-(\alpha+\nu r\beta)\,w] = 0.$$

Introducing $\operatorname{tg}\varphi_0 = -\alpha$ from expression (3) and remembering that again $w^{\cdot\cdot} = w'^{\cdot\cdot} = v^{\cdot} = 0$ at the boundaries $x = \pm 1$, the boundary conditions may be written as:

At $x = \pm 1$

$$v = 0, \quad w + \alpha u = 0, \quad w'' = 0, \quad \alpha w''' + \gamma u' - \gamma(\alpha + \nu r \beta) w = 0. \tag{39}$$

At $y = \pm 1$, similarly

$$u = 0, \quad w + r\beta v = 0, \quad w^{\cdot\cdot} = 0, \quad r^3 w^{\cdot\cdot\cdot} + \gamma v^{\cdot} - \gamma(r\beta + \nu\alpha) w = 0. \tag{40}$$

b) Functions $g(y)$ satisfying the boundary conditions at $y = \pm 1$

Similarly to Eq. (36) we assume

$$u = f_1(x)(1-y^2)^2, \quad v = f_2(x)\tfrac{1}{2}(c_1 y - y^3), \quad w = f_3(x)(c_2 - 1.2y^2 + 0.2y^4), \tag{41}$$

where c_1 and c_2 are constants[10]) to be determined such that the boundary conditions (40) are satisfied. Introducing (41) in (40) we find

$$c_1 = 1 + \frac{2}{1 + r\beta(r\beta + \nu\alpha)} - \frac{9.6 r^3 \beta}{1 + r\beta(r\beta + \nu\alpha)} \frac{1}{\gamma} \frac{f_3(x)}{f_2(x)}$$

and $$c_2 = 1 - \frac{r\beta}{1 + r\beta(r\beta + \nu\alpha)} \frac{f_2(x)}{f_3(x)} + \frac{4.8 r^4 \beta^2}{1 + r\beta(r\beta + \nu\alpha)} \frac{1}{\gamma}.$$

Substituting these expressions in (41) we find

$$\left.\begin{aligned} u &= f_1(x)(1-y^2)^2, \\ v &= f_2(x)\tfrac{1}{2}[(1+D)y - y^3] - f_3(x) F y, \\ w &= f_3(x)[(1 + r\beta F) - 1.2y^2 + 0.2y^4] - f_2(x)\frac{r\beta}{2} D, \\ \text{where} \quad D &= \frac{2}{1 + r\beta(r\beta + \nu\alpha)} \\ \text{and} \quad F &= \frac{4.8 r^3}{1 + r\beta(r\beta + \nu\alpha)} \frac{1}{\gamma}. \end{aligned}\right\} \tag{42}$$

The expressions (42) satisfy the boundary conditions (40) as may easily be checked. The factor F, however, is very small since it contains $\frac{1}{\gamma} = \frac{t^2}{12a^2}$. In the expression for w it is readily seen that $r\beta F$ may be neglected against 1; in the expression for v it is not so obvious that F can be put equal to zero. We may, however, predict that $f_3(x)$ will not be much more than 10—20 times greater than $f_2(x)$. In such cases, and if $\beta = \alpha = 1/5$, $\nu = 0$, $\gamma = 2.10^5$, say, it is seen that here also the influence of F will be negligible.

We have thus the following functions satisfying the boundary conditions at $y = \pm 1$ with reasonable accuracy:

---

[10]) I. e. constants w. r. t. y.

$$\begin{aligned} u &= f_1(x)\,(1-y^2)^2, \\ v &= f_2(x)\,\tfrac{1}{2}\,[(1+D)\,y-y^3], \\ w &= f_3(x)\,[1-1.2\,y^2+0.2\,y^4]-f_2(x)\frac{r\beta}{2}D, \end{aligned} \qquad (43)$$

where $$D = \frac{2}{1+r\beta\,(r\beta+\nu\,\alpha)}.$$

c) Solution

Substituting the expressions (43) in the basic differential Eq. (7) and integrating (13) using the approximate functions

$$g_i = (1-y^2)^2, \qquad g_2 = \tfrac{1}{2}\,(3\,y-y^3), \qquad g_3 = (1-1.2\,y^2+0.2\,y^4),$$

as weighting functions, we get as before the matrix equation

$$\frac{\partial}{\partial x} Z = A\,Z + B, \qquad (44)$$

where $$A = \begin{bmatrix} \cdot & \cdot & \cdot & a_{14} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{25} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & a_{36} & \cdot & \cdot \\ a_{41} & \cdot & \cdot & \cdot & a_{45} & a_{46} & \cdot & \cdot \\ \cdot & a_{52} & a_{53} & a_{54} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{67} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{78} \\ \cdot & a_{82} & a_{83} & a_{84} & \cdot & \cdot & a_{87} & \cdot \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_8 \end{bmatrix}. \qquad (44)$$

The coefficients of A and B are:

$$\begin{aligned} a_{14} &= +1, \\ a_{25} &= +1, \\ a_{36} &= +1, \\ a_{41} &= +1.5\,(1-\nu)\,r^2, \\ a_{45} &= -(0.1876+0.3281\,D)\,(1+\nu)-0.6562\,r\,\beta\,D\,(\alpha+\nu\,r\,\beta), \\ a_{46} &= +1.1\,(\alpha+\nu\,r\,\beta), \\ a_{52} &= +2.4\,r^2\,N, \\ a_{53} &= -1.5543\,r^2\,(r\,\beta+\nu\,\alpha)\,N, \\ a_{54} &= +0.6857\,r^2\,(1+\nu)\,N, \\ a_{67} &= +1, \\ a_{78} &= +1, \end{aligned}$$

$$
\begin{aligned}
a_{82} &= +(0.2722+0.6351\,D)\,(r\,\beta+\nu\,\alpha)\,\gamma \\
&\quad +(\alpha^2+2\,\nu\,r\,\alpha\,\beta+r^2\,\beta^2)\,M\,\gamma+M\,a_{52}\,S\,, \\
a_{83} &= -(\alpha^2+2\,\nu\,r\,\alpha\,\beta+r^2\,\beta^2)\,\gamma-6.0973\,r^4+M\,a_{53}\,S\,, \\
a_{84} &= +0.8872\,(\alpha+\nu\,r\,\beta)\,\gamma+M\,a_{54}\,(S+a_{41})\,, \\
a_{87} &= +4.9360\,r^2+M\,(a_{53}+a_{54}\,a_{46})
\end{aligned}
$$

and $$b_8 = +1.2703\,(1-\nu^2)\,\gamma\frac{a}{t}\,\frac{Z}{E}\,.$$

Here
$$
\begin{aligned}
2\,a &= \text{Base length in } x \text{ direction}\,, \\
r &= a/b\,, \\
t &= \text{thickness}\,, \\
Z &= \text{external, uniformly distributed normal load}\,, \\
E &= \text{Young's modulus}\,, \\
\nu &= \text{Poisson's ratio}\,, \\
\alpha &= -\frac{2\,h_1}{a}\,, \\
\beta &= -\frac{2\,h_2}{b}\,, \\
\gamma &= \frac{12\,a^2}{t^2}\,, \\
D &= \frac{2}{1+r\,\beta\,(r\,\beta+\nu\,\alpha)}\,, \\
M &= 0.6351\,r\,\beta\,D\,, \\
N &= \frac{1}{(0.08571+0.2\,D)\,(1-\nu)}\,, \\
S &= a_{52}+a_{54}\,a_{45}\,.
\end{aligned}
$$

Putting the slope at the edges $\alpha=\beta=0$ [11]), we obtain the A matrix for the case where the diaphragms are perpendicular to the shell surface (37). The solution of (44) is as before

$$Z\,(x) = G\,(x)\,Z_0+\tilde{Z}\,(x)\,, \tag{23}$$

where
$$
\begin{aligned}
Z_0 &= -k_1\,[k_2\,G_2\,(1.0)\,k_1]^{-1}\,k_2\,\tilde{Z}\,(1.0)\,, \\
G\,(x) &= e^{Ax} \quad \text{and} \quad \tilde{Z}\,(x) = -(I-e^{Ax})\,A^{-1}\,B\,.
\end{aligned} \tag{28}
$$

---

[11]) The terms $(\alpha+\nu\,r\,\beta)$ and $(r\,\beta+\nu\,\alpha)$ due to the curvatures must not be put equal to zero!

Here k_1 is the same as in previous cases:

$$k_1 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}. \tag{25}$$

k_2 has to be determined according to the four conditions (39) at $x = +1$. Substituting (43) in (39) it is readily seen that the boundary conditions only can be satisfied at given points $y =$ constant. Choosing the point $y = 0$ as the most significant, we get:

At $x = +1$, $y = 0$

$$\begin{aligned} v &= 0 \quad (\text{i.e. } f_2 = 0 \text{ at all points } (1, y)), \\ \alpha u + w &= \alpha f_1 + f_3 - \frac{r\beta}{2} D f_2 = 0, \\ w'' &= f_3'' - \frac{r\beta}{2} D f_2'' = 0, \end{aligned} \tag{45}$$

$$\frac{\alpha}{\gamma} w''' + u' - (\alpha + \nu r \beta) w = \frac{\alpha}{\gamma} f_3''' - \frac{\alpha}{\gamma}\frac{r\beta}{2} D f_2''' + f_4 - (\alpha + \nu r\beta) f_3 + (\alpha + \nu r \beta)\frac{r\beta}{2} D f_2 = 0.$$

Here $f_2'' = f_5'$ and $f_2''' = f_5''$ are given by (44). The k_2 matrix is hence:

$$k_2 = \begin{bmatrix} \cdot & k_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ k_{21} & k_{22} & k_{23} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & k_{32} & k_{33} & k_{34} & \cdot & \cdot & k_{37} & \cdot \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} & \cdot & k_{48} \end{bmatrix}, \tag{46}$$

where

$$\begin{aligned} k_{12} &= +1, \\ k_{21} &= +\alpha, \\ k_{22} &= -\frac{r\beta}{2} D, \\ k_{23} &= +1, \\ k_{32} &= -\frac{r\beta}{2} D a_{52}, \\ k_{33} &= -\frac{r\beta}{2} D a_{53}, \\ k_{34} &= -\frac{r\beta}{2} D a_{54}, \\ k_{37} &= +1, \end{aligned}$$

$$k_{41} = -\frac{\alpha}{\gamma}\frac{r\beta}{2} D\, a_{54}\, a_{41}, \tag{46}$$

$$k_{42} = +\frac{r\beta}{2} D\,(\alpha + \nu r \beta),$$

$$k_{43} = -(\alpha + \nu r \beta),$$

$$k_{44} = +1,$$

$$k_{45} = -\frac{\alpha}{\gamma}\frac{r\beta}{2} D\,(a_{52} + a_{54}\, a_{45}),$$

$$k_{46} = -\frac{\alpha}{\gamma}\frac{r\beta}{2} D\,(a_{53} + a_{54}\, a_{46}),$$

$$k_{48} = +\frac{\alpha}{\gamma}.$$

Putting $\alpha = \beta = 0$ we obtain the k_2 matrix for the case where the diaphragms are perpendicular to the shell surface (38).

d) Improvements to the solution

1. Having found the functions $f_1 \dots f_8$, we assume similar to (29):

$$u = c_1 f_1(x) f_2(y), \qquad v = c_2 f_2(x) f_1(y), \qquad w = c_3 f_3(x) f_3(v) + c_4 f_2(x) f_2(y). \tag{47}$$

Here the constants c_i are not arbitrary, but have to be determined such that they satisfy the boundary conditions at $x = \pm 1$, $y = 0$ and at $x = 0$, $y = \pm 1$. Using the conditions (45) already imposed on the functions $f_1 \dots f_8$ (for the argument $+1$), we obtain by substituting (47) in (39):

$$c_1 = \delta\, c_3$$

and
$$c_4 = -\frac{r\beta}{2} D\, \delta\, c_3, \tag{48}$$

where
$$\delta = \frac{f_3(0)}{f_2(0)}.$$

The first and the fourth conditions being identically satisfied. Substituting (47) in (40), the first and the third conditions are identically satisfied. The second gives

$$c_2 = -\frac{\alpha}{r\beta}\delta\, c_3 \tag{49}$$

and the fourth takes the form

$$\frac{\beta r^3}{\gamma} f_3'''(1) - \frac{\beta r^3}{\gamma}\frac{r\beta}{2} D f_2'''(1) + \frac{\alpha}{r\beta} f_4 - (r\beta + \nu\alpha) f_3(1) + (r\beta + \nu\alpha)\frac{r\beta}{2} D f_2(1) = 0.$$

Comparing this with the imposed condition (45d) it is seen that the fourth boundary conditions in this case can be identically satisfied only if

and
$$\boxed{\begin{aligned} r &= 1 \\ \alpha &= \beta \end{aligned}}\,, \tag{50}$$

which means that the shell must have a square base and that the parabolae in the x and y direction must be identical.

With these restrictions we obtain, by substituting (48) and (49) in (47)

$$\begin{aligned} u &= \delta c_3 f_1(x) f_2(y), \\ v &= \delta c_3 f_2(x) f_1(y), \\ w &= c_3 f_3(x) f_3(y) - \delta c_3 \frac{\beta}{2} D f_2(x) f_2(y), \end{aligned}$$

where
$$\delta = \frac{f_3(0)}{f_2(0)}.$$

Similar to (30) it will be readily seen that c_3 will have the approximate value of $\frac{+1}{f_3(0)}$. With this we have as a third approximation

$$\begin{aligned} u &= c \frac{1}{f_2(0)} f_1(x) f_2(y), \\ v &= c \frac{1}{f_2(0)} f_2(x) f_1(y), \\ w &= c \left[\frac{1}{f_3(0)} f_3(x) f_3(y) - \frac{\beta}{2} D \frac{1}{f_2(0)} f_2(x) f_2(y) \right], \end{aligned} \tag{52}$$

where c is a constant with the approximate value of $+1$.

2. Instead of determining the best value of c by considering the energy, we will now determine c such that the errors made by introducing (52) in the basic Eqs. (7) are minimum (method of least squares):

$$\iint_S [\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2]\, dx\, dy = \text{minimum}$$

i.e.
$$\frac{\partial}{\partial c} \iint \sum_{i=1}^{i=3} \epsilon_i^2\, dx\, dy = 0$$

or
$$\underline{\int_{-1}^{+1} \int_{-1}^{+1} \sum_{i=1}^{i=3} \epsilon_i \frac{\partial \epsilon_i}{\partial c}\, dx\, dy = 0.} \tag{53}$$

Here, for instance,

$$\epsilon_1 = \frac{c}{f_2(0)} \left[f_1'' f_2 + \tfrac{1}{2}(1-\nu) f_1 f_2'' + \tfrac{1}{2}(1+\nu) f_1' f_2' - \frac{\alpha}{\delta}(1+\nu) f_3' f_3 + \frac{\alpha^2}{2} D (1+\nu) f_2' f_2 \right].$$

Proceeding as in § 4e, we can integrate (53) numerically with a digital computer and thus find the wanted constant c. Having found c we can compute all quantities of interest using the known values of $f_1 \ldots f_8$.

It must be remembered that the boundary conditions are satisfied only at the points $(\pm 1, 0)$ and $(0, \pm 1)$ and that this solution only is valid for shells with a square base and where the generatrix and the directrix have the same rise (Eq. (50)).

7. Numerical Example

Since the solution for the case with diaphragms perpendicular to the surface has been extensively treated by Dr. Dikovich [9], it will be of interest to test the accuracy of our method by comparing the results obtained for a particular example:

Consider a spherical cap over a square base, simply supported on diaphragms perpendicular to the surface, with the following dimensions and properties:

$$a = b = 11^m,$$

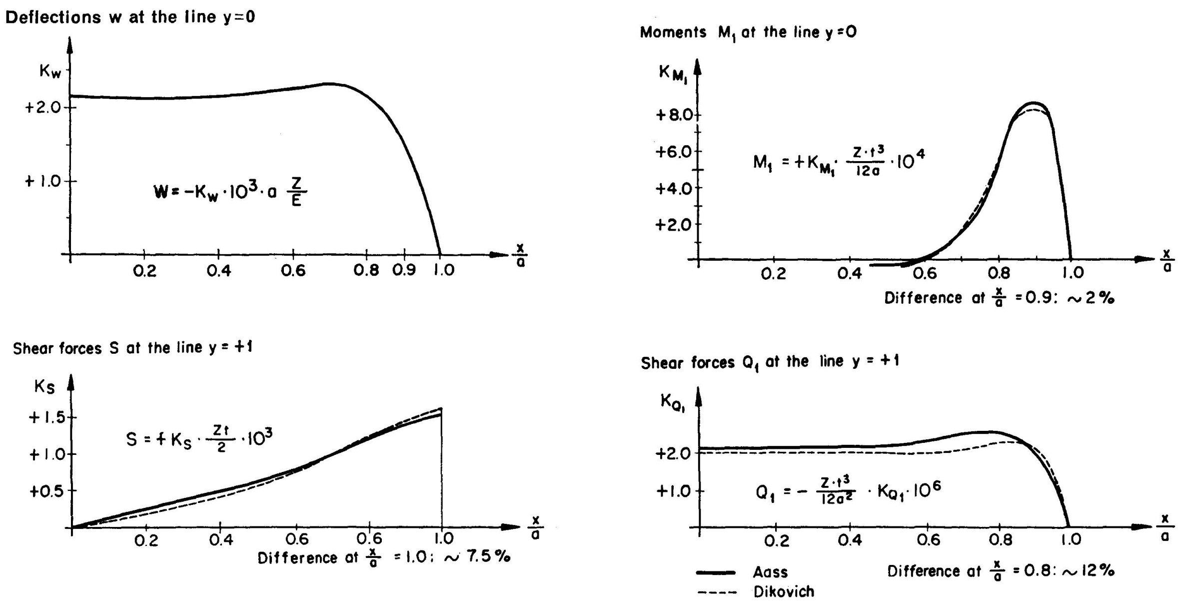
$$h_1 = h_2 = 1.43^m,$$

$$\frac{1}{k_1} = \frac{1}{k_2} = R = 42.3^m,$$

$$\text{Thickness} = t = 8^{cm},$$

$$v = 0.$$

The results are as follows:



The curve for w practically coincides with the one given by Dr. Dikovich.

8. Conclusions

On the basis of the numerical example given in § 7, one may conclude that the method will give sufficiently accurate results for practical purposes. Since the actual computing time is very short [12]), the procedure will provide a cheap means for analysing EP shells.

For the special case where $r = a/b = 1$ and $\alpha = \beta$ (i. e. $k_1 = k_2$), it is also possible to take the slope at the edges into consideration by formulating the boundary conditions (vertical edge members!). By given values of Poisson's ratio, the wanted functions $f_1 \ldots f_8$ and the unknown constant c may easily be tabulated for different values of the two parameters $\alpha = -\frac{2h_1}{a}$ and $\gamma = \frac{12a^2}{t^2}$. In a forthcoming publication a study of numerical values for this case will be presented.

Introducing given displacements along the boundaries, it will also be possible to establish "edge load" tables for calculation of EP shells supported on elastic edge members. This will be the subject of a paper to be published shortly.

Acknowledgment

The writer wishes to express his gratitude to Mr. H. Tottenham [13]) for his inspiring assistance and good advice while carrying out the presented work. He is also greatly indebted to the British Council, London and Norges Teknisk Naturvitenskapelige Forskningsråd, Oslo, who have supported him economically during his stay at Southampton University.

Notation

$2a$ = Base length in x direction
$2b$ = Base length in y direction
f_i = Function of x
g_i = Function of y
h_1 = Maximum rise of the parabola in the x direction
h_2 = Maximum rise of the parabola in the y direction

$$\left.\begin{aligned} k_1 &= -\frac{\partial^2 w}{\partial x^2} \\ k_2 &= -\frac{\partial^2 w}{\partial y^2} \\ k_{12} &= -\frac{\partial^2 w}{\partial x\, \partial y} \end{aligned}\right\} \text{Curvatures}$$

[12]) The programme consisted of approximately 150 matrix instructions.

[13]) Reader in Structural Engineering at the University of Southampton, England.

$r \quad = a/b$

$t \quad =$ shell thickness

$\left.\begin{matrix} u \\ v \\ w \end{matrix}\right\} =$ displacements in the direction of the axes

$\left.\begin{matrix} x \\ y \\ z \end{matrix}\right\} =$ co-ordinate axes

$$\left.\begin{aligned} \alpha &= -\frac{2h_1}{a} \\ \beta &= -\frac{2h_2}{b} \\ \gamma &= +\frac{12a^2}{t^2} \end{aligned}\right\} \text{shell parameters}$$

$\epsilon \quad =$ Error

$\nu \quad =$ Poisson's ratio

$A \quad =$ Matrix (8×8)

$B \quad =$ Matrix (8×1)

$C \quad =$ Arbitrary constant

$E \quad =$ Young's modulus

$G(x) = e^{Ax}$ Distribution matrix

$I \quad =$ Unit matrix

$k_1 \quad =$ Initial value matrix (at $x = 0$)

$k_2 \quad =$ Boundary restraint matrix (at $x = +1$)

$L_i \quad =$ Linear differential operator

$\left.\begin{matrix} X \\ Y \\ Z \end{matrix}\right\} =$ External loads

$Z(x) =$ Action matrix (8×1), containing the unknown functions $f_1 \ldots f_8$

$Z_0 \quad = Z(x)_{x=0}$

$\tilde{Z}(x) = -(I - e^{Ax})A^{-1}B =$ Loading solution matrix

$N_1, N_2, S, M_1, M_2, M_{12}, Q_1, Q_2 =$ Stress resultants according to Fig. 2

$(\ )'' = \dfrac{\partial^2}{\partial x^2}$

$(\ )^{\cdot\cdot} = \dfrac{\partial^2}{\partial y^2}$

$\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

$\nabla^4 = \nabla^2 \nabla^2$

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Summary

The writer presents a variational method for the analysis of elliptic paraboloidal shells with the following boundary conditions:

1. Edges rigidly clamped.
2. Edges simply supported on shear diaphragms perpendicular to the surface.
3. Edges simply supported on vertical shear diaphragms.

In an example the accuracy of the method is tested by comparing the results with those given by B. B. DIKOVICH [9].

Résumé

L'auteur présente une méthode variationelle pour l'étude des voiles minces en forme de paraboloïde elliptique. Il considère les conditions au contour suivantes:

1. Bords encastrés.
2. Bords simplement appuyés sur des tympans perpendiculaires à la surface.
3. Bords simplement appuyés sur des tympans verticaux.

Dans un exemple, il vérifie l'exactitude de la méthode en comparant ses résultats avec ceux donnés par B. B. DIKOVICH [9].

Zusammenfassung

Der Autor beschreibt eine Variationsmethode zur Berechnung von elliptischen Paraboloidschalen mit den folgenden Randbedingungen:

1. Die Ränder sind fest eingespannt.
2. Die Ränder sind frei drehbar auf Randscheiben gelagert, die senkrecht zur Schalenfläche angeordnet sind.
3. Die Ränder sind frei drehbar auf vertikalen Randscheiben gelagert.

Die Genauigkeit der Methode wird an einem Beispiel, durch Vergleich mit den Ergebnissen der Methode B. B. DIKOVICH [9], geprüft.