Energy Method for Analyzing All Stresses in Rigid Trusses

Une méthode énergétique pour la détermination de toutes les contraintes dans les treillis à nœuds rigides

Energiemethode für die Analyse aller Beanspruchungen in Fachwerken mit starren Knotenverbindungen

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Introduction

Methods of analyzing pin-connected trusses have inappropriately remained in use though the latter have long become obsolete. They were first superseded by riveted trusses about half a century ago. Since World War II, welded trusses have gained increasing acceptance. All these modern rigidly-connected trusses, whether with or without internal or external redundancy, are, by their inherent nature, highly statically-indeterminate rigid frames. The rigidity of the joints constitutes the main cause for end moments and transverse shear in each member.

Including MANDERLA's [1] first enunciation of a method 85 years ago, at least nine independent methods have been developed, for the solution of the so-called "secondary stresses" — stresses caused by conditions ignored in the conventional analysis of "primary stresses". The problem of secondary stress has, in reality, arisen from inappropriate solution of rigidly-connected truss, rather than from it being truly secondary in nature. By analyzing a rigidly-connected truss under a given loading as an assemblage or chain of rigid frames, only one

[1] MANDERLA, HEINRICH: «Welche Spannungen entstehen in den Streben eines Fachwerks dadurch, daß die Winkel der Fachwerkdreiecke durch die Belastung eine Änderung erleiden?» Preisarbeit, Jahresbericht der Technischen Hochschule München, 1878—1879, P. 18.

set of perfectly normal genuine stresses will be found existing in such a truss, thus dispelling the heretofore misnomer of "secondary stresses".

To achieve this elegant ideal of solving all genuine stresses including secondary stresses in each member of a rigidly-connected truss of any configuration with any redundancy under any externally applied loading, a matrix-energy formulation for executing the solution is proposed herein. The method enables the determination of all genuine stresses in a unified single set-up; it adapts to programmed electronic computation; and it provides both exact and approximate solutions.

Basic Concepts

A rigidly-connected truss, under a given loading, is equivalent to an otherwise ideal, pin-connected version not only identically loaded but also acted by couples on the bar ends, equal to the internal resisting moments thereat.

In the most general case, if (1) the internal resisting moments at the ends of the members, (2) the axial stresses in the redundant members, and (3) the redundant reactions at the supports were all known, a rigidly-connected indeterminate truss of any redundancy would be completely determined by statics. These three types of quantities are treated as unknowns in the proposed method. To ensure that all unknowns are statically independent, equations of static equilibrium must be fully applied to eliminate dependent unknowns. Consequently, the number of statically independent unknowns is just equal to the degree of statical indeterminateness of the truss viewed as an assemblage of rigid frames. All stresses (internal forces and moments) therein, and hence the total strain energy of the truss, can be expressed in terms of the externally applied panel loads and the said unknowns. By appropriate partial differentiations, all the necessary simultaneous equations will be evolved. The following development in its operative sequence is founded on the above basic concepts.

Constituent Strain-Energy Matrix

The laterally exaggerated elastic curve of any truss member $I-J$ is represented in Fig. 1.

Notations for any truss member $I-J$ are defined as follows: M_{ij} and M_{ji} are respectively the unknown internal resisting end moment in kip-in at the I- and J-end; N_{ij}, the total axial force in kip; Q_{ij}, the transverse shear in kip;

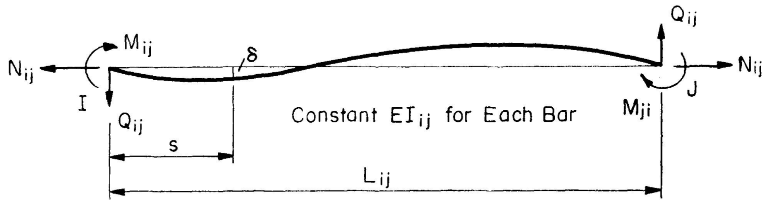

Fig. 1.

A_{ij}, the cross-sectional area in sq. in.; I_{ij}, the moment of inertia in in^4; L_{ij}, the length in in.; s, the distance from I-end in in.; δ, the displacement in in. at a distance s from the I-end, normal to the original centroidal axis; E, modulus of elasticity of the material in ksi; G, modulus of rigidity of the material in ksi; μ, Poisson's ratio of the material which may be taken as equal to 0.3 for structural steel; U_{ij}, V_{ij}, and W_{ij}, the strain energy in in-kip respectively due to bending moment, transverse shear, and axial force.

Let the sign convention be defined such that (1) positive end moments produce clockwise rotation of the ends; (2) positive axial forces are in tension; and (3) a positive pair of end shears forms a counterclockwise couple.

The matrix of constituent strain-energy expressions may now be formulated. In Fig. 1, recognizing that the moment due to axial force and deviation from original centroidal axis is usually negligibly small, the true moment about any point at a distance s from the I-end, that is

$$M_s = M_{ij} - Q_{ij}\, s - N_{ij}\,\delta \tag{1}$$

may take the simplified form of

$$M_s = M_{ij} - Q_{ij}\, s, \tag{2}$$

where

$$Q_{ij} = \frac{M_{ij} + M_{ji}}{L_{ij}}. \tag{3}$$

Hence, the constituent strain-energy matrix of any member $I - J$ may be written according to MÉNABRÉA [2]) as

$$\begin{bmatrix} W_{ij} \\ U_{ij} \\ V_{ij} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \int_0^{L_{ij}} \frac{N_{ij}^2}{E A_{ij}}\, d s \\ \int_0^{L_{ij}} \frac{M_s^2}{E I_{ij}}\, d s \\ \int_0^{L_{ij}} \frac{Q_{ij}^2}{G A_{ij}}\, d s \end{bmatrix} = \frac{1}{E} \begin{bmatrix} \frac{1}{2} \frac{L_{ij}}{A_{ij}} N_{ij}^2 \\ \frac{L_{ij}}{6 I_{ij}} (M_{ij}^2 - M_{ij} M_{ji} + M_{ji}^2) \\ \frac{1+\mu}{A_{ij} L_{ij}} (M_{ij} + M_{ji})^2 \end{bmatrix} \tag{4}$$

in the last of which $G = E/2(1+\mu)$.

Summing up $\{W\ U\ V\}_{ij}$ for all members of the truss, the total strain energy, U, of any truss is then

$$U = \sum_1^m \lfloor 1 \quad 1 \quad 1 \rfloor \{W_{ij} \quad U_{ij} \quad V_{ij}\}, \tag{5}$$

where m is the number of members in the truss.

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[2]) MENABREA, L. F.: A memoir presented to the Academy of Sciences in Turin, 1857. (Containing the earliest suggestion in the use of the expression for the strain energy of the truss.)

The Matrix Equation of Unknowns and Their Solution

Under the usual conditions of truss analysis, taking all joints as infinitely rigid, all components as ideally fit, and all supports as unyielding, the application of CASTIGLIANO's second theorem[3]), or the theorem of least work, to the present problem, involving trusses with any degree of redundancy, will give the following relations:

$$\left\{\frac{\partial U}{\partial M} \ \frac{\partial U}{\partial N} \ \frac{\partial U}{\partial R}\right\} = \{0 \ \ 0 \ \ 0\}, \tag{6}$$

where M is the statically independent unknown end moment; N, the unknown axial force in the redundant member, if any; R, the unknown redundant reaction, if any. While Eqs. (6) represent minimization of strain energy or zero "relative" displacements, the last of Eqs. (6) also denotes the condition of zero settlement of support. In case of non-zero, then according to CASTIGLIANO's first theorem[3]), $\frac{\partial U}{\partial M}$ would be equal to the rotation; $\frac{\partial U}{\partial N}$, an over-run or under-run; and $\frac{\partial U}{\partial R}$, the support settlement.

The unknowns M's, N's, and R's of any loaded plane truss of any configuration may be generalized as the unknown column vector $\{X_i\}$. Repeated application of $\frac{\partial U}{\partial X_i} = 0$ will yield a set of n non-homogeneous simultaneous algebraic linear equations of the type

$$\lfloor a_{ik} \rfloor \{X_k\} = \{C_i\} \tag{7}$$

as the ith equation, in which both i and $k = 1, 2, \ldots, i, \ldots, j, \ldots, n$, and the constant term C_i has been transposed to the right-hand side.

It follows directly from MAXWELL's theorem of reciprocity[4]) that the coefficient a_{ji} of X_i in the jth equation is identical both in sign and magnitude as the coefficient a_{ij} of X_j in the ith equation, and by virtue of this well-known fact,

$$a_{ij} = a_{ji}, \tag{8}$$

where $i \neq j$, giving a symmetric coefficient matrix, analogous to the "flexibility matrix"[5]). Hence, in abbreviated matrix form, the set of equations becomes

$$[a_{ij}]\{X_i\} = \{C_i\}, \tag{9}$$

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[3]) CASTIGLIANO, ALBERTO: «Nuova teoria interno dell'equilibrio dei sistemi elastici». Atti delle Academia delle Scienze, Torino, 1875.

[4]) MAXWELL, JAMES CLERK: "On the Calculation of the Equilibrium and the Stiffness of Frames". Phil. Mag., Series 4, Vol. 27, 1864, P. 294.

[5]) ARGYRIS, J. H.: A series of articles in Aircraft Engineering (London) between October 1954 and May 1955.

which always has a general solution by inverting $[a_{ij}]$ unless the matrix $[a_{ij}]$ is singular. That is to say, if $|a_{ij}| \neq 0$, the solution is

$$\{X_i\} = [a_{ij}]^{-1}\{C_i\}. \tag{10}$$

Since $[a_{ij}]^{-1}$ is uniquely satisfying $[a_{ij}]^{-1}[a_{ij}] = U$ (unit matrix), the vector of solutions given by $[a_{ij}]^{-1}\{C_i\}$ constitutes the only solutions. In Eqs. (9), because of symmetry, only $\frac{1}{2}n(n+1)$ coefficients are to be evaluated and consequently the computer time for inverting the matrix will be correspondingly reduced. In inverting large matrices, an efficient and fast method such as LI's algorisms[6]) is recommended.

The Illustrative Example

To exemplify the numerical process and compare the results with those obtained by using recognized methods, let the example given by SUTHERLAND and BOWMAN[7]) be solved by the proposed matrix-energy method.

Statement of the Problem

It is desired to find all genuine stresses at the ends of each member of the rigidly-connected truss shown in Fig. 2 due to vertical loads of 166 kips at

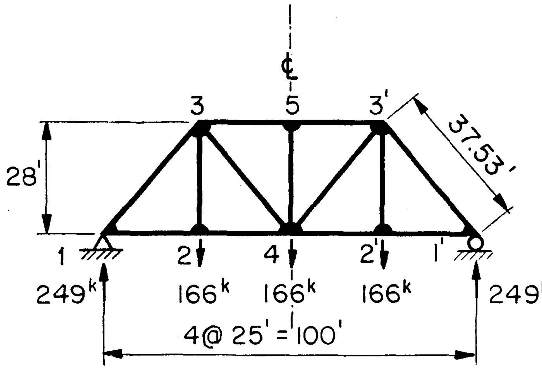

Fig. 2.

each lower panel point except at supports. The makeup of the members is given in Table 1. For simplicity, assume centroidal axes of members intersect at theoretical panel points, thus eliminating eccentric moments. Poisson's ratio, μ, is taken as 0.3.

6) LI, SHU-T'IEN: "Converging Matric Algorisms for Solving Systems of Linear Equations". Trans. of the November 1962 Convention of the Chinese Association for the Advancement of Science, Taipei, China, Vol. 1, November 1962, PP. 16—22.

7) SUTHERLAND, HALE and BOWMAN, HARRY LAKE: "Structural Theory". 4th Edition, 1950, Seventh Printing, 1961; John Wiley & Sons, Inc., New York; PP. 351—357.

Table 1. Makeup of Members

Bar	Section	Area in.^{2}	I in.^{4}	L in.	I/c in.^{3}	Sketch
1 – 3	2 – [15 × 33.9 1 – *Pl* 18 × $^7/_{16}$	27.68	961.0	450.44	167.5 99.1	
3 – 5	2 – [15 × 33.9 1 – *Pl* 18 × $^3/_8$	26.55	922.8	300.00	156.0 97.6	
1 – 2 2 – 4	4 – ⌊s 6 × 3$^1/_2$ × $^1/_2$	18.00	175.3	300.00	27.5	
2 – 3	4 – ⌊s 6 × 3$^1/_2$ × $^7/_{16}$	15.88	153.8	336.00	24.1	
3 – 4	4 – ⌊s 6 × 3$^1/_2$ × $^3/_8$	13.68	131.8	450.44	20.7	
4 – 5	4 – ⌊s 5 × 3 × $^3/_8$	11.44	79.1	336.00	14.7	

Solution

In general, for an asymmetrical rigidly-connected truss of m members under asymmetrical loading, there will be $2m$ unknown end moments. In a symmetrical rigidly-connected truss and under symmetrical loading, if n is the number of joints, the number (N) of statically independent unknown end moments is given by

$$N = \tfrac{1}{2}(2m - n) = m - \tfrac{1}{2}n. \tag{11}$$

In the present case, $m = 13$, $n = 8$, therefore, $N = 13 - \frac{1}{2}(8) = 9$; that is, the present truss is determinate when pin-connected, but becomes indeterminate to the 9th degree when rigidly connected.

Let the nine statically independent unknown end moments be represented, element for element, by the matrix:

$$\begin{bmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{bmatrix} = \begin{bmatrix} M_{13} & M_{21} & M_{24} \\ M_{31} & M_{32} & M_{35} \\ M_{42} & M_{43} & M_{53} \end{bmatrix} = - \begin{bmatrix} M_{1'3'} & M_{2'1'} & M_{2'4'} \\ M_{3'1'} & M_{3'2'} & M_{3'5'} \\ M_{4'2'} & M_{4'3'} & M_{5'3'} \end{bmatrix}. \tag{12}$$

Then, by $\sum M = 0$ at joints, 1, 2, 3 and 1′, 2′, 3′, six of the remaining dependent unknown end moments can be expressed, thus

$$\begin{bmatrix} M_{12} \\ M_{23} \\ M_{34} \end{bmatrix} = - \begin{bmatrix} M_{1'2'} \\ M_{2'3'} \\ M_{3'4'} \end{bmatrix} = - \begin{bmatrix} X_1 \\ X_2 + X_3 \\ X_4 + X_5 + X_6 \end{bmatrix}. \tag{13}$$

And by symmetry, we have

$$\{M_{45} \;\; M_{54} \;\; Q_{45}\} = \{0 \;\; 0 \;\; 0\}. \tag{14}$$

The total axial stress in each member is readily determined by the "extended method of moments, shears, or joints", which are illustrated for members 1—2, 1—3, and 2—3 as follows:

A. Extended Method of Moments

Passing a section just to the left of member 2—3 and considering the equilibrium of the free body to the left, as shown in Fig. 3, we have,

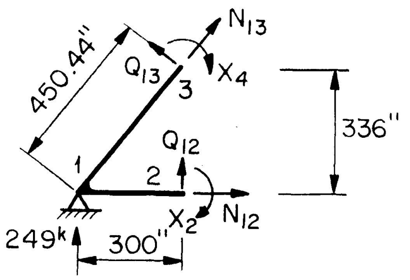

Fig. 3.

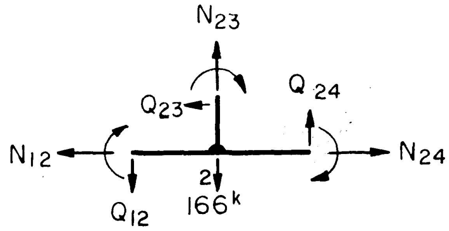

Fig. 4.

by $\sum M_3 = 0$, $\quad \lfloor -N_{12}\ X_2\ 249\ X_4 \rfloor \{336\ 1\ 300\ 1\} = 0.$

Hence, $\quad N_{12} = \lfloor X_2\ X_4\ 1 \rfloor \{0.002976\ 0.002976\ 222.321\}.$

B. Extended Method of Shears

Taking the same free body as shown in Fig. 3, we have

$$\{Q_{12}\ Q_{13}\} = \left\{\frac{-X_1+X_2}{300.00}\ \frac{X_1+X_4}{450.44}\right\}$$

and by $\sum Y = 0$,

$$\lfloor N_{13}\ X_1+X_4\ -X_1+X_2\ 249 \rfloor \left\{\frac{28}{37.53}\ \frac{25}{450.44\,(37.53)}\ \frac{1}{300}\ 1\right\} = 0.$$

Therefore

$$N_{13} = \lfloor X_1\ -X_2\ -X_4\ -1 \rfloor \{0.002486\ 0.004469\ 0.001982\ 333.808\}.$$

C. Extended Method of Joints

Passing a horseshoe section around joint 2, as shown in Fig. 4, we have,

by $\sum Y = 0$, $\quad \lfloor N_{23}\ Q_{24}\ Q_{12}\ 166 \rfloor \{1\ 1\ -1\ -1\} = 0.$

Substituting the values of Q_{24} and Q_{12}, we get

$$\lfloor N_{23}\ X_3+X_7\ -X_1+X_2\ 166 \rfloor \left\{1\ \frac{1}{300}\ -\frac{1}{300}\ -1\right\} = 0$$

or

$$N_{23} = \lfloor -X_1\ X_2\ -X_3\ -X_7\ 1 \rfloor \{0.003333\ 0.003333\ 0.003333\ 0.003333\ 166.000\}.$$

Similarly, other total axial stresses may be found as listed in Table 2. The constant term in each N expression is exactly equal to the heretofore so-called "primary stress" in the same bar if it were pin-jointed.

Constants in the strain-energy expressions of Eqs. (4) are given in Table 3. With the aid of Tables 2 and 3, Eqs. (4) give the strain energy multiplied by E in each member as shown in Table 4.

Table 2. Total Axial Stresses

Member	Total Axial Stresses
1—2	$0.002976X_2+0.002976X_4+222.321$
1—3	$0.002486X_1-0.004469X_2-0.001982X_4-333.808$
2—3	$-0.003333X_1+0.003333X_2-0.003333X_3-0.003333X_7+166.000$
2—4	$-0.002976X_3+0.002976X_4+0.002976X_5+222.321$
3—4	$0.004469X_3-0.001982X_4-0.001982X_5+0.002486X_6+0.004469X_7+$ $+0.001982X_8+0.004469X_9+111.269$
3—5	$-0.002976X_7-0.002976X_8-0.002976X_9-296.429$
4—5	$-0.006667X_6-0.006667X_9$

Table 3. Constants in the Strain-Energy Expressions

Property	Member						
	1—2	1—3	2—3	2—4	3—4	3—5	4—5
L/A	16.66667	16.27311	21.15869	16.66667	32.92689	11.29944	29.37063
$L/6I$	0.285225	0.078120	0.364109	0.285253	0.569600	0.054183	0.707965
$2(1+\mu)/AL$	0.000481	0.000209	0.000487	0.000481	0.000422	0.000326	0.000676

Table 4. E Times Strain Energy

Member	E Times Strain Energy in Member $I-J=E(W_{ij}+U_{ij}+V_{ij})$
1—2	$\frac{1}{2}(16.66667)\,(0.002976X_2+0.002976X_4+222.321)^2+$ $+0.285225(X_1^2+X_1X_2+X_2^2)+\frac{1}{2}(0.000481)\,(-X_1+X_2)^2$
1—3	$\frac{1}{2}(16.27311)\,(0.002486X_1-0.004469X_2-0.001982X_4-333.808)^2+$ $+0.078120\,(X_1^2-X_1X_4+X_4^2)+\frac{1}{2}(0.000209)\,(X_1+X_4)^2$
2—3	$\frac{1}{2}(21.15869)\,(-0.003333X_1+0.003333X_2-0.003333X_3-0.003333X_7+$ $+166.000)^2+0.364109\,[(-X_2-X_3)^2+(X_2+X_3)X_5+X_5^2]+$ $+\frac{1}{2}(0.000487)\,(-X_2-X_3+X_5)^2$
2—4	$\frac{1}{2}(16.66667)\,(-0.002976X_3+0.002976X_4+0.002976X_5+222.321)^2+$ $+0.285253\,(X_3^2-X_3X_7+X_7^2)+\frac{1}{2}(0.000481)\,(X_3+X_7)^2$
3—4	$\frac{1}{2}(32.92689)\,(0.004469X_3-0.001982X_4-0.001982X_5+0.002486X_6+$ $+0.004469X_7+0.001982X_8+0.004469X_9+111.269)^2+$ $+0.569600\,[(-X_4-X_5-X_6)^2+(X_4+X_5+X_6)X_8+X_8^2]+$ $+\frac{1}{2}(0.000422)\,(-X_4-X_5-X_6+X_8)^2$
3—5	$\frac{1}{2}(11.29944)\,(-0.002976X_7-0.002976X_8-0.002976X_9-296.429)^2+$ $+0.054183\,(X_6^2-X_6X_9+X_9^2)+\frac{1}{2}(0.000326)\,(X_6+X_9)^2$
4—5	$\frac{1}{2}(29.37063)\,(-0.006667X_6-0.006667X_9)^2$

By repeated application of $\frac{1}{2}E\frac{\partial U}{\partial X_i}=0$, the following matrix equation is obtained:

$$\{X_i\} = [a_{ij}]^{-1}\{C_i\},$$

where $$\{X_i\} = \{X_1\ X_2\ X_3\ X_4\ X_5\ X_6\ X_7\ X_8\ X_9\}$$

$[a_{ij}]^{-1} =$

$$\begin{bmatrix}
0.727716 & 0.284328 & 0.000235 & -0.077992 & 0 & 0 & 0.000235 & 0 & 0 \\
 & 1.300346 & 0.728471 & 0.000292 & 0.363622 & 0 & -0.000235 & 0 & 0 \\
 & & 1.300734 & -0.000439 & 0.363183 & 0.000366 & -0.283879 & 0.000292 & 0.000658 \\
 & & & 1.296559 & 1.139899 & 1.139460 & -0.000292 & 0.569049 & -0.000292 \\
 & & & & 1.868605 & 1.139460 & -0.000292 & 0.569049 & -0.000292 \\
 & & & & & 1.249171 & 0.000366 & 0.569340 & -0.052838 \\
 \multicolumn{3}{l}{a_{ij} \text{ below main diagonal}} & & & & 0.571981 & 0.000392 & 0.000758 \\
 \multicolumn{3}{l}{= a_{ji} \text{ above it}} & & & & & 1.139852 & 0.000392 \\
 & & & & & & & & 0.110103
\end{bmatrix}^{-1}$$

$\{C_i\} =$

$\{25.2144, -47.0097, 6.3636, -25.5609, -3.7656, -9.1098, -14.6329, -17.2309, -26.3407\}$.

Table 5. Bending Stresses at Member Ends

Member	End	End Moment (k-in.) Cross Method	End Moment (k-in.) Proposed Method	I/c (in.^{3})	Bending Stress (ksi)
1	2	−67	−66.20	27.5	2.262
	3	67	66.20	167.5 99.1	0.395 (Top) 0.668 (Bottom)
2	1	−85	−84.47	27.5	3.072
	3	45	45.28	24.1	1.879
	4	40	39.19	27.5	1.425
3	1	−11	−13.41	167.5 99.1	0.080 (Top) 0.135 (Bottom)
	2	43	42.50	24.1	1.763
	4	12	11.45	20.7	0.553
	5	−44	−40.54	156.0 97.6	0.260 (Top) 0.415 (Bottom)
4	2	− 5	− 5.803	27.5	0.211
	3	− 9	− 9.309	20.7	0.450
	5	0	0	14.7	0
5	3	−263	−258.7	156.0 97.6	1.658 (Top) 2.651 (Bottom)
	4	0	0	14.7	0

The solution of $\{X_i\}$ in kip-in by electronic digital computer or otherwise is recorded, element for element, as

$$\begin{bmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{bmatrix} = \begin{bmatrix} 66.20 & -84.47 & 39.19 \\ -13.41 & 42.50 & -40.54 \\ -5.803 & -9.309 & -258.8 \end{bmatrix} = \begin{bmatrix} M_{13} & M_{21} & M_{24} \\ M_{31} & M_{32} & M_{35} \\ M_{42} & M_{43} & M_{53} \end{bmatrix}. \quad \text{(A)}$$

Dividing the end moments by their respective section moduli (I/c) given in Table 1, bending stresses in ksi at member ends are as recorded in Table 5. These correspond to the so-called "secondary-stresses". The values of end moments as found by SUTHERLAND and BOWMAN by CROSS' method for the same truss are reproduced in the first column of Table 5 for comparison.

Total axial stresses and transverse shears are obtained by substituting the values of X_i respectively into Table 2 and Eq. (3). They are recorded in Table 6. Unit axial stresses are also calculated.

Table 6. Values of N_{ij}, N_{ij}/A_{ij}, and Q_{ij}

Member	1—2	1—3	2—3	2—4	3—4	3—5	4—5
N_{ij} (kips)	222.030	-333.239	165.387	222.291	110.085	-295.614	1.996
N_{ij}/A_{ij}(ksi)	12.335	-12.039	10.415	12.350	8.047	-11.134	0.174
Q_{ij} (kips)	-0.502	0.118	0.261	0.111	0.005	-0.998	0

The Simplified Method

A study of the equation obtained from $\frac{\partial U}{\partial X_i}=0$ suggests a simplified method which saves much time in writing the energy expressions and in evaluating the elements of the matrix $[a_{ij}]$. Considering the process for obtaining the first equation from

$$\begin{aligned} \tfrac{1}{2} E \frac{\partial U}{\partial X_1} = 0 = {} & 0.285225\,(2X_1+X_2) + \underline{0.000481\,(-1)\,(-X_1+X_2)} \\ & + \underline{16.27311\,(0.002486)\,(0.002486\,X_1 - 0.004469\,X_2 - 0.001982\,X_4} - 333.808) \\ & + 0.078120\,(2X_1 - X_4) + \underline{0.000209\,(X_1+X_4)} \\ & + \underline{21.15869\,(0.003333)^2\,(X_1 - X_2 + X_3 + X_7} - 166.000/0.003333)\,, \end{aligned}$$

the values of the non-underlined terms are about thousand times of those of the underlined. An approximate solution sufficient for engineering accuracy can, therefore, be most expediently obtained by deleting each strain energy term due to transverse shears in writing the energy expressions, and only retaining constituents of axial stresses not dependent on end moments and shears after partial differentiation. Thus, the simplified form of the first equation becomes

$$\lfloor 0.7262 \quad 0.2852 \quad -0.0781 \rfloor \; \{X_1 \; X_2 \; X_4\} = 25.214$$

and the symmetric matrix equation reduces to:

$$\begin{bmatrix} 0.7262 & 0.2852 & 0 & -0.0781 & 0 & 0 & 0 & 0 & 0 \\ & 1.2987 & 0.7282 & 0 & 0.3641 & 0 & 0 & 0 & 0 \\ & & 1.2988 & 0 & 0.3641 & 0 & -0.2853 & 0 & 0 \\ & & & 1.2954 & 1.1392 & 1.1392 & 0 & 0.5696 & 0 \\ & & & & 1.8674 & 1.1392 & 0 & 0.5696 & 0 \\ & & & & & 1.2476 & 0 & 0.5696 & -0.0542 \\ \multicolumn{6}{l}{a_{ij} \text{ below main diagonal}} & 0.5705 & 0 & 0 \\ \multicolumn{6}{l}{= a_{ji} \text{ above it}} & & 1.1392 & 0 \\ & & & & & & & & 0.1084 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \\ X_9 \end{bmatrix} = \begin{bmatrix} 25.214 \\ -47.010 \\ 6.364 \\ -25.561 \\ -3.766 \\ -9.110 \\ -14.633 \\ -17.231 \\ -26.341 \end{bmatrix}$$

whose solution by electronic digital computer yields

$$\begin{bmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{bmatrix} = \begin{bmatrix} 66.9 & -84.9 & 39.0 \\ -10.7 & 43.4 & -44.5 \\ -6.15 & -9.25 & -265 \end{bmatrix} = \begin{bmatrix} M_{13} & M_{21} & M_{24} \\ M_{31} & M_{32} & M_{35} \\ M_{42} & M_{43} & M_{53} \end{bmatrix} \qquad (A')$$

after which, any axial, bending, and shearing stress in each member of the truss can be determined by statics. The accuracy of the simplified method can be seen by comparing Eq. (A′) with Eq. (A).

Conclusion

The energy method proposed herein will yield the solution of axial, bending, and shearing stresses in all members of a truss in one unified single set-up. The rigidly-connected truss is treated as an assemblage of rigid frames. With wide-spread use of electronic computer, the entire process can be programmed from given data to end results. It provides both exact and approximate methods to suit the needs of special investigations and ordinary purposes. The mysterious category of "secondary" stresses is henceforth dispelled.

Summary

An energy method is formulated for analyzing all genuine stresses, including secondary stresses, in rigidly-connected trusses. Such trusses, by virtue of having axial stresses coexisting with flexural and shearing stresses (heretofore called "secondary stresses"), constitute, in reality, an assemblage of a chain of rigid frames. To unify their solution into one single system of strain energy and matrix procedure, formulae expressing constituent strain energy for plane trusses of any configuration with any redundancy are developed. A simple illustrative example showing exact and approximate solutions is given. Conclusions are stated.

Résumé

Les auteurs présentent une méthode énergétique permettant de déterminer toutes les contraintes effectives, y compris les contraintes secondaires, dans les treillis à nœuds rigides. Ces treillis, sollicités simultanément par des efforts axiaux et par des moments de flexion et des efforts tranchants (sollicitations appelées jusqu'ici «contraintes secondaires»), constituent en réalité une succession de portiques rigides. Pour obtenir la solution à l'aide d'un système unique d'énergie de déformation et de calcul matriciel, les auteurs développent des formules exprimant l'énergie de déformation pour des poutres planes de configuration et de degré d'hyperstaticité quelconque. Ils donnent un exemple simple et illustratif, montrant la solution exacte et la solution approximative et présentent des conclusions.

Zusammenfassung

Es wird eine Energiemethode für die Bestimmung aller auftretenden Spannungen, einschließlich der Nebenspannungen, in steifknotigen Fachwerken dargestellt. Solche Fachwerke, die sowohl Axial- als auch Biege- und Querkräftebeanspruchung (die sogenannten Nebenspannungen) aufweisen, bilden tatsächlich eine kettenförmige Verbindung starrer Rahmen. Um die Lösung dieses Problems in ein einziges System von Formänderungsarbeit und Matrizenberechnung zu vereinigen, werden Formeln entwickelt, die die Formänderungsarbeit für ebene Fachwerkträger von beliebiger Form und beliebigem Grad der statischen Unbestimmtheit ausdrücken. Ein einfaches anschauliches Beispiel wird gegeben, das sowohl die genaue wie eine Annäherungslösung zeigt. Schlußfolgerungen sind angegeben.