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Vibrations of Systems with Curved Members

Vibrations des systèmes formés d'éléments courbes

Schwingungen der Stabwerke mit krummen Stäben

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In the practice of civil engineering it is often necessary to investigate structural systems set up of rigid frames, some members of which are curved. The classic exact methods for investigating dynamic behaviour of such structures subjected to the action of pulsating forces or to that of rolling loads are very tedious, while the approximate solutions used up to date furnish results the accuracy of which can be not considered satisfactory, the more so, as even an estimate of the error is very often difficult, if not impossible.

In this paper a new method of solution is discussed, which the author developped with the aim of reducing the amount of numerical work involved, while permissible inaccuracy of the results is attained. The method as described here works with slope-deflection equations, but also any modification of the strain energy method may be applied, if the basic assumptions are appropriately changed.

The values of end-forces and end-moments for the individual members of the system have to be first determined, if the members are subjected to unit end-displacements. Exact calculation of these values is very difficult, and they can be expressed in closed form only in some special cases, if the variance of cross section and uneven mass distribution is to be taken into account.

The dynamic end-forces are caused by inertial resistance, which is in direct proportion to the respective acceleration. For harmonic vibrations the acceleration is directly proportional to the respective amplitude, so that the distribution of inertia forces along the system depends distinctly on the respective shape of vibration. Forced vibrations caused by dynamic loading of low frequency display vibrational shapes, that differ only slightly from the respective shapes of the static deflection curve. The discrepancy between the curves of static and dynamic deflection becomes distinct only if higher frequencies are to be considered, and the range of discrepancy becomes rapidly more pronounced if the frequency of vibrations is being raised. The variation of this interrelation can best be studied considering a beam of uniform section, where it is possible to express the end-forces and end-moments by means of the frequency functions, the values of which have been tabulated by the author.

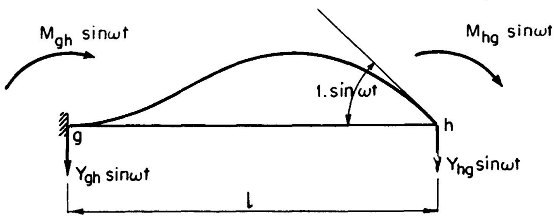

Fig. 1. Prismatic bar, the right-hand end of which is subject to periodic rotation (positive sense of end-forces and end-moments is marked by arrows).

If a beam of uniform section is vibrating as shown in Fig. 1, we can express the end-forces and end-moments by their respective amplitudes as follows:

$$\begin{aligned} M_{gh} &= \frac{EJ}{l} F_1(\lambda), \qquad & M_{hg} &= \frac{EJ}{l} F_2(\lambda), \\ Y_{gh} &= \frac{EJ}{l^2} F_3(\lambda), \qquad & Y_{hg} &= \frac{EJ}{l^2} F_4(\lambda), \end{aligned} \tag{1}$$

where

$$\lambda = l \sqrt[4]{\frac{\mu \omega^2}{EJ}}.$$

The values of the frequency functions $F_1(\lambda) \ldots F_4(\lambda) \ldots$ are tabulated in [1]) and [2]).

The exact values of the functions are

$$\begin{aligned} F_1(\lambda) &= -\lambda \frac{\sinh\lambda - \sin\lambda}{\cosh\lambda\cos\lambda - 1}, \\ F_2(\lambda) &= -\lambda \frac{\cosh\lambda\sin\lambda - \sinh\lambda\cos\lambda}{\cosh\lambda\cos\lambda - 1}, \\ F_3(\lambda) &= -\lambda^2 \frac{\cosh\lambda - \cos\lambda}{\cosh\lambda\cos\lambda - 1}, \\ F_4(\lambda) &= \lambda^2 \frac{\sinh\lambda\sin\lambda}{\cosh\lambda\cos\lambda - 1}. \end{aligned} \tag{2}$$

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[1]) V. Kolоušek: «Baudynamik der Durchlaufträger und Rahmen». Leipzig 1953, p. 149—161.

[2]) V. Koloušek: «Calcul des efforts dynamiques dans les ossatures rigides». Dunod, Paris 1959, p. 223—235.

If the static curves of deflection are used for determining the inertia forces, as shown in Fig. 2, we can write[3])

$$M_{gh} = M_{gh\,stat.} + \Delta M_{gh} \cong M_{gh\,stat.} - \mu\,\omega^2 \int_0^l v_1 v_2\,d\,x = \frac{2\,E\,J}{l} + \frac{1}{140}\mu\,\omega^2\,l^3,$$

so that we obtain approximate values for the function $F_1(\lambda)$ as follows:

$$F_1(\lambda) \cong 2 + \frac{1}{140}\lambda^4.$$

In a similar way we obtain:

$$F_2(\lambda) \cong 4 - \frac{1}{105}\lambda^4,$$

$$F_3(\lambda) \cong 6 + \frac{13}{420}\lambda^4, \qquad (3)$$

$$F_4(\lambda) \cong -6 + \frac{11}{210}\lambda^4.$$

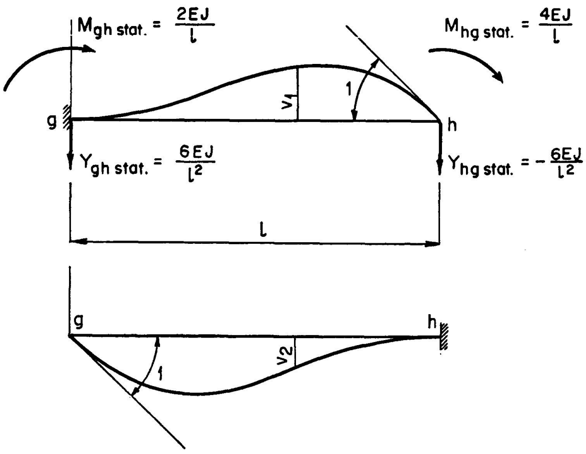

Fig. 2. Static deflection of prismatic bar.

The formulae (3) may be obtained also if the Eqs. (2) are expressed by power series, and only the first two terms of the respective series are considered[3]).

The discrepancy between the values as given by Eqs. (2) and (3) is growing with rising frequency, that is with higher values of λ. Close analysis of the variation of the exact and approximate values of $F(\lambda)$ shows, that the approximate expressions correspond to the exact expressions that are true for slightly lower values of λ. For example the values for $\lambda = 2$ correspond approximately to the exact values for $\lambda = 1{,}98$. Similarly, the approximate values for $\lambda = 3$ correspond to exact values valid for $\lambda = 2{,}9$.

Thus if numerical investigation of the rigid frame system is carried out using the approximate expressions for $F(\lambda)$, the values of natural frequences

[3]) See [1]), p. 61 and 147, or [2]), p. 61 and 221.

obtained will be higher than the true values. A sound estimate of the error may, however, be based in the value of $\lambda = l\sqrt[4]{\frac{\mu\omega^2}{EJ}}$. For example if $\lambda = 2$ is obtained for all the members of the system the true value of λ will be approximately 1% lower. The error of the natural frequency will be approximately 2%, as the frequency is in proportion to the second power of λ. Similarly if $\lambda \simeq 3$ is obtained for all the members of the system, the true respective values of λ and of the natural frequency will be about 3% and 6% lower.

The possibility for estimating the error in the just described manner enables us to correct the obtained results. Thus the investigation of a rigid frame system is feasible even for relatively high values of λ, and no fictive joints need be introduced into the system for computation purposes, and for artificial decrease of the λ values. If the value of λ is approximately the same for all members of the particular system, the results obtained in the above described way may be considered to be sufficiently accurate even for $\lambda = 4$. With systems where the respective values of λ differ distinctly for the individual members, the estimate of error is more difficult, and thus only $\lambda = 3$ may be regarded as the upper limit of appliability of the presented approximate investigation[4]).

The use of the approximate expressions for $F(\lambda)$ simplifies the numerical calculations to a large extent as the matrix of the set of slope-deflection equations has in this case the form

$$\boldsymbol{A} + \boldsymbol{B}\lambda^4$$ [5]).

The described method of investigation may be applied also to rigid frame systems with curved bars. Two alternative modifications of the method are feasible in this case. The first modification is based on the idea of introducing fictive joints and substituting the chord polygon for the actual curved member. The curved member is thus devided up into a large number of short beams, which are then considered to be straight and prismatic between the assumed fictive joints[6]). The matrix of the described substitute system attains thus a considerably higher order, and the method is practically applicable only if automatic computers are used. The idea of the modification just described is to approximate as closely as possible the true variation of both actual curvature and cross section.

The second modification works — as a rule — without any fictive joints, and the investigation is in this case based on considerations similar to those that hold true for straight bars. The assumption is made for curved bars, that

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[4]) The above limits for the error hold true for the first 6 functions tabulated in [2]), p. 149—161 and in [3]), p. 223—235. For the other tabulated functions the error is considerably larger.

[5]) V. Fiřt: "Vlastní kmitání rámů v rovině a prostoru". Aplikace matematiky, Praha 1962.

[6]) See [2]), p. 4 or [3]), p. 52—53.

for lower frequencies there is only slight difference between the curves of dynamic and static deflection. The values of λ are determined using average values of μ and J, while the limiting values of λ are approximately the same as for prismatic bars. Thus the end-forces for the curved members are directly determined from the ordinates of the curves of static deflection, which — as a rule — are known from the previous statical analysis of the system. Let us consider the curved member as shown in Fig. 3a. The static curve of deflection caused by unit displacement of the right-hand end is shown in Fig. 3c.

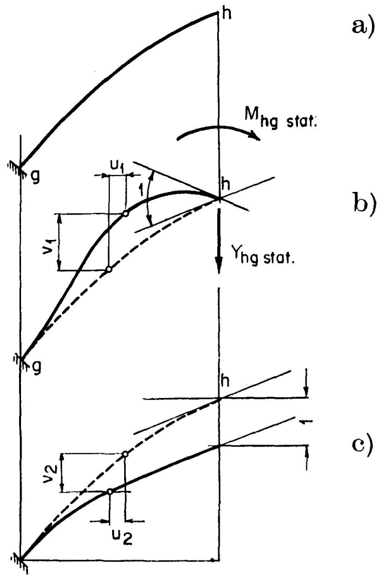

Fig. 3. Static deflection of curved bar.

With the exception of sign the curve represents also the line of influence for the vertical component of the end-force acting at point h. Fig. 3b shows the curve of static deflection produced by unit rotation of the right-hand joint. The amplitude of the end-force that corresponds to the right-hand end rotating with angular amplitude $\gamma_h = 1$ is, according to Maxwell's Reciprocal Theorem, given as follows:

$$Y_{hg} \cong Y_{hg\,stat.} + \varDelta Y_{hg} = Y_{hg\,stat.} - \omega^2 \int_0^l \mu\, v_1 v_2\, d\,s - \omega^2 \int_0^l \mu\, u_1 u_2\, d\,s\,.$$

For the amplitude of the end-moment M_{hg} we obtain similarly

$$M_{hg} \cong M_{hg\,stat.} + \varDelta M_{hg} = M_{hg\,stat.} - \omega^2 \int_0^l \mu\, v_1^2\, d\,s - \omega^2 \int_0^l \mu\, u_1^2\, d\,s\,.$$

In both these formulae the subscript "stat." denotes the magnitude of the end-force $Y_{hg\,stat}$ or of the end-moment $M_{hg\,stat}$ which would act at the end "h" if this end were subjected to a unit static deformation. The integrals are evaluated numerically, deviding the member into strips and replacing the integrals by sums.

The numerical example which follows shall illustrate the procedure.

Numerical example

The system as shown in Fig. 4 had been investigated by the author in his previously published paper [7]). The mentioned investigation had been carried out using stepwise approximation. The previously obtained results may be thus compared with the results obtained by the method of solution discussed in this paper. The new method requires only a fraction of numerical work which had been necessary for stepwise approximation. The simplified structural system used for "computation scheme" is schematically represented in Fig. 5. The system is symmetric with respect to the vertical line through the

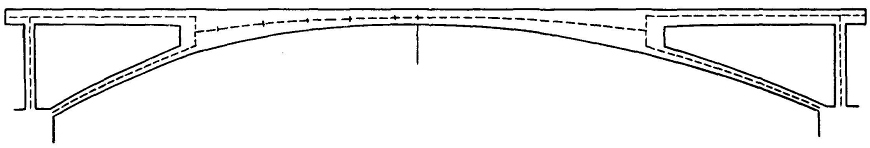

Fig. 4. Bridge structure — general lay out.

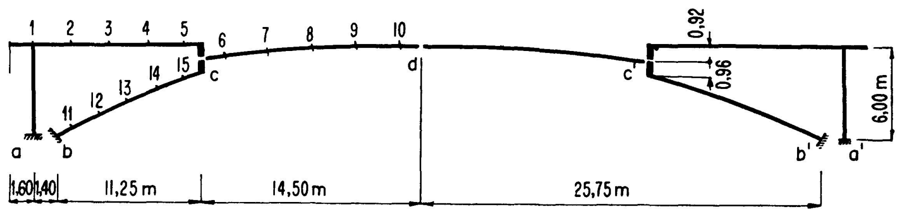

Fig. 5. "Computation scheme" of the bridge structure shown in Fig. 4.

midpoint "d", each half of the system is set up of three elements, the centre lines of which are as follows: the centre line of the element $a-c$ is a polygon, while the centre lines of the other two elements $b-c$ and $c-d$ are curved.

In this paper only the natural frequency for free, symmetric vibrations of the first mode shall be given. Only very small horizontal displacements are taking place with this mode of vibration so that the influence of horizontal components of the inertia forces upon the horizontal end-forces will be taken into account only approximately, and the effect of horizontal inertia forces upon the end-moments and vertical end-forces shall be neglected altogether. The horizontal components of the end-forces will be determined on the simplifying assumption that the mass of the elements $b-c$ and $c-d$ is concentrated at both respective ends of each element, while the mass of the horizontal beam belonging to the element $a-c$ is assumed to vibrate in coincidence with the point "c". It shall be demonstrated, that if the effect of horizontal inertia forces is altogether neglected, this affects the final results only very slightly. The simplified procedure appears thus to be justified. The lines of influence

[7]) V. KOLOUŠEK: «Schwingungen der Brücken aus Stahl und Stahlbeton». Publications IABSE, Vol. XVI, 1956, p. 301—332.

for the end-forces "X" and "Y" and for the end-moment "M" that hold true for the respective elements, were available from the previous statical analysis, and they are shown in Figs. 6, 7 and 8. According to the above made assumptions we shall consider only the respective vertical ordinates. In Figs. 6, 7 and 8 also the end-forces and end-moments that correspond to unit static deformations of the ends are given. The end-forces and end-moments produced by the forces of inertia [$\Delta X, \Delta Y, \Delta M$] are evaluated in Tables I, II, and III.

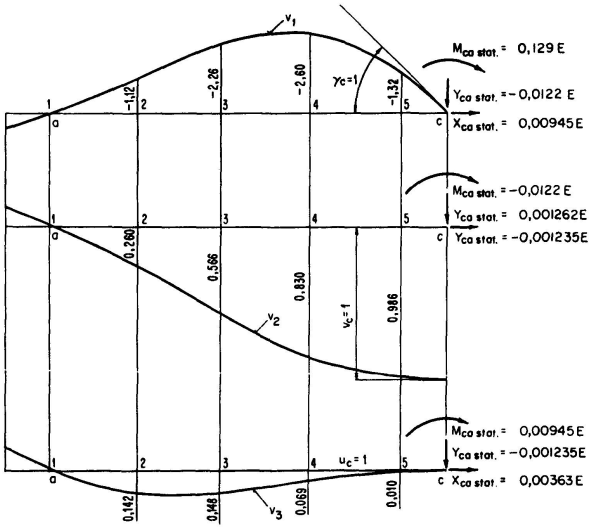

Fig. 6. Deflection curves for unit end-deformations of member "$a-c$" (vertical ordinates).

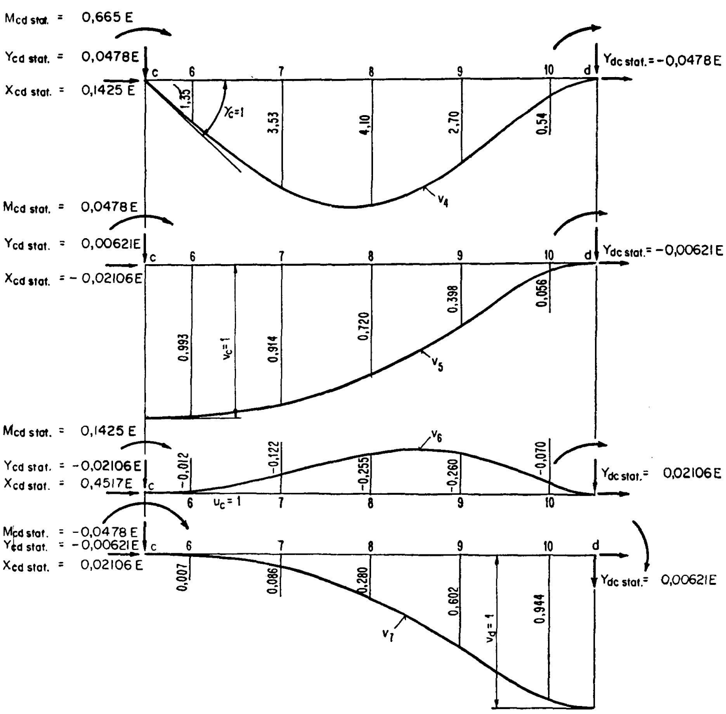

Fig. 7. Deflection curves for unit end-deformations of member "$c-d$" (vertical ordinates).

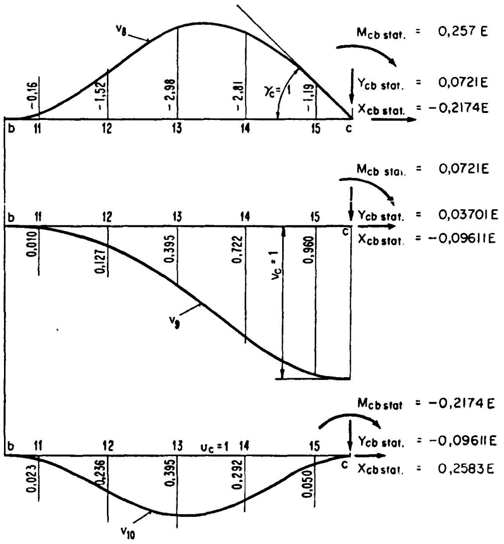

Fig. 8. Deflection curves for unit end-deformations of member "$b-c$" (vertical ordinates).

Table I. Evaluation of End-Forces for the Bar $a-c$ (Effect Due to Forces of Inertia)

1	2	3	4	5	6	7	8	9	10	11
Point i	Mass m_i	Coordinate			$m_i v_{1i}^2$	$m_i v_{1i} v_{2i}$	$m_i v_{1i} v_{3i}$	$m_i v_{2i}^2$	$m_i v_{2i} v_{3i}$	$m_i v_{3i}^2$
		v_{1i}	v_{2i}	v_{3i}						
1	10,84	—	—	—	—	—	—	—	—	—
2	5,77	-1,12	0,260	0,142	7,2	- 1,68	-0,92	0,39	0,213	0,1162
3	5,77	-2,26	0,566	0,148	29,5	- 7,38	-1,93	1,85	0,483	0,1263
4	6,60	-2,60	0,830	0,069	44,6	-14,23	-1,18	4,55	0,377	0,0314
5	7,60	-1,32	0,986	0,010	13,2	- 9,90	-0,10	7,39	0,075	0,0008
Σ	36,58				94,5	-33,19	-4,13	14,18	1,148	0,2747

With regard to the assumptions concerning horizontal inertia forces, we obtain e.g. for the element "$c-a$" the following expressions, valid for unit rotation of the point "c", if we replace the integrals by sums:

$$\Delta M_{ca} = -\omega^2 \sum m_1 v_{1i}^2, \qquad \Delta Y_{ca} = -\omega^2 \sum m_i v_{1i} v_{2i} \quad \text{etc.}$$

The amplitudes of the end-moments and end-forces are thus given as follows.

a) For unit rotation $\gamma_c = 1$ of point "c":

$$\begin{aligned} M_{ca} &= 0{,}129\,E - 94{,}5\,\omega^2 \\ M_{cd} &= 0{,}665\,E - 325{,}4\,\omega^2 \\ M_{cb} &= 0{,}257\,E - 53{,}6\,\omega^2 \\ \hline \sum M_c &= 1{,}051\,E - 473{,}5\,\omega^2 = a_{11} E + b_{11}\omega^2 \end{aligned}$$

Table II. Evaluation of End-Forces for the Bar c–d (Effect Due to Forces of Inertia)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Point i	Mass m_i	Coordinate				$m_i v_{4i}^2$	$m_i v_{4i} v_{5i}$	$m_i v_{4i} v_{6i}$	$m_i v_{4i} v_{7i}$	$m_i v_{5i}^2$	$m_i v_{5i} v_{6i}$	$m_i v_{5i} v_{7i}$	$m_i v_{6i}^2$	$m_i v_{6i} v_{7i}$	$m_i v_{7i}^2$
		v_{4i}	v_{5i}	v_{6i}	v_{7i}										
6	11,73	1,35	0,993	-0,012	0,007	21,4	15,7	-0,19	0,11	11,57	-0,140	0,082	0,002	-0,001	—
7	9,40	3,53	0,914	-0,122	0,086	117,2	30,3	-4,05	2,86	7,86	-1,047	0,739	0,140	-0,099	0,07
8	7,98	4,10	0,720	-0,255	0,280	134,2	23,6	-8,34	9,17	4,13	-1,462	1,608	0,519	-0,569	0,63
9	6,96	2,70	0,398	-0,260	0,602	50,7	7,5	-4,88	11,30	1,10	-0,720	1,666	0,470	-1,090	2,52
10	6,45	0,54	0,056	-0,070	0,944	1,9	0,2	-0,24	3,29	0,02	-0,025	0,341	0,032	-0,426	5,75
Σ	42,52					325,4	77,3	-17,70	26,73	24,68	-3,394	4,436	1,163	-2,185	8,97

Table III. Evaluation of End-Forces for the Bar b – c (Effect Due to Forces of Inertia)

1	2	3	4	5	6	7	8	9	10	11
Point i	Mass m_i	Coordinate			$m_i v_{8i}^2$	$m_i v_{8i} v_{9i}$	$m_i v_{8i} v_{10i}$	$m_i v_{9i}^2$	$m_i v_{9i} v_{10i}$	$m_i v_{10i}^2$
		v_{8i}	v_{9i}	v_{10i}						
11	2,00	-0,16	0,010	0,023	0,05	—	-0,01	—	—	0,001
12	2,36	-1,52	0,127	0,236	5,46	- 0,46	-0,85	0,038	0,071	0,131
13	2,54	-2,98	0,395	0,395	22,55	- 2,99	-2,99	0,396	0,396	0,396
14	2,69	-2,81	0,722	0,292	21,23	- 5,45	-2,21	1,403	0,567	0,229
15	3,03	-1,19	0,960	0,050	4,29	- 3,46	-0,18	2,790	0,146	0,008
Σ	12,62				53,58	-12,36	-6,24	4,627	1,180	0,765

$$
\begin{aligned}
Y_{ca} &= -0{,}0122\,E + 33{,}2\,\omega^2 \\
Y_{cd} &= 0{,}0478\,E - 77{,}3\,\omega^2 \\
Y_{cb} &= 0{,}0721\,E + 12{,}4\,\omega^2 \\
\hline
\sum Y_c &= 0{,}1077\,E - 31{,}7\,\omega^2 = a_{12}\,E + b_{12}\,\omega^2
\end{aligned}
$$

$$
\begin{aligned}
X_{ca} &= 0{,}0095\,E + 4{,}13\,\omega^2 \\
X_{cd} &= 0{,}1425\,E + 17{,}70\,\omega^2 \\
X_{cb} &= -0{,}2174\,E + 6{,}24\,\omega^2 \\
\hline
\sum X_c &= -0{,}0654\,E + 28{,}07\,\omega^2 = a_{13}\,E + b_{13}\,\omega^2
\end{aligned}
$$

$$Y_{dc} = -0{,}0478\,E - 26{,}7\,\omega^2 = a_{14}\,E + b_{14}\,\omega^2$$

b) For unit vertical displacement $v_c = 1$ of point "c":

$$\sum M_c = 0{,}1077\,E - 31{,}7\,\omega^2 = a_{12}\,E + b_{12}\,\omega^2$$

$$
\begin{aligned}
Y_{ca} &= 0{,}00126\,E - 14{,}18\,\omega^2 \\
Y_{cd} &= 0{,}00621\,E - 24{,}68\,\omega^2 \\
Y_{cb} &= 0{,}03701\,E - 4{,}63\,\omega^2 \\
\hline
\sum Y_c &= 0{,}04448\,E - 43{,}49\,\omega^2 = a_{22}\,E + b_{22}\,\omega^2
\end{aligned}
$$

$$
\begin{aligned}
X_{ca} &= -0{,}00124\,E - 1{,}15\,\omega^2 \\
X_{cd} &= -0{,}02106\,E + 3{,}39\,\omega^2 \\
X_{cb} &= -0{,}09611\,E - 1{,}18\,\omega^2 \\
\hline
\sum X_c &= -0{,}11841\,E + 1{,}06\,\omega^2 = a_{23}\,E + b_{23}\,\omega^2
\end{aligned}
$$

$$Y_{dc} = -0{,}00621\,E - 4{,}44\,\omega^2 = a_{24}\,E + b_{24}\,\omega^2$$

c) For unit horizontal displacement of point "c" we take into account also the horizontal end-force ΔX_c^h, which is produced by the horizontal components of the inertia forces. The value of ΔX_c^h has to be determined from the respective masses of the individual elements; thus $m_1 - m_5$ pertain to the element $c - a$; m_6, m_7, $\frac{1}{2} m_8$ pertain to the element $c - d$; the masses m_{15}, m_{14}, and $\frac{1}{2} m_{13}$ pertain to the element $c - b$.

Thus we have

$$\Delta X_c^h = -[36{,}58 + (11{,}73 + 9{,}40 + 3{,}99) + (3{,}03 + 2{,}69 + 1{,}27)]\,\omega^2 = -68{,}69\,\omega^2$$

and

$$
\begin{aligned}
\sum M_c &= -0{,}0654\,E + 28{,}07\,\omega^2 = a_{13}\,E + b_{13}\,\omega^2 \\
\sum Y_c &= -0{,}11841\,E + 1{,}06\,\omega^2 = a_{23}\,E + b_{23}\,\omega^2
\end{aligned}
$$

$$
\begin{aligned}
X_{ca} &= 0{,}0036\,E - 0{,}28\,\omega^2 \\
X_{cd} &= 0{,}4517\,E - 1{,}16\,\omega^2 \\
X_{cb} &= 0{,}2583\,E - 0{,}77\,\omega^2 \\
\Delta X_c^h &= -68{,}69\,\omega^2 \\
\hline
\sum X_c &= 0{,}7136\,E - 70{,}90\,\omega^2 = a_{33}\,E + b_{33}\,\omega^2
\end{aligned}
$$

$$Y_{dc} = 0{,}02106\,E + 2{,}185\,\omega^2 = a_{34}\,E + b_{34}\,\omega^2$$

d) Finally for unit vertical displacement $v_d = 1$ of point "d" we have

$$\begin{aligned} M_{cd} &= a_{14}\,E + b_{14}\,\omega^2 \\ Y_{cd} &= a_{24}\,E + b_{24}\,\omega^2 \\ X_{cd} &= a_{34}\,E + b_{34}\,\omega^2 \\ Y_{dc} &= 0{,}00621\,E - 8{,}97\,\omega^2 = a_{44}\,E + b_{44}\,\omega^2 \end{aligned}$$

The slope-deflection equations are obtained from the conditions of equilibrium of point "c" and from the equilibrium of vertical forces at point "d". The equations are given in Table IV and Table IV a.

Table IV

	γ_c	v_c	u_c	v_d	
M_c	$a_{11} + b_{11}\dfrac{\omega^2}{E}$	$a_{12} + b_{12}\dfrac{\omega^2}{E}$	$a_{13} + b_{13}\dfrac{\omega^2}{E}$	$a_{14} + b_{14}\dfrac{\omega^2}{E}$	$= 0$
Y_c	$a_{12} + b_{12}\dfrac{\omega^2}{E}$	$a_{22} + b_{22}\dfrac{\omega^2}{E}$	$a_{23} + b_{23}\dfrac{\omega^2}{E}$	$a_{24} + b_{24}\dfrac{\omega^2}{E}$	$= 0$
X_c	$a_{13} + b_{13}\dfrac{\omega^2}{E}$	$a_{23} + b_{23}\dfrac{\omega^2}{E}$	$a_{33} + b_{33}\dfrac{\omega^2}{E}$	$a_{34} + b_{34}\dfrac{\omega^2}{E}$	$= 0$
Y_d	$a_{14} + b_{14}\dfrac{\omega^2}{E}$	$a_{24} + b_{24}\dfrac{\omega^2}{E}$	$a_{34} + b_{34}\dfrac{\omega^2}{E}$	$a_{44} + b_{44}\dfrac{\omega^2}{E}$	$= 0$

Table IV a

γ_c	v_c	u_c	v_d	
$1{,}051 - 474\dfrac{\omega^2}{E}$	$0{,}1077 - 31{,}7\dfrac{\omega^2}{E}$	$-0{,}0654 + 28{,}1\dfrac{\omega^2}{E}$	$-0{,}0478 - 26{,}7\dfrac{\omega^2}{E}$	$= 0$
$0{,}1077 - 31{,}7\dfrac{\omega^2}{E}$	$0{,}0445 - 43{,}5\dfrac{\omega^2}{E}$	$-0{,}1184 + 1{,}06\dfrac{\omega^2}{E}$	$-0{,}00621 - 4{,}44\dfrac{\omega^2}{E}$	$= 0$
$-0{,}0654 + 28{,}1\dfrac{\omega^2}{E}$	$-0{,}1184 + 1{,}06\dfrac{\omega^2}{E}$	$0{,}714 - 70{,}90\dfrac{\omega^2}{E}$	$0{,}0211 + 2{,}185\dfrac{\omega^2}{E}$	$= 0$
$-0{,}0478 - 26{,}7\dfrac{\omega^2}{E}$	$-0{,}00621 - 4{,}44\dfrac{\omega^2}{E}$	$0{,}0211 + 2{,}185\dfrac{\omega^2}{E}$	$0{,}00621 - 8{,}97\dfrac{\omega^2}{E}$	$= 0$

Setting the determinant of the equations equal to zero we obtain

$$\frac{\omega^2}{E} = 2{,}47 \cdot 10^{-4}\,\mathrm{s}^{-2}\,\mathrm{t}^{-1}\,\mathrm{m}^2$$

and for $E = 4{,}2 \cdot 10^6\,\mathrm{t/m^2}$ we have

$$\omega = 32{,}2\,\mathrm{s}^{-1}.$$

Thus we find also the frequency of the first mode of symmetric free vibrations, which is

$$n_1 = 5{,}12 \text{ Hz}.$$

(If we neglect the effect of horizontal inertia forces, we obtain the frequency of $n_1 = 5{,}17$ Hz. The negligible influence of these forces becomes thus evident.)

The mean values of λ that hold true for the individual members are

$$\lambda_{ca} = l \sqrt[4]{\frac{\mu\,\omega^2}{E J}} = 12{,}65 \sqrt[4]{\frac{2{,}89 \cdot 2{,}47 \cdot 10^{-4}}{0{,}31}} = 2{,}76,$$

(only the horizontal part considered)

$$\lambda_{cd} = 14{,}60 \sqrt[4]{\frac{2{,}92 \cdot 2{,}47 \cdot 10^{-4}}{1{,}0}} = 2{,}39,$$

$$\lambda_{cb} = 12{,}96 \sqrt[4]{\frac{0{,}98 \cdot 2{,}47 \cdot 10^{-4}}{0{,}15}} = 2{,}60.$$

The corresponding first angular frequency that had been previously obtained by the author[8]) had the value

$$\omega = 33{,}0\,\text{s}^{-1}.$$

The value obtained by application of the present approximate procedure is about 2% lower. According to the above described considerations the present solution should furnish a value slightly higher than that obtained by stepwise approximation. The expected difference has, however, not been obtained, but this is not due to any deficiency of the method presented in this paper. The influence line that served as basic data for the method of stepwise approximation had been obtained by the Beggs method, using a celluloid model of the structure, while the lines of influence as shown in Figs. 6, 7, and 8, have been obtained by statical analysis. It appears, that the model had been of slightly larger stiffness than would correspond to the theoretical scheme of the structure, used for the "computation scheme".

Table of Notations

a or b	coefficients of equations.
g or h	left-hand and right-hand supports of the bar $g-h$.
l	length of bar.
m	mass.
m_i	mass of the i-th part of bar.
n	frequency per second.
u	horizontal displacement.

[8]) See [7]), p. 318.

v	vertical displacement.
$\boldsymbol{A}$ or $\boldsymbol{B}$	matrix of set of equations.
E	modulus of elasticity.
$F(\lambda)$	frequency functions.
J	moment of inertia of cross section.
M_{gh}	end-moment at the end "g" of bar $g-h$.
$M_{gh\,stat.}$	statical value of end-moment.
ΔM_{gh}	increment of end-moment due to inertia forces.
X	horizontal force.
Y	vertical force.
γ	angle of rotation.
λ	$= l \sqrt[4]{\frac{\mu \omega^2}{E J}}$.
μ	mass per unit length.
ω	angular frequency.

The units used are taken in the metric system, i.e. metric tons, m, sec.

Summary

The classical methods for investigating the vibrations of rigid frame systems with curved members are — as a rule — very tedious. In this paper the author describes a new method of solution, which — while relatively simple — may be regarded as sufficiently accurate. Slope-deflection equations are used in the procedure, but the end forces and end-moments for the curved members are determined directly from the static influence lines. Even higher modes of natural vibrations as well as forced vibrations may be investigated in a relatively simple manner. The procedure is illustrated by a numerical example, in which the lowest natural frequency is determined for a bridge, the main structural element of which is composed of a rigid arch stiffened by a rigid frame. The results obtained by application of the procedure are then compared with results obtained previously by stepwise approximation. While both results are in good agreement, the method of stepwise approximation requires a considerably larger amount of numerical work.

Résumé

Les méthodes classiques utilisées pour l'étude des vibrations dans les ossatures formées d'éléments courbes sont en général lentes et fastidieuses. Dans la présente contribution, l'auteur développe une nouvelle méthode de calcul qui, bien que relativement simple, donne cependant une précision acceptable. Il utilise la méthode des déformations mais détermine les moments et les efforts aux extrémités des éléments curvilignes directement à partir des lignes d'influence statique. On peut ainsi déterminer aisément même les fréquences d'ordre supérieur des vibrations naturelles ainsi que les vibrations forcées. Un exemple numérique sert à illustrer la méthode: on y détermine la fréquence fondamentale d'un pont dont le système porteur est formé d'un arc raidi par un cadre. Le problème avait déjà été résolu par une méthode d'approximations successives, ce qui permet une comparaison des résultats donnés par les deux procédés: la nouvelle méthode proposée, beaucoup plus rapide, donne une précision très satisfaisante.

Zusammenfassung

Die klassischen Verfahren, die bis jetzt für die Ermittlung von Schwingungen in Systemen mit krummen Stäben angewendet werden, sind im allgemeinen sehr mühsam und zeitraubend. In vorliegender Arbeit wird eine neue Berechnungsmethode entwickelt, welche — obwohl verhältnismäßig einfach — Ergebnisse von genügender Genauigkeit ergibt. Bei der Berechnung wurde die Deformationsmethode benützt. Die Endstabskräfte und Momente der krummen Glieder wurden jedoch direkt aus den statischen Einflußlinien ermittelt. Die Methode kann für die Berechnung der Eigenschwingungen der ersten und höheren Ordnung sowie für die Lösung der erzwungenen Schwingungen benützt werden. Das Verfahren wird in einem Zahlenbeispiel gezeigt, wo die erste Eigenschwingungszahl einer Brücke berechnet wird. Das Tragsystem der Brücke besteht aus einem Bogen, der durch einen Rahmen versteift wird. Dasselbe System wurde vorher mit Hilfe der schrittweisen Näherungen berechnet, und man konnte deshalb die Ergebnisse der beiden Methoden vergleichen. Der Unterschied ist ohne Bedeutung. Die Methode der schrittweisen Näherungen ist jedoch mit viel größerer Arbeit verbunden.