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Matrix Analysis of Indeterminate Space Trusses

Analyse matricielle de treillis tridimensionnels hyperstatiques

Matrizenanalyse von statisch unbestimmten Raumfachwerken

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Introduction

Although the general analysis of indeterminate space frames in matrix or other forms has not too infrequently appeared in technical literature, while a "Formulation of Equilibrium Equations for Pin-Jointed Structures" was contributed by Professor M. Naruoka to the 1961 Volume of these Publications, very little has been published, however, on the general analysis of indeterminate space trusses, especially in the form of using matrix methods directly. This paper has been developed to fill the gap as well as the need in the wake of increasingly more applications of indeterminate trusses in space. Exactly the same procedure would apply to the simpler Naruoka case.

An indeterminate space truss, as herein construed, may consist of any three-dimensional trussed structure of direct-stress-carrying members, having redundant reactions or redundant members, or both, such that the three-dimensional equations of statics alone, namely $\sum X=0$, $\sum Y=0$, $\sum Z=0$, $\sum M_x=0$, $\sum M_y=0$, $\sum M_z=0$, are incapable of giving a solution.

It is the aim of this paper to develop a general matrix analysis, by taking the mathematical advantages of dealing with arrays of row (in $\lfloor\ \rfloor$) and column (in $\{\}$) vectors, of indeterminate space trusses of any geometrics, under any loading conditions, and with any status of supports. Matrix vectors are particularly adapted to represent 3-dimensional member-length projections, and displacement, stress, and reaction components.

The treatment has been extended to multiple loadings through matrices of unknown vectors and loading vectors via the inversion of coefficient matrix of displacement components. The matrix formulation applies as well to indeterminate space trusses with yielding and elastic supports.

A. Axial Stresses of Members

a) Deformation-Displacement Equations

The axial stress in any member in a space truss may be conveniently expressed in matrix form in terms of the deformation, and the linear displacements of the ends, of the member.

Let $A\,B$ in Fig. 1 be any member in a space truss. The ends A and B have orthogonal coordinates denoted by space vectors $(x_a\,y_a\,z_a)$ and $(x_b\,y_b\,z_b)$ with reference to some origin 0. The linear displacements of the ends (A, B), parallel to the axes, may be indicated by vectors $(u_a\,v_a\,w_a)$ and $(u_b\,v_b\,w_b)$ respectively. The linear projections of $A\,B$ on the axes will be denoted by $(X_{ab}\,Y_{ab}\,Z_{ab})$, thus

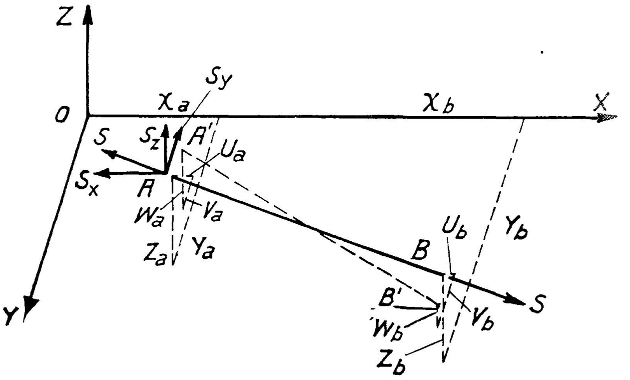

Fig. 1.

$$\begin{bmatrix} X_{ab} \\ Y_{ab} \\ Z_{ab} \end{bmatrix} = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} - \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} . \tag{1}$$

Deleting subscripts for the member $A\,B$, unless two or more of its kind are under consideration, we have, by virtue of orthogonal projections, its length (L) defined by

$$\begin{aligned} L^2 &= \lfloor X\ Y\ Z \rfloor \{X\ Y\ Z\}, \\ &= X^2 + Y^2 + Z^2 \end{aligned} \tag{2}$$

and, following first derivative of the terms, we get

$$L\,(\varDelta\,L) = \lfloor X\ Y\ Z \rfloor \{\varDelta\,X\ \varDelta\,Y\ \varDelta\,Z\} \tag{3}$$

after cancelling the common scalar factor 2.

But since

$$\begin{bmatrix} \varDelta\,X \\ \varDelta\,Y \\ \varDelta\,Z \end{bmatrix} = \begin{bmatrix} u_a \\ v_a \\ w_a \end{bmatrix} - \begin{bmatrix} u_b \\ v_b \\ w_b \end{bmatrix} \tag{4}$$

it follows that

$$\varDelta\,L = \frac{1}{L} \lfloor X\ Y\ Z \rfloor \begin{bmatrix} u_a - u_b \\ v_a - v_b \\ w_a - w_b \end{bmatrix} . \tag{5}$$

Moreover, according to basic principles of theory of elasticity, if S be the axial stress in any member AB, A its cross-sectional area, and E the modulus of elasticity of the material, its length deformation is given by

$$\varDelta L = \frac{S\,L}{A\,E}. \tag{6}$$

Hence
$$S = \frac{A\,E}{L}\varDelta L = \frac{A\,E}{L^2}\lfloor X\;Y\;Z\rfloor \begin{bmatrix} u_a - u_b \\ v_a - v_b \\ w_a - w_b \end{bmatrix}. \tag{7}$$

As the stress components are proportional to the length projections, we have the relation

$$\begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix} = \frac{S}{L}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{8}$$

and, therefore,
$$\begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix} = \frac{A\,E}{L^3}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}\lfloor X\;Y\;Z\rfloor \begin{bmatrix} u_a - u_b \\ v_a - v_b \\ w_a - w_b \end{bmatrix}. \tag{9}$$

It may then be stated that *the stress components of any member of given A, E, L, are a matrix-vector function of the length projections and differences of components of joint displacements.*

b) Joint Equilibrium Equations

At any joint of members, there may be externally applied loads as well as internally transmitted stresses acting. Static equilibrium of the joint requires that there must be no unbalanced components in any direction of the orthogonal axes. If the components of an external load applied at the joint are denoted by the matrix vector $(P_x P_y P_z)$, then we have for all external loads, and internal stresses in members JK, JL, etc., at joint J,

$$\begin{bmatrix} \sum P_x \\ \sum P_y \\ \sum P_z \end{bmatrix} = \begin{bmatrix} P_{j1x} + P_{j2x} + \cdots \\ P_{j1y} + P_{j2y} + \cdots \\ P_{j1z} + P_{j2z} + \cdots \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \sum S_x \\ \sum S_y \\ \sum S_z \end{bmatrix} = \begin{bmatrix} S_{jkx} + S_{jlx} + \cdots \\ S_{jky} + S_{jly} + \cdots \\ S_{jkz} + S_{jlz} + \cdots \end{bmatrix}.$$

To satisfy the condition of joint equilibrium at J, it is necessary that

$$\begin{bmatrix} \sum S_x \\ \sum S_y \\ \sum S_z \end{bmatrix} - \begin{bmatrix} \sum P_x \\ \sum P_y \\ \sum P_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{10}$$

When stress-component vectors $(S_x S_y S_z)$ are further substituted by Eq. (9), we have "deformation-displacement-equilibrium equations", called "compatibility equations".

c) Compatibility Equations

Compatibility equations at a joint are such that they satisfy three-dimensional equilibrium at the joint, are congruous with the length deformations of the members meeting there, and agree with the displacements of the joint and the far-end joints of the members. In other words, they are "inter-joint harmonious equations". If we denote by the subscript "*i*" any member connecting *J* with any far-end joint, *K*, *L*, etc. whose generalized subscript is also "*i*", the substitution of Eq. (9) into Eq. (10) will yield, by changing the member subscripts (a, b) into the generalized (j, i), a set of compatibility equations. Thus,

$$E \sum_{i=jk,\, jl,\, \text{etc.}} \frac{A_i}{L_i^3} \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} \lfloor X_i Y_i Z_i \rfloor \underset{i=k,\, l,\, \text{etc.}}{\begin{bmatrix} u_j - u_i \\ v_j - v_i \\ w_j - w_i \end{bmatrix}} - \sum_{m=1,\, 2,\, \text{etc.}} \begin{bmatrix} P_{jmx} \\ P_{jmy} \\ P_{jmz} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{11}$$

where the vector $\{u_j v_j w_j\}$ does not vary under the summation sign, and *E* would be within the latter if it were not constant throughout.

By calling

$$\alpha_i = \frac{A_i}{L_i^3} \tag{12}$$

and computing

$$\begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} A'_{jk} \\ B'_{jk} \\ C'_{jk} \\ D'_{jk} \\ E'_{jk} \\ F'_{jk} \end{bmatrix} + \begin{bmatrix} A'_{jl} \\ B'_{jl} \\ C'_{jl} \\ D'_{jl} \\ E'_{jl} \\ F'_{jl} \end{bmatrix} + \text{etc., where} \quad \begin{array}{c} \begin{bmatrix} A' & D' & E' \\ 0 & B' & 0 \\ 0 & F' & C' \end{bmatrix} = \begin{bmatrix} X^2 & XY & XZ \\ 0 & Y^2 & 0 \\ 0 & YZ & Z^2 \end{bmatrix} \alpha \\ \| \\ \begin{bmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{bmatrix} \begin{bmatrix} X & Y & Z \\ 0 & Y & 0 \\ 0 & Y & Z \end{bmatrix} \alpha, \end{array} \tag{13}$$

for each member meeting at the joint, we will get as the set of compatibility equations from the form of Eq. (11) into the following convenient working system:

$$\begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix} \begin{bmatrix} u_j \\ v_j \\ w_j \end{bmatrix} - \sum_{i=k,\, l,\, \text{etc.}} \left\{ \begin{bmatrix} A'_{ji} & D'_{ji} & E'_{ji} \\ D'_{ji} & B'_{ji} & F'_{ji} \\ E'_{ji} & F'_{ji} & C'_{ji} \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} \right\} = \sum_{m=1,\, 2,\, \text{etc.}} \begin{bmatrix} P_{jmx} \\ P_{jmy} \\ P_{jmz} \end{bmatrix}. \tag{14}$$

d) Simultaneous Equations of Displacement Components

The orthogonal components of any displacement at each joint are represented by a set of 3 equations as formulated in Eq. (14). If an indeterminate space truss has *n* joints free to move in any direction, there are *n* sets of such equations forming a system of $3\,n$ simultaneous equations with orthogonal components of joint displacements as Unknows. The solution of the system of equations will give a unique determination of the unknown displacement components, as there are 3 such equations for the 3 unknowns at each joint.

When the number (n) of joints becomes large, the solution of a system of simultaneous equations 3 times as large will be arduous, requiring a fast method such as the author's "Matrix Algorisms*) for the solution of Non-Homogeneous Simultaneous Algebraic Linear Equations" presented orally to the 1960 national meeting of the Association for Computing Machinery.

After the displacement-component vectors ($u\,v\,w$) of all joints have been determined, the stress in each member may be readily computed with Eq. (7).

e) Consistent Coordinates, Sign Convention, and Stress Senses

In order to get automatically from Eq. (7) the stress in any member AB as tension when positive and compression when negative, systematic care must be exercised to keep coordinates and sign convention consistent throughout the numerical analysis. This may be accomplished in the following sequence:

1. Establish the space position vector ($x\,y\,z$) of each joint consistent with reference to the orthogonal axes.

2. Obtain the length-projection vector ($X\ Y\ Z$) for each member consistent with the position vectors of its end joints as stated in Eq. (1) by subtracting the position vector for the far-end joint of each member from that of the joint for which the compatibility equations are being written. That is, Eq. (1) is in consistent form for writing equations for joint A. The same set will take the form

$$\begin{bmatrix} X_{ba} \\ Y_{ba} \\ Z_{ba} \end{bmatrix} = \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} - \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} = - \begin{bmatrix} X_{ab} \\ Y_{ab} \\ Z_{ab} \end{bmatrix} \qquad (15)$$

for writing equations for joint B, that is, the sign of the length (L) projection vector of any member AB for joint B is opposite to that for joint A. By recognizing this fact, consistency will be achieved and computation simplified.

3. Fix proper signs for the length-projection vector ($X\ Y\ Z$) of each member and substitute its numerical value into the equations.

By adhering to the above sequence systematically, the stress-component vector ($S_{ax}\,S_{ay}\,S_{az}$) acting upon the member AB at joint A will have senses consistent with the directions of the orthogonal axes. The stress (S) in the member AB as given by Eq. (7) will be tension when "+" and compression when "–".

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*) Its complete version in English entitled "Converging Matric Algorisms for Solving Systems of Linear Equations", by Shu-t'ien Li, appears in Trans., Chinese Association for the Advancement of Science (CAAS). Taipeh, Vol. 3, No. 1, November 1962, pp. 16—22.

f) Illustrative Example

To exemplify the numerical process, let the stresses for all members of the trussed bracket in Fig. 2 be determined by means of Eqs. (1) to (15) with simplified modifications made possible by the round-number space position

D, *E*, *F*, *G* are fixed supports

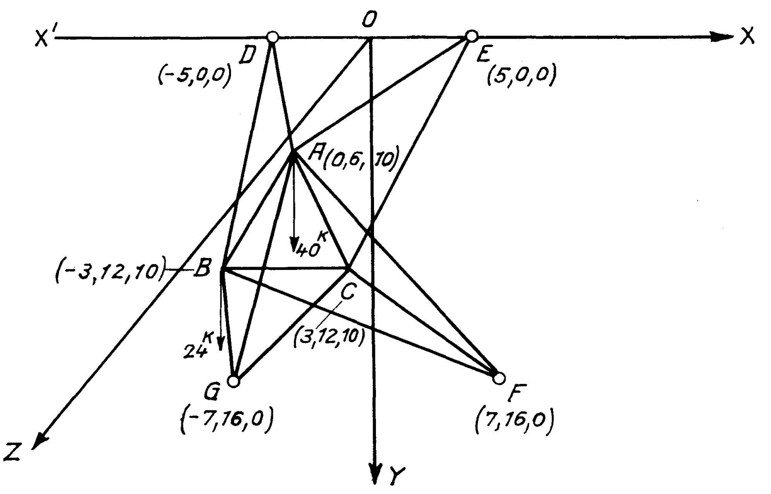

Fig. 2.

Table 1. Space Member Constants

Member	X ft	Y ft	Z ft	X^2 ft^2	Y^2 ft^2	Z^2 ft^2	L^2 ft^2	L ft	XY ft^2	XZ ft^2	YZ ft^2	A in.^{2}	100α in.^{2}/ft^3
AB	3	–6	0	9	36	0	45	6.7082	–18	0	0	1.5	0.49690
C	–3	–6	0	9	36	0	45	6.7082	18	0	0	1.5	0.49690
D	5	6	10	25	36	100	161	12.6886	30	50	60	2.5	0.122377
E	–5	6	10	25	36	100	161	12.6886	–30	–50	60	2.5	0.122377
F	–7	–10	10	49	100	100	249	15.7797	70	–70	–100	4.0	0.101803
G	7	–10	10	49	100	100	249	15.7797	–70	70	–100	4.0	0.101803
BA	–3	6	0	9	36	0	45	6.7082	–18	0	0	1.5	0.49690
C	–6	0	0	36	0	0	36	6.0000	0	0	0	1.5	0.69444
D	2	12	10	4	144	100	248	15.7480	24	20	120	2.5	0.064012
F	–10	–4	10	100	16	100	216	14.6969	40	–100	–40	1.5	0.047251
G	4	–4	10	16	16	100	132	11.4891	–16	40	–40	1.5	0.098908
CA	3	6	0	9	36	0	45	6.7082	18	0	0	1.5	0.49690
B	6	0	0	36	0	0	36	6.0000	0	0	0	1.5	0.69444
E	–2	12	10	4	144	100	248	15.7480	–24	–20	120	2.5	0.064012
F	–4	–4	10	16	16	100	132	11.4891	16	–40	–40	1.5	0.098908
G	10	–4	10	100	16	100	216	14.6969	–40	100	–40	1.5	0.047251

where $$L = [\lfloor X\ Y\ Z \rfloor \{X\ Y\ Z\}]^{1/2}$$

and $$100\alpha = \frac{100A}{L^2 L},$$

"100" times being introduced to avoid small numbers in later calculations.

vectors. They and sectional areas of members indicated in Fig. 2 yield length projections, projection squares, length squares, lengths, projection products, and α's as computed in Table 1.

Recognizing D, E, F, G being fixed supports in this case, we have

$$\{u\,v\,w\}_{d,e,f,g}=\{0\;0\;0\} \tag{16}$$

Hence, the formulation of the system of equations may be simplified as set forth in the symmetric matrix operations below:

$$\left[\begin{array}{c}[0.49690]\\ \times \\ \left[\begin{array}{c}\begin{bmatrix}\mathbf{9} & -18 & 0\\ -18 & \mathbf{36} & 0\\ 0 & 0 & \mathbf{0}\end{bmatrix}_{ab}\\ + \\ \begin{bmatrix}\mathbf{9} & 18 & 0\\ 18 & \mathbf{36} & 0\\ 0 & 0 & \mathbf{0}\end{bmatrix}_{ac}\end{array}\right]\end{array} + \begin{array}{c}[0.122377]\\ \times \\ \left[\begin{array}{c}\begin{bmatrix}\mathbf{25} & 30 & 50\\ 30 & \mathbf{36} & 60\\ 50 & 60 & \mathbf{100}\end{bmatrix}_{ad}\\ + \\ \begin{bmatrix}\mathbf{25} & -30 & -50\\ -30 & \mathbf{36} & 60\\ -50 & 60 & \mathbf{100}\end{bmatrix}_{ae}\end{array}\right]\end{array} + \begin{array}{c}[0.101803]\\ \times \\ \left[\begin{array}{c}\begin{bmatrix}\mathbf{49} & 70 & -70\\ 70 & \mathbf{100} & -100\\ -70 & -100 & \mathbf{100}\end{bmatrix}_{af}\\ + \\ \begin{bmatrix}\mathbf{49} & -70 & 70\\ -70 & \mathbf{100} & -100\\ 70 & -100 & \mathbf{100}\end{bmatrix}_{ag}\end{array}\right]\end{array}\right] \begin{bmatrix}u_a\\ v_a\\ w_a\end{bmatrix}$$

$$-\left[\begin{array}{c}[0.49690]\\ \times \\ \left[\begin{array}{c}\begin{bmatrix}\mathbf{9} & -18 & 0\\ -18 & \mathbf{36} & 0\\ 0 & 0 & \mathbf{0}\end{bmatrix}_{ab}\begin{bmatrix}u_b\\ v_b\\ w_b\end{bmatrix}\\ + \\ \begin{bmatrix}\mathbf{9} & 18 & 0\\ 18 & \mathbf{36} & 0\\ 0 & 0 & \mathbf{0}\end{bmatrix}_{ac}\begin{bmatrix}u_c\\ v_c\\ w_c\end{bmatrix}\end{array}\right]\end{array}\right] = \begin{bmatrix}0\\ 40\\ 0\end{bmatrix}_a \frac{100}{E}, \tag{17}$$

$$\left[\begin{array}{c}[0.49690]\\ \times \\ \begin{bmatrix}\mathbf{9} & -18 & 0\\ -18 & \mathbf{36} & 0\\ 0 & 0 & \mathbf{0}\end{bmatrix}_{ba}\end{array} + \begin{array}{c}[0.69444]\\ \times \\ \begin{bmatrix}\mathbf{36} & 0 & 0\\ 0 & \mathbf{0} & 0\\ 0 & 0 & \mathbf{0}\end{bmatrix}_{bc}\end{array} + \begin{array}{c}[0.064012]\\ \times \\ \begin{bmatrix}\mathbf{4} & 24 & 20\\ 24 & \mathbf{144} & 120\\ 20 & 120 & \mathbf{100}\end{bmatrix}_{bd}\end{array} + \begin{array}{c}[0.047251]\\ \times \\ \begin{bmatrix}\mathbf{100} & 40 & -100\\ 40 & \mathbf{16} & -40\\ -100 & -40 & \mathbf{100}\end{bmatrix}_{bf}\end{array}\right.$$

$$\left. + \begin{array}{c}[0.098908]\\ \times \\ \begin{bmatrix}\mathbf{16} & -16 & 40\\ -16 & \mathbf{16} & -40\\ 40 & -40 & \mathbf{100}\end{bmatrix}_{bg}\end{array}\right] \begin{bmatrix}u_b\\ v_b\\ w_b\end{bmatrix} - \left[\begin{array}{c}[0.49690]\\ \times \\ \begin{bmatrix}\mathbf{9} & -18 & 0\\ -18 & \mathbf{36} & 0\\ 0 & 0 & \mathbf{0}\end{bmatrix}_{ba}\begin{bmatrix}u_a\\ v_a\\ w_a\end{bmatrix}\\ + \\ [0.69444]\\ \times \\ \begin{bmatrix}\mathbf{36} & 0 & 0\\ 0 & \mathbf{0} & 0\\ 0 & 0 & \mathbf{0}\end{bmatrix}_{bc}\begin{bmatrix}u_c\\ v_c\\ w_c\end{bmatrix}\end{array}\right] = \begin{bmatrix}0\\ 24\\ 0\end{bmatrix}_b \frac{100}{E}, \tag{18}$$

$$\left[\begin{matrix}[0.49690] \\ \times \\ \begin{bmatrix} \mathbf{9} & 18 & 0 \\ 18 & \mathbf{36} & 0 \\ 0 & 0 & \mathbf{0} \end{bmatrix}_{ca}\end{matrix} + \begin{matrix}[0.69444] \\ \times \\ \begin{bmatrix} \mathbf{36} & 0 & 0 \\ 0 & \mathbf{0} & 0 \\ 0 & 0 & \mathbf{0} \end{bmatrix}_{cb}\end{matrix} + \begin{matrix}[0.064012] \\ \times \\ \begin{bmatrix} \mathbf{4} & -24 & -20 \\ -24 & \mathbf{144} & 120 \\ -20 & 120 & \mathbf{100} \end{bmatrix}_{ce}\end{matrix} + \begin{matrix}[0.098908] \\ \times \\ \begin{bmatrix} \mathbf{16} & 16 & -40 \\ 16 & \mathbf{16} & -40 \\ -40 & -40 & \mathbf{100} \end{bmatrix}_{cf}\end{matrix}\right.$$

$$\left. + \begin{matrix}[0.047251] \\ \times \\ \begin{bmatrix} \mathbf{100} & -40 & 100 \\ -40 & \mathbf{16} & -40 \\ 100 & -40 & \mathbf{100} \end{bmatrix}_{cg}\end{matrix}\right] \begin{bmatrix} u_c \\ v_c \\ w_c \end{bmatrix} - \left[\begin{matrix}[0.49690] \\ \times \\ \begin{bmatrix} \mathbf{9} & 18 & 0 \\ 18 & \mathbf{36} & 0 \\ 0 & 0 & \mathbf{0} \end{bmatrix}_{ca} \begin{bmatrix} u_a \\ v_a \\ w_a \end{bmatrix} \\ + \\ [0.69444] \\ \times \\ \begin{bmatrix} \mathbf{36} & 0 & 0 \\ 0 & \mathbf{0} & 0 \\ 0 & 0 & \mathbf{0} \end{bmatrix}_{cb} \begin{bmatrix} u_b \\ v_b \\ w_b \end{bmatrix}\end{matrix}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_c \frac{100}{E}. \tag{19}$$

All the necessary and sufficient conditions of deformation-displacement-equilibrium are written into Eqs. (17) to (19) for the solution of unknown displacement-component vectors $(u\,v\,w)_{a,b,c}$ at joints A, B, C. As the physical space truss in this case has "symmetric geometry", we get, as it should be, a "symmetric coefficient matrix" for the unknown vector, even though any or all of the axes were given any other parallel location. After reducing the coefficients, Eqs. (17) to (19) are arrayed into one single simultaneous symmetric matrix equation:

$$\begin{bmatrix} \mathbf{25.039} & 0 & 0 & -4.4721 & 8.9442 & 0 & -4.4721 & -8.9442 & 0 \\ 0 & \mathbf{64.949} & -5.6756 & 8.9442 & -17.8880 & 0 & -8.9442 & -17.8880 & 0 \\ 0 & -5.6756 & \mathbf{44.836} & 0 & 0 & 0 & 0 & 0 & 0 \\ -4.4721 & 8.9442 & 0 & \mathbf{36.036} & -7.1004 & 0.51146 & -25.000 & 0 & 0 \\ 8.9442 & -17.8880 & 0 & -7.1004 & \mathbf{29.445} & 1.83512 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.51146 & 1.83512 & \mathbf{21.017} & 0 & 0 & 0 \\ -4.4721 & -8.9442 & 0 & -25.000 & 0 & 0 & \mathbf{36.036} & 7.1004 & -0.51146 \\ -8.9442 & -17.8880 & 0 & 0 & 0 & 0 & 7.1004 & \mathbf{29.445} & 1.83510 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.51146 & 1.83510 & \mathbf{21.017} \end{bmatrix} \begin{bmatrix} u_a \\ v_a \\ w_a \\ u_b \\ v_b \\ w_b \\ u_c \\ v_c \\ w_c \end{bmatrix} = \begin{bmatrix} 0 \\ 40 \\ 0 \\ 0 \\ 24 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \frac{100}{E}. \tag{20}$$

Solving the above simultaneous displacement-component matrix equation by one of the author's matrix algorisms previously mentioned or any other fast method, we arrive at the desired array of results as follows:

$$\begin{bmatrix} u_a & u_b & u_c \\ v_a & v_b & v_c \\ w_a & w_b & w_c \end{bmatrix} = \frac{100}{E}\begin{bmatrix} -0.31906 & 0.18566 & 0.28789 \\ 1.30089 & 1.75693 & 0.62694 \\ 0.16467 & -0.15793 & -0.04774 \end{bmatrix}. \tag{21}$$

Substituting these displacement components at ends of each member and its $(X\,Y\,Z)$ into Eq. (7), and using proper signs according to Eq. (15), we get stresses in lb. after multiplying the matrix by the scalar 10^5.

$$\begin{bmatrix} S_{ab} \\ S_{ac} \\ S_{ad} \\ S_{ae} \\ S_{af} \\ S_{ag} \\ S_{bc} \\ S_{bd} \\ S_{bf} \\ S_{bg} \\ S_{ce} \\ S_{cf} \\ S_{cg} \end{bmatrix} = \begin{bmatrix}
1.5/\ 45\lfloor & 3 & -6 & 0\rfloor\{-0.50472 & -0.45604 & 0.32260\} \\
1.5/\ 45\lfloor & -3 & -6 & 0\rfloor\{-0.60695 & 0.67395 & 0.21241\} \\
2.5/161\lfloor & 5 & 6 & 10\rfloor\{-0.31906 & 1.30089 & 0.16467\} \\
2.5/161\lfloor & -5 & 6 & 10\rfloor\{-0.31906 & 1.30089 & 0.16467\} \\
4.0/249\lfloor & -7 & -10 & 10\rfloor\{-0.31906 & 1.30089 & 0.16467\} \\
4.0/249\lfloor & 7 & -10 & 10\rfloor\{-0.31906 & 1.30089 & 0.16467\} \\
1.5/\ 36\lfloor & -6 & 0 & 0\rfloor\{-0.10223 & 1.12999 & -0.11019\} \\
2.5/248\lfloor & 2 & 12 & 10\rfloor\{\ 0.18566 & 1.75693 & -0.15793\} \\
1.5/216\lfloor & -10 & -4 & 10\rfloor\{\ 0.18566 & 1.75693 & -0.15793\} \\
1.5/132\lfloor & 4 & -4 & 10\rfloor\{\ 0.18566 & 1.75693 & -0.15793\} \\
2.5/248\lfloor & -2 & 12 & 10\rfloor\{\ 0.28789 & 0.62694 & -0.04774\} \\
1.5/132\lfloor & -4 & -4 & 10\rfloor\{\ 0.28789 & 0.62694 & -0.04774\} \\
1.5/216\lfloor & 10 & -4 & 10\rfloor\{\ 0.28789 & 0.62694 & -0.04774\}
\end{bmatrix} [10^5] =$$

$$\begin{bmatrix} +\ 4{,}074 \\ -\ 7{,}410 \\ +12{,}200 \\ +17{,}154 \\ -14{,}665 \\ -21{,}840 \\ +\ 2{,}556 \\ +20{,}035 \\ -\ 7{,}266 \\ -\ 8{,}937 \\ +\ 6{,}522 \\ -\ 4{,}701 \\ -\ \ \ 74 \end{bmatrix}_{lb} \tag{22}$$

Applying Eq. (8), we obtain stress components. The premultiplication of a unit row vector gives summations. Thus, we have

$$\begin{array}{lcccccccc}
\text{Joint} & S/L & [\ X & Y & Z] & & [\ S_x & S_y & S_z\ ]
\end{array}$$

$$\begin{array}{l}
\mathbf{A}\ \ B\,[1\ 1\ 1\ 1\ 1\ 1]\vphantom{\dfrac{1}{1}}\\
\quad C\vphantom{\dfrac{1}{1}}\\
\quad D\vphantom{\dfrac{1}{1}}\\
\quad E\vphantom{\dfrac{1}{1}}\\
\quad F\vphantom{\dfrac{1}{1}}\\
\quad G\vphantom{\dfrac{1}{1}}
\end{array}
\left[\begin{array}{rrrrcrrr}
\dfrac{4074}{6.708}\,[& 3 & -6 & 0] & = & [\ 1822 & -3644 & 0\ \ ]\\
\dfrac{-7410}{6.708}\,[& -3 & -6 & 0] & = & [\ 3314 & 6628 & 0\ \ ]\\
\dfrac{12200}{12.689}\,[& 5 & 6 & 10] & = & [\ 4807 & 5769 & 9615]\\
\dfrac{17154}{12.689}\,[& -5 & 6 & 10] & = & [-6759 & 8111 & 13519]\\
\dfrac{-14665}{15.780}\,[& -7 & -10 & 10] & = & [\ 6505 & 9293 & -9293]\\
\dfrac{-21840}{15.780}\,[& 7 & -10 & 10] & = & [-9688 & 13840 & -13840]
\end{array}\right] \tag{23}$$

$$\begin{array}{lccc}
 & \parallel & \parallel & \parallel\\
[\sum S_x\ \sum S_y\ \sum S_z] \text{ should } = [\ 0\ \ 40{,}000\ \ 0]\text{ vs. }[& 1 & 39{,}997 & 1\ ]
\end{array}$$

$$\begin{array}{l}
\mathbf{B}\ \ A\,[1\ 1\ 1\ 1\ 1]\vphantom{\dfrac{1}{1}}\\
\quad C\vphantom{\dfrac{1}{1}}\\
\quad D\vphantom{\dfrac{1}{1}}\\
\quad F\vphantom{\dfrac{1}{1}}\\
\quad G\vphantom{\dfrac{1}{1}}
\end{array}
\left[\begin{array}{rrrrcrrr}
\dfrac{4074}{6.708}\,[& -3 & 6 & 0] & = & [-1822 & 3644 & 0\ \ ]\\
\dfrac{2556}{6.000}\,[& -6 & 0 & 0] & = & [-2556 & 0 & 0\ \ ]\\
\dfrac{20035}{15.748}\,[& 2 & 12 & 10] & = & [\ 2544 & 15267 & 12722]\\
\dfrac{-7226}{14.697}\,[& -10 & -4 & 10] & = & [\ 4917 & 1967 & -4917]\\
\dfrac{-8937}{11.489}\,[& 4 & -4 & 10] & = & [-3112 & 3112 & -7779]
\end{array}\right] \tag{24}$$

$$\begin{array}{lccc}
 & \parallel & \parallel & \parallel\\
[\sum S_x\ \sum S_y\ \sum S_z] \text{ should } = [\ 0\ \ 24{,}000\ \ 0]\text{ vs. }[& -29 & 23{,}990 & 26\ ]
\end{array}$$

$$\begin{array}{l}
\mathbf{C}\ \ A\,[1\ 1\ 1\ 1\ 1]\vphantom{\dfrac{1}{1}}\\
\quad B\vphantom{\dfrac{1}{1}}\\
\quad E\vphantom{\dfrac{1}{1}}\\
\quad F\vphantom{\dfrac{1}{1}}\\
\quad G\vphantom{\dfrac{1}{1}}
\end{array}
\left[\begin{array}{rrrrcrrr}
\dfrac{-7410}{6.708}\,[& 3 & 6 & 0] & = & [-3314 & -6628 & 0\ \ ]\\
\dfrac{2556}{6.000}\,[& 6 & 0 & 0] & = & [\ 2556 & 0 & 0\ \ ]\\
\dfrac{6522}{15.748}\,[& -2 & 12 & 10] & = & [\ -828 & 4970 & 4141\ ]\\
\dfrac{-4701}{11.489}\,[& -4 & -4 & 10] & = & [\ 1637 & 1637 & -4092]\\
\dfrac{-74}{14.697}\,[& 10 & -4 & 10] & = & [\ -50 & 20 & -50]
\end{array}\right] \tag{25}$$

$$\begin{array}{lccc}
 & \parallel & \parallel & \parallel\\
[\sum S_x\ \sum S_y\ \sum S_z] \text{ should } = [\ 0\ \ 0\ \ 0]\text{ vs. }[& 1 & -1 & -1\ ].
\end{array}$$

The discrepancies are due to rounding-off and other computational errors.

B. Reactions at Supports

a) Equilibria and Reactions at Supports

To satisfy the condition of equilibrium at supports, the orthogonal reaction components at each of them must be equal in magnitude, but opposite in direction, to the summations of stress components of members transmitting loads thereto. Thus, the components of any reaction and its magnitude are given by

$$\lfloor R_x \; R_y \; R_z \rfloor = \lfloor -\sum S_x \; -\sum S_y \; -\sum S_z \rfloor, \tag{26}$$

$$R = [\lfloor R_x \; R_y \; R_z \rfloor \{R_x \; R_y \; R_z\}]^{1/2} \tag{27}$$

and its direction cosines are expressed by

$$\lfloor \cos\theta_x \; \cos\theta_y \; \cos\theta_z \rfloor = \frac{1}{R} \lfloor R_x \; R_y \; R_z \rfloor . \tag{28}$$

Eqs. (26) to (28) completely define the reaction at any support, provided we have no other external loads applied there. In cases where applied loads act at any support, Eq. (26) becomes

$$\lfloor R_x R_y R_z \rfloor = -\lfloor \sum (S_x + P_x) \; \sum (S_y + P_y) \; \sum (S_z + P_z) \rfloor . \tag{29}$$

The sign of each component of $(\sum P_x \; \sum P_y \; \sum P_z)$ should be respectively the same as, or opposite to, that of $(\sum S_x \; \sum S_y \; \sum S_z)$, depending upon whether they are in the same, or opposite, direction.

If the resultant of member stresses and applied loads at a support is toward it, the reaction at the same support will be also toward it but in the opposite direction in space; and vice versa. Consequently, when stress signs are "+" for tension and "−" for compression, the signs for reactions will be "−" away from a support and "+" toward it.

b) Illustrative Example

Let the reactions at supports D, E, F, G of Section A (*f*), Fig. 2, be determined, where $(\sum P_x \; \sum P_y \; \sum P_z)_{d,e,f,g} = (0 \;\; 0 \;\; 0)$ because of no applied loads acting at any of the supports. Members transmitting loads thereto are grouped below:

Joint D,	DA	Joint E,	EA	Joint F,	FA	Joint G,	GA
	DB		EC		FB		GB
					FC		GC

With stress-component vectors already obtained at the close of Section A (*f*), reaction-component vectors, magnitudes of reactions, and their direction cosines are computed by Eqs. (26) to (28) as follows:

$$\begin{array}{lcl}
 & & \qquad \lfloor S_x \quad S_y \quad S_z \rfloor \qquad\qquad\qquad \lfloor R_x \quad\quad R_y \quad\quad R_z \rfloor \\
\lfloor R_x\, R_y\, R_z \rfloor_d & = & \lfloor -1 \;\; -1 \;\;\;\; \rfloor \left[\begin{array}{rrr} 4,807 & 5,769 & 9,615 \\ 2,544 & 15,267 & 12,722 \end{array}\right]\begin{matrix} a \\ b \end{matrix} = \lfloor \;\; -7,351 \quad -21,036 \quad -22,337 \rfloor_d, \\
\lfloor R_x\, R_y\, R_z \rfloor_e & = & \lfloor -1 \;\; -1 \;\;\;\; \rfloor \left[\begin{array}{rrr} -6,759 & 8,111 & 13,519 \\ -828 & 4,970 & 4,141 \end{array}\right]\begin{matrix} a \\ c \end{matrix} = \lfloor \;\;\; 7,587 \quad -13,081 \quad -17,660 \rfloor_e, \\
\lfloor R_x\, R_y\, R_z \rfloor_f & = & \lfloor -1 \;\; -1 \;\; -1 \rfloor \left[\begin{array}{rrr} 6,505 & 9,293 & -9,293 \\ 4,917 & 1,967 & -4,917 \\ 1,637 & 1,637 & -4,092 \end{array}\right]\begin{matrix} a \\ b \\ c \end{matrix} = \lfloor -13,059 \quad -12,897 \quad 18,302 \rfloor_f, \qquad (30) \\
\lfloor R_x\, R_y\, R_z \rfloor_g & = & \lfloor -1 \;\; -1 \;\; -1 \rfloor \left[\begin{array}{rrr} -9,688 & 13,840 & -13,840 \\ -3,112 & 3,112 & -7,779 \\ -50 & 20 & -50 \end{array}\right]\begin{matrix} a \\ b \\ c \end{matrix} = \lfloor \;\; 12,850 \quad -16,972 \quad 21,669 \rfloor_g,
\end{array}$$

whence

$$\begin{array}{c}
 \\
\left[\begin{array}{c} R_d \\ R_e \\ R_f \\ R_g \end{array}\right]
\end{array}
=
\begin{array}{c}
R_x \qquad R_y \qquad R_z \qquad\qquad R_x \qquad R_y \qquad R_z \\
\left[\begin{array}{l}
(\lfloor -7,351 \;\; -21,036 \;\; -22,337 \rfloor \; \{ -7,351 \;\; -21,036 \;\; -22,337 \})^{1/2} \\
(\lfloor \;\; 7,587 \;\; -13,081 \;\; -17,660 \rfloor \; \{ \;\; 7,587 \;\; -13,081 . -17,660 \})^{1/2} \\
(\lfloor -13,059 \;\; -12,897 \;\;\; 18,302 \rfloor \; \{ -13,059 \;\; -12,897 \;\;\; 18,302 \})^{1/2} \\
(\lfloor \;\; 12,850 \;\; -16,972 \;\;\; 21,669 \rfloor \; \{ \;\; 12,850 \;\; -16,972 \;\;\; 21,669 \})^{1/2}
\end{array}\right]
\end{array}
=
\begin{array}{c}
lb \\
\left[\begin{array}{r} -{}^{*}31,551 \\ -23,250 \\ +25,920 \\ +30,376 \end{array}\right]\begin{matrix} d \\ e \\ f \\ g \end{matrix}
\end{array}
\qquad (31)$$

* "–" indicates away from support.

and

$$\begin{array}{lcl}
 & & \qquad\quad 1/R \qquad\quad R_x \qquad\quad R_y \qquad\quad R_z \qquad\qquad\qquad \theta_x \qquad\quad \theta_y \qquad\quad \theta_z \\
\lfloor \theta_x\, \theta_y\, \theta_z \rfloor_{R_d} & = & \cos^{-1}\left(\dfrac{1}{31,551} \lfloor -7,351 \;\; -21,036 \;\; -22,337 \rfloor\right) = \lfloor 256^\circ\text{-}32' \;\; 228^\circ\text{-}11' \;\; 134^\circ\text{-}56' \rfloor_d, \\
\lfloor \theta_x\, \theta_y\, \theta_z \rfloor_{R_e} & = & \cos^{-1}\left(\dfrac{1}{23,250} \lfloor \;\; 7,587 \;\; -13,081 \;\; -17,660 \rfloor\right) = \lfloor 289^\circ\text{-}03' \;\; 235^\circ\text{-}46' \;\; 139^\circ\text{-}26' \rfloor_e, \\
\lfloor \theta_x\, \theta_y\, \theta_z \rfloor_{R_f} & = & \cos^{-1}\left(\dfrac{1}{25,920} \lfloor -13,059 \;\; -12,897 \;\;\; 18,302 \rfloor\right) = \lfloor 239^\circ\text{-}45' \;\; 119^\circ\text{-}50' \;\;\; 45^\circ\text{-}05' \rfloor_f, \qquad (32) \\
\lfloor \theta_x\, \theta_y\, \theta_z \rfloor_{R_g} & = & \cos^{-1}\left(\dfrac{1}{30,376} \lfloor \;\; 12,850 \;\; -16,972 \;\;\; 21,669 \rfloor\right) = \lfloor 295^\circ\text{-}02' \;\; 123^\circ\text{-}58' \;\;\; 44^\circ\text{-}30' \rfloor_g.
\end{array}$$

C. Different Loadings and Their Combinations

a) Different Loadings

An indeterminate space truss may be subjected to dead load, live load, live load impact, wind load, wind load on live load, tractive force, centrifugal force, frictional resistance, rib shortening, shrinkage (if of reinforced concrete members), thermal rise, thermal fall, and seismic force (inertia force), etc. Whenever there are more than one loading besides dead load, it is desirable to compute the stresses and reactions separately for different loadings.

In the extension of Eq. (14) to relate all unknowns in the system in the form of Eq. (20), if A_n is the coefficient matrix, X_n^m the unknown-vector matrix, and C_n^m the loading-vector matrix, the generalized matrix equation will be

$$[A]_n [X]_n^m = [C]_n^m, \tag{33}$$

where m is the number of different loading conditions. It means there are m unknown vectors corresponding to m loading vectors, but the coefficient matrix remains constant for all loading conditions if the geometrical and dimensional properties of the space truss are held constant.

Whenever $m > 2$, it is more advantageous to solve Eq. (33) by inversion, using the appropriate one of the author's matrix algorisms previously referred to, or any other fast method, whence

$$[X]_n^m = [A]_n^{-1} [C]_n^m. \tag{34}$$

Stress and reaction vectors may then be obtained as before from the results of the displacement-component vectors given by Eq. (34).

b) Combination of Stresses and Reactions

Within the range of strains and stresses that Hooke's law applies, any desired combination of stresses for any group of loadings may be validly obtained by applying the "principle of superposition", that is, for any member,

$$S_{comb.} = \sum_G S, \tag{35}$$

where G denotes any desired group of loadings; and similarly, for any reaction,

$$\begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix}_{comb.} = \sum_G \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix}. \tag{36}$$

D. Yielding and Elastic Supports

a) Yielding and Elastic Displacements

In the present treatment, let a yielding displacement vector $(u\,v\,w)_y$ at any support be defined as such that an equilibrium is attained on reaching such component displacements but they will wholly or partially remain upon release of loading.

On the other hand, an elastic displacement vector $(u\,v\,w)_e$ at any support will be defined as such that an equilibrium is attained on reaching such component displacements but they will completely disappear upon release of loading.

Any case of yielding supports for an indeterminate space truss requires evaluated or observed displacements for computing stresses and reactions, while any case of elastic supports can be analyzed once the elastic properties of the supports are defined.

A fixed support will have all 3 displacement components equal to "zero", while a yielding or elastic support will have at least one displacement component not equal to "zero".

b) Stresses and Reactions due to Support Displacements

The treatment in Sections $A\,(a)$ to $A\,(d)$ is perfectly general under all conditions of loading as well as joint and support displacements. The illustrative example of Section $A\,(f)$, however, has all supports D, E, F, G fixed in space. Predetermined yielding displacements affect stresses and reactions but do not increase the number of equations in a system like Eq. (20). For each elastic support, 3 more equations in the said system will have to be formulated.

Yielding and elastic support displacements should be written into the matrix equations in the same way as joint displacements except that yielding displacements have already appraised known values while elastic displacements are unknowns as joint displacements.

Having determined all unknown joint and support displacements, stresses and stress components of truss members are given by Eqs. (7) and (9); and reaction components, reactions, and their direction angles by Eqs. (26) to (29).

General Applicability and Advantages

1. The method as presented in the foregoing is capable of general application to the analysis of any indeterminate space truss whose reactions cannot be determined by three-dimensional equations of statics, namely $\sum X=0$, $\sum Y=0$, $\sum Z=0$, $\sum M_x=0$, $\sum M_y=0$, $\sum M_z=0$.

2. Its applicability is not restricted by the degree of redundancy, external or internal.

3. It is operative for any geometrics, under any loading conditions, and with any status of supports.

4. The coefficient matrix of joint displacement-component vectors, once formulated, holds true under all loading conditions, and its "inverse" affords a ready solution for all unknown vectors corresponding to their loading vectors.

5. In the case of symmetrical space truss about one or more orthogonal axes, the resulting symmetric matrix further reduces the time of inverting a large matrix almost to one half.

6. With an indeterminate space truss, it is more expedient to solve stresses

first than reactions first, and the fact that there are generally more members than joints, makes it the most advantageous choice to determine joint (including support) displacement-component vectors first as adopted in the foregoing method.

7. The matrix formulation considerably simplifies the entire numerical analysis by dealing with large arrays of vectors instead of individual quantities, and enables electronic digital computation in matrix operations almost without specific programming.

8. In particular, the general applicability of the method to any indeterminate space truss, may best be shown in the case of dome skeleton structures*) of which all the known types of highly indeterminate, efficient systems: a) meridional ribs with latitudinal rings, b) lamellar, c) curvilinear-lamellar, d) lattice, e) geodesic, and even f) Kaiser geodesic space truss, g) inverted (suspension), may be analyzed by the method presented, though the nature of the last two types would need slight supplementary analyses.

9. For programmed automatic computation, automatic logical checks may be instituted at appropriate stages to verify the fulfillment of the equilibrium criterion

$$\sum \lfloor F_x \, F_y \, F_z \rfloor = \lfloor 0 \;\; 0 \;\; 0 \rfloor \tag{37}$$

for any tested joint or support, where F components are those of all stresses, loads, or reactions at a joint or a support; and finally to see whether the computed direction cosines satisfy

$$\lfloor \cos\theta_x \;\; \cos\theta_y \;\; \cos\theta_z \rfloor \{\cos\theta_x \;\; \cos\theta_y \;\; \cos\theta_z\} = [1] \tag{38}$$

as they should.

Summary

A matrix formulation for the analysis of any indeterminate space truss is presented. All deformation-displacement equations, joint equilibrium equations, compatibility equations, and simultaneous equations of displacement components have been developed in matrices. And in this form are treated equilibria and reactions at supports, different loadings and their combinations, yielding and elastic supports. Numerical matrices for determining stresses and reactions are given.

Résumé

L'auteur montre comment on peut formuler sous forme matricielle l'étude de n'importe quel treillis tridimensionnel hyperstatique. Toutes les équations de déformation-déplacement, les équations d'équilibre aux articulations, les

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*) "Metallic Dome-Structure Systems", by Shu-t'ien Li, Proc., ASCE, Vol. 88, No. ST6, Journal of the Structural Division, December 1962, Paper 3358, pp. 201—226.

équations de compatibilité et les équations simultanées des composantes des déplacements ont été mises sous forme matricielle. C'est sous cette forme que sont étudiés les équilibres et les réactions aux appuis, les différentes charges et leurs combinaisons, les appuis plastiques et élastiques.

L'auteur donne des matrices numériques pour la détermination des efforts et des réactions.

Zusammenfassung

Der Verfasser zeigt die Anwendung von Matrizen auf die Berechnung von statisch unbestimmten Raumfachwerken. Die Verformungs-Verschiebungs-gleichungen, die Gleichgewichtsbedingungen in den Knotenpunkten, die Verträglichkeitsbedingungen und die Simultangleichungen der Verschiebungskomponenten werden alle in Matrizenform entwickelt. In dieser Form werden Gleichgewichte und Reaktionen an den Auflagern, verschiedene Belastungen und deren Kombinationen, plastische und elastische Auflager untersucht. Es werden zur Bestimmung der Beanspruchungen und Reaktionen numerische Matrizen angegeben.