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On Structural Concrete and the Theorems of Limit Analysis [1])

Application des théorèmes fondamentaux de l'analyse limite aux constructions en béton

Die Anwendung der Fundamentaltheoreme der plastischen Berechnungsmethode (Limit Analysis) auf den Eisenbeton

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Introduction

Load carrying capacity or ultimate strength calculations are taking on a more and more important role in the design of structures of steel and of reinforced and prestressed concrete [1] [2]). The limit theorems for a steel structure idealized to behave in a perfectly plastic manner have a sound theoretical basis [2] and their limitations are well understood [3]. Excellent agreement is obtained with the behavior of actual framed structures of steel [4], [5]. Concrete, however, is brittle in compression and very weak in tension. Although the ability to carry compression makes it suitable for heavy piers and footings, inability to take appreciable tension would seem on first principles to prohibit its use in beams, frames, and slabs.

In the design and analysis of beams it is said that reinforcing steel takes the tension and concrete the compression in bending. Design then proceeds on the basis of beam theory. Should the shear stresses be found too large the diagonal tension is taken by additional steel. However, tensile stresses around and between the stirrups or the bent-up steel are not considered explicitly nor are computations made of the ever present large tensile stresses corresponding to the shearing stresses of bond. It is true that at ordinary loads the average

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[2]) Numbers in brackets refer to the Bibliography at the end of the paper.

tensile stresses produced by bond or transverse shear are small enough to be carried by sound concrete without cracking. Also, appreciable tension can be taken on oblique planes even when shrinkage and longitudinal tension cause transverse cracking.

Nevertheless, when the load factor N (factor of safety against reaching the load carrying capacity) is considered, it is the conditions applying at and near collapse which are of importance along with the subsequent behavior at working loads once an overload has been applied. Tensile cracking is likely to occur on oblique as well as on transverse planes of a beam which is designed in accordance with present day practice.

There is then some point in starting from the beginning and considering the analysis and the design of a concrete and steel structure on the safe assumption that concrete is unable to take any tension. Complete inability to carry tensile stress does fall within the scope of limit theory as discussed in earlier applications to soil mechanics and Voussoir arches [6]. Fig. 1 shows the Mises or octahedral shear stress criterion of yielding in principal stress space with tension cut-off planes. Fig. 2 gives the corresponding picture for the Tresca

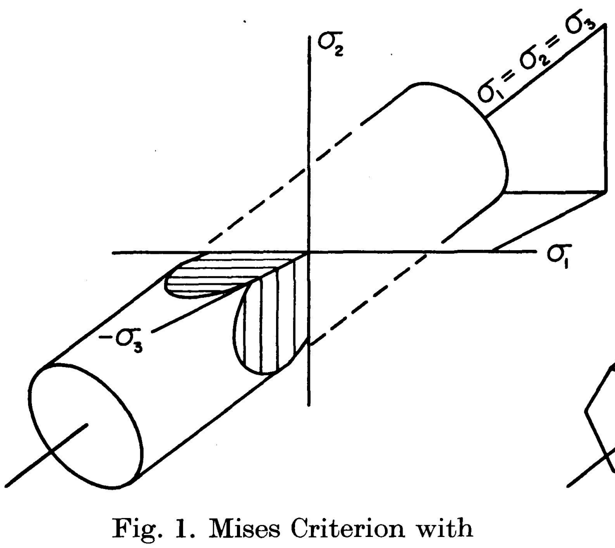


Fig. 1. Mises Criterion with Tension Cut-Off.

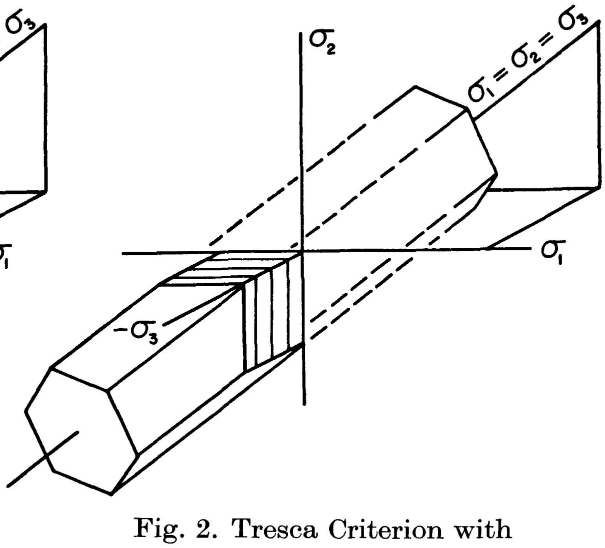


Fig. 2. Tresca Criterion with Tension Cut-Off.

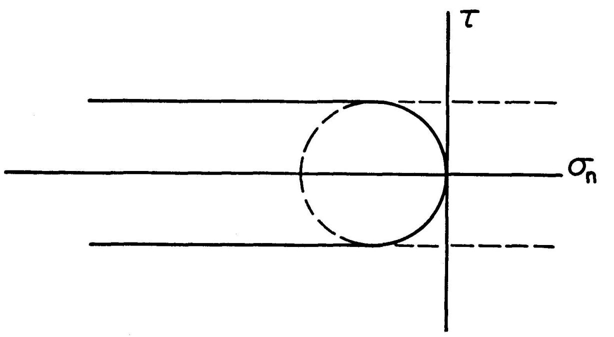


Fig. 3. Mohr's Circle Envelope with Tension Cut-Off.

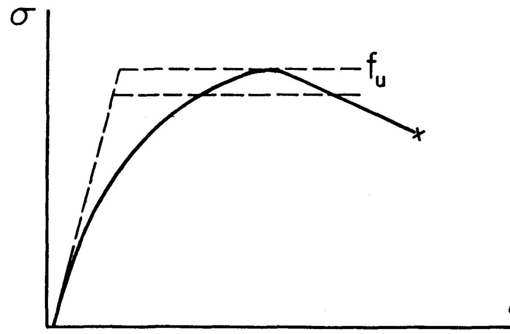


Fig. 4. Stress-Strain Curve for Concrete and Perfectly Plastic Idealizations.

criterion of maximum shearing stress. Fig. 3 is the representation of the envelope of Mohr's circles familiar from soil mechanics in which the requirement of zero tension is met by the circle termination as shown. No energy is dissipated in the formation of a simple tension crack; both normal and shearing stresses are zero on the plane of separation.

Unfortunately, brittleness in compression and a falling stress-strain curve after a maximum strength is reached, Fig. 4, are not compatible with limit theory. Professor WINTER's statements on page 50 of his paper [7] are most pertinent:

"There is a certain paradox in the fact that recent tendencies to use ever higher concrete strengths coincide with simultaneous tendencies toward utilization of the ductility of concrete in ultimate design. ... With broader acceptance of ultimate strength design a more realistic attitude toward concrete strength is likely to establish itself."

Suppose temporarily that the deformability of concrete in compression prior to appreciable fall-off of stress is sufficient to permit the applicability of limit theory with concrete idealized as perfectly plastic at a yield stress in compression close to the ultimate stress, f_u. Also suppose the steel to have a flat yield region at stress s_0 or to be approximated reasonably well or safely by such a perfectly plastic idealization. The two limit theorems [2] then may be phrased as:

Lower Bound. If an equilibrium distribution of stress can be found in the concrete and the steel which is nowhere tensile in the concrete and is everywhere at or below yield, the structure will not collapse or will just be at the point of collapse.

Upper Bound. The structure will collapse if there is any compatible pattern of plastic deformation for which the rate of work of the external loads exceeds the rate of internal dissipation.

Simply Supported Rectangular Beams as Illustrations

It is of interest to consider the design of simply supported rectangular beams $b \times d$ from the viewpoint of the limit theorems and to pretend that beam theory had not been invented. A uniformly loaded beam and a beam of negligible weight supporting a "concentrated" load at mid-span spread out just sufficiently to avoid local crushing at collapse are shown in Figs. 5a and 5b, respectively. Concrete, which cannot carry tension, must behave as a very flat arch [8]. The outward thrust of the arch is shown with a slight blurring of geometry as taken by a steel tension tie between two end plates bearing on the concrete. The steel tie is unbonded. Efficient use of the material would seem at first to dictate that at the ultimate or collapse load both steel and concrete be at their yield stresses s_0 and f_u, respectively. Furthermore, the steel should be at the $\frac{3}{4}$ depth position to provide the maximum possible resisting moment equal to that of a homogeneous beam with yield stress f_u

$$M_{max} = f_u b d^2/4 = s_0 A_s d/2 \tag{1}$$

as

$$s_0 A_s = f_u b d/2. \tag{2}$$

The approach so far has been by means of the lower bound or equilibrium theorem and so might underestimate the strength of the beam. Fig. 6a is a

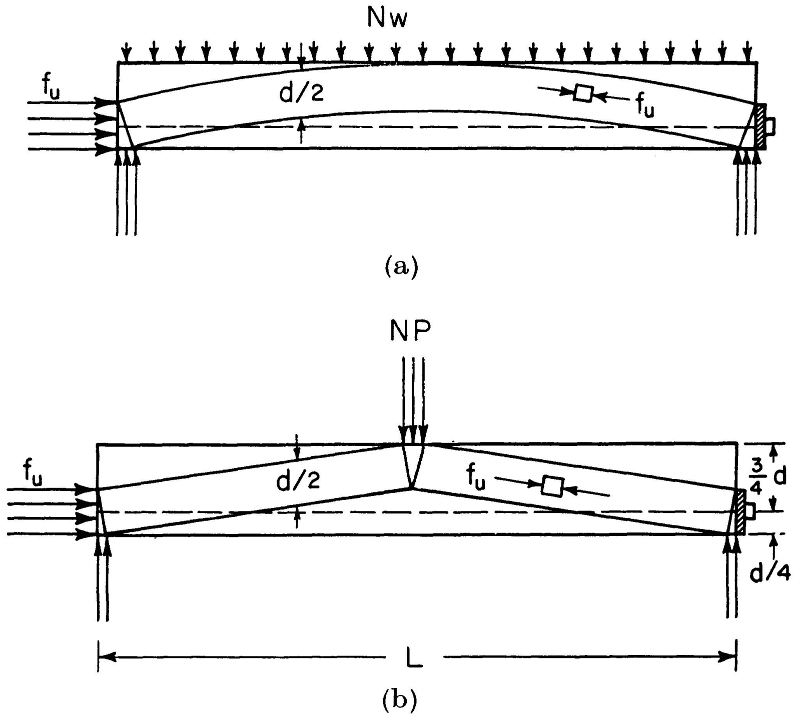


Fig. 5. Arch Action. A Lower Bound or Equilibrium Picture.
Note: Little triangles shown are under equal biaxial compression f_u.

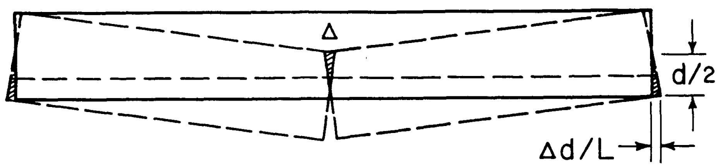


Fig. 6a. Kinematic Picture of Collapse.
No work is done in tensile cracking. Upper bound equals lower bound of Fig. 5, Eq. (3).

kinematic picture which gives an upper bound equal to the lower bound and, therefore, the correct answer for the idealization. Very closely,

$$N w L = 2 N P = 2 f_u b d^2/L = 4 s_0 A_s d/L \tag{3}$$

where N is the load factor, w is the uniform load per unit of length, L is the span, and P is the "concentrated" load.

The kinematic picture as drawn, showing rotation about the mid-depth position, can be interpreted in two independent ways. One is that the steel tie does not stretch and the shaded areas of concrete at the ends as well as in the center are crushed plastically. If the maximum rate of deflection is denoted by Δ, the maximum crushing rate for the shaded triangle at each end is $\Delta\, d/L$, and is $2\Delta\, d/L$ for the center shaded triangle. Equating external rate of work, $N w L \Delta/2$ or $N P \Delta$ to internal rate of dissipation $[2(b\,d/2)(\Delta\, d/2\,L) + (b\,d/2)(\Delta\, d/L)]\, f_u$ leads to the f_u form of Eq. (3). The kinematic picture may be thought of instead as representing plastic stretching of the steel tie rod at a rate of $\Delta\, d/2\,L$ at each end plus plastic crushing of the concrete at the center of the span to the same depth $d/2$ as before but without deformation of the concrete at the ends of the beam. There is no change in the external rate of doing work but the internal dissipation rate becomes $[2 A_s(\Delta\, d/2\,L)]\, s_0 + [(b\,d/2)(\Delta\, d/L)]\, f_u$ which again leads to Eq. (3) upon substitution of Eq. (2).

Of course, the difference in the possible control of quality of steel and of concrete would dictate a larger factor of safety for the concrete than for the steel. Not only would the value of f_u be chosen conservatively but the continued increase of strength of concrete with the months and years would also be ignored. Ductile steel therefore would be the actual determining factor in the load carrying capacity given by the s_0 form of Eq. (1) with a little larger lever arm than $d/2$. The lower bound picture would show a tied arch as in Fig. 5 (with compressive stress f_u) but $s_0 A_s/b\, f_u$ deep instead of $d/2$. The upper bound picture would be similar to Fig. 6a with the rotation taking place about a pivot $s_0 A_s/b\, f_u$ from the top of the beam. The steel tie would be taken to elongate plastically; the concrete at the ends of the beam would not crush.

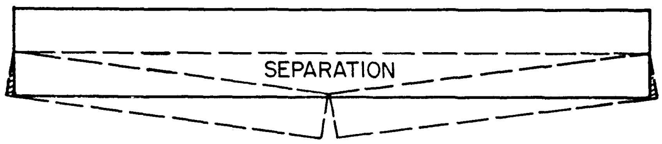


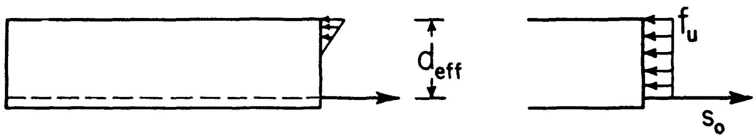
Fig. 6b. Kinematic Picture of Collapse for Load Applied at Mid-Height. Load carrying capacity is halved.

The vertical location of the point of application of the vertical load is of great importance in a material unable to take any tension. No downward load at all could be carried if it were applied to the bottom instead of the top of the beam. Half the load only could be applied at the mid-height. This is seen from lower bound considerations because the arch would have only half the rise. Fig. 6b, the upper bound picture of separation along the mid-height plane plus midspan cracking also gives half the dissipation of Fig. 6a. Agreement of upper and lower bounds proves that the load carrying capacity is exactly halved.

Ability to take some tension does, of course, mean that the consequences are not as drastic in reinforced or prestressed concrete as in the idealized material. It does emphasize, however, the greater function of shear reinforcement when beams are loaded by cross-beams framing into them than when loads are applied to the top surface of the beam.

Beam Theory

Beam theory, with its plane cross-sections remaining plane, leads to quite different answers. Steel is thought of as at the very bottom of the effective depth d_{eff} of the beam and is covered primarily for fire protection. In idealized elastic behavior, Fig. 7a, the strain and the stress distribution are linear. In idealized plastic behavior, with both steel and concrete yielding, the limit



(a) Elastic (b) Limiting Plastic
$s_0 A_s = f_u b d_{eff}$

Fig. 7. Beam Theory.

picture of plastic moment would look as shown in Fig. 7b. Once more, however, steel governs in practice because of the greater uncertainty in the properties of concrete. Nevertheless, full use of the concrete would appear to require almost twice the amount of steel as in (2)

$$s_0 A_s = f_u b\, d_{eff} \tag{4}$$

with a consequent doubling to $f_u b\, d_{eff}^2/2$ of the moment carrying capacity found previously. However, the percentage of steel would be twice that for the tied arch picture, Eq. (2)

$$\frac{A_s}{b\,d} = \frac{f_u}{2\,s_0}, \tag{5}$$

which is 5% based on the gross area with f_u of 3000 psi and s_0 of 30,000 psi, and 1% for f_u of 5000 and s_0 of 250,000.

Utilization of the full plastic moment carrying capacity of a beam would not only require twice this excessive reinforcement but would also lead to very large shear and bond stress. The accompanying tensile stress could not be dealt with by ordinary reinforcing bars or wires.

Beam theory in fact provides neither a lower bound nor an upper bound of much value in computing load carrying capacity if a material is unable to take tension. There is no valid lower bound because there is no consideration of transmission of force from the steel to the concrete without causing any tensile stress in the idealized material or without causing appreciable tensile stress over an appreciable area in the actual concrete. The upper bound given by ordinary beam theory is likely to be far too high when the usual plane section remaining plane mode of deformation is chosen.

As is well known, much can depend upon small changes in the geometry of the structure. An illustration is the addition of haunched ends which may prove extremely effective in improving arch action and providing proper anchorage for the steel. Details must be watched carefully also. For example, reinforcing bars should pass through or terminate in a region in which at least biaxial compression is present or can be induced by the bar.

Geometry Changes

Not only is the fully plastic beam moment essentially unobtainable but the full tied arch carrying capacity also can not be achieved without difficulty. The arch is extremely flat so that its length does not exceed its horizontal projection by much of a margin. For the case of the concentrated load, the rise of $d/2$ in a distance of approximately $L/2$ makes each half of the arch about $d^2/4\,L$ longer than $L/2$. If the sum of the strain in the concrete plus the strain in the steel exceeds

$$e = (d^2/4\,L)\,/\,(L/2) = \tfrac{1}{2}\,(d/L)^2 \qquad (6)$$

the arch will flatten out completely. Deflections will be excessive long before this stage is reached. Large irreversible deformations of the concrete also will take place much too early in the loading history. Post-stressing of the steel suggests itself immediately as a means of eliminating almost all of the strain in the steel under subsequent application of load. However, this alone is unlikely to reduce the danger of premature brittle collapse sufficiently. The solution toward which the designer is forced seems to be a reduction in the percentage of steel and a consequent partial rather than full use of the available concrete. As shown in Fig. 8, the effective rise of the arch is increased from $d/2$ by dropping the position of the steel below the $\tfrac{3}{4}$ height position and not depending upon as much of the concrete to participate in the arch action. If

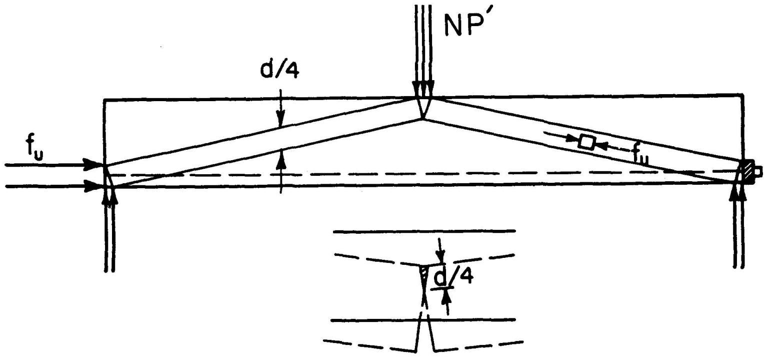


Fig. 8. Less Steel and Less Dependence on Concrete.

half the steel is used and is placed at $\frac{7}{8}d$, the load carrying capacity is still $\frac{3}{4}$ as large as in Fig. 5. This result is obtained from both the lower bound picture of Fig. 8 and from the upper bound sketch which is similar to Fig. 6 with the axis of rotation at $d/4$ from the top of the beam instead of $d/2$. Post-stressing retains its value in preventing flattening of the arch.

The emphasis on arch action arises from the approach of limit theory which looks at the equations of equilibrium in the undeformed configuration of the structure. If the geometry changes [9] of the beam are followed, it may be seen that as the beam deflects the arch of concrete flattens, and the steel tie bows downward and maintains the moment carrying capacity. If the strain in the concrete is not large enough to cause brittle fracture, such geometry change aids continuous beams and flat slabs in the carrying of load. Continuous beams usually are too deep compared to their span to take much advantage of this membrane type of action. Flat slabs, on the other hand, with d/L ratios of $^1/_{20}$ have an arch flattening limiting e of 0.00125 for the sum of the strains for steel and concrete as given by Eq. (6). Membrane action will therefore be very strong from almost the very start of loading. The steel reinforcement required to make the best use of membrane stress is not the same as for bending alone because the concrete does not contribute its share in regions of membrane tension.

Although membrane behavior increases the load carrying capacity and so adds to the safety of the flat slab, it may be misleading in the comparison of experiment and prediction of limit theory. If, for example, the limit load is found by an upper bound technique such as the rupture line procedure described by JOHANSEN [10], it may well be appreciably too high. In an experimental check of the result, however, the large membrane stresses induced in the model or prototype may mask the over-estimate completely and convince the experimenter of the validity of the answer. Other flat slabs of the same geometry in plan and with the same or smaller values of d/L will behave properly. However, a flat slab of the same shape in plan but with appreciably greater depth to span ratio may fail at a load well below the load carrying capacity predicted by the upper bound. Furthermore, whereas strain-hardening has a secondary strengthening effect in thick plates of steel or other metals, the falling off of the stress-strain curve in compression of concrete has a weakening effect instead.

Flat Slab Limit Analysis

Despite the great importance of membrane stresses, it is of interest to compute limit loads for flat slabs on the basis of the original or undeflected geometry. The arch action equilibrium state for a beam becomes a dome or compression membrane or shell for a slab and gives lower bounds. A uniformly loaded circular slab of diameter L with balanced or over-ample steel appears in section exactly as Fig. 5a. Calling the design load per unit area p and the load factor N

$$N p \geqq 4 f_u d^2/L^2. \tag{7}$$

The result for a homogeneous plate of yield stress f_u is $6 f_u d^2/L^2$ which provides an upper bound

$$N p \leqq 6 f_u d^2/L^2. \tag{8}$$

For a square plate of side L, the lower bound corresponding to two way arch action and the upper bound corresponding to diagonal rupture lines in a homogeneous plate are the same as (7) and (8), respectively. Although the square cannot be stronger than the circle and is likely to be appreciably weaker, both satisfy

$$4 f_u d^2/L^2 \leqq N p \leqq 6 f_u d^2/L^2. \tag{9}$$

Just as for the beam, a lower percentage of steel than needed to develop the concrete fully will reduce the load carrying capacity. In distinction to the beam, however, the effects of shear and bond are less critical, and the upper bounds given by plate theory or the rupture line approach are useful even when the percentage of steel is high. Also, the need for fire protection insures that the placement of steel is not too close to the surface of the slab.

Concluding Remarks

An introductory exposition has been presented of the limit analysis of structures composed of two materials. Beam theory is seen to be inappropriate in concept for the computation of the load carrying capacity of steel and concrete structures in which the concrete can carry no tension at all. The limit theorems do apply, however, if the deformability of the concrete is sufficient. Although upper and lower bounds coincided in the beam examples chosen for illustration, they will not do so in general. Sufficiently close bounds can be found, however, without excessive effort. Upper bound calculations will require clever extensions of the kinematic patterns involving separation and cracking.

The limit approach may well offer some help to the designer in achieving structures which more nearly utilize steel and concrete in accordance with their qualities and in determining ultimate loads for existing structures. A detailed analytical and spot-checking experimental study of shear reinforcement, of haunches at the ends of beams, of the placement and curvature of tensile reinforcement, and of the optimum strength of concrete all should prove fruitful, although many of the results will turn out to be well known.

Much more also should be done on plastic analysis beyond limit analysis for flat slabs and other indeterminate plate and shell structures of reinforced concrete which develop large membrane stress prior to failure.

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Summary

Although concrete is a material of very limited deformability, indications are that the load carrying capacity of reinforced and prestressed structures will, in time, be computed on the basis of the limit theorems of plastic analysis and design. A discussion is given, therefore, of the significance of the theorems in terms of the real behavior of structural concrete and its idealizations. The postulated inability of concrete to take appreciable tension does fall within the scope of limit analysis. The appropriate statement of the limit theorems is given, and some simple results are computed for rectangular beams and flat slabs. A quite different point which falls within existing plasticity theory

but not limit analysis is that appreciable membrane stresses are developed in flat slabs and similar structural elements long before the onset of collapse or structural distress. Comparison of experiment and limit theory then may be misleading.

Résumé

Bien que le matériau béton possède une capacité de déformation très restreinte, l'auteur pense que la capacité portante des constructions en béton armé et en béton précontraint se calculera à l'avenir en appliquant les théorèmes fondamentaux de l'analyse limite. Il examine la signification de ces théorèmes eu égard au comportement réel du béton et à ses idéalisations. Dans l'analyse limite, il est possible d'inclure l'hypothèse que le béton n'a pas de résistance appréciable à la traction. L'auteur indique la manière appropriée de formuler les théorèmes fondamentaux et donne quelques résultats simples, concernant des poutres rectangulaires et des dalles. Un tout autre phénomène — le fait que, dans les dalles et les éléments similaires, il se développe un état membranaire appréciable bien avant le début de la rupture ou de la mise hors service — peut être considéré dans la théorie actuelle de la plasticité mais pas dans l'analyse limite. La comparaison de résultats expérimentaux avec ceux donnés par l'analyse limite pourra donc conduire à des conclusions erronées.

Zusammenfassung

Obgleich Beton ein Baustoff von sehr beschränkter Verformbarkeit ist, weisen verschiedene Anzeichen darauf hin, daß mit der Zeit die Tragfähigkeit von Stahlbeton- und Vorspannkonstruktionen mit Hilfe der Fundamentaltheoreme der plastischen Berechnungsmethode berechnet wird. Der Autor untersucht die Bedeutung dieser Theoreme unter Berücksichtigung des tatsächlichen sowie des idealisierten Verhaltens des Betons. Die Annahme, Beton könne keine nennenswerte Zugspannung aufnehmen, kann in die Voraussetzungen der plastischen Berechnungsmethode eingegliedert werden. Der Autor zeigt uns die für Betonkonstruktionen zutreffende Formulierung der Fundamentaltheoreme sowie einige an rechteckigen Balken und Platten ermittelten Berechnungsresultate. Eine ganz andere Tatsache ist hingegen, daß in dünnen Platten und ähnlichen Konstruktionen, lange vor Eintritt des Bruches oder des Unbrauchbarkeitszustandes, recht ansehnliche Membranspannungen entstehen können. Diese Tatsache ist wohl mit der bestehenden Plastizitätstheorie, doch nicht mit der plastischen Berechnungsmethode zu vereinbaren. So könnten denn Vergleiche zwischen plastischer Berechnungsmethode und Versuchsresultaten irreführend sein.

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