Zeitschrift: IABSE publications = Mémoires AIPC = IVBH Abhandlungen
Band: 21 (1961)

Artikel: Dynamic response of beams to moving loads
Autor: Gesund, Hans / Young, Dana
DOI: https://doi.org/10.5169/seals-18248

Dynamic Response of Beams to Moving Loads

Le comportement dynamique des poutres sous l'action de charges en mouvement

Das dynamische Verhalten von Balken unter bewegter Last

HANS GESUND
M ASCE, Associate Professor of Structural Engineering University of Kentucky, Lexington, Kentucky

DANA YOUNG
F ASCE, Dean, School of Engineering, Yale University, New Haven, Connecticut

Introduction

The investigation reported in this paper is concerned with the problem of the dynamic response of a beam to moving mass loads. A finite difference procedure is developed which is suitable for solution on a digital computer. In particular, the method includes the effect of modes of vibration of the beam higher than the first, and also includes the effect of several moving loads in tandem.

The analysis is based upon the ordinary theory of flexural vibrations which neglects the rotatory inertia of the beam element and also the effect of shearing deflections. It is also assumed that the moving mass loads always remain in contact with the beam. The derivation of the theory and the specific examples are restricted to uniform simply supported beams and to concentrated loads which move across the beam with constant velocity. However, the method can readily be extended to eliminate these restrictions.

The solution yields the deflection and bending moment as functions of time and position along the beam. Results are given for several examples in the form of curves of deflection and bending moment versus time at several specific stations along the beam.

Previous work. The work previously done can be divided into several categories: (a) experimental investigations, either of whole bridges or of beams, with attempts at correlation with simplified theories or empirical formulas,

(b) analytical investigations making simplifying assumptions regarding the basic problem, (c) analytical investigations in which the differential equations are simplified and solved only approximately, (d) analytical investigations in which the differential equations are simplified and solved rigorously, and (e) analytical investigations in which the differential equations are set up rigorously (within the limits of the Bernoulli-Euler assumptions) and are solved approximately. A complete solution of the rigorous equation does not yet exist.

As might be expected, the first attempts to discover the effects of a moving load on a beam were of an experimental nature. The results of the very first investigation, apparently, were published by BECKER [1] in 1848. He measured cast iron railroad bridge deflections for passages at various speeds, of a locomotive. At almost the same time, a Royal Commission in England was also looking into the problem, and in 1849 WILLIS [2], reported on its experiments. He also wrote a simplified equation for the problem, omitting the mass of the beam as being small in comparison with that of the load. WILLIS suggested a successive approximation solution for his equation. STOKES [4], in his paper the same year, however, gave a power series solution for WILLIS' equation, which gave the trajectory of the load.

STOKES also mentioned the limitations implicit in neglecting the mass of the beam, and tried to bracket the physical problem by obtaining a solution to the case of the constant force crossing a beam which has mass. He was able to do this only approximately by simplifying the differential equation, using the assumption that all the forces acting on the beam are uniformly distributed.

In 1855, PHILLIPS [5], wrote a more complete equation than his predecessors, but was only able to solve it by successive approximations, similar to WILLIS' [2]. His equations, however, still neglected the centrifugal and the Coriolis accelerations of the load due to the curvature the load causes in the beam. PHILLIPS was followed by BRESSE [6], and in 1883 by SAINT-VENANT [8], and BOUSSINESQ [9], who extended PHILLIPS' work. SAINT-VENANT also showed that the method of successive approximations used by his predecessors may diverge. The last author to use WILLIS' method was JEFFCOTT [17] in 1929, who applied it to the equivalent of Eq. (1). In 1934, STEUDING [19] proved that this method of successive approximations must always diverge for the case of a moving concentrated load on a beam.

STOKES' [4] solution of the massless beam problem was rederived by ZIMMERMANN [11] in 1896 and elaborated by STEUDING [19] in 1934. The latter found the trajectories of 2 masses in succession in series form. The problem was last discussed by DELPUECH [25] in 1951, who pointed out that the system can vibrate in as many modes as there are loads on the beam.

As was mentioned above, STOKES [4] attempted a solution of the problem of the constant force crossing a beam whose mass is considered. His solution was very approximate. In 1899, LEBERT [12] was the first to attempt to use trigonometric series in the general problem and it is possible to isolate the complete solution for the constant force case from his approximate solution for Eq. (1). After him, KRYLOFF [13], TIMOSHENKO [14], SCHMIDT [16], INGLIS [18] and LOWAN [20] all presented the solution to the constant force case, obtaining it by a variety of methods.

The last big category of solutions is that in which the differential equations were written completely, but the solutions were found approximately, by means other than leaving out the mass of the beam or of the load. In 1937, SCHALLENKAMP [21] gave a complete solution in series form for the trajectory of a single concentrated load and indicated a solution for two loads in tandem. This was checked experimentally by AYRE, JAKOBSEN, and HSU [27] in 1951.

Approximations to the complete solution of Eq. (1) were made as early as 1861 by RENAUDOT [7]. In his work, and in most of the work following him, a shape was assumed for the deflected elastic curve of the beam, thus neglecting higher modes of vibration than the first. SAINT-VENANT [8] and MELAN [10] both mentioned this approach. In 1899, LEBERT [12] used Fourier series for the first time on the problem. TIMOSHENKO [14] and INGLIS [18] followed him, using a sine curve for the deflection curve. INGLIS was able to draw approximate graphs of center deflection against time. In 1944, LOONEY [22] introduced a step by step method for deflection with respect to time in combination with an assumed deflection curve based on a sinusoidal representation of the elastic curve. In 1948, ÖDMAN [23] considered the problem quite generally but was unable to solve the general equation. Next, HILLERBORG [24, 26] made use of LOONEYS' method and also INGLIS' method to find bending moments, using the accelerations of the beam and the load. Regressing to a simpler model, STÜSSI [30], in 1953, reduced the system to a single degree of freedom. In 1956, BIGGS, SUER and LOUW [33, 35] also assumed a single degree of freedom together with a sinusoidal deflection curve for the beam, but assumed the concentrated load to be partially sprung. TUNG, GOODMAN, CHEN and NEWMARK [34] studied the same problem but with more variables, including damping. They partially followed HILLERBORG's methods, but also used the assumption that the dynamic moment diagram would have the same shape as the static one, while its magnitude would be a function of time. The latest refinements in this, and many numerical examples were carried out by WEN [36] in 1957.

Notation

The notation used in this paper is as follows:

E modulus of elasticity of the beam material.

I moment of inertia of the beam cross-section (assumed constant).

x distance from left support to any section of the beam.

y lateral deflection of a point on the beam axis (taken positive downwards).
w weight per unit length of the beam.
W $w\,L$ total weight of the beam.
L length of the beam.
$F(x,t)$ moving distributed load per unit length.
P concentrated load.
t time, measured from the instant the first load crosses the left support.
v velocity of the moving load (assumed constant).
K $48\,E\,I/L^3$.
g acceleration of gravity.

Governing Equations

Based on the ordinary theory of flexure (the Bernoulli-Euler theory), the governing differential equation is

$$E\,I\frac{\partial^4 y}{\partial x^4} = -\frac{w}{g}\frac{\partial^2 y}{\partial t^2} + F(x,t)\left[1 - \frac{1}{g}\frac{d^2 y}{d t^2}\right]. \tag{1}$$

In this equation $\partial^2 y/\partial t^2$ is the acceleration of a point on the beam axis, while $d^2 y/d t^2$ is the acceleration of the moving load. The latter is given by

$$\frac{d^2 y}{d t^2} = \frac{\partial^2 y}{\partial x^2}\left(\frac{d x}{d t}\right)^2 + 2\frac{\partial^2 y}{\partial x\,\partial t}\frac{d x}{d t} + \frac{\partial y}{\partial x}\frac{d^2 x}{d t^2} + \frac{\partial^2 y}{\partial t^2}, \tag{2}$$

where $d\,x/d\,t$ is the velocity of the load. For $d\,x/d\,t = v = \text{const.}$ Eq. (2) reduces to

$$\frac{d^2 y}{d t^2} = v^2\frac{\partial^2 y}{\partial x^2} + 2\,v\frac{\partial^2 y}{\partial x\,\partial t} + \frac{\partial^2 y}{\partial t^2}. \tag{3}$$

Assume a finite difference net as shown in Fig. 1 and let y_{ij} be the deflection at the i^{th} station along the beam at the j^{th} time interval. The derivatives in (1) and (3) can be approximated by the following finite difference operators:

$$\begin{aligned} \frac{\partial^4 y}{\partial x^4} &\doteq \frac{1}{(\Delta x)^4}A, & \frac{\partial^2 y}{\partial t^2} &\doteq \frac{1}{(\Delta t)^2}B, \\ \frac{\partial^2 y}{\partial x\,\partial t} &\doteq \frac{1}{4\,\Delta x\,\Delta t}C, & \frac{\partial^2 y}{\partial x^2} &\doteq \frac{1}{(\Delta x)^2}D, \end{aligned} \tag{4}$$

where
$$\begin{aligned} A &= y_{(i-2)j} - 4\,y_{(i-1)j} + 6\,y_{ij} - 4\,y_{(i+1)j} + y_{(i+2)j}, \\ B &= -y_{i(j-3)} + 4\,y_{i(j-2)} - 5\,y_{i(j-1)} + 2\,y_{ij}, \\ C &= y_{(i+1)(j-2)} - 4\,y_{(i+1)(j-1)} + 3\,y_{(i+1)j} - y_{(i-1)(j-2)} + 4\,y_{(i-1)(j-1)} - 3\,y_{(i-1)j}, \\ D &= y_{(i+1)j} + y_{(i-1)j} - 2\,y_{ij}. \end{aligned} \tag{5}$$

In the above expressions, central difference operators are used for derivatives in the x-direction while backward difference operators are used for derivatives in the t-direction. The magnitude of the error in each of the operators is of the order of the interval squared.

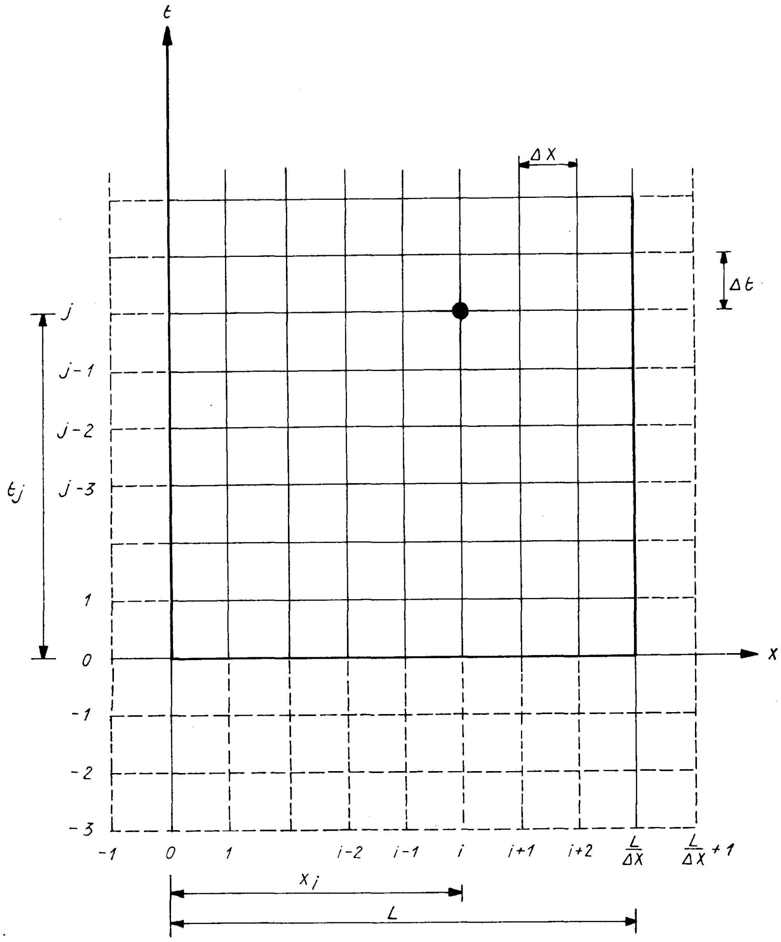

Fig. 1. Finite Difference Net.

Substituting these operators into (1) and (3) gives, after some rearrangement,

$$\frac{K}{48\,W}\left(\frac{L}{\varDelta x}\right)^3 [A] = -\frac{1}{g}\,\frac{x}{L}\,\frac{1}{(\varDelta t)^2}[B] + F\,(x_{ij}\,t_j)\,\varDelta x\left\{1 - \frac{1}{g}\,\frac{v^2}{(\varDelta x)^2}[D] - \frac{1}{g}\,\frac{v}{2\varDelta x\varDelta t}[C] - \frac{1}{g}\,\frac{1}{(\varDelta t)^2}[B]\right\}. \tag{6}$$

The factor $F\,(x_i\,,t_j)\,\varDelta x$ may be interpreted as the total load between $\left(x_i - \frac{\varDelta x}{2}\right)$ and $\left(x_i + \frac{\varDelta x}{2}\right)$ at the time t_j. Consider a concentrated load P_k which is moving across the beam with velocity v so that at time t_j its position is $x_k = v\,t_j$. If, at time t_j, P_k is between x_i and x_{i+1} (or x_{i-1}) we will assume that P_k is divided between x_i and x_{i+1} (or x_{i-1}) in proportion to its distance from the other station. That is, we take

$$F\,(x_{i,t_j})\,\varDelta x = S_i\,P_k\,, \tag{7}$$

where
$$
\begin{aligned}
S_i &= 1 && \text{if } x_k = x_i, \\
S_i &= \frac{x_{i+1}-x_k}{\Delta x} && \text{if } x_i < x_k < x_{i+1}, \\
S_i &= \frac{x_k - x_{i-1}}{\Delta x} && \text{if } x_{i-1} < x_k < x_i, \\
S_i &= 0 && \text{elsewhere.}
\end{aligned}
\tag{8}
$$

Substituting (7) into (6) gives the following difference equation for the case of a moving concentrated mass load,

$$
\begin{aligned}
\frac{K}{48\,W}\left(\frac{L}{\Delta x}\right)^3 & [y_{(i-2)j} - 4\,y_{(i-1)j} + 6\,y_{ij} - 4\,y_{(i+1)j} + y_{(i+2)j}] = \\
& \frac{\Delta x}{L}\,\frac{1}{g}\,\frac{1}{(4x)^2}[y_{i(j-3)} - 4\,y_{i(j-2)} + 5\,y_{i(j-1)} + 2\,y_{ij}] \\
& + \frac{S_i P_k}{W} + \frac{S_i P_k}{W g}\,\frac{v^2}{(\Delta x)^2}[-y_{(i+1)j} + 2\,y_{ij} - y_{(i-1)j}] \\
& + \frac{S_i P_k}{W g}\,\frac{v}{2\,\Delta x\,\Delta t}[-y_{(i+1)(j-2)} + 4\,y_{(i+1)(j-1)} - 3\,y_{(i+1)j} \\
& \qquad + y_{(i-1)(j-2)} - 4\,y_{(i-1)(j-1)} + 3\,y_{(i-1)j}] \\
& + \frac{S_i P_k}{W g}\,\frac{1}{(\Delta t)^2}[y_{i(j-3)} - 4\,y_{i(j-2)} + 5\,y_{i(j-1)} - 2\,y_{ij}].
\end{aligned}
\tag{9}
$$

If there are several concentrated loads in tandem, it is only necessary to superpose an equation of the above type for each load, with the proper shift in the x coordinates.

For a simply supported beam, the boundary conditions expressed in the finite difference form are

$$
y_{0,j} = 0; \qquad y_{L/\Delta x, j} = 0; \qquad y_{1,j} = -y_{-1,j};
$$

and
$$
y_{(L/\Delta x-1)j} = -y_{(L/\Delta x+1)j}.
$$

Assuming that the beam is initially at rest, the initial conditions were approximated by taking

$$
y_{i,0} = y_{i,-1} = y_{i,-2} = y_{i,-3} = 0.
$$

Due to storage space limitations in the IBM 650, it is necessary to make Δt much smaller than $\Delta x/v$ in order to achieve the greatest possible accuracy. This means that it will take several j steps to move the load one i step, and the load will therefore be between two net points a large part of the time.

Solution of the difference equations. It is now possible to proceed with the computations outlined by Eq. (9). The computations are started at $i=1$ and $j=1$. The expression for y_{31} has the form

$$
y_{31} = c_1 y_{11} + c_2 y_{21} + c_3,
$$

where the c_n are constants. Next, using $i=2$, $j=1$, another expression will be obtained

$$y_{41} = f(y_{11} y_{21} y_{31}).$$

Using the value of y_{31} already computed, this has the form

$$y_{41} = c_4 y_{11} + c_5 y_{21} + c_6.$$

Similarly, the other $y_{i,1}$ can be calculated up to

$$y_{(L/\Delta x+1),1} = c_7 y_{11} + c_8 y_{21} + c_9.$$

The boundary conditions at the right end are

$$y_{L/\Delta x,1} = 0, \tag{10}$$

$$-y_{(L/\Delta x-1),1} = y_{(L/\Delta x+1),1}, \tag{11}$$

which can be expressed in terms of y_{11}, y_{21} and c_n.

Eqs. (10) and (11) can be solved simultaneously to yield values of y_{11} and y_{21} which can then be used to find the $y_{i,1}$. After all the $y_{i,1}$ are found, the $y_{i,2}$ may be obtained similarly, and so on until all the $y_{i,j}$ desired have been found.

Bending moments can be obtained at each net point after the deflections have been calculated by using the central difference expression

$$M = \frac{EI}{(\Delta x)^2}[y_{(i+1)j} + y_{(i-1)j} - 2y_{ij}]. \tag{12}$$

These calculations were programmed for the IBM 650 in such a way that up to six loads in succession may cross the beam, spaced any multiple of $2\Delta x$ apart. Both loads and spacing may vary among the six, and the numerical limit stems from the limitations of storage space on this particular machine. Unfortunately, it takes a little more than three minutes per j step when $\Delta x/L = 0.1$ on the IBM 650, so that a complete run takes a long time. The usual number of j steps required was 200, so that the whole run took about 12 hours of machine time.

Errors

1. *General.* The use of finite difference methods of solving differential equations involves errors whose magnitude cannot always be determined analytically. These errors are due to a variety of causes. In the analysis used here, the main sources of errors are (a) the assumptions made in the choice of initial and boundary conditions, (b) the use of the difference operators in place of derivatives, (c) the assumption that the load acts only at net points, and (d) the truncation error in the machine arithmetic.

The magnitude of the errors due to the first three causes diminishes as $\frac{\Delta x}{L}$ and $\frac{\Delta t}{L/v}$ are made smaller, and can, theoretically, be made as small as desired. As these errors are diminished, however, the truncation error grows in magnitude. Unless special programming is adopted, which is very wasteful in terms of both storage space and time, the IBM 650 does not round off the results of numerical calculations, but instead, simply neglects all digits to the right of the last one it can handle. In these calculations, since they have to be performed in floating point arithmetic, only eight digits can be used, with all digits beyond that, being lost. Double precision routines are available which allow the use of eighteen digits, but the program written for the solution of Eq. (9) is too long to permit the use of these routines with it on the IBM 650.

It is necessary to choose values of $\frac{\Delta x}{L}$ and $\frac{\Delta t}{L/v}$ which are so small that the first three causes of error are negligible, and yet not so small that the truncation error is appreciable. Since it was not possible to determine the best values of the differences analytically, the method was checked against two known solutions, the case of very slowly moving loads, and the case of moving constant forces.

Slowly moving loads. The computations were carried out for a very small velocity. The case chosen consisted of two concentrated loads, each half the weight of the beam and spaced $0.4\,L$ apart, crossing a 360 inch long beam for which $K/W = 3.6545472$ per inch, at a crawl speed of 0.01 inches per second. $\frac{\Delta x}{L}$ was taken as 0.1, and $\frac{\Delta t}{L/v}$ was 0.005. The bending moment obtained at each tenth point of the beam at each time step was divided by the bending moment at that point and time, due to the same loads applied statically. The largest quotient obtained was 1.00082 and the smallest was 0.99956. This indicates that the moments obtained were correct to at least three significant figures. By reference to Eq. (9), it can be seen that the magnitudes of v and of Δt represented in this case are such that this is essentially a test of (a) the accuracy of the central difference operator in representing the fourth derivative, (b) the magnitude of the truncation error, and (c) the accuracy of the central difference operator in representing the second derivative.

Constant force series solutions. To check against the constant force case, the equation obtained by A. N. KRYLOFF [13], (see also Eq. (145), Reference [32]), was programmed for the IBM 650 both in the form given by KRYLOFF, as well as differentiated twice with respect to x to give bending moments.

These equations are

$$y = \frac{2\,L^4 g}{\pi^2}\,\frac{P}{W}\sum_{i=1}^{\infty}\frac{\sin\frac{i\,\pi\,x}{L}}{(i^2\,\pi^2\,b^2 - v^2\,L^2)\,i^2}\left[\sin\frac{i\,\pi\,v\,t}{L} - \frac{L}{i\,\pi}\,\frac{v}{b}\sin\frac{i^2\,\pi^2\,b\,t}{L^2}\right], \tag{13}$$

$$M = 2\,L\,P\,b^2\sum_{i=1}^{\infty}\frac{\sin\frac{i\,\pi\,x}{L}}{(i^2\,\pi^2\,b^2 - v^2\,L^2)}\left[\sin\frac{i\,\pi\,v\,t}{L} - \frac{L}{i\,\pi}\,\frac{v}{b}\sin\frac{i^2\,\pi^2\,b\,t}{L^2}\right], \tag{14}$$

where $b^2 = \frac{E\,I\,g\,L}{W}$. For deflections, 30 terms were taken for the series, and for bending moments, 60 terms were used. The case of multiple loads was handled by superposition, which is valid here. Thus, for two loads on the beam, y and M are found by adding the deflection or bending moment for the two $v\,t/L$.

The results were obtained in such a form, that for any given set of force and beam characteristics, deflection and bending moment curves could be drawn at any instant, or the variation of deflection or bending moment at any point on the beam could be plotted against time.

To obtain constant force solutions from Eq. (9), the terms containing P_k/g are omitted from the calculations, which could be done by changing just one instruction in the main program. Comparisons between the results obtained from the two methods are discussed below.

Comparison of constant force solutions. For the great number of terms used in the series solution, the results are quite accurate, and so it would be expected that the results of the difference equation method should converge toward them as $\frac{\Delta t}{L/v}$ and $\frac{\Delta x}{L}$ are made smaller. As a first trial, the values chosen were $\frac{\Delta x}{L}=0.1$ and $\frac{\Delta t}{L/v}=0.01$, and for the second trial $\frac{\Delta x}{L}=0.1$ and $\frac{\Delta t}{L/v}=0.005$. The results obtained for the record of center bending moment against time for a case where $L=360$ in., $v=803.019$ in./sec, $\frac{K}{W} = 3.6545472$/inch with a single force $\frac{P}{W}=0.5$, are shown in Fig. 2. As can be seen, the difference equation

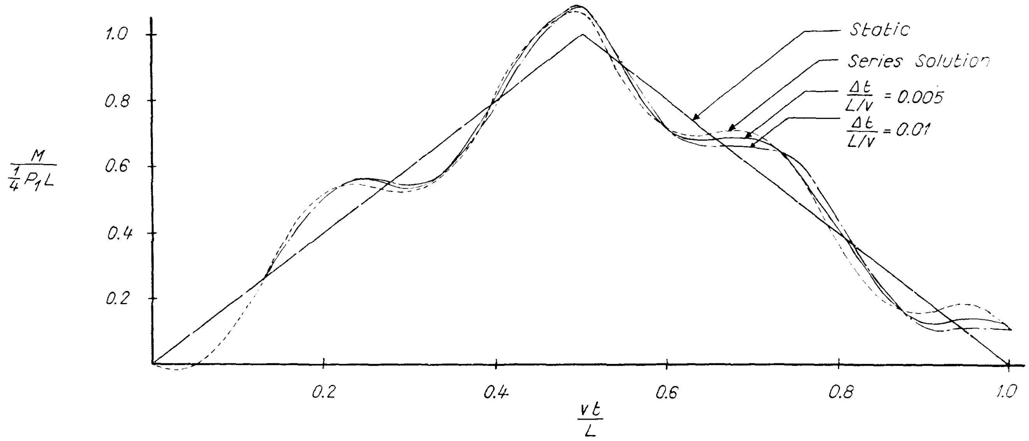

Fig. 2. Bending Moment at $x/L=0.5$ (plotted against time).
L = 360 in., v = 803.019 in./sec, K/W = 3.6545472/in., P/W = 0.5.
Comparison of constant force series solution with solutions obtained from the difference equation for constant force.

solutions converge toward the series solution. Similar results were obtained with two forces in tandem. Complete bending moment diagrams and deflection curves drawn for various instants using the two methods, showed similar good agreement and convergence. Attempts to use intervals smaller than $\frac{\Delta x}{L}=0.1$ and $\frac{\Delta t}{L/v}=0.005$ gave very erratic results, indicating that they had been affected by the truncation error. The above intervals were therefore chosen as standard for the investigation.

Experimental check. As a check against an experimental result (which is the only check available for the moving mass load), the physical constants used in an experiment by TUNG, GOODMAN, CHEN and NEWMARK, Reference [34], Fig. 4, were used in a run on the computer. The computed results, checking a center bending moment versus time diagram, were so close to their experimental curve, that the two curves could not be separated in a figure of the size possible in this publication. They are shown, however, in Fig. 19 of Reference [37].

Results

General. As was previously shown, the successive solutions of Eq. (9) give the deflection at every net point and, using Eq. (12), the bending moment at the same time and place. The program was written so that all these deflections and bending moments were printed out as answers.

One method of presenting the results is to plot the deflection and bending

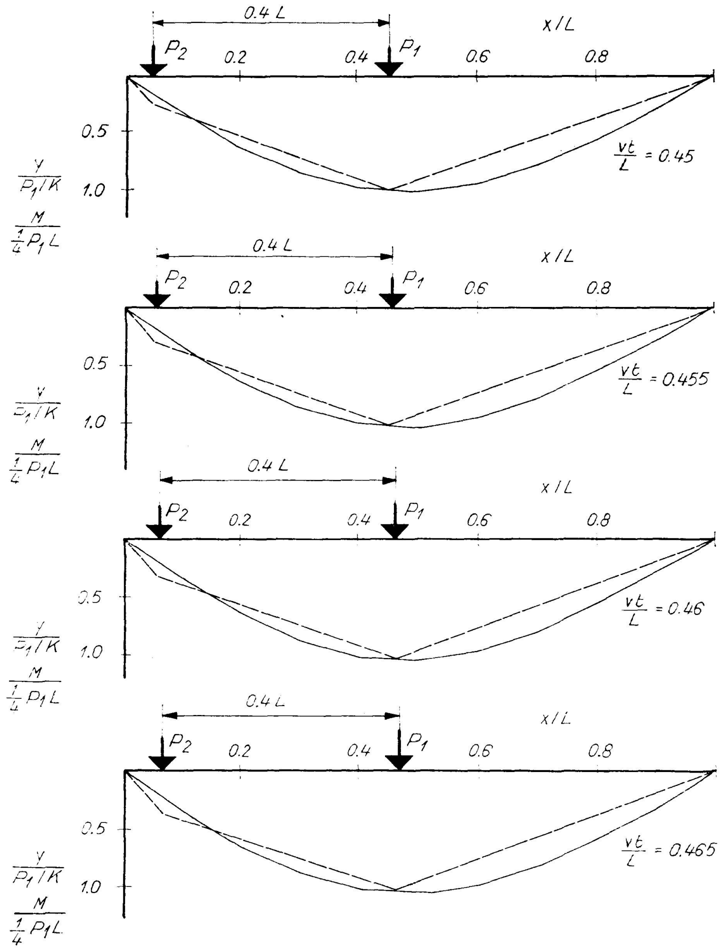

Fig. 3. Successive Instantaneous Bending Moment Diagrams --- and Deflection Curves —— (obtained from the difference equations)
$L = 360$ in., $v = 803.019$ in./sec, $K/W = 3.6545472$/in., $P_1/W = P_2/W = 0.5$.

moment for all stations along the beam at a given time. Samples of this method of plotting are shown in Fig. 3 which gives the curves for the case of two loads a distance $0.4\,L$ apart and at four successive times beginning at $vt/L = 0.45$.

One of the most interesting phenomena observed from the results is how closely the shape of both the instantaneous elastic curve and the instantaneous bending moment diagram resemble those due to static loading (see Fig. 3). For loads greater than or equal to $^1/_2$ the weight of the beam, the mass of the beam seems to have negligible effect on the shape of these curves.

Fundamental mode of vibration. Fig. 4 is the record of center bending moment

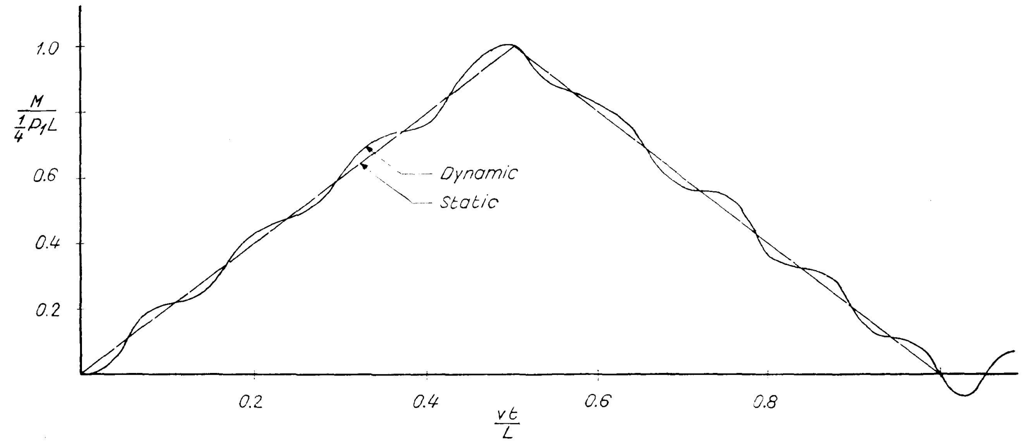

Fig. 4. Bending Moment at $x/L = 0.5$ (plotted against time).
$L = 1127.5$ in., $v = 960$ in./sec, $P/W = 0.5$, $K/W = 3.6545472$/in.

versus time for a case of a single concentrated load in which $L = 1127.5$ in., $v = 960$ in./sec., $P/W = 0.5$, $K/W = 3.6545472$/inch. This figure shows the change in the period of vibration of the beam as the load proceeds across it. When the load is at the beginning and at the end of the beam, the period is $0.105\frac{L}{v}$ (the calculated value for the unloaded beam is 0.100) whereas when the load is near the center, the period is $0.145\frac{L}{v}$.

Higher modes. A beam traversed by two concentrated loads spaced a significant portion of the length of the beam apart, may undergo vibrations in which the second mode predominates. One of the main purposes of this investigation was to develop a method of obtaining these higher mode vibrations for such a case. It was found, that a good way of picturing the second mode, was to plot bending moment versus time diagrams for the points $x/L = 0.3$ and $x/L = 0.7$ on the beam. Fig. 5 shows such plots for the case of $L = 360$ in., $v = 850$ in./sec., $K/W = 3.6545472$ in., $P_1/W = P_2/W = 0.5$ and distance between loads $= 0.4\,L$ and Fig. 6 for the same beam and loads with $v = 803.019$ in./sec. and distance between loads $= 0.6\,L$. Fig. 7 shows the center bending moments for these two cases.

Fig. 5 and 6 clearly show the presence of the second mode superimposed on the first, and using Fig. 7, some idea of its amplitude may be gained. The effect of the second mode apparently is to increase the curvature first near one end of the beam, and then near the other end.

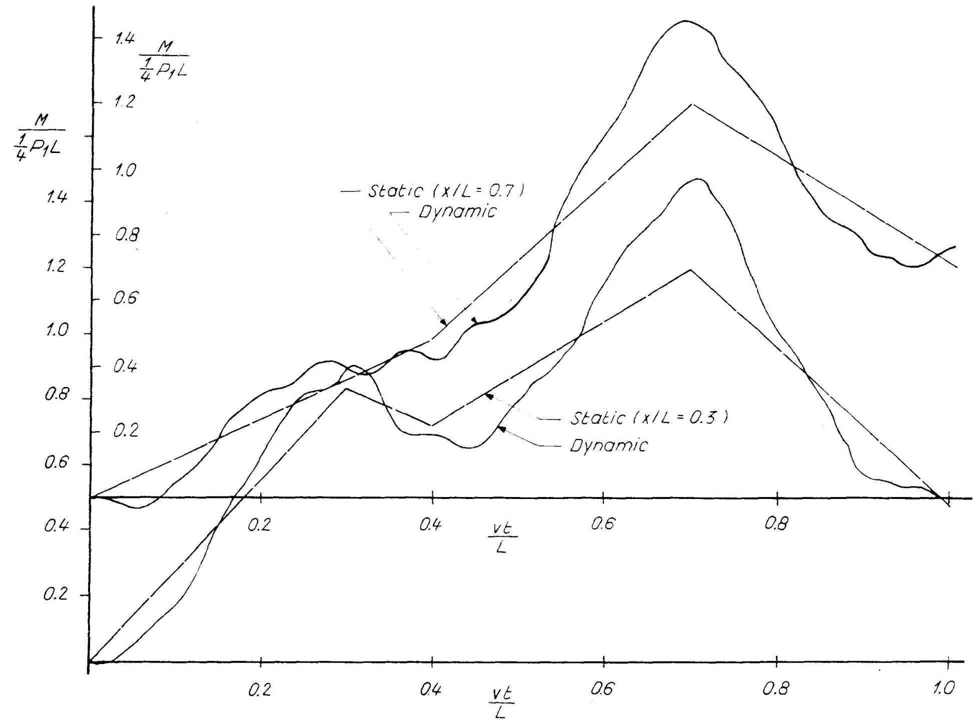

Fig. 5. Bending Moment at $x/L = 0.3$ and $x/L = 0.7$ (plotted against time). $L = 360$ in., $v = 850$ in./sec, $K/W = 3.6545472$/in., $P_1/W = P_2/W = 0.5$, Distance between Loads $= 0.4\ L$.

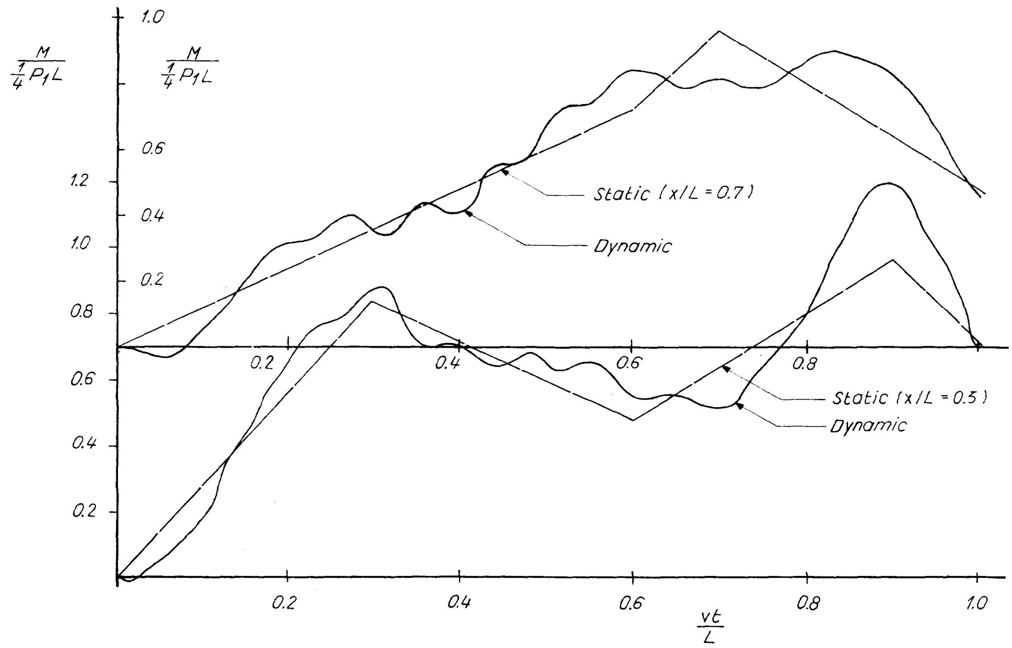

Fig. 6. Bending Moment at $x/L = 0.3$ and $x/L = 0.7$ (plotted against time). $L = 360$ in., $v = 803.019$ in./sec, $K/W = 3.6545472$/in., $P_1/W = P_2/W = 0.5$, Distance between Loads $= 0.6\ L$.

Resonance. It is conceivable, that for certain velocities, magnitudes and spacing of the two loads, quasi resonance conditions may occur. Indeed, the results of one case, $L = 360$ in., $v = 803.019$ in./sec., $K/W = 3.6545472$, $P_1/W = P_2/W = 0.5$ and distance between loads $= 0.4\,L$, showed very large second mode effects. The bending moments at the $x/L = 0.3$ and $x/L = 0.7$ points became twice as large as the static center bending moment. However, this condition of resonance was incredibly sensitive to the velocity, occurring only within a range of 0.1% velocity change about the 803.019 in./sec. Investigations using the series solution for the constant force case, also showed that this resonance would be very sensitive to the spacing of the loads.

Resonance effects in the first mode have been obtained by numerous other investigators. In the constant force case, where superposition applies, it is evident that a succession of forces crossing a beam may produce very large dynamic effects. When the mass of the load is taken into account, superposition can no longer be used, but a successive buildup in amplitude is still possible and was actually found to occur in some cases. Here again, resonance was very sensitive to changes in the physical characteristics of the system.

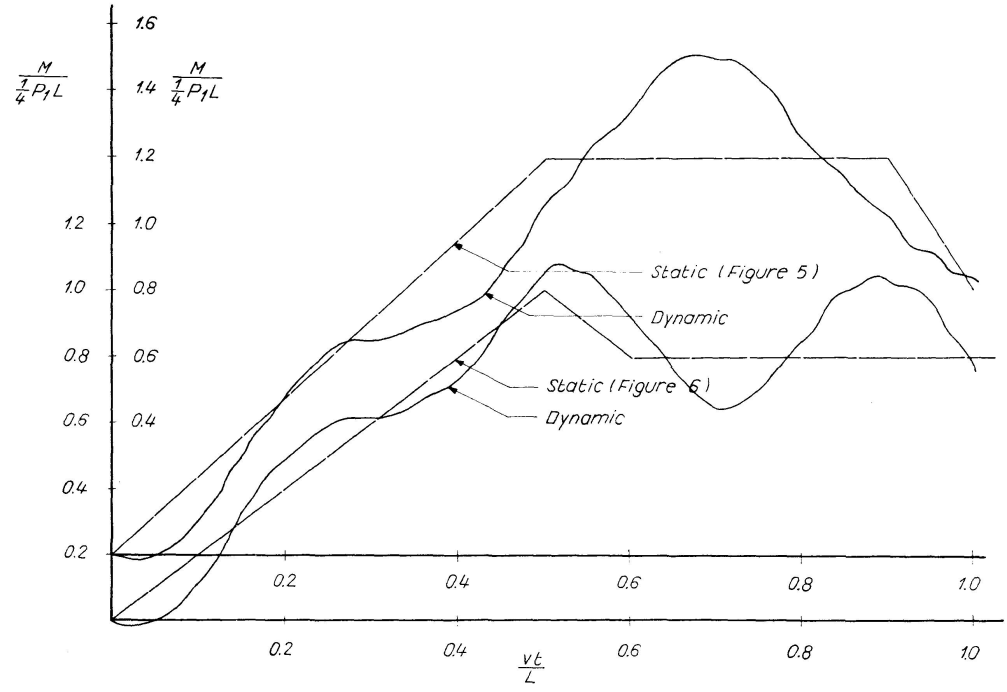

Fig. 7. Bending Moment at $x/L = 0.5$ (plotted against time). For the cases of Figs. 5 and 6.

Conclusions

A method has been presented for calculating the behavior of a beam under a moving load system. It is possible to include in Eq. (9), if necessary, additional terms to take into account any spring action and damping of the loads, as well as surface roughness of the beam, and terms which will take into account the damping, shear and rotatory inertia of the beam. It would not be wise to attempt to do this on an IBM 650, which took about 12 hours per standard run for the calculations made so far, but would be quite feasible on a larger and faster machine such as the IBM 704. Also, on the larger machine, the errors could be reduced. In particular, (a) the truncation error could be reduced with a double precision routine, giving 18 digit accuracy, (b) the $\frac{\Delta x}{L}$ could be made small enough so that the loads would always be on the net points, and (c) the accuracy of the difference representation of the differential equation and the boundary and initial conditions could be improved with smaller intervals. Alternatively, more accurate difference operators and initial and boundary conditions could be used.

The results of the examples which were solved indicate that neglecting modes of vibration higher than the first may be dangerous under certain conditions. The results also indicate that the dynamic effect of two loads in tandem may be greater than that of a single load, depending on the spacing. In the time available for the study reported here and using only the IBM 650 computer, it was not possible to calculate enough cases to determine the range of the variables for which these effects become significant. It is hoped that further studies can be made, using a higher speed computer, to extend the present data.

Bibliography

1. M. Becker, «Die gußeisernen Brücken der Badischen Eisenbahn». Ingenieur, v. 1, 1848.
2. R. Willis, "Report of the Commissioners Appointed to Inquire into the Application of Iron to Railway Structures". Appendix B, London 1849.
3. H. Cox, "On the Dynamical Deflection and Strain of Railway Girders". Civil Engineer and Architect's Journal, v. 11, p. 258, London 1848.
4. G. G. Stokes, "Discussion of a Differential Equation Relating to the Breaking of Railway Bridges". Trans. Cambridge Philosophical Society, v. 8, p. 707, 1849.
5. E. Phillips, «Résistance des poutres droites sous l'action d'une charge en mouvement». Annales des Mines, v. 7, p. 467, 1855.
6. J. A. C. Bresse, «Cours de mécanique appliquée». 2nd Edition, v. 1, p. 375, Paris 1866.
7. Renaudot, «Etude de l'influence des charges en mouvement sur la résistance des ponts métalliques à poutres droites». Annales des Ponts et Chaussées, v. 1, p. 145, 1861.
8. B. de Saint-Venant, «L'élasticité des corps solides de Clebsch». p. 597, Paris 1883.
9. J. Boussinesq, «Sur le mouvement d'une charge roulante, le long d'une barre élastique horizontale appuyée à ses deux bouts et dont la masse est beaucoup plus petite que la sienne». Comptes rendus des séances de l'Académie des sciences à Paris. July-December 1883, p. 897.
10. J. Melan, «Über die dynamische Wirkung bewegter Lasten auf Brücken». Zeitschrift des Österreichischen Ingenieur- und Architekten-Vereins, 1893, p. 293.
11. H. Zimmermann, «Die Schwingungen eines Trägers mit bewegter Last». Berlin 1896.

12. E. LEBERT, «Etude des mouvements vibratoires dans les ponts à poutres droites à une travée et dans les ponts suspendus à tablier continu simplement appuyés aux culées». Annales des Ponts et Chaussées, v. 1899-III, p. 215.
13. A. N. KRYLOFF, «Über die erzwungenen Schwingungen von gleichförmigen elastischen Stäben». Mathematische Annalen, v. 61, p. 211, 1905.
14. S. TIMOSHENKO, «Erzwungene Schwingungen prismatischer Stäbe». Bulletin of the Polytechnic Institute in Kiev, 1908, also Zeitschrift für Mathematik und Physik, v. 59, p. 163, 1911.
15. C. E. INGLIS, "Theory of Transverse Oscillations in Girders and its Relation to Live Load and Impact Allowances". Proceedings Institution of Civil Engineers, v. 218, p. 225, 1924.
16. H. SCHMIDT, «Zur Berechnung von Schwingungen elastischer Systeme unter dem Einfluß beweglicher Belastungen». Proceedings of the Third International Congress for Applied Mechanics, v. 3, p. 138, 1930.
17. H. H. JEFFCOTT, "On the Vibration of Beams under the Action of Moving Loads". Philosophical Magazine, Series 7, v. 8, p. 66, 1929.
18. C. E. INGLIS, "A Mathematical Treatise on Vibrations in Railway Bridges". Cambridge University Press, 1934.
19. H. STEUDING, «Die Schwingung von Trägern bei bewegten Lasten». Ingenieur-Archiv, v. 5, p. 275, 1934; v. 6, p. 265, 1935.
20. A. N. LOWAN, "On Transverse Oscillations of Beams Under the Action of Moving Variable Loads". Philosophical Magazine Series 7, v. 19, p. 708, 1935.
21. A. SCHALLENKAMP, «Schwingungen von Trägern bei bewegten Lasten». Ingenieur-Archiv, v. 8, p. 182, 1937.
22. C. T. G. LOONEY, "Impact on Railway Bridges". University of Illinois Engineering Experiment Station Bulletin No. 352, 1944
23. S. T. A. ÖDMAN, "Differential Equation for Calculation of Vibrations Produced in Load Bearing Structures by Moving Loads". Preliminary Publications, International Association for Bridge and Structural Engineering, Third Congress, Liege, Belgium, 1948, p. 669.
24. A. HILLERBORG, "A Study of Dynamic Influences of Moving Loads on Girders". Preliminary Publications, International Association for Bridge and Structural Engineering, Third Congress, Liege, Belgium 1948, p. 661.
25. P. DELPUECH, "Flexion dynamique et oscillations des ponts". Annales des Ponts et Chaussées, v. 121, p. 1, 225, 321, 1951.
26. A. HILLERBORG, "Dynamic Influences of Smoothly Running Loads on Simply Supported Girders". Institution of Structural Engineering and Bridge Building of the Royal Institute of Technology, Stockholm, Sweden, 1951.
27. R. S. AYRE, L. S. JACOBSEN, and C. S. HSU, "Transverse Vibration of One and of Two Span Beams Under the Action of a Moving Mass Load". Proceedings, First U. S. National Congress of Applied Mechanics, p. 81, 1951.
28. W. H. MILLEY, "Experimental Studies of Dynamic Effects of Smoothly Rolling Loads on Simply Supported Beams". University of Illinois, 1952.
29. E. H. LEE, "On a Paradox in Beam Vibration Theory". Quarterly of Applied Mathematics, v. 10, p. 290, 1952.
30. F. STÜSSI, «Trägerschwingungen unter bewegter Last». Publications of the International Association for Bridge and Structural Engineering, v. 13, p. 339, 1953.
31. R. H. BOEHNING, "Single and Tandem Axle Dynamic Effects on a Highway Bridge Model". University of Illinois, 1953.
32. S. TIMOSHENKO, "Vibration Problems in Engineering". 3rd Edition, New York 1955.
33. J. M. BIGGS, H. S. SUER, and J. M. LOUW, "Progress Report No. 2 Bridge Vibration".

Massachusetts Institute of Technology and Massachusetts Department of Public Works Research Report No. 19, Cambridge, Massachusetts, 1956.
34. T. P. TUNG, L. E. GOODMAN, T. Y. CHEN, and N. M. NEWMARK, "Highway Bridge Impact Problems". Highway Research Board Bulletin No. 124, National Academy of Sciences National Research Council Publication No. 411, 1956.
35. J. M. BIGGS, H. S. SUER, and J. M. LOUW, "The Vibration of Simple Span Highway Bridges". Proceedings of the American Society of Civil Engineers, Paper No. 1186, 1957.
36. R. K. L. WEN, "Dynamic Behaviour of Simple Span Highway Bridges Traversed by Two Axle Vehicles". University of Illinois, 1957.
37. H. GESUND, "The Dynamic Response of Beams to Moving Mass Loads". Yale University, 1958.

Summary

A finite difference procedure is developed for solving the problem of the dynamic response of a beam to moving mass loads. In particular, the method includes the effect of the higher modes of vibration, and also the effect of several moving concentrated loads in tandem.

Résumé

A l'aide de la méthode aux différences finies, les auteurs résolvent le problème du comportement dynamique d'une poutre sous l'action de forces massiques en mouvement. En particulier, ils tiennent compte de l'influence des modes de vibration supérieurs et de celle d'une paire de charges concentrées.

Zusammenfassung

Zur Lösung des Problems von Trägerschwingungen unter bewegter Last wurde eine mit endlichen Differenzen arbeitende Methode entwickelt. Insbesondere wurde mit dieser Methode der Einfluß der Oberschwingungen sowie das Verhalten des Trägers bei Durchfahrt von zwei gekoppelten Einzellasten untersucht.