An application of Donell's theory of circular cylindrical shells to the analysis of curved edge disturbances

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An Application of Donnell's Theory of Circular Cylindrical Shells to the Analysis of Curved Edge Disturbances

Application de la théorie de Donnell concernant les voiles cylindriques circulaires à l'étude des perturbations marginales sur le bord incurvé

Anwendung der Donnellschen Theorie der Kreiszylinderschalen für die Untersuchung von Randstörungen am gekrümmten Rand

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Introduction

The growing use of cylindrical shell structures in recent times has made methods for the design of such structures desirable. Interest has so far been chiefly centred on the calculation of straight edge disturbances, because of their great importance to the design of concrete shell roofs.

Proper methods for the analysis of curved edge disturbances were first given by MIESEL [1]. At about the same time FLÜGGE [2] established the three partial differential equations of the complete theory of cylindrical shells and developed the corresponding characteristic equation for curved edge disturbances. The subject was also treated by AAS-JAKOBSEN [4] and by OLSEN [7]. OLSEN gave the roots of the characteristic equation of FLÜGGE in closed form and used his theoretical results particularly to investigate continuous shells.

The papers mentioned so far base the methods of design on the complete theory. SJÖSTRÖM [5] neglected relatively unimportant terms in FLÜGGE's equation. Thereby he arrived at a characteristic equation identical with that obtained from the DONNELL [3] theory. Later on, HOFF [8], AAS-JAKOBSEN [10], SCHMIDT [11], PARME [12] and others have given methods of design based directly on the DONNELL theory.

The complete shell theory applied to the calculation of curved edge disturbances has also been treated by the author [13]. In that paper the partial differential equations of FLÜGGE are solved by the aid of a stress function, and it is shown how closed expressions for the edge value relations and the damping relations may be established, when terms of the order of magnitude h/R are neglected. The final formulas are fairly simple, but the developments are lengthy and complicated. The expressions found reduce to those pertaining to the DONNELL theory when $1/m^2$ is neglected compared with unity. Thus the DONNELL theory will in many practical cases be of sufficient accuracy. A corresponding development based on this theory is given in the present paper.

1. Basic Theory

Fig. 1 shows an element of a shell with radius R and thickness h. The sides of the element are $dx = R\,d\xi$ and $dy = R\,d\varphi$. On this element are acting the forces shown in figs. 2, 3 and 4. As edge loads only will be considered, no surface loads are assumed to act on the element. The displacements of the middle surface are denoted as shown in fig. 5.

In the DONNELL theory the moments and transverse forces are expressed by the displacement w of the middle surface as known from the theory of laterally loaded plates

$$\begin{aligned} M_x &= K\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right), & Q_x &= K\frac{\partial \nabla^2 w}{\partial x}, \\ M_\varphi &= K\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right), & Q_\varphi &= K\frac{\partial \nabla^2 w}{\partial y}. \\ M_{x\varphi} &= M_{\varphi x} = K(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y}, \end{aligned} \tag{1}$$

where

$$K = \frac{E h^3}{12(1-\nu^2)},$$

$$\nu = \text{Poissons's ratio},$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \text{Laplace's operator}.$$

The extensional forces may be expressed by an Airy's stress function Φ as follows:

$$N_x = \frac{\partial^2 \Phi}{\partial y^2}, \qquad N_\varphi = \frac{\partial^2 \Phi}{\partial x^2}, \qquad N_{x\varphi} = -\frac{\partial^2 \Phi}{\partial x\,\partial y}. \tag{2}$$

Hooke's law gives the equations

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} = (N_x - \nu N_\varphi)\frac{1}{E h}, \\ \epsilon_\varphi &= \frac{\partial v}{\partial y} + \frac{w}{R} = (N_\varphi - \nu N_x)\frac{1}{E h}, \\ \gamma_{x\varphi} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2(1+\nu) N_{x\varphi}\frac{1}{E h}. \end{aligned} \tag{3a–c}$$

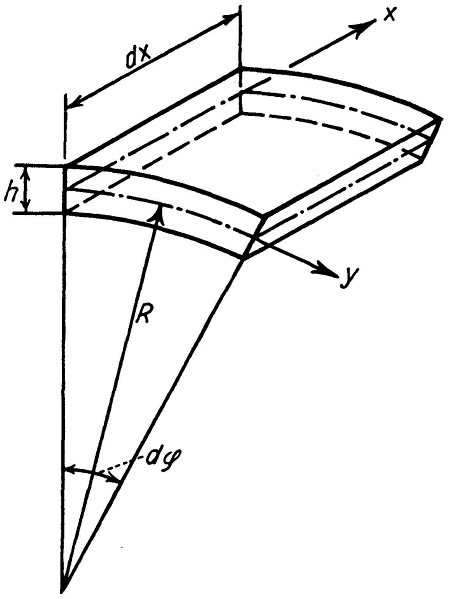

Fig. 1. Shell Element.

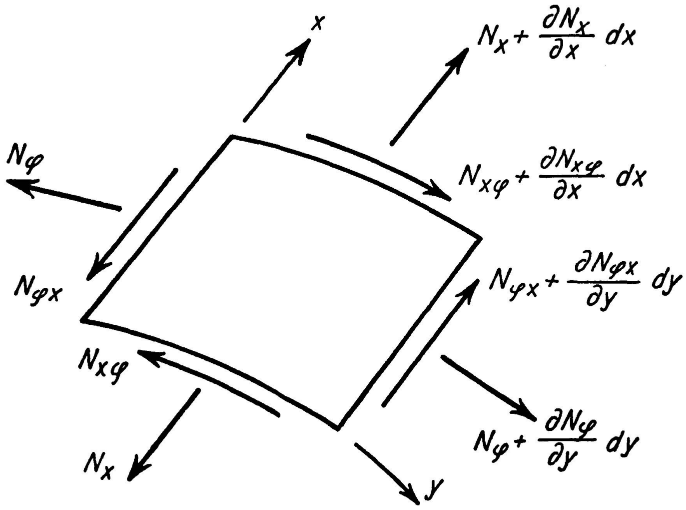

Fig. 2. Extensional Forces.

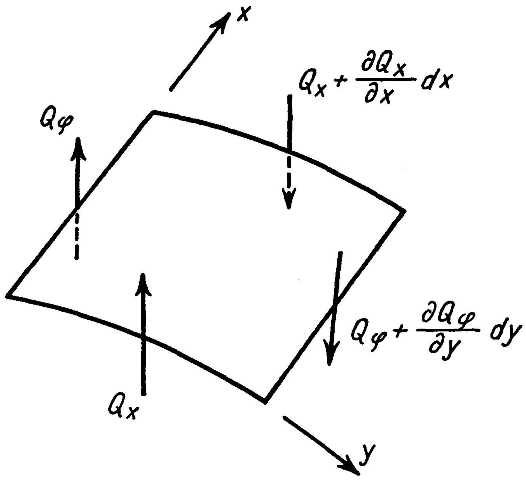

Fig. 3. Transverse Forces.

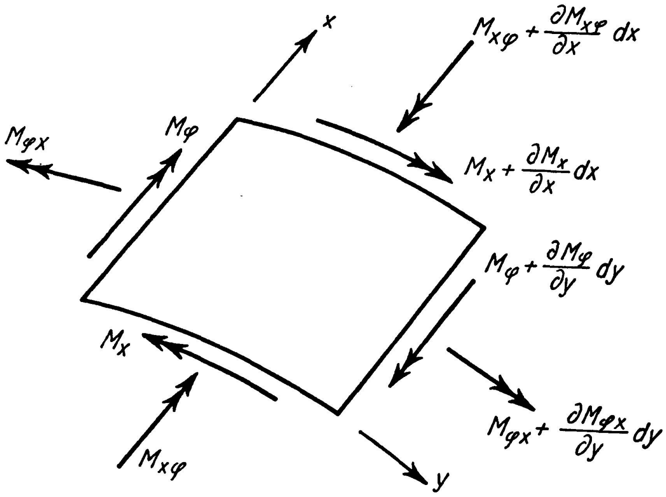

Fig. 4. Moments.

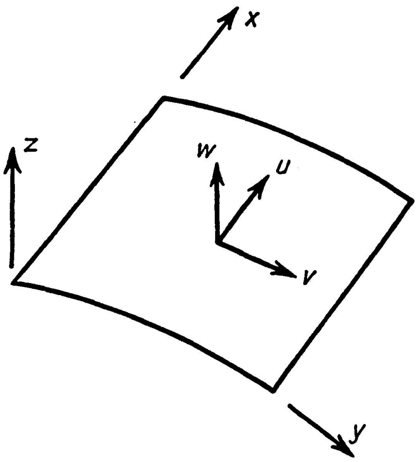

Fig. 5. Displacements of the Middle Surface.

Eqs. (2), (3a) and (3c) yield

$$\frac{\partial u}{\partial x} = \left(\frac{\partial^2 \Phi}{\partial y^2} - \nu \frac{\partial^2 \Phi}{\partial x^2}\right) \frac{1}{E h},$$
$$\frac{\partial^2 v}{\partial x^2} = - \frac{\partial}{\partial y}\left[\nabla^2 \Phi + (1+\nu) \frac{\partial^2 \Phi}{\partial x^2}\right] \frac{1}{E h}. \tag{4}$$

Eq. (3b) then gives the differential equation

$$\nabla^4 \Phi = \frac{E h}{R} \frac{\partial^2 w}{\partial x^2}, \tag{5}$$

where

$$\nabla^4 \Phi = \nabla^2 \nabla^2 \Phi.$$

From figs. 2 and 3 the condition of equilibrium normal to the shell surface is found to be

$$\frac{\partial Q_\varphi}{\partial y} + \frac{\partial Q_x}{\partial x} + \frac{N_\varphi}{R} = 0, \tag{6}$$

which gives the differential equation

$$\nabla^4 w = - \frac{1}{K R} \frac{\partial^2 \Phi}{\partial x^2}. \tag{7}$$

The two equations (5) and (7), which are symmetrically constructed, constitute the differential equations of the shell. When R increases indefinitely, the right side terms reduce to zero, and the corresponding equations of the plane plate result.

If Φ is eliminated from eqs. (5) and (7), one equation of the eighth order is obtained:

$$\nabla^8 w + \frac{E h}{K R^2} \frac{\partial^4 w}{\partial x^4} = 0. \tag{8}$$

2. The Characteristic Equation

A solution of eq. (8) is obtained by the substitution

$$w = \sum w_m(\xi) \sin m \varphi, \tag{9}$$

where

$$\xi = x/R$$

and m is an arbitrary number. For each of the unknown functions $w_m(\xi)$, eq. (8) yields the differential equation

$$\left(\frac{\partial^2}{\partial \xi^2} - m^2\right)^4 w_m(\xi) + 4 c^4 \frac{\partial^4 w_m(\xi)}{\partial \xi^4} = 0, \tag{10}$$

where

$$c^4 = \frac{R^2 E h}{4 K} = 3(1-\nu^2) \frac{R^2}{h^2}. \tag{11}$$

Eq. (10) is solved by substituting

$$w_m(\xi) = C\, e^{\lambda\xi}, \tag{12}$$

which gives the characteristic equation

$$(\lambda^2 - m^2)^4 + 4\, c^4 \lambda^4 = 0. \tag{13}$$

This equation is easily solved explicitly. From (13)

$$\lambda^2 \pm (1 \pm i)\, c\, \lambda - m^2 = 0. \tag{14}$$

This second degree equation gives the eight roots

$$\hat{\lambda} = \pm(\alpha_1 \pm i\,\beta_1), \qquad \hat{\lambda} = \pm(\alpha_2 \pm i\,\beta_2), \tag{15}$$

where $\hat{\lambda}$ is a reduced root

$$\hat{\lambda} = \frac{\lambda}{c} \tag{16}$$

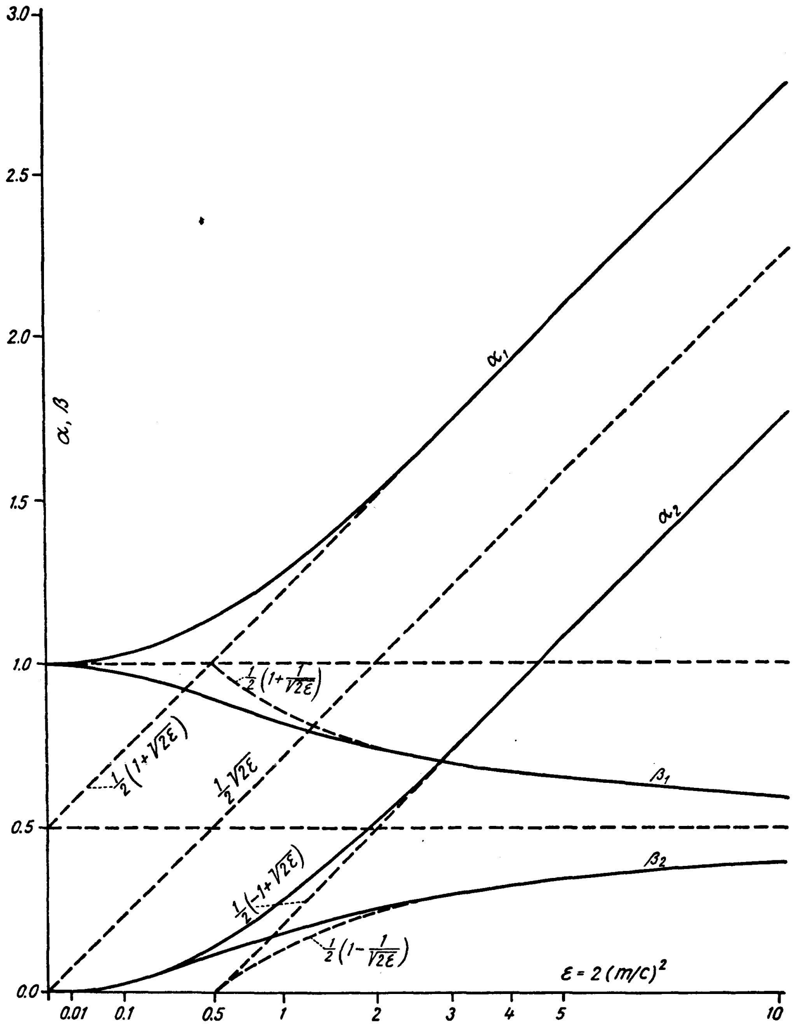

Fig. 6. Roots of the Characteristic Equation.

and furthermore

$$\begin{aligned} \alpha_1 &= \tfrac{1}{2}\{\ \ 1+[(1+\epsilon^2)^{1/2}+\epsilon]^{1/2}\}, & \beta_1 &= \tfrac{1}{2}\{1+[(1+\epsilon^2)^{1/2}-\epsilon]^{1/2}\}, \\ \alpha_2 &= \tfrac{1}{2}\{-1+[(1+\epsilon^2)^{1/2}+\epsilon]^{1/2}\}, & \beta_2 &= \tfrac{1}{2}\{1-[(1+\epsilon^2)^{1/2}-\epsilon]^{1/2}\}, \end{aligned} \tag{17}$$

$$\epsilon = 2\left(\frac{m}{c}\right)^2. \tag{18}$$

A diagram of the roots is given in fig. 6.

Eq. (13) was first established by SJÖSTRÖM [5] by neglecting relatively unimportant terms in FLÜGGE's equation [2]. SJÖSTRÖM also gave roots corresponding to those of eq. (17). The same equation has also been developed by AAS-JAKOBSEN [10], HOFF [8], SCHMIDT [11] and PARME [12] from the DONNELL theory, in a manner similar to the one used here.

Each of the two groups of roots given in eq. (15) contains one root in each quadrant. As the principal roots are chosen

$$\hat{\lambda}_1 = -\alpha_1 + i\beta_1, \qquad \hat{\lambda}_2 = -\alpha_2 + i\beta_2. \tag{19}$$

The eight roots are then:

$$\hat{\lambda}_1, \quad -\hat{\lambda}_1, \quad \bar{\hat{\lambda}}_1, \quad -\bar{\hat{\lambda}}_1, \quad \hat{\lambda}_2, \quad -\hat{\lambda}_2, \quad \bar{\hat{\lambda}}_2, \quad -\bar{\hat{\lambda}}_2,$$

where $\bar{z}$ denotes a number conjugate to the complex number z. The principal root is chosen with a negative real part to get the corresponding function $e^{c\hat{\lambda}\xi}$ as a damped wave.

In the further developments, powers of the principal roots will occur. It is more easily seen how these expressions may be simplified, if eq. (17) is written in a different form. Let

$$a = \operatorname{Ar}\operatorname{Sin}\epsilon. \tag{20}$$

Then

$$\begin{aligned} \alpha_1 &= \frac{e^{a/2}+1}{2} = e^{a/4}\operatorname{Cos}\left(\frac{a}{4}\right), & \beta_1 &= \frac{e^{-a/2}+1}{2} = e^{-a/4}\operatorname{Cos}\left(\frac{a}{4}\right), \\ \alpha_2 &= \frac{e^{a/2}-1}{2} = e^{a/4}\operatorname{Sin}\left(\frac{a}{4}\right), & \beta_2 &= \frac{-e^{-a/2}+1}{2} = e^{-a/4}\operatorname{Sin}\left(\frac{a}{4}\right). \end{aligned} \tag{17a}$$

3. Characteristic Coefficients (Coefficients to the Constants of Integration)

The solution of the differential equation (10) is

$$\begin{aligned} w_m(\xi) = {} & C_1 e^{\lambda_1\xi} + C_2 e^{\lambda_2\xi} + C_3 e^{-\lambda_1\xi} + C_4 e^{-\lambda_2\xi} + C_5 e^{\bar{\lambda}_1\xi} + \\ & + C_6 e^{\bar{\lambda}_2\xi} + C_7 e^{-\bar{\lambda}_1\xi} + C_8 e^{-\bar{\lambda}_2\xi}. \end{aligned} \tag{21}$$

$C_1 - C_8$ are 8 constants of integration to be determined from the edge conditions. These constants and the exponential functions are complex numbers, whereas the deflection w is a real number. With new constants C, eq. (21) may then also be written

$$w_m(\xi) = R\{C_1 e^{\lambda_1 \xi} + C_2 e^{\lambda_2 \xi} + C_3 e^{-\lambda_1 \xi} + C_4 e^{-\lambda_2 \xi}\}, \tag{22}$$

where
$$R\{z\} = \text{real part of } z.$$

Eqs. (21) and (22) are of the same form as those used by LUNDGREN [6] for calculating straight edge disturbances.

For each term of the series (9) the solution of eq. (8) is then

$$w = \sin m\varphi \, R\{C_1 e^{\lambda_1 \xi} + C_2 e^{\lambda_2 \xi} + C_3 e^{-\lambda_1 \xi} + C_4 e^{-\lambda_2 \xi}\}. \tag{23}$$

This expression is substituted for w in eq. (5), yielding

$$\begin{aligned}\Phi = \frac{EhR}{2c^2} \sin m\varphi \, R\Big\{ & C_1 2\hat{\lambda}_1^2 \Big(\hat{\lambda}_1^2 - \frac{\epsilon}{2}\Big)^{-2} e^{\lambda_1 \xi} + C_2 2\hat{\lambda}_2^2 \Big(\hat{\lambda}_2^2 - \frac{\epsilon}{2}\Big)^{-2} e^{\lambda_2 \xi} + \\ & + C_3 2\hat{\lambda}_1^2 \Big(\hat{\lambda}_1^2 - \frac{\epsilon}{2}\Big)^{-2} e^{-\lambda_1 \xi} + C_4 2\hat{\lambda}_2^2 \Big(\hat{\lambda}_2^2 - \frac{\epsilon}{2}\Big)^{-2} e^{-\lambda_2 \xi}\Big\}.\end{aligned} \tag{24}$$

This formula is most easily simplified when the formulas (17a) are used for the roots. Then

$$\begin{aligned}\hat{\lambda}_1^2 &= (-\alpha_1 + i\beta_1)^2 = \frac{\epsilon}{2} + \operatorname{Sin}\Big(\frac{a}{2}\Big) - i\Big[\operatorname{Cos}\Big(\frac{a}{2}\Big) + 1\Big], \\ \hat{\lambda}_2^2 &= (-\alpha_2 + i\beta_2)^2 = \frac{\epsilon}{2} - \operatorname{Sin}\Big(\frac{a}{2}\Big) - i\Big[\operatorname{Cos}\Big(\frac{a}{2}\Big) - 1\Big], \\ \Big(\hat{\lambda}_1^2 - \frac{\epsilon}{2}\Big)^2 &= -2\Big[\operatorname{Cos}\Big(\frac{a}{2}\Big) + 1\Big] - i\Big[\epsilon + 2\operatorname{Sin}\Big(\frac{a}{2}\Big)\Big], \\ \Big(\hat{\lambda}_2^2 - \frac{\epsilon}{2}\Big)^2 &= 2\Big[\operatorname{Cos}\Big(\frac{a}{2}\Big) - 1\Big] + i\Big[\epsilon - 2\operatorname{Sin}\Big(\frac{a}{2}\Big)\Big],\end{aligned} \tag{25}$$

from which

$$2\hat{\lambda}_1^2 \Big(\hat{\lambda}_1^2 - \frac{\epsilon}{2}\Big)^{-2} = i, \qquad 2\hat{\lambda}_2^2 \Big(\hat{\lambda}_2^2 - \frac{\epsilon}{2}\Big)^{-2} = -i. \tag{26}$$

The formulas (26) are easily seen to agree with eq. (13). Hence, eq. (24) may be simplified to

$$\Phi = \Big(\frac{EhR}{2c^2}\Big) \sin m\varphi \, R\{C_1 i e^{\lambda_1 \xi} - C_2 i e^{\lambda_2 \xi} + C_3 i e^{-\lambda_1 \xi} - C_4 i e^{-\lambda_2 \xi}\}. \tag{27}$$

Substitution of the expression (23) for w in eqs. (1), and the expression (27) for Φ in eqs. (2) and (4), yields similar expressions for all statical quantities. For an arbitrarily chosen quantity H the expression obtained may be written

$$H = [H] \, R\{C_1 \hat{H}_1 e^{\lambda_1 \xi} + C_2 \hat{H}_2 e^{\lambda_2 \xi} \pm C_3 \hat{H}_1 e^{-\lambda_1 \xi} \pm C_4 \hat{H}_2 e^{-\lambda_2 \xi}\}. \tag{28}$$

$[H]$ is a multiplier containing the function of φ ($\cos m\varphi$ or $\sin m\varphi$) and factors depending on the dimensions and elastic properties of the shell. $\hat{H}_1$ and $\hat{H}_2$ are reduced quantites, which will be named characteristic coefficients. This term was introduced by LUNDGREN [6] in the theory of edge disturbances from the straight edge. As to the double signs, + should be used for even quantities, i. e. quantities derived an even number of times with respect to ξ, and $-$ should be used for odd quantities. Thus, from eq. (23)

$$\hat{w}_1 = 1\,, \qquad \hat{w}_2 = 1$$

and from eq. (27)

$$\hat{\Phi}_1 = i\,, \qquad \hat{\Phi}_2 = -i\,. \tag{29}$$

As eq. (28) contains the undetermined constants of integration, all multipliers may be multiplied by the same factor without changing the result. It is found convenient to have

$$[N_x] = \sin m\varphi\,.$$

To obtain this, all multipliers obtained by the procedure mentioned above have been multiplied by the common factor

$$\frac{2\,R\,c^2}{E\,h\,m^2}\,.$$

Furthermore, the notation

$$b = \frac{\epsilon}{4} = \frac{m^2}{2\,c^2} \tag{30}$$

is introduced. The detailed derivations needed to find all multipliers and characteristic coefficients are omitted here. The result is given in table 1 where the following quantities needed in the edge disturbance calculations are also included:

The resulting transverse edge forces, found as in the theory of plates:

$$R_x = Q_x + \frac{\partial\,M_{x\varphi}}{\partial\,y}\,, \qquad R_\varphi = Q_\varphi + \frac{\partial\,M_{x\varphi}}{\partial\,x}\,. \tag{31}$$

The angles of rotation in the direction of x or φ respectively:

$$\vartheta_x = \frac{\partial\,w}{\partial\,x}\,, \qquad \vartheta_\varphi = \frac{\partial\,w}{\partial\,y}\,. \tag{32}$$

The expressions in table 1 may also be derived from the formulas given by Hoff [8] and by Aas-Jakobsen [10], when simplifications similar to those of eqs. (25) and (26) are used. The only difference remaining is that the authors mentioned give their formulas in real form.

In a previous work [13] the author has given characteristic coefficients deduced from the theory of Flügge [2] by neglecting quantities of the order of magnitude h/R. These coefficients reduce to those given above when $1/m^2$ is neglected when compared with 1. Hence, the errors of Donnell's theory are of the order of magnitude h/R and $1/m^2$. The errors of the order of magnitude h/R are of no importance whatever to the design of thin shells, whereas a more exact theory may be needed in case of small values of m.

Table 1. Multipliers and Characteristic Coefficients

Quantity	Multiplier	$\hat{H}_1$	$\hat{H}_2$
N_x	$\sin m\varphi$	$-i$	i
N_φ	$2\frac{c^2}{m^2}\sin m\varphi$	$\frac{1}{2}(\alpha_1+\beta_1)+i[b+\frac{1}{2}(\alpha_1-\beta_1)]$	$-\frac{1}{2}(\alpha_2-\beta_2)-i[b-\frac{1}{2}(\alpha_2+\beta_2)]$
$N_{x\varphi}$	$\frac{c}{m}\cos m\varphi$	$\beta_1+i\,\alpha_1$	$-\beta_2-i\,\alpha_2$
M_x	$\frac{R}{m^2}\sin m\varphi$	$(1-\nu)\,b+\frac{1}{2}(\alpha_1-\beta_1)-i\,\frac{1}{2}(\alpha_1+\beta_1)$	$(1-\nu)\,b-\frac{1}{2}(\alpha_2+\beta_2)-i\,\frac{1}{2}(\alpha_2-\beta_2)$
M_φ	$\frac{R}{m^2}\sin m\varphi$	$-b+\nu[b+\frac{1}{2}(\alpha_1-\beta_1)-i\,\frac{1}{2}(\alpha_1+\beta_1)]$	$-b+\nu[b-\frac{1}{2}(\alpha_2+\beta_2)-i\,\frac{1}{2}(\alpha_2-\beta_2)]$
$M_{x\varphi}$	$\frac{1-\nu}{2}\,\frac{R}{mc}\cos m\varphi$	$-\alpha_1+i\,\beta_1$	$-\alpha_2+i\,\beta_2$
Q_x	$\frac{c}{m^2}\sin m\varphi$	$-b+\beta_1+i\,(b+\alpha_1)$	$b-\beta_2+i\,(b-\alpha_2)$
Q_φ	$\frac{1}{m}\cos m\varphi$	$\frac{1}{2}(\alpha_1-\beta_1)-i\,\frac{1}{2}(\alpha_1+\beta_1)$	$-\frac{1}{2}(\alpha_2+\beta_2)-i\,\frac{1}{2}(\alpha_2-\beta_2)$
R_x	$\frac{c}{m^2}\sin m\varphi$	$\beta_1+b\,(\alpha_2-\nu\,\alpha_1)+i\,[\alpha_1+b\,(\beta_2+\nu\,\beta_1)]$	$-\beta_2+b\,(\alpha_1-\nu\,\alpha_2)-i\,[\alpha_2-b\,(\beta_1+\nu\,\beta_2)]$
R_φ	$\frac{1}{m}\cos m\varphi$	$(1-\nu)\,b+(2-\nu)\,[\frac{1}{2}(\alpha_1-\beta_1)-i\,\frac{1}{2}(\alpha_1+\beta_1)]$	$(1-\nu)\,b-(2-\nu)\,[\frac{1}{2}(\alpha_2+\beta_2)-i\,\frac{1}{2}(\alpha_2-\beta_2)]$
u	$\frac{R}{Eh}\,\frac{2c^3}{m^4}\sin m\varphi$	$[-\beta_2+\nu\,\beta_1+i\,(\alpha_2+\nu\,\alpha_1)]\,b$	$[\beta_1-\nu\,\beta_2-i\,(\alpha_1+\nu\,\alpha_2)]\,b$
v	$\frac{R}{Eh}\,\frac{2c^2}{m^3}\cos m\varphi$	$-\frac{1}{2}(\alpha_2-\beta_2)-i\,[(1+\nu)\,b+\frac{1}{2}(\alpha_2+\beta_2)]$	$\frac{1}{2}(\alpha_1+\beta_1)+i\,[(1+\nu)\,b-\frac{1}{2}(\alpha_1-\beta_1)]$
w	$\frac{R}{Eh}\,\frac{2c^2}{m^2}\sin m\varphi$	1	1
ϑ_x	$\frac{1}{Eh}\,\frac{2c^3}{m^2}\sin m\varphi$	$-\alpha_1+i\,\beta_1$	$-\alpha_2+i\,\beta_2$
ϑ_φ	$\frac{1}{Eh}\,\frac{2c^2}{m}\cos m\varphi$	1	1

4. Edge Value Relations and Damping Relations in Closed Form

When eq. (28) and the characteristic coefficients of table 1 are used, relatively simple expressions are obtained for all statical quantities. These expressions contain 4 complex constants of integration, that is 8 real ones, which must be determined from the edge conditions.

At each edge there are 8 quantities which may occur in the edge conditions, namely:

the forces N_x, $N_{x\varphi}$ and R_x;
the flexural moment M_x;
the displacements u, v and w;
the angle of rotation ϑ_x.

Only 4 of these quantities are independent of each other. When any 4 of them are given, the rest are determined by the equations of equilibrium and compatibility. For each edge 4 equations of continuity may be established to determine the constants of integration. When the characteristic coefficients are known, these equations may be formed and solved numerically. However, it is advantageous to proceed in a different manner. The first two exponential functions in eq. (28) are found to represent waves damping out with increasing values of ξ, whereas the last two functions represent waves increasing with increasing values of ξ. The latter group of waves may be considered to originate from the opposite edge. They are in many cases of little importance at the edge considered and may at a preliminary stage be neglected. If necessary, they are taken into account afterwards by superposition. Hence, eq. (28) may be replaced by

$$\hat{H} = \frac{H}{[H]} = R\{C_1 \hat{H}_1 e^{\lambda_1 \xi} + C_2 \hat{H}_2 e^{\lambda_2 \xi}\}. \tag{33}$$

The reduced value of H at the edge $\xi = 0$ is then

$$\hat{H}_0 = R\{C_1 \hat{H}_1 + C_2 \hat{H}_2\}. \tag{34}$$

The quantities N_x, $N_{x\varphi}$, ϑ_x and w are now assumed to be given at the edge. The constants of integration may then be expressed by these edge quantities. The edge quantities mentioned are chosen because they give the most simple expressions. Eq. (34) yields

$$\begin{aligned} &R\{C_1 \hat{N}_{x1} + C_2 \hat{N}_{x2}\} = \hat{N}_{x0}, \qquad && R\{C_1 \hat{N}_{x\varphi 1} + C_2 \hat{N}_{x\varphi 2}\} = \hat{N}_{x\varphi 0}, \\ &R\{C_1 \hat{\vartheta}_{x1} + C_2 \hat{\vartheta}_{x2}\} = \hat{\vartheta}_{x0} && R\{C_1 \hat{w}_1 + C_2 \hat{w}_2\} = \hat{w}_0. \end{aligned} \tag{35}$$

Let

$$\begin{aligned} C_1 &= A_1 + i\,B_1, \\ C_2 &= A_2 + i\,B_2. \end{aligned} \tag{36}$$

When the expressions of table 1 are used, eqs. (35) are then transformed into the following equations

$$\begin{aligned}
&& B_1 && - B_2 &= \hat{N}_{x0}, \\
\beta_1 A_1 & -\alpha_1 B_1 & -\beta_2 A_2 & +\alpha_2 B_2 && = \hat{N}_{x\varphi 0}, \\
-\alpha_1 A_1 & -\beta_1 B_1 & -\alpha_2 A_2 & -\beta_2 B_2 && = \hat{\vartheta}_{x0}, \\
A_1 && + A_2 &&& = \hat{w}_0 .
\end{aligned} \tag{37}$$

or, in matrix notation

$$\begin{bmatrix} 0 & 1 & 0 & -1 \\ \beta_1 & -\alpha_1 & -\beta_2 & \alpha_2 \\ -\alpha_1 & -\beta_1 & -\alpha_2 & -\beta_2 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} \hat{N}_{x0} \\ \hat{N}_{x\varphi 0} \\ \hat{\vartheta}_{x0} \\ \hat{w}_0 \end{bmatrix}. \tag{37a}$$

The solution of eq. (37a) is easily found to be

$$\begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (\alpha_2+\beta_2) & 1 & -1 & -(\alpha_2-\beta_2) \\ -(\alpha_2-\beta_2) & -1 & -1 & -(\alpha_2+\beta_2) \\ -(\alpha_1-\beta_1) & -1 & 1 & (\alpha_1+\beta_1) \\ -(\alpha_1+\beta_1) & -1 & -1 & -(\alpha_1-\beta_1) \end{bmatrix} \begin{bmatrix} \hat{N}_{x0} \\ \hat{N}_{x\varphi 0} \\ \hat{\vartheta}_{x0} \\ \hat{w}_0 \end{bmatrix}. \tag{38}$$

When these constants of integration are used in eq. (33), the values of the reduced quantities

$$\hat{N}_x, \ \hat{N}_{x\varphi}, \ \hat{\vartheta}_x \text{ and } \hat{w}$$

are found at an arbitrary section expressed by the values of the same quantities at the edge. The expressions obtained are of the form

$$\begin{aligned}
\hat{N}_x &= s_{11}\hat{N}_{x0} + s_{12}\hat{N}_{x\varphi 0} + s_{13}\hat{\vartheta}_{x0} + s_{14}\hat{w}_0, \\
\hat{N}_{x\varphi} &= s_{21}\hat{N}_{x0} + s_{22}\hat{N}_{x\varphi 0} + s_{23}\hat{\vartheta}_{x0} + s_{24}\hat{w}_0, \\
\hat{\vartheta}_x &= s_{31}\hat{N}_{x0} + s_{32}\hat{N}_{x\varphi 0} + s_{33}\hat{\vartheta}_{x0} + s_{34}\hat{w}_0, \\
\hat{w} &= s_{41}\hat{N}_{x0} + s_{42}\hat{N}_{x\varphi 0} + s_{43}\hat{\vartheta}_{x0} + s_{44}\hat{w}_0
\end{aligned} \tag{39}$$

or, in matrix notation

$$S = [s_{ik}]\, S_0 \tag{39a}$$

where the column vector

$$S = \{\hat{N}_x, \ \hat{N}_{x\varphi}, \ \hat{\vartheta}_x, \ \hat{w}\}, \tag{40}$$

and S_0 is the value of S at the edge. The matrix $[s_{ik}]$, expressing the damping of the vector S, may be called a damping matrix. Proceeding in the manner described, one may find the elements of this matrix to be

$$\begin{bmatrix} s_{11} \\ s_{21} \\ s_{31} \\ s_{41} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -(\alpha_2-\beta_2) & (\alpha_2+\beta_2) & (\alpha_1+\beta_1) & (\alpha_1-\beta_1) \\ 2b & -2b & -2b & -2b \\ -2b & -2b & 2b & -2b \\ (\alpha_2+\beta_2) & (\alpha_2-\beta_2) & -(\alpha_1-\beta_1) & (\alpha_1+\beta_1) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}, \tag{41a}$$

$$\begin{bmatrix} s_{12} \\ s_{22} \\ s_{32} \\ s_{42} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & 1 \\ (\alpha_1+\beta_1) & -(\alpha_1-\beta_1) & -(\alpha_2-\beta_2) & -(\alpha_2+\beta_2) \\ -(\alpha_1-\beta_1) & -(\alpha_1+\beta_1) & (\alpha_2+\beta_2) & -(\alpha_2-\beta_2) \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}, \tag{41b}$$

$$\begin{bmatrix} s_{13} \\ s_{23} \\ s_{33} \\ s_{43} \end{bmatrix} = \begin{bmatrix} -s_{42} \\ -s_{32} \\ s_{22} \\ s_{12} \end{bmatrix}, \quad (41\text{c}) \qquad \begin{bmatrix} s_{14} \\ s_{24} \\ s_{34} \\ s_{44} \end{bmatrix} = \begin{bmatrix} -s_{41} \\ -s_{31} \\ s_{21} \\ s_{11} \end{bmatrix}, \quad (41\text{d})$$

where

$$\begin{aligned} f_1 &= e^{-\alpha_1 c \xi} \cos \beta_1 c \xi, & f_3 &= e^{-\alpha_2 c \xi} \cos \beta_2 c \xi, \\ f_2 &= e^{-\alpha_1 c \xi} \sin \beta_1 c \xi, & f_4 &= e^{-\alpha_2 c \xi} \sin \beta_2 c \xi. \end{aligned} \tag{42}$$

Eqs. (41a)—(41d) show that only 6 of the 16 matrix elements, namely s_{11}, s_{41}, s_{12}, s_{22}, s_{32} and s_{42} need to be computed as a sum of 4 damped waves. Then

$$s_{21} = -2b\, s_{12}, \qquad s_{31} = -2b\, s_{42} \tag{43}$$

and the remaining elements are numerically equal to those already calculated.

If the distribution of S is wanted with intervals ξ_1 along the shell, the matrix $[s_{ik}]$ needs to be calculated for one interval ξ_1 only. Then, at $\xi = \xi_1$

$$S_1 = [s_{ik}]\, S_0. \tag{44}$$

The arc at $\xi = \xi_1$ may now be considered a new edge. Hence, at $\xi = 2\xi_1$

$$S_2 = [s_{ik}]\, S_1 \tag{45}$$

and so on. This method corresponds to one proposed by Zunz [9] for calculating edge disturbances from the straight edge.

The value of S at an arbitrary arc being known, the other quantities may be calculated from S by relations independent of ξ. For $\xi = 0$ eq. (34) may be used to express the remaining quantities by S_0. As the origin of ξ is chosen arbitrarily, the relations obtained are valid not at $\xi = 0$ only, but at an arbitrary arc. The resulting expressions may be written

$$\begin{bmatrix} \hat{u} \\ \hat{v} \\ -\hat{M}_x \\ \hat{R}_x \end{bmatrix} \begin{bmatrix} -(\alpha_1+\alpha_2)b & -(1-\nu)b & 0 & (\beta_1-\beta_2)b \\ -(1-\nu)b & -\frac{1}{2}(\alpha_1+\alpha_2) & \frac{1}{2}(\beta_1-\beta_2) & 1 \\ 0 & \frac{1}{2}(\beta_1-\beta_2) & \frac{1}{2}(\alpha_1+\alpha_2) & (1+\nu)b \\ (\beta_1-\beta_2)b & 1 & (1+\nu)b & (\alpha_1+\alpha_2)b \end{bmatrix} \begin{bmatrix} \hat{N}_x \\ \hat{N}_{x\varphi} \\ \hat{\vartheta}_x \\ \hat{w} \end{bmatrix}, \tag{46}$$

$$\begin{bmatrix} \hat{N}_\varphi \\ \hat{M}_\varphi \\ \hat{R}_\varphi \\ \hat{\vartheta}_\varphi \\ \hat{Q}_x \\ \hat{Q}_\varphi \\ \hat{M}_{x\varphi} \end{bmatrix} = \begin{bmatrix} b & \frac{1}{2}(\alpha_1+\alpha_2) & -\frac{1}{2}(\beta_1-\beta_2) & 0 \\ 0 & -\frac{\nu}{2}(\beta_1-\beta_2) & -\frac{\nu}{2}(\alpha_1+\alpha_2) & -(1+\nu)\,b \\ 0 & -\left(1-\frac{\nu}{2}\right)(\beta_1-\beta_2) & -\left(1-\frac{\nu}{2}\right)(\alpha_1+\alpha_2) & -4\left(1-\frac{\nu}{2}\right)b \\ 0 & 0 & 0 & 1 \\ (\beta_1-\beta_2)\,b & 1 & 2b & (\alpha_1+\alpha_2)\,b \\ 0 & -\frac{1}{2}(\beta_1-\beta_2) & -\frac{1}{2}(\alpha_1+\alpha_2) & -2b \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{N}_x \\ \hat{N}_{x\varphi} \\ \hat{\vartheta}_x \\ \hat{w} \end{bmatrix}. \tag{47}$$

In abbreviated notation eq. (46) may be written

$$T = [r_{ik}]\,S, \tag{46a}$$

where T is the column vector

$$T = \{\hat{u},\ \hat{v},\ -\hat{M}_x,\ \hat{R}_x\} \tag{48}$$

and $[r_{ik}]$ is the matrix in eq. (46). The vectors S and T contain the 8 quantities which may occur in the edge conditions. Eq. (46) shows that the matrix $[r_{ik}]$ is symmetrical. This is a consequence of the theorem of reciprocal work.

In the case of interaction between two edges, the relation between the vector T at an arbitrary generator and the vector $S = S_0$ at the edge is needed. This relation is found from eqs. (39a) and (46a) to be

$$T = [r_{ik}]\,[s_{ik}]\,S_0 = [t_{ik}]\,S_0, \tag{49}$$

where the new notation

$$t_{ik} = [r_{ik}]\,[s_{ik}] \tag{50}$$

is introduced.

The proceeding of a shell design if one edge only is taken into account, will then be as follows:

1. The actual edge conditions are used to form 4 equations between the edge quantities S_0 and T_0.
2. The matrix $[r_{ik}]$ (see eqs. (46) and (46a) is evaluated numerically, and the quantities T_0 in the equations are expressed by S_0.
3. The resulting 4 equations with 4 unknowns are solved, giving the quantities S_0. Most frequently the equations are easily solved by iteration.
4. The damping matrix $[s_{ik}]$ (see eqs. (39)—(41) is evaluated for a chosen interval ξ_1, and the vector S is calculated with intervals ξ_1 as far away from the edge as may be found necessary.
5. The remaining quantities wanted are found from S by using eqs. (46) and (47).

If edge disturbances from two edges interfere, the total influence is found by superimposing the influences from both edges. Then 8 equations with 8 unknowns result. In this case too, the equations are often easily solved by iteration.

Notations

R	shell radius
h	shell thickness
$x, y, \xi = x/R, \varphi = y/R$	coordinates
M_x, M_φ	bending moments
$M_{x\varphi}, M_{\varphi x}$	torsional moments
Q_x, Q_φ	transverse forces
R_x, R_φ	resulting transverse edge forces
N_x, N_φ	normal forces
$N_{x\varphi}, N_{\varphi x}$	shear forces
$\epsilon_x, \epsilon_\varphi, \gamma_{x\varphi}$	unit deformations
u, v, w	displacements
$\vartheta_x, \vartheta_\varphi$	angles of rotation
E	modulus of elasticity
ν	Poisson's ratio
$K = \frac{E h^3}{12(1-\nu^2)}$	flexural rigidity
Φ	stress function
$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$	Laplace's operator
m	arbitrary number
λ	root of the characteristic equatio
α, β	real and imaginary parts of λ
$\epsilon = 2\left(\frac{m}{c}\right)^2$	dimensionless constant
$a = \mathrm{Ar\,Sin}\,\epsilon$	dimensionless constant
$b = \frac{\epsilon}{4} = \frac{1}{2}\left(\frac{m}{c}\right)^2$	dimensionless constant
$c = \sqrt{\frac{R}{h}\sqrt[4]{3(1-\nu^2)}}$	dimensionless constant
H	arbitrary statical quantity
$[H]$	multiplier of H
$\hat{H}$	reduced value of H
H_0	edge value of H
A, B, C	constants of integration
f_1, f_2, f_3, f_4	damped trigonometric functions (see eq. (42))
$S = \{\hat{N}_x, \hat{N}_{x\varphi}, \hat{\vartheta}_x, \hat{w}\}$	column vector of statical quantities
$T = \{\hat{u}, \hat{v}, -\hat{M}_x, \hat{R}_x\}$	column vector of statical quantities
$[r_{ik}]$	matrix defined by $T = [r_{ik}]\, S$
$[s_{ik}]$	matrix defined by $S = [s_{ik}]\, S_0$
$[t_{ik}]$	matrix defined by $T = [t_{ik}]\, S_0$

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Summary

The relations between forces and displacements in the DONNELL theory are given, and the differential equations of the shell are deduced. This theory is used for a Fourier analysis of edge disturbances from the curved edge. By a further analytical treatment of the solution all statical quantities are expressed in closed form by the edge quantities at one disturbing edge. Edge disturbances from two edges may be treated by superposition. Matrix notation is used to simplify the formulas.

Résumé

A partir des relations indiquées dans la théorie de Donnell entre les efforts et les déformations, l'auteur établit les équations différentielles du voile. Cette théorie est appliquée à l'analyse par la méthode de Fourier des perturbations qui se produisent sur le bord incurvé. Une autre transformation analytique de la solution permet d'exprimer toutes les grandeurs statiques sous forme finie, par l'intermédiaire des valeurs marginales d'un bord perturbé. Les perturbations marginales de deux bords peuvent être traitées par superposition. L'écriture matricielle est employée pour simplifier les formules.

Zusammenfassung

Aus den angegebenen Beziehungen zwischen Kräften und Deformationen der Donnellschen Theorie werden die Differentialgleichungen der Schale abgeleitet. Diese Theorie wird für eine Fourier-Analyse von Störungen am gekrümmten Rande verwendet. Durch eine weitere analytische Umformung der Lösung werden alle statischen Größen in geschlossener Form durch die Randwerte des einen gestörten Randes ausgedrückt. Randstörungen von beiden Rändern können durch Superposition behandelt werden. Die Matrizen-Schreibweise wird zur Vereinfachung der Formeln angewendet.