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Secondary Moments, End Rotations, Inflection Points and Elastic Buckling Loads of Truss Members

Moments secondaires, angles de rotation, points d'inflexion et charges de flambage des barres de treillis

Sekundäre Biegespannungen, Stabdrehwinkel, Wendepunkte und Knicklast von Fachwerkstäben

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Introduction

The buckling loads of members in a truss with rigid joints are not as easily predictable as those of members in a truss with frictionless pin joints. For the latter case, each compression member is a simple supported column, for which the critical buckling load can easily be found and, any combination of loading, which will cause the stress in any of the compression members to reach its critical value, will cause failure of the whole truss if statically determinate. For a rigid jointed truss, because of the secondary moments involved, the compression members will not act independently. The buckling loads of the compression members are not reached until the combination of loading on the truss causes instability of the whole truss. In this study, the word "truss" is to be understood as referring to a statically determinate truss with rigid joints.

The truss buckling problem has been studied by various authors among them JAMES [1][1]), LUNDQUIST [2], HOFF [3,4], NILES and NEWELL [5], WESSMAN and KAVANAGH [6]. The basic equations involved can be traced back to MANDERLA's paper in 1880 [7,8]. MANDERLA studies the problem of secondary moments in truss members taking into consideration the effects of axial forces in members. The relationship between the truss buckling problem and the secondary moment problem, however, has not been fully explored. The purposes of this study are (I) to clarify conceptions related to

[1]) Referring to the number of the reference listed at the end of the paper.

secondary moments and the buckling phenomenon and (II) to study the variation of such moments and the physical behavior of truss members under increasing load up to the point of buckling.

The loads considered in this study are fixed in both direction and in position. A group of such loads acting on a truss can be represented by their magnitudes, $P_0, P_1, P_2 \ldots$ etc. Take the magnitude of any load, say P_0, as a reference quantity. This reference load P_0 shall be referred to as "the load" on the truss considered. An "increase of the load" P_0 from a certain initial value P_{0i} to $\lambda P_{0i} (\lambda > 1)$ shall be taken to mean the multiplication of the initial values of all the loads simultaneously by the same λ.

The study is based on the following basic assumptions: (I) the truss is perfectly elastic and (II) the deflections of the members are small (III) the members are initially perfectly straight and without end eccentricities. The implications of these assumptions are clarified in the following discussions.

The first assumption implies that plastic buckling is not considered. Although inelastic buckling has been considered by several authors [5, 6]. no satisfactory solution has been found. Previous studies are generally based on reduced moduli of elasticity. Such studies are valid only if the compression member is straight up to the point of buckling. This is not true for truss members, as they are bent by the secondary moments. The problem has also been approached from the point of view that the buckling load of a member can be taken as the load causing yielding stress at any point of the member. If yielding starts at one end or at both ends, such yielding may cause local buckling and plastic hinges to form. But it may not cause either the general buckling of the member or the general buckling of the truss. Therefore, to give a realistic picture, the member has to be considered as partly elastic and partly plastic. As a satisfactory solution has not been found for this case, the present study is limited to the scope of perfect elasticity.

The second assumption involves the governing differential equations which expresses the relationship between bending moment and deflection. Let us examine the following expressions:

$$\frac{M}{EI} = -\frac{d^2 y}{dx^2}, \tag{1}$$

$$\frac{M}{EI} = -\frac{d^2 y}{dx^2} \Big/ \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}, \tag{1a}$$

where M = bending moment about a principal axis of a member, I = moment of inertia about the same axis, E = modulus of elasticity, y = deflection, x = length measured along the longitudinal axis of the member from some reference point on the axis. The more exact relationship is given by eq. (1a). But eq. (1) is a good approximation if the deflection is small. The second assumption implies that the eq. (1), not eq. (1a) is used for the analysis.

Theoretically, eq. (1) is invalid for large deflections and it will give a less perfect prediction of behavior of an elastic structure than eq. (1a). However, it can be shown that eq. (1) is sufficient for all practical purposes. For example, consider a prismatic bar of length L with simply supported ends and subjected to an axial compressive force P. The critical load or the Euler load of this bar is $P_e = \frac{\pi^2 EI}{L^2}$. Now investigate the case when the bar is given a small lateral displacement Δ at center of span and is subjected to an axial force $P > P_e$. Based on eq. (1), the resisting moment $\left(EI\frac{d^2y}{dx^2}\right)$ will be smaller than the bending moment $P\Delta$. This means that the deflection would increase indefinitely. Using eq. (1a), however, it can be shown that the bar will be stable in a bent form. Nevertheless, when P is only 1.5% greater than P_e, the bar has a deflection of $0.11\,L$ at center span and a 20° slope at the ends [9]. No practical structure can remain useful under such deflection. Therefore, although eq. (1a) may give theoretically correct results, it is considered as an unnecessary refinement, and this study is based on eq. (1).

An estimation of the slope at ends of the members and an investigation of the effect of the differences between arc length and chord length of the members have been made in the course of this study.

The third assumption that the members are initially straight and without end eccentricities also limits the scope of the present work.

Basic Equations

The basic equations relating secondary moments, axial load, deflection curve, and end rotations, based on MANDERLA's solution, follow [7, 8]. These equations are more or less well known. They are given in order to introduce the concept, as well as the sign conventions and notations.

1. For Compression Members

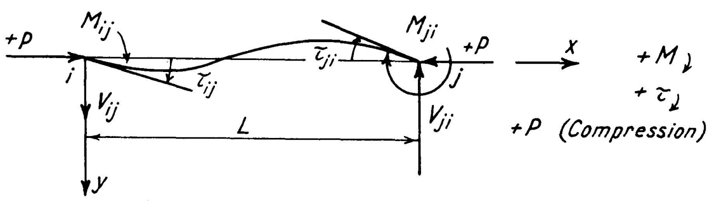


Fig. 1,

Referring to fig. 1,

$$M = M_{ij} + Py - V_{ij}\,x, \tag{2}$$

$$EI\frac{d^2y}{dx^2} = -M = -M_{ij} - Py + V_{ij}\,x. \tag{3}$$

Let

$$\phi = \frac{L}{2}\sqrt{\frac{P}{EI}} \tag{4}$$

and
$$p = \frac{P}{A} = \frac{P}{I/r^2}, \tag{5}$$
where A = cross-sectional area and r = radius of gyration of the member.

Let
$$P_e = p_e A = \frac{\pi^2 E I}{L^2} = \frac{\pi^2 E A}{(L/r)^2} \quad \text{(Euler Load)}, \tag{6}$$
then
$$\phi = \frac{\pi}{2}\sqrt{\frac{P}{P_e}}. \tag{7}$$
The solution of (3) with the boundary conditions $y=0$ when $x=0$ and $x=L$, is
$$y = \left(\frac{M_{ij}}{P}\tan\phi - \frac{V_{ij} L}{P \sin 2\phi}\right)\sin\frac{2\phi x}{L} + \frac{M_{ij}}{P}\cos\frac{2\phi x}{L} - \frac{M_{ij}}{P} + \frac{V_{ij} x}{P}. \tag{8}$$
Since $V_{ij} = \frac{M_{ij}+M_{ji}}{L}$, eq. (8) can be reduced to the following form:
$$y = \frac{M_{ij}}{P}\,\frac{\sin 2\phi\left(1-\frac{x}{L}\right)}{\sin 2\phi} - \frac{M_{ji}}{P}\,\frac{\sin 2\phi \frac{x}{L}}{\sin 2\phi} - \frac{M_{ij}}{P}\left(1-\frac{x}{L}\right) + \frac{M_{ji} x}{PL}. \tag{9}$$
Since
$$\left.\frac{dy}{dx}\right]_{x=0} = \tau_{ij}, \qquad \left.\frac{dy}{dx}\right]_{x=L} = \tau_{ji} \tag{10}$$
the following expression can be derived
$$M_{ij} = \frac{2EI}{L}(2a_c \tau_{ij} + b_c \tau_{ji}), \tag{11}$$
where
$$a_c = \frac{\phi}{4}\left(\frac{\phi}{1-\phi\cot\phi} + \cot\phi\right) \tag{12a}$$
and
$$b_c = \frac{\phi}{2}\left(\frac{\phi}{1-\phi\cot\phi} - \cot\phi\right). \tag{12b}$$
The above expressions are to be used in this paper. But for small values of ϕ, the following expressions may be useful:
$$a_c = 1 - \frac{(2\phi)^2}{30} - \frac{11\,(2\phi)^4}{25\,000} - \cdots \tag{12c}$$
and
$$b_c = 1 + \frac{(2\phi)^2}{60} + \frac{13\,(2\phi)^4}{25\,000} + \cdots \tag{12d}$$
$$\text{limit } \phi = 0, \qquad a_c = b_c = 1. \tag{12e}$$

2. For Tension Members

Changing P to $-P$ in eq. (2) and (3), the following solutions can be found
$$y = -\frac{M_{ij}}{|P|}\,\frac{\sinh 2\phi\left(1-\frac{x}{L}\right)}{\sinh 2\phi} + \frac{M_{ji}}{|P|}\,\frac{\sinh\frac{2\phi x}{L}}{\sinh 2\phi} + \frac{M_{ij}}{|P|}\left(1-\frac{x}{L}\right) - \frac{M_{ji}}{|P|}\,\frac{x}{L}, \tag{13}$$
where
$$\phi = \frac{L}{2}\sqrt{\frac{|P|}{EI}}, \tag{14}$$
and $|P|$ = absolute value of P, since it is always taken as positive.

$$M_{ij} = \frac{2EI}{L}(2a_t\tau_{ij} + b_t\tau_{ji}), \tag{15}$$

where
$$a_t = \frac{\phi}{4}\left(\frac{\phi}{\phi\coth\phi - 1} + \coth\phi\right), \tag{16a}$$

and
$$b_t = \frac{\phi}{2}\left(\frac{\phi}{\phi\coth\phi - 1} - \coth\phi\right). \tag{16b}$$

The following expressions may be useful for small values of ϕ

$$a_t = 1 + \frac{(2\phi)^2}{30} - \frac{11(2\phi)^4}{25000} + \cdots \tag{16c}$$

and
$$b_t = 1 - \frac{(2\phi)^2}{60} + \frac{13(2\phi)^4}{25000} - \cdots \tag{16d}$$

$$\text{limit } \phi = 0, \qquad a_t = b_t = 1. \tag{16e}$$

3. Members with End Displacements

Let one end of the member be displaced with respect to the other end by an amount Δ in a direction perpendicular to the length. Fig. 2 shows such a member with resulting rotation $R = \Delta/L$ in a positive direction.

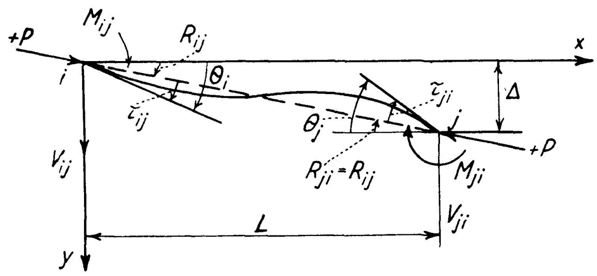


Fig. 2,

Then
$$\tau_{ij} = \theta_i - R_{ij}, \qquad \tau_{ji} = \theta_j - R_{ij} \tag{17a, b}$$

The resulting equations for the end moment M_{ij} are:

For compression members

$$\begin{aligned} M_{ij} &= \frac{2EI}{L}[2a_c(\theta_i - R_{ij}) + b_c(\theta_j - R_{ij})], \\ &= \frac{2EI}{L}[2a_c\theta_i + b_c\theta_j - (2a_c + b_c)R_{ij}], \\ &= \frac{2EI}{L}[2a_c\theta_i + b_c\theta_j - c_c R_{ij}]. \end{aligned} \tag{18a}$$

For tension members

$$M_{ij} = \frac{2EI}{L}[2a_t\theta_i + b_t\theta_j - c_t R_{ij}], \tag{18b}$$

where
$$c_c = 2a_c + b_c, \qquad c_t = 2a_t + b_t. \tag{19a, b}$$

It should be noted that eqs. (18a) and (18b) correspond exactly to ordinary slope deflection equations for the case $\phi = 0$. To find the secondary moments in a given truss, we may first write down eqs. (18a, b) for every member. The R values are known, as they can be found by drawing a Williot Diagram or by other means (see later example). Since the joints are rigid, each joint has only one rotation angle θ. We may then set $\sum M = 0$ at each joint of the truss. For a truss of n joints, this results in n equations with n values of θ as unknowns. The equations are of the following form:

$$\begin{aligned} g_{11}\theta_1 + g_{12}\theta_2 + g_{13}\theta_3 + \ldots &= h_1, \\ g_{21}\theta_1 + g_{22}\theta_2 + g_{23}\theta_3 + \ldots &= h_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{20a}$$

or

$$\sum g_{ij}\theta_j = h_i, \tag{20b}$$

where

$ij = 1, 2, 3 \ldots$ etc.,
g_{ij} are numerical coefficients of θ_j,
h_i are constants equal to $\sum c R$ at joint i where $c = c_c$ or c_t.

In matrix notation, eq. (20b) becomes

$$G\theta = H. \tag{20c}$$

From Maxwell's reciprocal theorem, the following relationship exists between the g values

$$g_{ij} = g_{ji} \quad \text{when} \quad i \neq j. \tag{21}$$

The secondary moments are found by solving for the θ values from eq. (20) and substituting these back into eq. (18) for the moments.

It should be noted that although R values are proportional to the external loading (if we neglect the differences between arc length and chord length of members), the values a_c, b_c, c_c, a_t, b_t, c_t are not directly proportional to the loading (i.e., the relationship between loading and these coefficients is not linear). Thus the values of the g's and h's are not directly proportional to loading. Therefore, there is no linear relationship between the loading and the secondary moments.

The Condition of Instability and Stiffness Coefficients

Since matrix H will not be equal to zero, the set of simultaneous eqs. (20) will have a unique set of solutions for θ if the determinant of the coefficients g_{ij} of θ is not equal to zero. If the determinant is equal to zero, then all θ's become infinity and the whole system becomes unstable. Thus the condition of instability (or the buckling condition) is the following familiar equation in elastic stability:

$$\text{Determinant} \quad G = |g_{ij}| = 0. \tag{22}$$

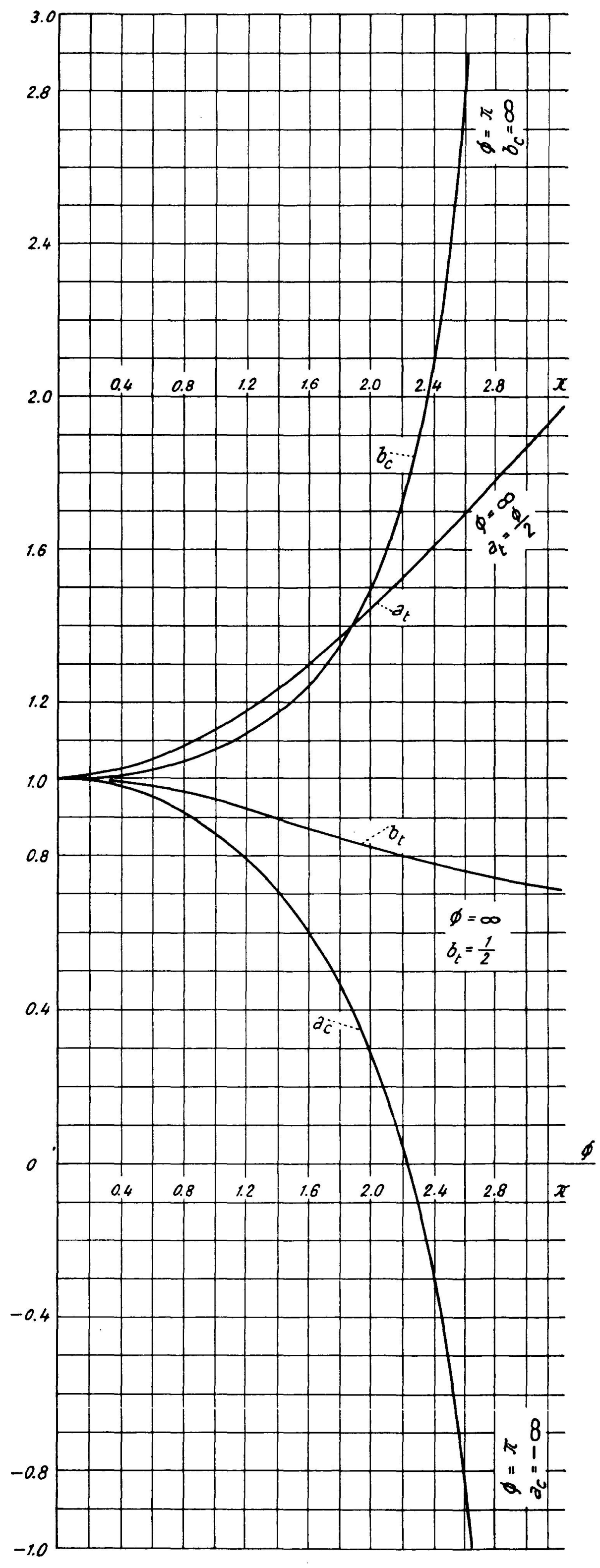


Fig. 3. Stiffness Coefficients a_c, b_c, a_t and b_t.

The value of g_{ij} depends on the coefficients a_c, b_c, and a_t, b_t which are referred to as the stiffness coefficients. These coefficients can be computed from formulas (12) and (16) or from given tables or diagrams [2, 5]. These tables and diagrams are not entirely satisfactory, however. The coefficients are given as stiffness and carry over factors. Due to the fact that at $\phi \approx 2.25$, $a_c = 0$, $b_c \neq 0$, the carry over factor b_c/a_c becomes infinite at this point. Thus computations making use of these published coefficients would not give accurate results in the neighborhood of $\phi = 2.25$.

Due to the above considerations, the writer prefers the direct use of the coefficients a_c, b_c, a_t and b_t. The values of these coefficients have been plotted in the diagram as shown in fig. 3. This diagram, besides giving a general picture of the variation of these coefficients, could be read to 2 decimal points and is found to be sufficiently accurate for slide rule computations.

Noting that since $\phi = \frac{L}{2}\sqrt{\frac{|P|}{EI}}$, ϕ increases as P increases. From fig. 3 the following observations can be made:

I. As the axial compression P increases, the stiffness coefficient a_c, decreases continuously from $+1$ at $\phi = 0$, becomes negative and equal to $-\infty$ when $\phi = \pi$. Meanwhile, the stiffness coefficient b_c increases continuously from $+1$ at $\phi = 0$ and becomes $+\infty$ when $\phi = \pi$.

II. As the axial tension P increases, the stiffness coefficient a_t increases continuously from $+1$ at $\phi = 0$ and approaches $\phi/2$ for large values of ϕ. Meanwhile the stiffness coefficient b_t decreases continuously from $+1$ at $\phi = 0$ and approaches the value $\frac{1}{2}$ asymptotically as ϕ approaches infinity.

III. The meaning of the stiffness coefficients, a_c, b_c, a_t and b_t can be best understood by the following facts: For a compression member ij, the moment at one end, say i, due to angular rotation θ_i at the *same* end is equal to $\frac{4EI}{L} a_c \theta_i$; meanwhile, the moment at the end i, due to angular rotation θ_j at the *other* end is equal to $\frac{2EI}{L} b_c \theta_j$. If the member ij is in tension instead of compression, we use a_t instead of a_c and b_t instead of b_c.

IV. It should be noted that no compression member in a truss could carry an axial load more than $P = 4P_e$ (4 times the Euler buckling load of a simply supported column) which is the buckling load of a column with fully fixed ends. Thus for any compression member in a truss, $\phi = \frac{\pi}{2}\sqrt{\frac{P}{P_e}}$ could never exceed π. (It is noted that at $\phi = \pi$, $a_c = -\infty$ and $b_c = +\infty$).

An Illustrative Example

The above discussion of fundamental concepts may be familiar to those who have been interested in stability problems. However, the main interest of the writer is to demonstrate by an illustrative example, the relationship between secondary moments and buckling load, and the variation of these moments as well as end rotations and inflection points in truss members under increasing load. The example is chosen so that it requires rather simple computations, yet it involves two members in compression and two members in tension, each behaving differently.

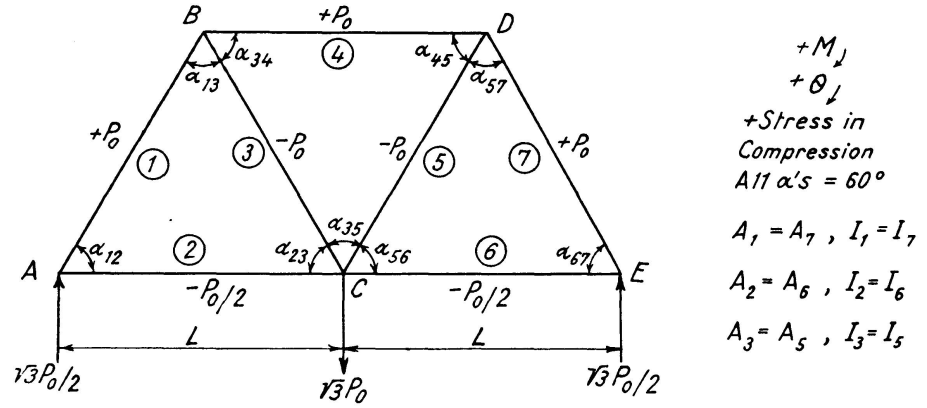


Fig. 4.

The truss considered is shown in fig. 4. The members are designated by end letters and also by numbers. Thus member AB is also designated by (1), its cross sectional area is A_1 and its moment of inertia I_1. Since the truss is symmetrical, we have the following correspondence between members: (1) and (7), (2) and (6), (3) and (5). The corresponding members have the same area and the same moment of inertia. The interior angles between members are designated by the members involved, for example α_{12} is the angle between the member (1) and (2). The loading and stresses in members are shown in the diagram with compressive stress as positive.

Table 1 gives the general equations involved. The subscripts 1, 2 . . . etc. of the coefficients a_{c1}, b_{t2} etc. designate the members (1), (2) . . . respectively. The equations are general in a sense that they can be applied to the general case of $A_1 \neq A_2 \neq A_3 \neq A_4$ and $I_1 \neq I_2 \neq I_3 \neq I_4$.

Now let us first study the special case of $A_1 = A_2 = A_3 = A_4$, $I_1 = I_2 = I_3 = I_4$, which is referred to as case I. The member proportions are not realistic. But this case is chosen for an exhaustive study because the two compression members involved, as will be shown later, are initially bent into different elastic curves.

Table 2 shows the computation of R's for case I. These values are computed in the following way. First compute the change of an angle α_{ij}, designated by $\delta\,\alpha_{ij}$, by the following formula. (The *sign* of the value of $\delta\,\alpha_{ij}$ is *positive* for an *increase* in α_{ij}).

$$\delta\,\alpha_{ij} = (S_i - S_k)\cot\alpha_{ik} + (S_j - S_k)\cot\alpha_{jk}, \qquad (23)$$

where members i, j, k are 3 sides of a triangle; α_{ij}, α_{ik}, α_{jk} are interior angles of the triangle between the sides i and j, i and k, j and k respectively. S_i, S_j,

Table 1. General Equations

By Symmetry $\theta_c = 0, \quad \theta_b = -\theta_d$
$M_{ab} = (2a_{c1}\theta_a + b_{c1}\theta_b - c_{c1}R_1)\frac{2EI_1}{L}$ $M_{ac} = (2a_{t2}\theta_a + 0 - c_{t2}R_2)\frac{2EI_2}{L}$
$M_{ba} = (2a_{c1}\theta_b + b_{c1}\theta_a - c_{c1}R_1)\frac{2EI_1}{L}$ $M_{bc} = (2a_{t3}\theta_b + 0 - c_{t3}R_3)\frac{2EI_3}{L}$ $M_{bd} = (2a_{c4} - b_{c4})\theta_b\frac{2EI_4}{L}$
$M_{cb} = (b_{t3}\theta_b - c_{t3}R_3)\frac{2EI_3}{L}$ $M_{ca} = (b_{t2}\theta_a - c_{t2}R_2)\frac{2EI_2}{L}$
$2\left[a_{c1} + a_{t2}\frac{I_2}{I_1}\right]\theta_a + b_{c1}\theta_b - \left[c_{c1}R_1 + c_{t2}R_2\frac{I_2}{I_1}\right] = \frac{L}{2EI_1}(M_{ab} + M_{ac}) = 0$ $b_{c1}\theta_a + \left[2a_{c1} + 2a_{c4}\frac{I_4}{I_1} + 2a_{t3}\frac{I_3}{I_1} - b_{c4}\frac{I_4}{I_1}\right]\theta_b - \left[c_{c1}R_1 + c_{t3}R_3\frac{I_3}{I_1}\right]$ $= \frac{L}{2EI_1}(M_{ba} + M_{bc} + M_{bd}) = 0$

Table 2. Computations for R's for Case I

$A_1 = A_2 = A_3 = A_4 = A, \quad Q = \frac{P_0}{2\sqrt{3}AE}$					
$\delta\alpha$ in terms of Q		R in terms of Q			
$\delta\alpha_{34} = \delta\alpha_{45}$	4	R_4 by Symmetry	0	$R_5 = R_3 + \delta\alpha_{35}$	-4
$\delta\alpha_{13} = \delta\alpha_{57}$	2	$R_3 = R_4 + \delta\alpha_{34}$	4	$R_6 = R_5 + \delta\alpha_{56}$	-11
$\delta\alpha_{12} = \delta\alpha_{67}$	5	$R_1 = R_3 + \delta\alpha_{13}$	6	$R_7 = R_6 + \delta\alpha_{67}$	-6
$\delta\alpha_{23} = \delta\alpha_{56}$	-7	$R_2 = R_1 + \delta\alpha_{12}$	11	$R_5 = R_7 + \delta\alpha_{57}$	-4
$\delta\alpha_{35}$	-8	$R_3 = R_2 + \delta\alpha_{23}$	4	$R_4 = R_5 + \delta\alpha_{45}$	0

S_k, are *unit* stresses in the members i, j, k respectively. Thus for angle α_{34} opposite to side (5).

$$\delta\alpha_{34} = \left[\frac{-P_0}{A_3} - \left(-\frac{P_0}{A_5}\right)\right]\frac{1}{\sqrt{3}E} + \left[\frac{P_0}{A_4} - \left(\frac{-P_0}{A_5}\right)\right]\frac{1}{\sqrt{3}E},$$

$$= 2\frac{P_0}{\sqrt{3}AE} = \frac{4P_0}{2\sqrt{3}AE} \quad (\text{for } A = A_3 = A_4 = A_5).$$

The R values are determined by assuming $R=0$ for one member. For any other member, R is determined from summation of $\delta\alpha$'s to reach that member. Referring to fig. 4, due to symmetry, obviously R of member (4) is zero. Thus we have

$$\begin{aligned} R_4 &= 0 \\ R_3 &= R_4 \pm \delta\alpha_{34} \\ R_1 &= R_3 \pm \delta\alpha_{13} \\ &\ldots\ldots\ldots\ldots \end{aligned} \tag{24}$$

Note that in the above equations, the $+$ sign should be used if an increase in α (positive $\delta\alpha$) results in clockwise rotation (positive R) of the member.

Table 3. Equations for Case I

$I_1=I_2=I_3=I_4=I, \quad a_{c1}=a_{c4}, \quad b_{c1}=b_{c4}, \quad c_{c1}=c_{c4}$ $\phi=\phi_1=\phi_3=\phi_4=\frac{L}{2}\sqrt{\frac{P_0}{EI}}, \quad \phi_2=\frac{L}{2}\sqrt{\frac{P_0}{2EI}}=\frac{\phi}{\sqrt{2}}$
$M_{ab}=(2a_{c1}\theta_a+b_{c1}\theta_b-6a_{c1}Q)\frac{2EI}{L}$ $M_{ac}=(2a_{t2}\theta_a-11c_{t2}Q)\frac{2EI}{L}$
$M_{ba}=(2a_{c1}\theta_b+b_{c1}\theta_a-6c_{c1}Q)\frac{2EI}{L}$ $M_{bc}=(2a_{t3}\theta_b-4c_{t3}Q)\frac{2EI}{L}, \quad M_{bd}=(2a_{c1}-b_{c1})\theta_b\frac{2EI}{L}$
$M_{cb}=(b_{t3}\theta_a-11c_{t3}Q)\frac{2EI}{L}, \quad M_{ca}=(b_{t2}\theta_a-11c_{t2}Q)\frac{2EI}{L}$
$2(a_{c1}+a_{t2})\theta_a+b_{c1}\theta_b=(6c_{c1}+11c_{t2})Q$ $b_{c1}\theta_a+(4a_{c1}+2a_{t3}-b_{c1})\theta_b=(6c_{c1}+4c_{t3})Q$

Table 3 shows the equations for case I, which can be obtained directly from table 1. Table 4 gives the coefficients for equations of case I and table 5 gives θ and M values as the results of the solution.

The above computations including those for obtaining the coefficient a_c, b_c, etc. were made on a desk calculator. With coefficients obtained from fig. 3, solutions were carried out by slide rule. It was found that for this particular problem, the slide rule solution was sufficiently accurate. The errors become large as ϕ becomes larger. But the maximum error for the case $\phi=2.2$ is only 5%.

Table 6 shows the computation of R values for case II, III, and IV. For all these cases $A_1=A_3=A_4=A$ and $A_2=A/2$. As can be seen, the proportions

Table 4. Coefficients for Equations of Case I

ϕ	a_{c1}	b_{c1}	$c_{c1}=2a_{c1}+b_{c1}$	a_{t2}	b_{t2}	$c_{t2}=2a_{t2}+b_{t2}$	a_{t3}	b_{t3}	$c_{t3}=2a_{t3}+b_{t3}$	$g_{ia}\theta_a+g_{ib}\theta_b=h_iQ$		
0	1	1	3	1	1	3	1	1	3	4 1	1 5	51 30
$\pi/4$	0.9147 $\cot\phi=1$	1.0445	2.8739	1.0403 $\phi_2=0.5554$, $\coth\phi_2=1.9820$	0.9797	3.0603	1.0792 $\coth\phi=1.5249$	0.9613	3.1197	3.9100 1.0445	1.0445 4.7727	50.9067 29.7222
$\pi/2$	0.6168 $\cot\phi=0$	1.2336	2.4672	1.1547 $\phi_2=1.1109$, $\coth\phi_2=1.2432$	0.9285	3.2379	1.2936 $\coth\phi=1.0904$	0.8744	3.4616	3.5430 1.2336	1.2336 3.8208	50.4231 28.6526
1.80	0.4655 $\cot\phi=-0.2333$	1.3509	2.2819	1.1998 $\phi_2=1.2730$, $\coth\phi_2=1.1701$	0.9105	3.3101	1.3742 $\coth\phi=1.0562$	0.8473	3.5957	3.3306 1.3509	1.3509 3.2595	50.1025 28.0742
2.10	0.1892 $\cot\phi=-0.5848$	1.6064	1.9848	1.2651 $\phi_2=1.4849$, $\coth\phi_2=1.1082$	0.8848	3.4150	1.4883 $\coth\phi=1.0304$	0.8070	3.7893	2.9086 1.6064	1.6064 2.1270	49.4738 27.0660
2.20	0.0648 $\cot\phi=-0.7279$	1.7310	1.8606	1.2886 $\phi_2=1.5559$, $\coth\phi_2=1.0933$	0.8758	3.4530	1.5280 $\coth\phi=1.0249$	0.8012	3.8572	2.7068 1.7310	1.7310 1.5842	49.1466 26.5964
2.251	-0.0062 $\cot\phi=-0.8089$	1.8086	—	1.3009 $\phi_2=1.5918$, $\coth\phi_2=1.0864$	0.8725	—	1.5487 $\coth\phi=1.0225$	0.7958	—	2.5894 1.8086	1.8086 1.2640	

Table 5. θ and M Values for Case I

ϕ $\phi=\frac{\pi}{2}\sqrt{\frac{P_0}{P_e}}$	P_0 in terms of $P_e=\frac{\pi^2EI}{L^2}$	θ in terms of $Q=\frac{P_0}{2\sqrt{3}AE}$ θ_a	 θ_b	Moments in terms of $\frac{2EI}{L}Q=\frac{P_0I}{\sqrt{3}AL}=\frac{P_0r^2}{\sqrt{3}L}$ M_{ab}	 M_{ac}	 M_{ba}	 M_{bc}	 M_{bd}	 M_{cb}	 M_{ca}
0	—	11.8421	3.6316	+9.316	-9.316	+1.105	-4.737	+3.632	-8.368	-21.158
$\pi/4$	0.25	12.0613	3.5879	+8.569	-8.569	+1.918	-4.734	+2.816	-9.030	-21.847
$\pi/2$	1.00	13.0925	3.2719	+5.381	-5.381	+5.381	-5.381	0	-10.986	-23.536
1.80	1.31	13.8835	2.8590	+3.096	-3.096	+7.725	-6.525	-1.200	-11.960	-23.770
2.10	1.79	17.1244	-0.2081	-5.763	+5.763	+15.521	-15.777	+0.256	-15.325	-22.413
2.20	1.96	24.6381	-10.1352	-25.514	+25.514	+30.171	-46.402	+16.231	-23.549	-16.405
2.251	2.05	$+\infty$	$-\infty$							

of areas of the members are more in accordance with those in an actual structure.

The assumptions for case II, in addition to the afore-mentioned ones (namely, $A_1=A_3=A_4=A$ and $A_2=A/2$) are $I_1=I_3=I_4=I$, and $I_2=I/2$. The solutions for case II are given in table 7.

The assumptions for case III are $A_1=A_3=A_4=A$, $A_2=A/2$, $I_1=I_4=I$, and $I_2=I_3=I/2$. Those for case IV are $A_1=A_3=A_4=A$, $A_2=A/2$, $I_1=I_4=I$, $I_2=I/4$, and $I_3=I/2$. The solutions for case III are given in table 8 and those for case IV in table 9.

The elastic curves of the members at various stages of loading, based on the results of case I, are shown in sketches in fig. 5. The values of end moments

Table 6. Computations for R's for Cases II, III and IV

$$A_1=A_3=A_4=A \quad A_2=\frac{A}{2} \quad Q=\frac{P_0}{2\sqrt{3}\,AE}$$

$\delta\alpha$ in terms of Q		R in terms of Q			
$\delta\alpha_{34}=\delta\alpha_{45}$	4	R_4 by Symmetry	0	$R_5=R_3+\delta\alpha_{35}$	-4
$\delta\alpha_{13}=\delta\alpha_{57}$	4	$R_3=R_4+\delta\alpha_{34}$	4	$R_6=R_5+\delta\alpha_{56}$	-12
$\delta\alpha_{12}=\delta\alpha_{67}$	4	$R_1=R_3+\delta\alpha_{13}$	8	$R_7=R_6+\delta\alpha_{67}$	-8
$\delta\alpha_{23}=\delta\alpha_{56}$	-8	$R_2=R_1+\delta\alpha_{12}$	12	$R_5=R_7+\delta\alpha_{57}$	-4
$\delta\alpha_{35}$	-8	$R_3=R_2+\delta\alpha_{23}$	4	$R_4=R_5+\delta\alpha_{45}$	0

Table 7. Equations for Case II and θ and M Values

$$I_1=I_3=I_4=I, \quad I_2=\frac{I}{2}, \quad \phi=\phi_1=\phi_2=\phi_3=\phi_4=\frac{L}{2}\sqrt{\frac{P_0}{EI}}$$

$$a_{c1}=a_{c4}, \; b_{c1}=b_{c4}, \; c_{c1}=c_{c4}, \quad a_{t2}=a_{t3}, \; b_{t2}=b_{t3}, \; c_{t2}=c_{t3}$$

$$(2a_{c1}+a_{t2})\,\theta_a+b_{c1}\theta_b=(8c_{c1}+6c_{t2})\,Q$$

$$b_{c1}\theta_a+(4a_{c1}+2a_{t2}-b_{c1})\,\theta_b=(8c_{c1}+4c_{t2})\,Q$$

ϕ $\phi=\frac{\pi}{2}\sqrt{\frac{P_0}{P_e}}$	P_0 in terms of $P_e=\frac{\pi^2EI}{L^2}$	θ in terms of $Q=\frac{P_0}{2\sqrt{3}AE}$		Moments in terms of $\frac{2EI}{L}Q=\frac{P_0I}{\sqrt{3}AL}=\frac{P_0r^2}{\sqrt{3}L}$						
		θ_a	θ_b	M_{ab}	M_{ac}	M_{ba}	M_{bc}	M_{bd}	M_{cb}	M_{ca}
0	—	12.43	4.71	+5.6	-5.6	-2.1	-2.6	+4.7	-7.3	-11.8
0.785	0.25	12.68	4.65	+5.0	-5.0	-1.2	-2.4	+3.6	-8.0	-12.6
1.57	1.00	13.91	4.30	+2.7	-2.7	+2.7	-2.7	0	-10.0	-14.7
1.80	1.31	15.03	3.78	+0.9	-0.9	+5.5	-4.0	-1.5	-11.2	-15.1
2.00	1.62	17.77	1.90	-3.5	+3.5	+11.1	-9.4	-1.7	-13.3	-15.0
2.10	1.79	23.07	-2.84	-11.7	+11.7	+20.2	-23.7	+3.5	-17.4	-15.4
2.18	1.93	+∞	-∞							

and end rotations are also shown on the sketches. These sketches are made in the following way. First, the original shape of the truss as shown in light solid lines is drawn. Then the primary deflected shape of the truss may be drawn, as shown in light dotted lines, considering only the lengthening and shortening of members. This primary deflected shape is highly exaggerated but it gives the R values of members in proper proportion to their actual values. Then from the joints at the primary deflected position, short solid lines are drawn making an angle equal to known values of θ with respect to the original direction of the member. (See fig. 2 for reference.) The angles θ are exaggerated in the same scale as the R's. The afore-mentioned short lines represent the tangents to the elastic curves at ends of members. With the direction of the end moments also known, the elastic curves can be sketched properly without difficulty.

The locations of the inflection points as shown on the sketches are determined by the following equations. These equations are obtained by putting $\frac{d^2 y}{d x^2}=0$ using y from eq. (9) and (13).

For compression members
$$\tan\frac{2\phi x}{L} = \frac{\sin 2\phi}{\beta+\cos 2\phi}. \qquad (25\,a)$$

For tension members
$$\tanh\frac{2\phi x}{L} = \frac{\sinh 2\phi}{\beta+\cosh 2\phi}. \qquad (25\,b)$$

For $\phi \to 0$, the above equations reduce to

$$\frac{2\phi x}{L} = \frac{2\phi}{\beta+1} \quad \text{or} \quad \frac{x}{L} = \frac{1}{1+\beta}, \qquad (25\,c)$$

Table 8. Equations for Case III and θ Values

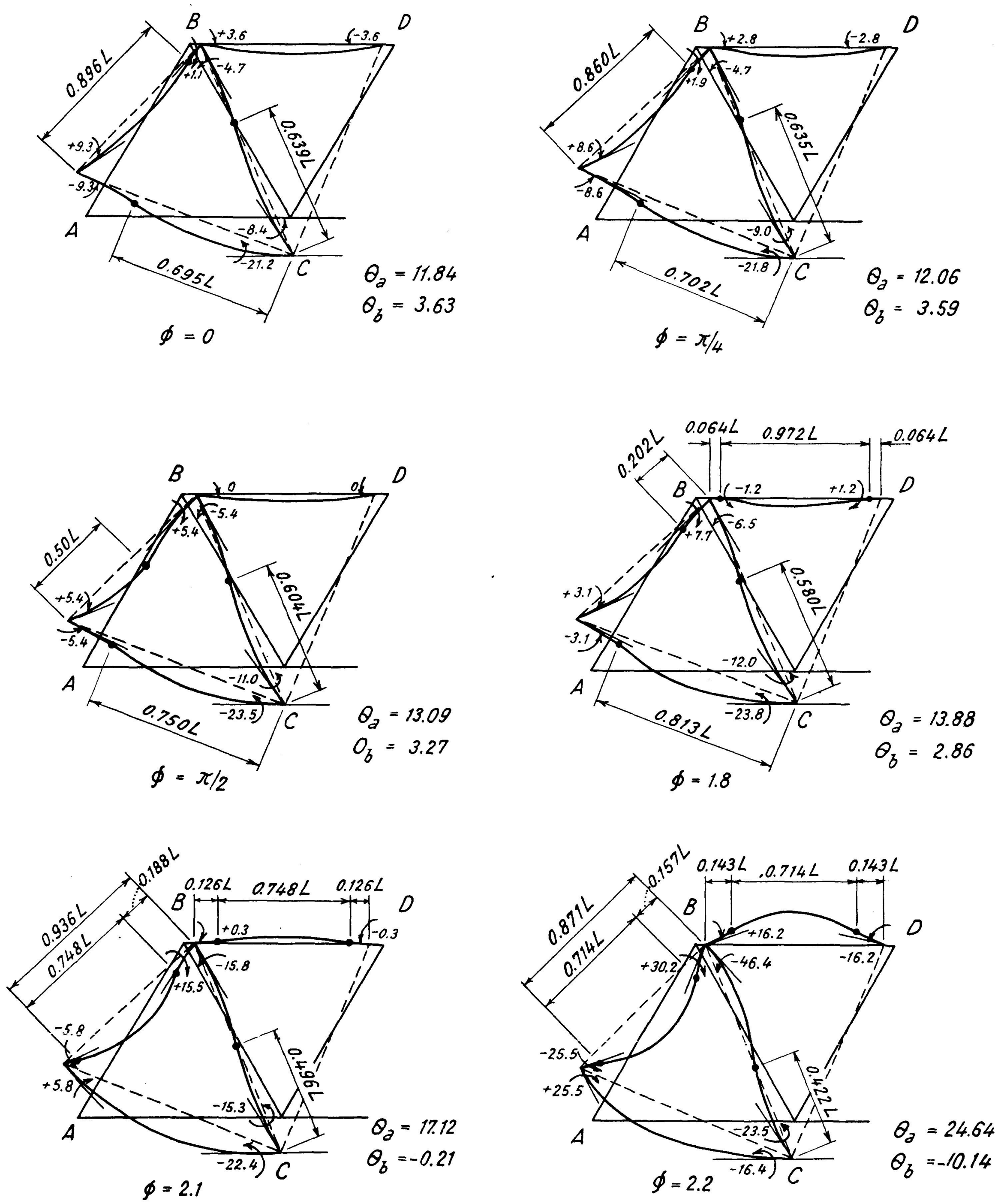
Equations	ϕ	P_0	θ in terms of Q	
$I_1=I_4=I,\ I_2=I_3=I/2,\ a_{c1}=a_{c4},\ b_{c1}=b_{c4},\ c_{c1}=c_{c4}$; $\phi_1=\phi_2=\phi_4=\phi,\quad \phi_3=\frac{L}{2}\sqrt{\frac{P_0}{EI/2}}=\sqrt{2}\,\phi$			θ_a	θ_b
$(2a_{c1}+a_{t2})\theta_a+b_{c1}\theta_b=(8c_{c1}+6c_{t2})Q$	0	—	12.55	4.36
$b_{c1}\theta_a+(4a_{c1}+a_{t3}-b_{c1})\theta_b=(8c_{c1}+2c_{t3})Q$	2.05	$1.70\,P_e$	$+\infty$	$-\infty$

Table 9. Equations for Case IV and θ Values

Equations	ϕ	P_0	θ in terms of Q	
$I_1=I_4=I,\ I_2=I/4,\ I_3=I/2,\ \phi_1=\phi_4=\phi,\ \phi_2=\phi_3=\sqrt{2}\,\phi$; $a_{c1}=a_{c4},\ b_{c1}=b_{c4},\ c_{c1}=c_{c4},\ a_{t2}=a_{t3},\ b_{t2}=b_{t3},\ c_{t2}=c_{t3}$			θ_a	θ_b
$(2a_{c1}+\frac{1}{2}a_{t2})\theta_a+b_{c1}\theta_b=(8c_{c1}+3c_{t2})Q$	0	—	11.33	4.66
$b_{c1}\theta_a+(4a_{c1}+a_{t2}-b_{c1})\theta_b=(8c_{c1}+2c_{t2})Q$	1.99	$1.60\,P_e$	$+\infty$	$-\infty$

where x is measured from the end with the larger end moment, and β is the ratio of the smaller end moment to the larger end moment of a member. (The moment is considered as + when rotating clockwise.)

Sketches such as those shown in fig. 5 for case I can be drawn for all other cases. The significance of these sketches will be discussed in detail in the next section.



θ in terms of $\frac{P_0}{2\sqrt{3}AE}$, M in terms of $\frac{P_0 r^2}{\sqrt{3}L}$, all Values based on Table 5

Fig. 5. Elastic Curves, End Moments and End Rotations for Case I.

Discussions of Results

Let us start the discussion by referring to fig. 5 which shows the results for case I.

1. *The phase of $\phi=0$.* The secondary moments and angular rotations obtained in this phase are the same as those in ordinary structural analysis and will be referred to as initial secondary moments and initial angular rotations, respectively. Although $\phi=\frac{\pi}{2}\sqrt{P_0/P_e}$, the condition $\phi=0$ does not mean that $P_0=0$, in which case there will be no axial stress and no secondary moment. It only means that the axial load P_0 in the members AB and BD is rather small in comparison with P_e which is the Euler buckling load of these members acting as simply supported columns.

It should be noted that in this initial case the two compression members are being bent into elastic curves of different shape. AB is being bent into an S-shaped curve while BD into one of single curvature. Both tension members are being bent into S-shaped curves. The locations of inflection points are also the same as those obtained by the ordinary structural analysis, as both methods employ the same formula, eq. (25c).

2. *The phase of $\phi=\pi/4$.* In this phase (I) the angular rotation θ_a becomes larger and θ_b becomes smaller than their respective initial values. (II) The inflection point in member BC moves toward C and that in AC toward A. The amounts of both shifts, however, are negligible. (III) The inflection point in AB is shifting toward the center of the member. (IV) The secondary moments at joints A and C do not differ very much from their initial moments. But the moments M_{ba} and M_{bd} at joint B differ from their initial values by appreciable amounts; M_{ba} increases by 72% and M_{bd} decreases by 22%.

3. *The phase of $\phi=\pi/2$ (i.e. $P_0=P_e$).* In this phase (I) θ_a continuously becomes larger and θ_b smaller. (II) Significant shifts of inflection points occur in the tension members: the inflection point in BC is moving toward C and that in AC toward A, continuing the former trend. (III) Member BD is bent into a single curvature with zero end moments. (IV) Member AB is bent into an antisymmetrical S shape with equal end moments rotating in the same direction.

It should be noted that at $\phi=\pi/2$ ($P_0=P_e$), there are only two stable forms possible for any compression member. Member BD exhibits typically one of the forms and member AB, the other. Theoretical explanations of the behavior of these two members are given below.

Take the case of member BD which is bent into single curvature. Suppose that it is bent by equal but opposite end moments $M_{bd}=-M_{db}$. Then eq. (9) could be reduced into the following form:

$$y = \frac{M_{bd}}{P_0} \frac{2 \sin\phi \left(1 - \frac{x}{L}\right) \sin\frac{\phi x}{L}}{\cos\phi}. \tag{26}$$

For $x = L/2$,
$$y_{max} = \frac{M_{bd}}{P_0} (\sec\phi - 1). \tag{26a}$$

(Note that eq. (26a) is the basis of the ordinary "secant formula".)

With $\phi = \pi/2$, $\sec\phi = \infty$, then $y_{max} = \infty$ when $M_{bd} \neq 0$. (26b)

Therefore, the member is not stable if $M_{bd} \neq 0$.

Let us examine the case when $M_{bd} = 0$. By differentiating eq. (26), and putting in the boundary conditions, $\frac{dy}{dx} = \theta_b$ when $x = 0$, and $\frac{dy}{dx} = -\theta_b$ when $x = L$, we have

$$\theta_b = \frac{2 M_{bd}}{P_0 L} \phi \tan\phi. \tag{27}$$

For $\phi = \pi/2$ and $M_{bd} = 0$, this is a form of $0 \times \infty$. Substituting eq. (27) into eq. (26), we get

$$y = \frac{\theta_b L}{2\phi} \frac{\cos\phi \left(1 - \frac{2x}{L}\right) - \cos\phi}{\sin\phi}. \tag{28}$$

For $\phi = \pi/2$ and $x = L/2$, $y_{max} = \frac{\theta_b L}{\pi}$ which is not equal to ∞. Therefore, the member is stable when $M_{bd} = 0$.

It might be of interest to obtain the bending moment in such a member:

$$M = -EI \frac{d^2 y}{dx^2} = \frac{2EI}{L} \theta_b \frac{\phi}{\sin\phi} \cos\phi \left(1 - \frac{2x}{L}\right). \tag{29}$$

For $x = L/2$,
$$M_{max} = \frac{2EI}{L} \theta_b \frac{\phi}{\sin\phi}. \tag{29a}$$

Eq. (29a) was discussed fully in the paper by PARCEL and MURER [10].

For $\phi = \pi/2$, $M_{max} = \frac{\pi EI}{L} \theta_b$ which is also of finite value.

The above results with regard to deflections and moments at $\phi = \pi/2$ should not be surprising to those familiar with the derivation of the Euler load. At the Euler load, a column with simply-supported ends (zero end moments) is in a state of neutral equilibrium. Thus it can be stable at deflected shapes consistent with bending moment curves due to the axial load. Only when with small increase of the axial load beyond the Euler load, if no restraining end moment developes, then the deflection would increase indefinitely. Fortunately, when P_0 is greater than P_e, restraining moments will develop in truss members as will be shown in the discussion of the next phase.

Take the case of the member AB which is bent into an antisymmetrical S shape. For $M_{ab} = M_{ba}$. Eq. (9) can be reduced into the following form:

$$y = \frac{M_{ab}}{P_0}\left[\frac{\sin\phi\left(1-\frac{2x}{L}\right)}{\sin\phi} - \left(1-\frac{2x}{L}\right)\right], \tag{30}$$

which is stable at $\phi=\pi/2$.

The formula for inflection points $\tan\frac{2\phi x}{L} = \frac{\sin 2\phi}{\beta+\cos 2\phi}$ is of a form 0/0 for $\phi=\pi/2$ since $\beta=+1$. Using l'Hospital's rule, we get

$$\tan\frac{2\phi x}{L} = -\frac{\cos 2\phi}{\sin 2\phi} = -\frac{\cos\pi}{\sin\pi} = \frac{+1}{0} = \infty \quad \text{or} \quad \frac{2\phi x}{L} = \frac{\pi x}{L} = \frac{\pi}{2}.$$

Hence $x=L/2$ is the location of the inflection point.

4. *The phase $\phi=1.8$.* In this phase, (I) the end rotations and (II) the inflection point in member BC continue the same trend as in the phase $\phi=\pi/2$. (III) The inflection point in the member AB has shifted from the center of the member toward B. (IV) Two inflection points symmetrical with respect to the mid-point of the member have appeared for the member BD. The end moments for this member have not only ceased to be zero but also occur in directions opposite to the initial moments. These new end moments are the restraining moments which prevent the member from buckling at $P_0=P_e$.

5. *The phase $\phi=2.1$.* In this phase (I) the angular rotations θ_a and θ_b continue the same trend as before to an increasing degree. θ_a has increased at a faster rate; θ_b has decreased likewise and becomes negative. (II) The inflection point in the member BC has shifted more toward C. (III) The inflection point in the member AC has disappeared and (IV) an extra inflection point has appeared in member AB near A. What happened in (III) and (IV) can be pictured as follows. The inflection point in the member AC has continued its former trend: it first shifted to A, then it continued to move around joint A in the same direction, and finally it appeared as an inflection point in member AB. (V) The original inflection point in member AB has moved toward B, continuing the former trend. (VI) The end moments at joint A have reversed in direction from the previous phase. This can be pictured as a consequence of the shifting of the inflection points described in (III) and (IV), which results in the reverse of the curvatures of the members in the neighborhood of joint A. (VII) The inflection points of the member BD have shifted toward the center. (VIII) At the same time, the moments at the ends of the member BD have reserved in direction from the previous phase, and as a result, the member has been bent into a shape in direct opposition to the previous phase. What happened in this member can be explained in the following way.

The end moments M_{bd} and M_{db} are given by the equation $M_{bd}=-M_{db}= =(2a_c-b_c)\,\theta_b\frac{2EI}{L}$. The moments would change signs if either the coefficient $(2a_c-b_c)$ or the angle θ_b changed signs. The coefficient $(2a_c-b_c)$, changes signs at $\phi = \frac{\pi}{2}\left(+\text{for}\,\phi < \frac{\pi}{2},\ -\text{for}\,\phi > \frac{\pi}{2}\right)$ but θ_b, though decreasing, remains positive when ϕ is somewhat greater than $\pi/2$. This first change of moment signs from positive to negative has been noted previously. Now as θ_b has continuously decreased, it has become negative (it would go to $-\infty$ at the buckling load) but the sign of $(2a_c - b_c)$ remains negative. Thus the end moments change signs from negative to positive again.

Now the curvature of the member BD in the present phase is extremely small due to small end rotations and end moments. In fact, as the load gradually increases from $\phi = 1.8$, the member tends to straighten, becoming perfectly straight when $\theta_b = -\theta_d = 0$, then bends into the opposite configuration. Theoretically, there are two inflection points in this member even at $\theta_b = -\theta_d = 0$ when the member is perfectly straight. These points must be considered as limiting positions with respect to the condition that the angular rotation $\theta_b\,(= -\theta_d)$ differs from zero by an infinitesimal amount.

6. *The phase* $\phi = 2.2$. In this phase (I) θ_a and θ_b continue their previous trends but more drastically: θ_a has increased and θ_b has decreased rather rapidly. (II) As before, the inflection point in BC has shifted more toward C. (III) Continuing the former trend, the inflection points in BD are shifting towards center, thus narrowing the distance between them. (IV) As before, both inflection points in AB are shifting toward B, but the inflection point near A is moving faster than the one near B. Consequently, the distance between them also becomes smaller.

7. *The phase* $\phi = 2.251$. At this phase, the determinant of the coefficients (referring to table 4) becomes equal to zero. (Actually, due to rounding off of the value of ϕ, the determinant becomes equal to 0.002). The θ's then become infinity and the final buckling condition has been reached. The buckling load for the compression members AB and BC is given by

$$P_0 = 2.05\,P_e = 2.05\frac{\pi^2 E I}{L^2} = \frac{\pi^2 E I}{L_r^2}, \tag{31}$$

where L_r is the "reduced length". In this case, $L_r = 0.70\,L$.

It might be of interest to compare this length L_r with the distances between inflection points of the members AB and BD in the phase of $\phi = 2.2$. In that phase, the distances between inflection points of AB and BD are both equal to $0.714\,L$. It becomes obvious that L_r is the limit of distance between inflection points at the buckling load. (It is not the distance between inflection points at any load.)

8. *The difference between a compression member with applied end moments or constant end eccentricities and one with restraining end moments.* At this point it might be of interest to discuss what is the difference between a compression member in a truss with secondary end moments and a compression member

with constant end moments or end eccentricities. For both kinds of members, the deflection y is given by eq. (9).

$$y = \frac{M_{ij}}{P} \frac{\sin 2\phi\left(1-\frac{x}{L}\right)}{\sin 2\phi} - \frac{M_{ji}}{P} \frac{\sin \frac{2\phi x}{L}}{\sin 2\phi} - \frac{M_{ij}}{P}\left(1-\frac{x}{L}\right) + \frac{M_{ji}}{P}\frac{x}{L}.$$

However, except for this similarity, the actions of these two kinds of members differ widely.

For members with constant end eccentricity or constant end moments M_{ij} and M_{ji}, with the exception of cases in which $M_{ij} = -M_{ji} = 0$ and $M_{ij} = = M_{ji} \neq 0$, y becomes infinity at $\phi = \pi/2$ (i.e. $P = P_e$) since $\sin 2\phi = 0$. The members with $M_{ij} = -M_{ji} = 0$ also have impending instability when $P = P_e$. Thus all such members, except the case in which $M_{ij} = M_{ji} \neq 0$, would buckle at $P = P_e$.

For compression members in a truss, the end moments M_{ij} and M_{ji} vary with the load in such a way that these members will not buckle at $\phi = \pi/2$, since at this instant, the end moments in all compression members become either $M_{ij} = -M_{ji} = 0$ or $M_{ij} = M_{ji} \neq 0$. The members with $M_{ij} = -M_{ji} = 0$ are stable in this case because restraining moments appear at the ends as P increases.

9. *The effect of changing areas and moments of inertia in members.* The proportions of members in case II ($A_1 = A_3 = A_4 = A$, $A_2 = A/2$, $I_1 = I_3 = I_4 = I$, $I_2 = I/2$) are more in accordance with those in actual structures than in case I ($A_1 = A_2 = A_3 = A_4 = A$, $I_1 = I_2 = I_3 = I_4 = I$). The results of computations for case II are shown in table 7. It can be observed that the results for this case, with few exceptions, are rather similar to the previous case. Note the following:

I. In this case, the two compression members, AB and BD, both are being bent initially into single curvature, since at $\phi = 0$, M_{ab} and M_{ba} are of opposite signs and M_{bd} and M_{db} are of opposite signs. This is different from case I where the member AB is initially bent into S shape while BD is bent into a single curvature. The same conclusion can be drawn for $\phi = \pi/4$.

II. The changes of end moments and end rotations occurring when $\phi = \pi/2$ and $\phi = 1.8$ are rather similar to those that occurred at corresponding ϕ values in case I.

III. At $\phi = 2.0$, the signs of the moments M_{ab} and M_{ac} have been changed from those at $\phi = 1.8$, while the signs of M_{bd} and M_{db} have not been changed. In case I, all these moments change signs almost at the same time as shown in the phase $\phi = 2.1$.

IV. At $\phi = 2.1$, the changes in θ and M occurring in the present case are rather similar to those in case I between $\phi = 2.1$ and $\phi = 2.2$.

The results of case III ($A_1 = A_3 = A_4 = A$, $A_2 = A/2$, $I_1 = I_4 = I$, $I_2 = I_3 = I/2$) and case IV ($A_1 = A_3 = A_4 = A$, $A_2 = A/2$, $I_1 = I_4 = I$, $I_2 = I/4$, $I_3 = I/2$) are shown in tables 8 and 9, respectively. From these results, it can be deduced that the shapes of elastic curves are rather similar to those for case II.

A comparison of results from cases I to IV are shown in table 10. In this table, θ_{a0} and θ_{b0} are initial end rotations (for $\phi=0$); ϕ_{cr} and P_{cr} are values of ϕ and P_0, respectively, at the point of buckling.

Table 10. Comparison of Results for Cases I, II, III and IV

Case	Cross-sectional Areas	Moments of Inertia	θ_{a0}/Q	θ_{b0}/Q	$\phi_{cr}=\frac{\pi}{2}\sqrt{\frac{P_{cr}}{P_e}}$	P_{cr}/P_e	%	$L_r=\frac{L}{\sqrt{P_{cr}/P_e}}$
I	$A_1=A_2=A_3=A_4=A$	$I_1=I_2=I_3=I_4=I$	11.84	3.63	2.25	2.05	100	0.70 L
II	$A_1=A_2=A_3=A$, $A_2=A/2$	$I_1=I_3=I_4=I$, $I_2=I/2$	12.43	4.71	2.18	1.93	94	0.72 L
III	$A_1=A_2=A_3=A$, $A_2=A/2$	$I_1=I_4=I$, $I_2=I_3=I/2$	12.55	4.36	2.05	1.70	83	0.77 L
IV	$A_1=A_2=A_3=A$, $A_2=A/2$	$I_1=I_4=I$, $I_2=I/4$, $I_3=I/2$	11.33	4.66	1.99	1.60	78	0.79 L

Apparently, the values of θ_{a0} and θ_{b0} depend on the relative stiffnesses of members meeting at a joint and relative lengthening and shortening of members in the truss. However, no definite trend can be seen from the results of various cases.

Nevertheless, a definite trend can be seen for the values of ϕ_{cr}, P_{cr} and the reduced length L_r. As the stiffness in any member of a truss decreases, the buckling load P_{cr} decreases accordingly. As P_{cr} decreases, ϕ_{cr} decreases and L_r increases. It should be noted that the AASHO[2]) specification of $L_r=0.75\,L$ for riveted members appears to lie between cases II and III.

10. *The variation of angular rotations.* The variation of angular rotations with respect to ϕ has been discussed previously. However, this variation can be shown more clearly by the curves of fig. 6. Fig. 6a shows the variation of θ with respect to ϕ and fig. 6b shows the variation of θ with respect to P_0 (in terms of P_e). Let $\theta=\mu\,P_0$ where μ is a coefficient. It can be seen that, for the example studied, μ remains practically constant up to $P_0=50\%\ P_{cr}$, which corresponds to $\phi=71\%\ \phi_{cr}$. At $P_0=0.5\ P_{cr}$, μ differs from its initial value (say μ_0) by about 10%. At $P_0=75\%\ P_{cr}$, corresponding to $\phi=87\%\ \phi_{cr}$, the difference is about 25%. The value of μ increases and decreases very rapidly when $P_0>75\%\ P_{cr}$.

It should be noted, however, that one should not draw the conclusion that μ always increases or decreases monotonically. Fig. 7 shows the variation of θ

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[2]) American Association of State Highway Officials.

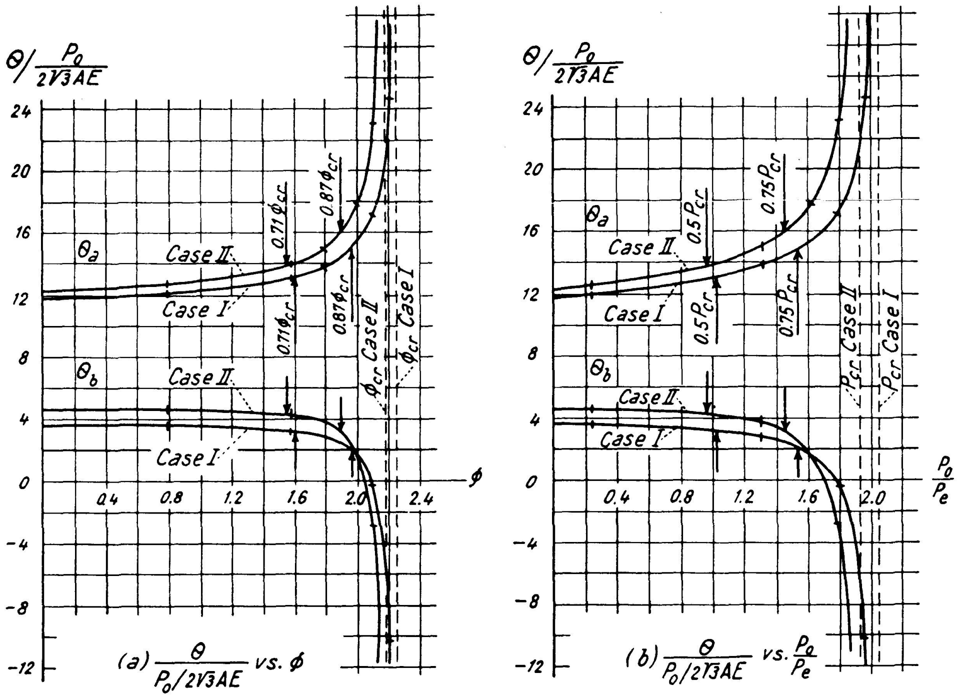

Fig. 6. θ Variations for the Example Shown on Fig. 4.

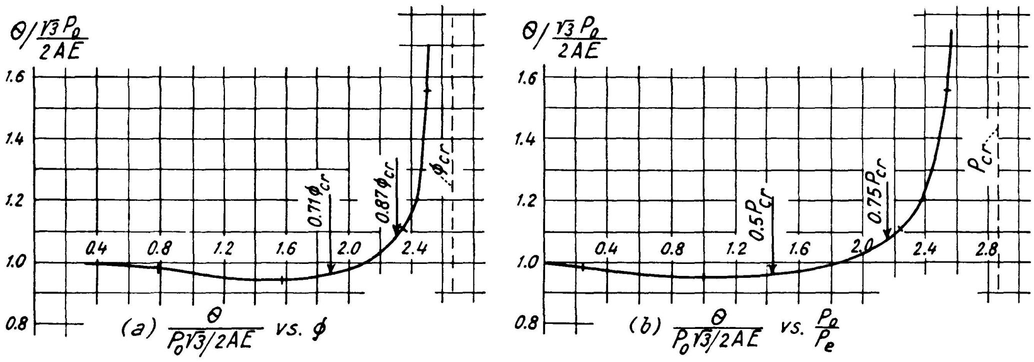

Fig. 7. θ Variations for the Example Shown on Fig. 8.

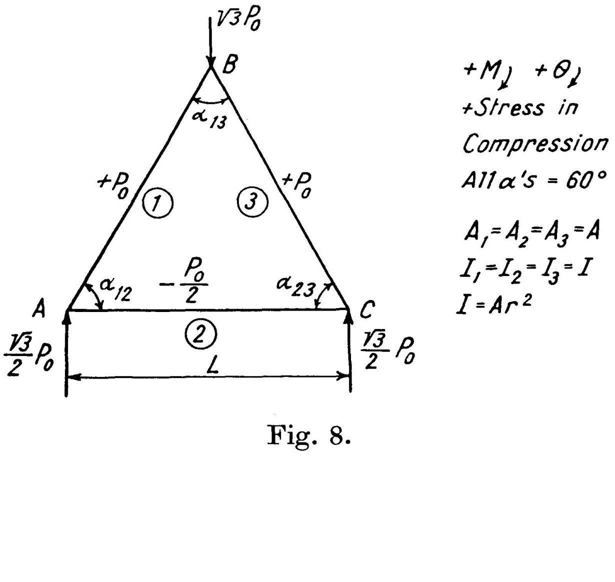

Fig. 8.

with respect to ϕ and P_0 for the example as shown in fig. 8. In this case, μ decreases slightly at first then increases to $+\infty$. At $P_0 = 50\%\ P_{cr}$, μ differs less than 5% from μ_0; and at $P_0 = 75\%\ P_{cr}$, the difference is less than 10%. But μ increases very rapidly when $P_0 > 75\%\ P_{cr}$.

11. *The maximum moment in a compression member.* Once the end moments of a member have been determined, the moment at any point of the member can be found by using the equation $M = -EI\frac{d^2y}{dx^2}$. By differentiating y given by eq. (9) twice, and by substituting $\left(\frac{2\phi}{L}\right)^2 = \frac{P}{EI}$, we have

Table 11. Equations for the Example in fig. 8

$$\delta\alpha_{12} = \delta\alpha_{23} = \left(-\frac{P_0}{2} - P_0\right)\frac{1}{\sqrt{3}AE} + (P_0 - P_0)\frac{1}{\sqrt{3}AE} = -\frac{\sqrt{3}P_0}{2AE}, \quad \delta\alpha_{13} = +\sqrt{3}\frac{P_0}{AE}$$

$$R_2 = 0, \quad R_1 = -\delta\alpha_{12} = -R_3 = R, \quad \theta_b = 0$$

$$M_{ab} = (2a_c\theta_a - c_cR)\frac{2EI}{L}$$

$$M_{ac} = (2a_t - b_t)\theta_a\frac{2EI}{L}$$

$$M_{ba} = (b_c\theta_a - c_cR)\frac{2EI}{L}$$

$$[2a_c + (2a_t - b_t)]\theta_a - c_cR = \frac{L}{2EI}(M_{ab} + M_{ac}) = 0$$

For a_c and b_c, $\phi_1 = \frac{L}{2}\sqrt{\frac{P_0}{EI}} = \frac{\pi}{2}\sqrt{\frac{P_0}{P_e}}$. Also $c_c = 2a_c + b_c$

For a_t and b_t, $\phi_2 = \frac{L}{2}\sqrt{\frac{P_0}{2EI}} = \frac{\phi_1}{\sqrt{2}}$. $\quad 2a_t - b_t = \phi_2 \coth\phi_2$

Table 12. Solutions for the Equations in Table 11

ϕ_1 $\phi_1 = \frac{\pi}{2}\sqrt{\frac{P_0}{P_e}}$	P_0 in terms of $P_e = \frac{\pi^2 EI}{L^2}$	Coefficients in the Equations of Table 11							θ_a in terms of $R = \frac{\sqrt{3}P_0}{2AE}$	M in terms of $\frac{\sqrt{3}P_0 r^2}{L}$		
		a_c	b_c	$\phi_2 = \frac{\phi_1}{\sqrt{2}}$	coth ϕ_2	$2a_t - b_t$	$2a_c + b_c$	$2a_c + 2a_t - b_t$		M_{ab}	M_{ac}	M_{ba}
0	—	1	1			1	3	3	1.000	-1	+1	-2
$\pi/4$	0.25	0.9147	1.0445	0.5554	1.9820	1.1008	2.8739	2.9302	0.9808	-1.0796	+1.0796	-1.8495
$\pi/2$	1.00	0.6169	1.2337	1.1107	1.2433	1.3809	2.4675	2.6147	0.9437	-1.3032	+1.3032	-1.3032
$3\pi/4$	2.25	0.1755	2.0051	1.6610	1.0749	1.7854	2.3561	2.1364	1.1028	-1.9690	+1.9690	-0.1449
2.50	2.53	-0.4733	2.3923	1.7678	1.0600	1.8739	1.4377	0.9273	1.5504	-2.9053	+2.9053	+2.2713
		cot $\phi_1 = -1.3387$										
2.661	2.87	-0.9862	3.1321	1.8816	1.0469	1.9710		≈ 0	$+\infty$			
		cot $\phi_1 = -1.9182$										

$$M = \frac{M_{ij} \sin 2\phi \left(1 - \frac{x}{L}\right)}{\sin 2\phi} - \frac{M_{ji} \sin \frac{2\phi x}{L}}{\sin 2\phi}. \tag{32}$$

To get the maximum moment, setting $\frac{dM}{dx} = 0$ we have

$$\cos 2\phi \left(1 - \frac{x}{L}\right) + \beta \cos \frac{2\phi x}{L} = 0, \tag{33}$$

where $\beta = M_{ji}/M_{ij}$, with the assumption that $M_{ji} \leqq M_{ij}$ and x is measured from the end with M_{ij}.

From eqs. (32) and (33) we get M_{max} as follows:

$$M_{max} = M_{ij} \frac{\sqrt{1 + 2\beta \cos 2\phi + \beta^2}}{\sin 2\phi}. \tag{34}$$

This equation with $\beta = -1$ (i. e. $M_{ij} = -M_{ji}$) can be reduced to the following form:

$$M_{max} = M_{ij} \sec \phi \quad \text{for } \beta = -1, \tag{34a}$$

which is the relationship used in deriving the ordinary "secant formula". Eq. (33) can be reduced to the following form:

$$\tan \frac{2\phi x}{L} = \frac{-(\beta + \cos 2\phi)}{\sin 2\phi}. \tag{33a}$$

As β varies from $+1$ to -1, we can put $\beta = -\cos 2\gamma$. By letting $\phi = \gamma + \delta$ and assuming δ to be small, it can be proved that if δ is negative, eq. (33a) would give a solution of $x/L > 1$, which is not possible. Therefore, in order for eq. (33a) to give valid solutions, δ must be positive, which establishes the following condition:

$$\left[2\phi = \frac{L}{r}\sqrt{\frac{P}{AE}}\right] > [2\gamma = \cos^{-1}(-\beta)]. \tag{35}$$

If eq. (35) is not satisfied then

$$M_{max} = M_{ij} \quad \text{if} \quad M_{ij} > M_{ji}; \qquad M_{max} = M_{ji} \quad \text{if} \quad M_{ji} > M_{ij}. \tag{36}$$

The eqs. (34), (34a), (35), and (36) form the bases of the AASHO [2]) column formulas which are based on a paper by YOUNG [11]. A summary of their derivations is given above in order to show the relationship between this study and the existing column formula.

12. *The effect of the differences between arc length and chord length of members.* In the foregoing investigations, no consideration has been given to the effect of the difference between arc length and chord length of each member. It remains to be seen whether such a difference would affect the results of the above computations. Let this difference for member ij be δL_{ij}. Then a good approximation of δL_{ij} is given by the following formula.

$$\delta L_{ij} = \frac{1}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx. \tag{37}$$

Using y from eqs. (9) and (13), we have the following results:

For compression members

$$\begin{aligned}\delta L_{ij} = {} & \frac{(M_{ij}+M_{ji})^2 \phi_{ij}^2}{P_{ij}^2 L_{ij} \sin^2 2\phi_{ij}}\left[1+\frac{\sin 2\phi_{ij}\cos 2\phi_{ij}}{2\phi_{ij}}\right] - \frac{(M_{ij}+M_{ji})^2}{2P_{ij}^2 L_{ij}} \\ & - \frac{M_{ij}M_{ji}\phi_{ij}^2}{P_{ij}^2 L_{ij}\cos^2\phi_{ij}}\left[1-\frac{\sin 2\phi_{ij}}{2\phi_{ij}}\right].\end{aligned} \tag{38a}$$

For tension members

$$\begin{aligned}\delta L_{ij} = {} & \frac{(M_{ij}+M_{ji})^2 \phi_{ij}^2}{P_{ij}^2 L_{ij} \sinh^2 2\phi_{ij}}\left[1+\frac{\sinh 2\phi_{ij}\cosh 2\phi_{ij}}{2\phi_{ij}}\right] - \frac{(M_{ij}+M_{ji})^2}{2P_{ij}^2 L_{ij}} \\ & + \frac{M_{ij}M_{ji}\phi_{ij}^2}{P_{ij}^2 L_{ij}\cosh\phi_{ij}}\left[1-\frac{\sinh 2\phi_{ij}}{2\phi_{ij}}\right].\end{aligned} \tag{38b}$$

Using θ and R as independent variables instead of M_{ij} and M_{ji}, the above formulas can be changed into the following forms:

For compression members

$$\delta L_{ij} = \frac{L_{ij}}{16}[C_1(\theta_i+\theta_j-2R_{ij})^2 + C_2(\theta_i-\theta_j)^2]. \tag{39a}$$

For tension members

$$\delta L_{ij} = \frac{L_{ij}}{16}[C_3(\theta_i+\theta_j-2R_{ij})^2 + C_4(\theta_i-\theta_j)^2]. \tag{39b}$$

In the eqs. (39),

$$\begin{aligned} C_1 &= \frac{\phi_{ij}^2\csc^2\phi_{ij}+\phi_{ij}\cot\phi_{ij}-2}{(1-\phi_{ij}\cot\phi_{ij})^2}, & C_2 &= \frac{\phi_{ij}^2\csc^2\phi_{ij}-\phi_{ij}\cot\phi_{ij}}{\phi_{ij}^2}, \\ C_3 &= \frac{\phi_{ij}^2\operatorname{csch}^2\phi_{ij}+\phi_{ij}\coth\phi_{ij}-2}{(\phi_{ij}\coth\phi_{ij}-1)^2}, & C_4 &= \frac{\phi_{ij}\coth\phi_{ij}-\phi_{ij}^2\operatorname{csch}^2\phi_{ij}}{\phi_{ij}^2}.\end{aligned} \tag{40}$$

Let

$$\theta_i = \xi_i\frac{P_0}{AE},\quad \theta_j = \xi_j\frac{P_0}{AE},\quad \text{and}\quad R_{ij} = \xi_R\frac{P_0}{AE}, \tag{41}$$

where ξ_i, ξ_j and ξ_R are coefficients; P_0 is the reference load, and A is the cross-sectional area of a reference member.

Since

$$\phi_{ij} = \frac{L_{ij}}{2}\sqrt{\frac{P_{ij}}{EI_{ij}}} = \frac{1}{2}\left(\frac{L_{ij}}{r_{ij}}\right)\sqrt{\frac{P_{ij}}{EA_{ij}}}$$

we have the following identity:

$$\frac{P_0}{AE} = \left(\frac{P_0 A_{ij}}{P_{ij}A}\right)^2 \frac{4\phi_{ij}^2}{(L_{ij}/r_{ij})^2}. \tag{42}$$

Let the change in length of the member ij caused by axial stress only be ΔL_{ij} i.e.

$$\Delta L_{ij} = \frac{P_{ij} L_{ij}}{A_{ij} E}. \tag{43}$$

Then
$$\frac{\delta L_{ij}}{\Delta L_{ij}} = \left(\frac{P_0 A_{ij}}{P_{ij} A}\right)^2 \frac{\phi_{ij}^2}{4(L_{ij}/r_{ij})^2} \left[\frac{C_1}{C_3}(\xi_i + \xi_j - 2\xi_R)^2 + \frac{C_2}{C_4}(\xi_i - \xi_j)^2\right]. \tag{44}$$

In the above equation, we use the coefficients C_1 and C_2 only for compression members and the coefficients C_3 and C_4 only for tension members.

The values of C_1, C_2, C_3, and C_4 for ϕ from 0 to π are given by the diagram shown in fig. 9.

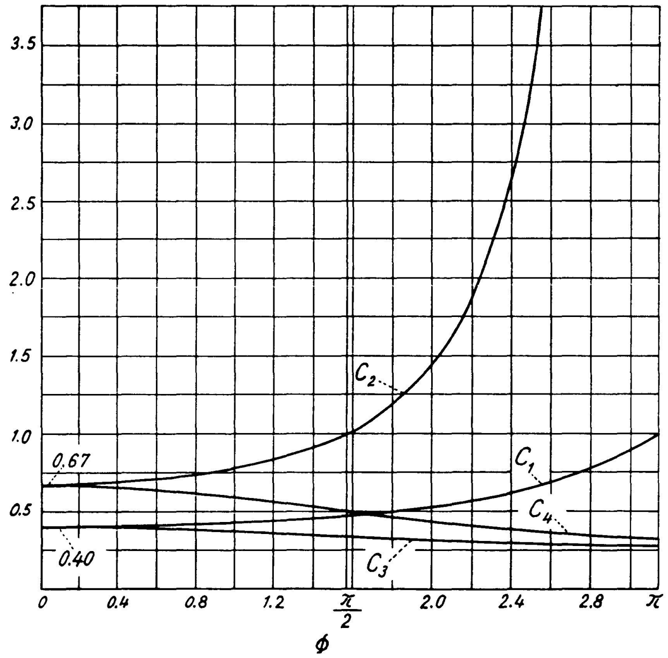

Fig. 9. Values of C_1, C_2, C_3 and C_4.

From fig. 9 it can be seen that the values of C_3 is in general smaller than C_1, and C_4 is much smaller than C_2. Therefore, for a member subjected to an axial force P, δL for P in tension will be smaller than δL for P in compression. Also C_2 is much larger than C_1. Therefore, a compression member with θ at two ends rotating in opposite direction will have a larger difference between arc length and chord length than the same with θ at both ends rotating in the same direction.

Some idea of the magnitudes of the values of θ and $\frac{\delta L}{\Delta L}$ can be obtained from the example shown in fig. 5. In this example, $\frac{P_0 A_{ij}}{P_{ij} A} = 1$ for the compression members AB and BD. For both members take $L/r = 100$ which is practically the minimum slenderness ration for the elastic analysis to be valid. Consider the case of $\phi = 2.2$. We have $4\phi^2/(L/r)^2 = 4 \times \overline{2.2}^2/(100)^2 = 19.36 \times 10^{-4}$, also $C_1 = 0.56$, $C_2 = 1.86$.

For member BD

$$\xi_i = -\frac{10.14}{2\sqrt{3}} = -2.93, \qquad \xi_j = +2.93, \qquad \xi_R = 0,$$

$$\theta_i = -0.325°, \qquad \frac{\delta L}{\Delta L} = 0.77\ \%.$$

For member AB

$$\xi_i = +\frac{24.64}{2\sqrt{3}} = +7.12, \qquad \xi_j = -2.93, \qquad \xi_R = \frac{+6}{2\sqrt{3}} = +1.73,$$

$$\theta_i = +0.790°, \qquad \frac{\delta L}{\Delta L} = 2.28\ \%.$$

It should be noted that the load corresponding to $\phi = 2.2$ is very close to the buckling load corresponding to $\phi = 2.251$. Yet at this load, the error in using ΔL instead of $(\Delta L + \delta L)$ for computing the axial deformation of a member is rather small. Therefore, neglecting the differences between chord length and arc length of a member would hardly affect the θ and M values in the above computations.

In fact, the axial stresses in bars in a statically determinate truss are independent with respect to the change in length in members $(\Delta L + \delta L)$, if one neglects the following effects: (I) the change in axial forces due to the shears in members caused by the secondary moments and (II) the change in geometry of the deflected structure. Since the axial stresses in members remains the same whether or not the effect of $\Delta L + \delta L$ is taken into consideration, the values of a_c, b_c, a_t and b_t remain unchanged. Thus the values of g_{ij} in eq. (22) remain unchanged and, therefore, the buckling load for a statically determinate truss remains the same.

The above conclusion, however, does not apply to statically indeterminate trusses. For these trusses, one must take into consideration the effect of $(\Delta L + \delta L)$. Consider the case of a redundant compression member in a statically indeterminate truss subjected to a certain initial loading. Now increase the loading by proportional increments. Considering only the effect of ΔL and neglecting the effect of δL, this member will always carry the same proportional amount of a reference load at any instant. If, due to δL, the axial deformation in this member is increasing at a greater rate than those in the other members meeting at a common joint, then this redundant member will carry proportionally less load and the other members will carry proportionally more load. Thus it can be seen that in a statically indeterminate truss, not only the effect of ΔL, but also the effect of δL should be considered. Since the axial stress in every member depends on its $(\Delta L + \delta L)$, the values of a_c, b_c, a_t, b_t and g_{ij} depend on the same. Hence the buckling load depends on the same. The effect of δL on buckling load in statically indeterminate trusses has been investigated by MASUR [12].

Conclusions

From the above studies, the following conclusions might be drawn. Most of the conclusions have been derived from the general principles. The few exceptions which are derived from the specific examples discussed are stated as such.

1. The secondary moments in truss members do not remain constant. As the load increases, they change not only in magnitude but also in sign.

Note that in a general case, "the load" refers to the magnitude of a reference load while all other loads on a truss can be represented by their respective proportional constants times the reference load. The load considered are fixed both in direction and in magnitude. An "increase of the load" by a certain proportion means the increase of all the loads simultaneously by the same proportion.

2. When the axial load in a compression member in a rigid jointed truss is equal to P_e (Euler buckling load for the member as if it were simply supported at ends), the member either bends in single curvature with zero end moments or in antisymmetrical double curvature (S shape) with equal end moments rotating in the same direction. Note that in a rigid jointed truss, P_e is always less than the actual buckling load, P_{cr}, for the member.

3. The buckling load of a truss is reached when the determinant of the coefficients in the set of simultaneous equations for angular rotations becomes zero. This condition implies that under critical load all angular rotations become infinity.

4. The curvatures of the members under increasing loads, as a consequence of changes in secondary moments, change not only in magnitude but also in direction. The initial deflection curve of a member, as determined by the ordinary method of secondary moment computations, may be of single curvature or of double curvature (S shape). But at and near the critical buckling load, P_{cr}, the compression members will all bend into a type of curve as shown by members AB and BD in fig. 5 with two inflection points.

5. The inflection points in members under increasing load, as a consequence of change in curvature, change not only in location but also in number. Initially, a compression member may have either zero or one inflection point. But, as noted before, at and near the critical load, any compression member will have two inflection points.

6. The reduced length L_r of a compression member is the distance between the two inflection points at the critical load (not at any other load). The portion of the member between the inflection points acts like a column with simply supported ends and the critical load of the member P_{cr} is given by the formula $P_{cr} = \frac{\pi^2 EI}{L_r^2}$.

7. The main difference between secondary moments in truss members and applied end moments or end eccentricities is that the secondary moments

arise from restraints to the deflection of the member, while applied end moments or end eccentricities in general exaggerate the deflection.

8. The difference between a compression member in rigid-jointed trusses and a compression member subjected to constant eccentricities or moments can also be seen from the fact that the former would not buckle at an axial load equal to P_e under any condition, while the latter will buckle unless the end moments are equal and rotating in the same direction.

9. The effect of reducing stiffness in any member of a truss is to lower the critical buckling load of the whole truss.

10. The relationship between the angular rotation θ and the axial load P in a member may be expressed by the formula $\theta = \mu P$ where μ is a coefficient. From the examples studied, it appears that μ remains practically constant up to $P = 0.5\, P_{cr}$ but it increases or decreases very rapidly when $P > 0.75\, P_{cr}$ where P_{cr} is the critical buckling load of the member.

11. In computing the values of angular rotations and secondary moments, the differences between arc length and chord length in members have been neglected. It has been shown that the effect of these differences is rather small. Also if one neglects the effects of shear in members and of the change in geometry of the undeflected structure, these differences in length would not affect the buckling load of a statically determinate truss. They should be considered, however, in finding the buckling load of a statically inderterminate truss.

12. The above conclusions are based on the assumption of perfect elasticity and small deflection theory. It might be useful in predicting the behavior of building or aircraft truss members of large L/r ratio. The problem of predicting the real behavior of most of the truss members in buildings and bridges, however, can only be solved after full development of the theory governing the elasto-plastic behavior of members. The main purpose of this paper is to furnish some insight into the behavior of statically determinate truss members under the conditions prescribed by the above assumptions.

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Summary

This paper presents a study of (I) the relationship between secondary moments and truss buckling load and (II) the variation of such moments and the physical behavior of members (such as end rotations and inflection points) in statically determinate trusses under increasing load up to the point of buckling. The study is based on the assumptions that the truss is perfectly elastic, the deflections are small, and the members are initially straight and without end eccentricities.

Résumé

L'auteur étudie la relation entre les moments secondaires de flexion et la charge de flambage d'un treillis, ainsi que la relation entre les variations de ces moments et les déformations (telles qu'angles de rotation des barres et points d'inflexion) des barres dans les treillis isostatiques sous une charge qui croît jusqu'au point d'instabilité.

Il est admis à titre de base que le treillis se comporte purement élastiquement, que les flexions restent faibles et que les barres sont initialement droites et sans excentricité à leurs extrémités.

Zusammenfassung

In dieser Arbeit wird die Beziehung zwischen den sekundären Biegungsmomenten und der Knicklast eines Fachwerks und die Beziehung zwischen der Variation dieser Momente und den Deformationen (wie Stabdrehwinkel und Wendepunkte) von Streben in statisch bestimmten Fachwerken unter einer Last, die bis zum Instabilitätspunkt anwächst, untersucht.

Es wird grundlegend angenommen, daß das Fachwerk rein elastisch wirkt, daß die Durchbiegungen klein bleiben und daß die Streben ursprünglich gerade und ohne Endexzentrizitäten sind.