Zeitschrift: IABSE publications = Mémoires AIPC = IVBH Abhandlungen

Band: 19 (1959)

Artikel: Theory of trusses

Autor: Asplund, S.O.

DOI: https://doi.org/10.5169/seals-16947

Theory of Trusses

Théorie du treillis

Fachwerktheorie

S. O. ASPLUND
Prof. Dr., Gothenburg, Sweden

1. General Considerations

Theoretically, trusses are elastic structures made up of bars that are connected to each other by end hinges or sockets, forming joints, upon which the loads are applied. The bar-forces of simple trusses are analyzed by Culmann's or Ritter's Method of Sections and by Maxwell's Force Diagram. The joint-displacements are analyzed by Williot's Displacement Diagram and by Mohr's Rotation Diagram. In the analysis of bar-forces in "complex" and in statically indeterminate trusses, Henneberg's bar exchange method and the force method for statically indeterminate structures are generally employed.

A matrix treatment of all these methods will be developed here. It will reveal that Maxwell's force diagram and Williot's displacement diagram, when properly applied to the same structure, are *dual* methods. Likewise, the method of sections for determining the support bar forces and Mohr's rotation diagram for satisfying the support displacement conditions are *dual* methods. Finally a bar exchange method for displacements, *dual* to Henneberg's bar exchange method, will be presented.

These dualities in particular will show up by the fact that the matrices that govern the corresponding dual force and displacement methods are transposes. The main effort in the application of any of these methods lies in the inversion of its matrix. If duality is taken advantage of, not more than one inversion needs to be done for both dual methods. In other words, after the inverse of the bar-force matrix is calculated, its transpose is applicable as an inverse of the joint-displacement matrix. Thereby, a large amount of work may be saved in the displacement calculation.

The analysis of bar-forces in loaded trusses in space can be generally treated by the method of writing three equilibrium equations for each joint. Including the support bars, a statically determinate truss contains three times as many bars as joints J. Consequently, for such trusses the number of joint equilibrium equations suffices for solving all bar forces. This general method of solution requires the inversion of a square matrix of the same order $3\,J$ as the number of bars in the truss. For a medium or large truss this number is high, so the method is generally numerically prohibitive to either hand or computer calculations. Other methods of solving the problem must be sought along the lines taken by Maxwell and Williot.

Fortunately, trusses used in structures follow or nearly follow a definite pattern that permits simple analysis. Such trusses therefore are called "simple". This pattern is the following: In the *plane*, from a "base" of two joints 1, 2 with a connecting bar, each additional joint j is fixed in a bipod manner by *two* bars from any two of the *previously* established joints 1, 2, 3, ., $j-1$, fig. 1.

In *space*, from a "base" of three joints 1, 2, 3 with three connecting rods, each additional joint j is fixed in a tripod manner by three bars from any *three* of the previously established joints 1, 2, ., $j-1$.

Under this definition the K-braced truss, fig. 2a, and the subdivided truss, fig. 2b, are simple trusses.

Whenever the build-up does *not* follow such a scheme, the truss is called *complex*. A complex truss may often be reduced to a tripod truss by taking away some bars and inserting the same number of new bars between appropriate joints. The latter bars are called "*exchange bars*". The complex structure may be analyzed by way of the simple structure so formed.

By supporting a free simple space truss or a "tripod truss" by six *support*

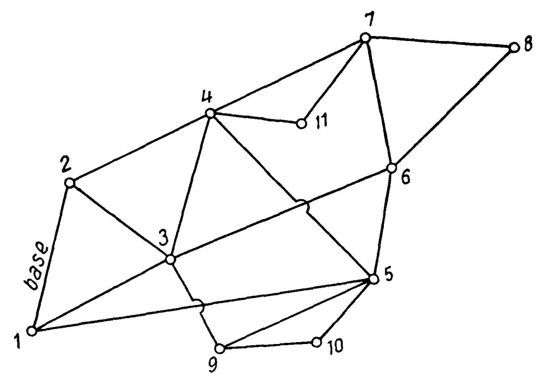

Fig. 1. Free Simple Truss.

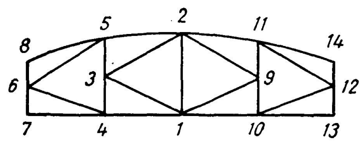

Fig. 2a. K-Braced Truss.

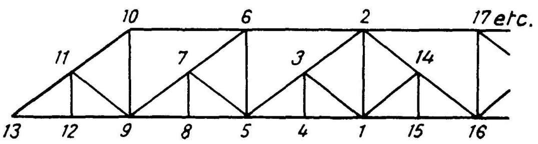

Fig. 2b. Subdivided Truss.

bars, a statically determinate simple structure is obtained. A free "bipod truss" in the plane must be supported by three support bars to form a statically determinate simple structure. Otherwise the treatment of plane trusses will follow the same general methods that will next be developed for trusses in space.

2a The Method of Joints

The joints in the free tripod truss are treated in the reverse order of build-up of the chain, starting with the highest numbered joint J and working toward the base. Renumbering, temporarily, this tip by d and the opposite ends of the connecting bars by a, b, c there will be three bars $a\,d$, $b\,d$, $c\,d$ with unknown forces, fig. 3.

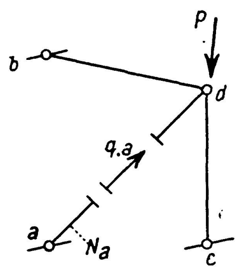

Fig. 3. The Method of Joints: Solving Forces in Three Bars.

The direction cosines $q_{.a}$, $(a\,b\,c)$[1] of the bars $a\,d$, $b\,d$, $c\,d$, are written as columns of a 3×3-matrix q. Joint d with the external load P is at rest when $P_i - \Sigma_{abc} q_{ia} N_a = 0$, $i = 1, 2, 3$, or in matrix form together with solution,

$$P - q\,N = 0, \qquad N = q^{-1}\,P, \tag{1}$$

that solves the bar-forces N_a, N_b, N_c. The components $q_{.a} N_a$ of the bar force N_a in the bar $a\,d$ are subtracted from the external load components P at joint d and added to the external load components at the joint a. The components of the bar forces N_b, N_c, are treated analogously.

Thereafter, the next highest numbered joint in the truss, $d = J - 1$, will contain only three unknown bar forces. Now it is possible to treat this joint in the same way as joint J was just treated. The whole tripod truss can be run through by this method. After all joints of the structure have been treated accordingly, it is evident that each of the three load components at every joint should be reduced to 0. This forms a substantial check upon the whole calculation as well as upon the calculation in article 3a of support forces by the method of sections.

---

[1]) The symbol $(a\,b\,c)$ indicates that three expressions are intended with a, b, c cyclically permuted.

2b. Joint Displacements by Williot's Method

To gain the advantage of using in the deformation analysis the same matrices as in the analysis of bar-forces, and particularly to gain the advantage of using their inversion already found, it is most appropriate to undertake the analysis of displacements in exactly the opposite order of joints as was used in the bar-force analysis. A simple truss that "cantilevers" from its base of three joints with zero or known displacements can be first treated by Williot's method.

The primary analysis of diplacements starts from the three "base" joints, 1, 2, 3, the diplacements of which are known (or else tentatively assumed) and it proceeds to find consecutively the displacements of joints 4, 5, etc. up to the highest numbered joint J of the truss. Thus, at one instance the displacement of the joints a, b, c, fig. 6a, is given by vectors $u_{.a}$, $(a\,b\,c)$, that have three components each: u_{1a}, u_{2a}, u_{3a}. The three joints are connected by bars to a fourth joint. The bars $a\,d$, fig. 4b, elongate by ΔL_a and the displacement of d is sought.

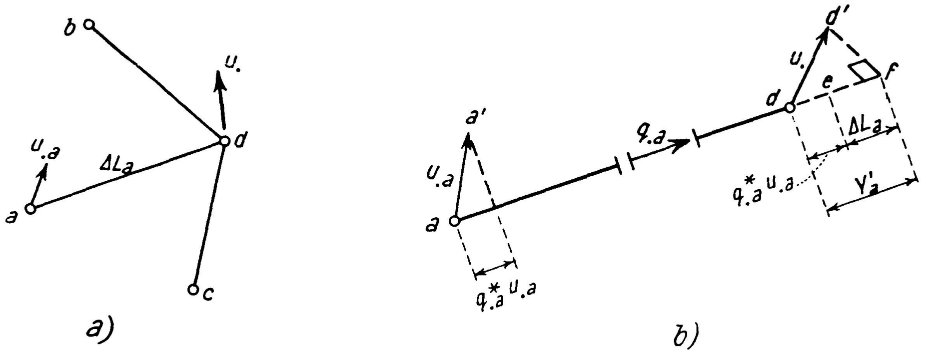

Fig. 4. Williot's Method, Solving the Displacements of a Joint.

The unit vector $q_{.a}$ leading from joint a towards joint d is the same as that which was used in the preceding analysis of bar-forces. The component of $u_{.a}$, fig. 4b, along the bar $a\,d$ is $q^*_{.a} u_{.a}$ where $q^*_{.a}$ denotes the transpose of the column vector $q_{.a}$. If the joint at d is unmade this is equivalent to a displacement of the bar-end from d to e. Adding the bar elongation ΔL_a, the bar-end moves to f. The distance $d\,f$ is also the projection upon $q_{.a}$ of the real movement $d\,d'$ or $u_{.d}$ of the joint d:

$$q^*_{.a} u_{.d} = q^*_{.a} u_{.a} + \Delta L_a = Y_a. \tag{2}$$

One such equation for each of the bars, $a\,d$, $b\,d$, $c\,d$, makes it possible to solve the three components $u_{.d}$ of the movement of the joint d, or in matrices

$$q^* u = Y, \qquad u = q^{-1*}\, Y. \tag{3}$$

Since the same joints and bars are used, q^* is the exact transpose of the matrix q of (1). The inverse calculated in (1) can also be used again in (3).

The calculation of joint-displacements in a simple truss that is not cantilevered from its base, as a rule must start by assuming three arbitrary movements for the three joints at the base end of the free tripod truss. The method can then be applied successively through to the highest numbered joint in the tripod chain. It yields in consecutive order all the joint displacements.

3a. The Method of Sections

In case the simple truss is not cantilevered from its base but is statically determinately supported by six bars connected at three to six arbitrary joints, the forces upon the truss from these six bars will first have to be determined and introduced as joint loads upon the free simple truss. Then the method 2a can again be carried through.

Consider a stable structure, statically determinately supported by six support bars with hinged ends. Each support bar, a, $(a\,b\,c\,d\,e\,f)$, fig. 5, is geometrically defined by a radius vector $q'_{.a}$ from an origin 0 to the support joint and by a direction unit vector $q_{.a}$ along the support bar. The loads

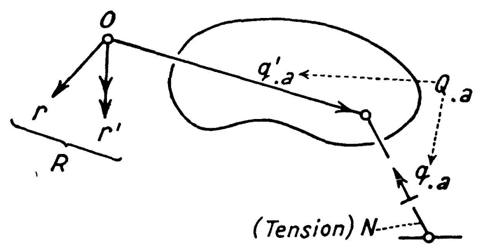

Fig. 5. To Calculate Six Support Bar Forces.

upon the truss are reduced to a force vector r and a moment vector r' at the origin 0 of a coordinate frame 0 1 2 3. This load upon the truss induces axial forces N_a in the six support bars. Their reactions upon the truss are included among the external loads P upon the free structure. Its equilibrium demands that[2])

$$r - qN = 0, \qquad r' - q'^0 qN = 0. \tag{4}$$

Writing

$$R = \begin{bmatrix} r \\ r' \end{bmatrix}, \qquad Q = \begin{bmatrix} q \\ q'^0 q \end{bmatrix} \tag{5}$$

makes it possible to combine (4) into

$$R = QN, \qquad N = Q^{-1} R, \tag{6}$$

---

[2]) The antisymmetric "vector matrix" Q^0, contains the three components of the vector Q in such a way that the product $Q^0 q$ becomes the vector product of the vectors Q and q.

$$Q^0 = \begin{bmatrix} 0 & -Q_3 & Q_2 \\ Q_3 & 0 & -Q_1 \\ -Q_2 & Q_1 & 0 \end{bmatrix}$$

by which the forces in all six support bars are solved. After this, the support bar reactions $-q_{.a} N_a$ are introduced among the external loads P upon the free space truss, that is now balanced in equilibrium by the set of given joint loads and their support joint loads.

3b. Mohr's Rigid Rotation

If a simple truss is not cantilevered from its base, the truss is made free from its supports. Compatible but otherwise arbitrary displacements of its base joints are assumed and the displacement u of all joints found by article 2b.

Mohr's rigid rotation, embodied in his plane rotation diagram, is then superposed upon the Williot displacements u in order to make a structure satisfy six displacement conditions at its supports.

This additional general rigid displacement, fig. 6, is a translation v and a rotation v' about an origin 0.

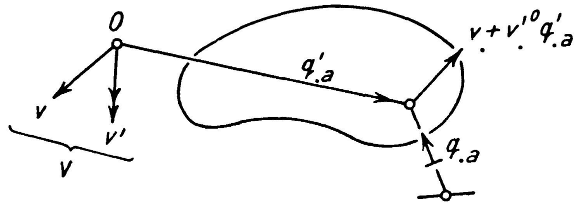

Fig. 6. Mohr's Rigid Displacement to Satisfy Six Support Conditions.

The support joints are defined by radius vectors $q'_{.a}$, and the direction of the support bars by the unit vectors $q_{.a}$. The movement of any support joint is given by the vector $v+v'^0 q'_{.a}$, $(a\,b\,c\,d\,e\,f)$. To that is added Williot's previously found displacement u. The resulting total movement *along* the bar must be consistent with the elongation ΔL_a of the support bar.

$$q^* (v - q'^0 v' + u) = \Delta L,$$

$$[q^*,\ (q'^0 q)^*] \begin{bmatrix} v \\ v' \end{bmatrix} = \Delta L - q^* u = Y, \qquad \begin{bmatrix} q \\ q'^0 q \end{bmatrix}^* V = Y, \qquad V = \begin{bmatrix} v \\ v' \end{bmatrix}$$

by (2), assuming the elongation ΔL of the support bar and $u_{.a}$ to be zero. This holds for six support bars. Then by (2)

$$Q^* V = Y, \qquad V = Q^{-1*} Y. \tag{7}$$

For the same support bars it is immediately seen that the matrix Q^* of this system of equations is the transpose of the matrix Q of the method of article 3a. The matrix equation is solved by inversion as indicated in (7). This inversion was already made under article 3a from which almost all numerical computations that are required in the solution of the problem of this article thus can be borrowed.

4a. Henneberg's Bar Exchange Method

The bar-force analysis according to article 3a, 2a cannot be carried through in a complex (non-tripod) truss. For example, Maxwell's force diagram cannot be drawn past the crown hinge of a three-hinged arch.

As a rule, it is possible by a comparatively small number of exchange bars, defined under article 1, to transform complex trusses, fig. 7a, that are used in structural engineering into tripod trusses. The introduction of each exchange or "Henneberg" bar implies that one bar i in the given complex truss is severed, fig. 5b, and that another rigid bar r is introduced between two appropriate joints.

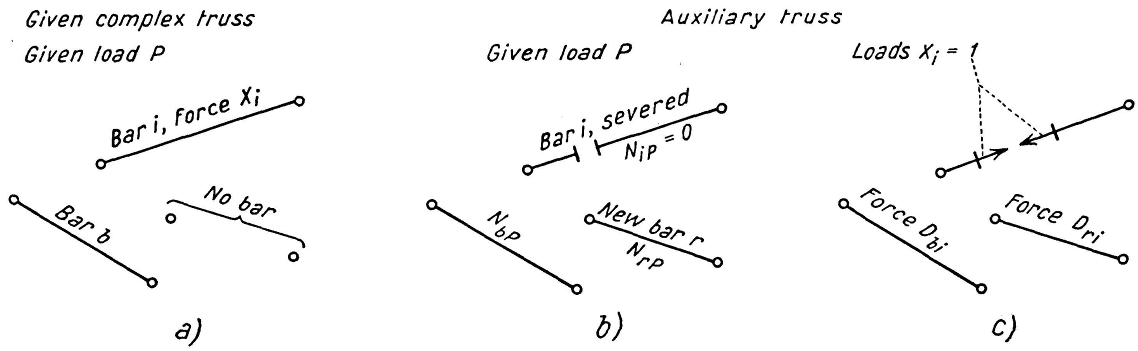

Fig. 7. Henneberg's Method: Solving Bar-Forces in Complex Trusses.

The loads P of the given complex truss are applied to this transformed simple "auxiliary truss", fig. 7b. Since the auxiliary truss is a tripod truss and possible of calculation, its bar forces N_{bP} are first calculated. Then the auxiliary structure is instead loaded by two equal and opposite axial forces at the ends of the severed bar i, fig. 7c, and its bar forces D_{bi} are calculated. This is repeated for each severed bar i.

The forces in the given truss will now be simulated in the auxiliary truss by loading this with the given loads P and by double axial forces X_i on the ends of all severed bars, X_i being the axial force in this bar in the given structure. Under such a loading, the forces N_r in all the added Henneberg bars become zero:

$$N_r = N_{rP} + \sum D_{ri} X_i = 0 \tag{8}$$

or in matrix form, together with solution,

$$N_P + DX = 0, \qquad X = -D^{-1} N_P. \tag{9}$$

After all values of X thus are solved, the force in any bar of the given complex truss is contained in the formula

$$N_b = N_{bP} + \sum D_{bi} X_i. \tag{10}$$

The treatment of bar-forces in statically indeterminate trusses will be deferred until the section 6 of this paper.

4b. Displacement Exchange Bars

In complex structures, the method of article 2b, 3b cannot be continuously applied through the structure. For example, Williot's displacement diagram cannot be drawn past the crown hinge in a three-hinged arch. In the analysis of bar forces, the corresponding difficulty can be overcome by the use of Henneberg exchange bars. In complete analogy to that method, *a dual method of displacement exchange bars exists.*

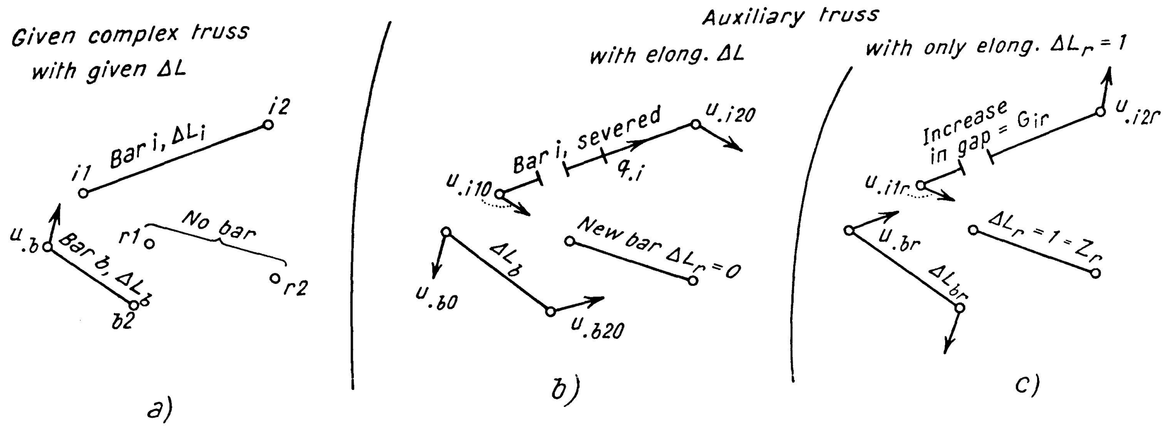

Fig. 8. The Method of Displacement Exchange Bars: To solve the Displacements in a Complex Truss.

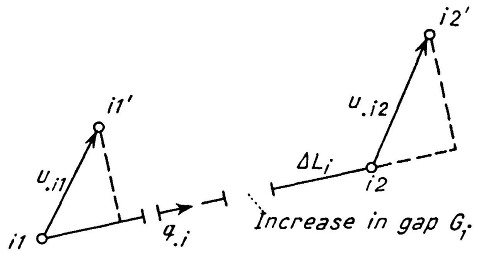

Fig. 9.

The sketch, fig. 8a, represents some bars in a complex truss. In order to make it a simple truss, suitable bars i are severed and the same number of other rigid bars, r, are introduced, fig. 8b. In order to utilize the same matrix inversion as in article 4, the same bars are again severed and introduced as there.

This auxiliary truss is now "deformed", fig. 8b, by the same bar elongations $\varDelta\, L_b$ as in the given truss. All joint displacements $u_{.b0}$ in the transformed auxiliary structure are determined by Williot's and Mohr's methods. It is now possible to determine the increase G_{i0} in the gap of every severed bar, i, by the equation, fig. 9,

$$G_{i0} = q_{.i}^{*}\,(u_{.i20} - u_{.i10}) - \varDelta\, L_i, \tag{11}$$

$i\,1$ and $i\,2$ being the end-joints of the bar i.

Secondly, the auxiliary truss is deformed, fig. 8c, by a sole bar-elongation $\Delta L_r = 1$. The joint displacements $u_{.br}$ are first constructed according to Williot-Mohr, and then the increases in gaps are calculated by the formula

$$G_{ir} = q_{.i}^* (u_{.i2r} - u_{.i1r}).$$

This is repeated for each displacement exchange bar r.

Note that Maxwell-Mohr's work equation, see article 5, applied to the two loading and deformation conditions of the identical auxiliary structures, fig. 5c and fig. 8c, gives

$$-1 \cdot G_{ir} + D_{ri} \cdot 1 = 0, \qquad G = D^*. \tag{12}$$

Thus the matrix G of this problem is the transpose of the matrix D of Henneberg's bar exchange method, article 4a.

Applying, instead of unit elongations, individual elongations Z_r to all the exchange bars, and superposing their corresponding deformation upon that caused by the given elongations ΔL_b in the auxiliary structure, the increase of the gaps in the severed bars are $G_{i0} + \sum G_{ir} Z_r$. Specifically, all U can be chosen so that all gaps will be increased by zero. This gives the matrix equation

$$G_0 + D^* Z = 0, \qquad Z = -D^{-1\,*} G_0 \tag{13}$$

with the solution as indicated. Its matrix was already inverted in article 4a. The exchange-bar elongations Y_r thus found, it is again possible to calculate the joint displacements continuously through the truss. Otherwise, the calculations already made can be used in conjunction with

$$u_{.b} = u_{.b0} + \sum u_{.br} Z_r$$

to find the displacements of each joint in the given truss. In the final stage of the application of this method the severed bars i can be mended again since their change in gap is zero; and all the introduced bars r can be removed since their elongations are fully consistent with the displacements of their end-joints.

5. Formulation of Maxwell-Mohr's Work Equation

A truss, fig. 10a, is loaded by given joint loads and by their support bar reactions P. All these loads induce bar-forces N in the truss. All loads P and bar forces N that act upon a joint will balance. Also, both forces N that act upon the ends of a bar will balance the bar.

Entirely independent of these loads and forces, the truss is assumed to undergo a deformation fig. 10b characterized by joint displacements u' and *consistent* bar elongations $\Delta L'$. In this deformation, the independent loads and forces previously discussed in fig. 10a will do virtual work. The work will be calculated and totaled for two different groupings of the same forces.

First, all forces and loads acting upon one joint are grouped together, fig. 10a, and moved by the independent displacement u' of that joint in fig. 10b. Since the forces are in equilibrium, their total virtual work will be zero. The total virtual work for all other joints will also be zero: There are no other loads or forces, so the total virtual work in the truss is zero: $V_1 = 0$.

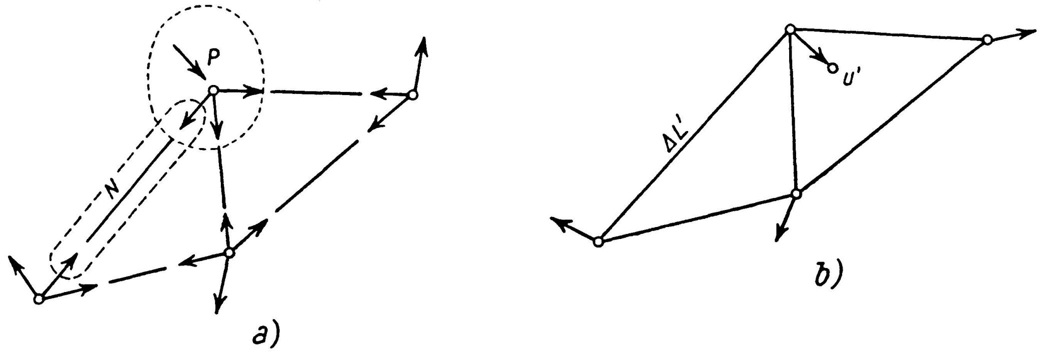

Fig. 10. Virtual Work in Truss.

Secondly, the forces N acting upon each end of one bar are grouped together, making for this bar a virtual work of $-N\Delta L'$, where $\Delta L'$ denotes the elongation of the bar (that is consistent with its joint displacements u'). This work is calculated and totaled for all bars, leaving behind only the external loads P that are applied at the joints. In their displacements u' these loads produce the virtual work $P^* u'$. The total virtual work in this second grouping will be $V_2 = -\sum N\Delta L' + \sum P^* u'$.

Arraying all load and displacement components in two corresponding $3J \times 1$-matrices P, u', and all bar-forces and bar-elongations in two corresponding $B \times 1$ matrices N, $\Delta L'$, and equating the total virtual work in both groupings, the equation is obtained $V_2 = 0$:

$$P^* u' = N^* \Delta L', \qquad V_e = V_i. \tag{14}$$

Equation (5, 3) is the famous work equation of Maxwell-Mohr. It tells that the total virtual work V_e of the external loads equals the total virtual strain work V_i of the internal bar forces.

6. Matrix Analysis of a Statically Indeterminate Truss

Finally, the matrix solution[3]) of a statically indeterminate truss by the force method will be explained.

1. Fig. 11 represents, abstractly, a statically indeterminate truss under various loads. Fig. 11a shows only two bars, i and b, of the given truss upon

[3]) S. O. Asplund, Matrix Formulation of Hyperstatic Analysis, Publication in Honour of Professor Carl Forssell, Stockholm 1956, p. 19.

which given external joint loads P, temperature changes t, and initial strains ϵ_0 are acting.

2. In the given truss, R redundant bars i are severed to obtain an auxiliary truss, fig. 11 b, that is statically determinate or simply calculable. For practical reasons, such bars are severed that the auxiliary truss will afterwards need a minimum number of exchange bars to be transformed into a tripod truss.

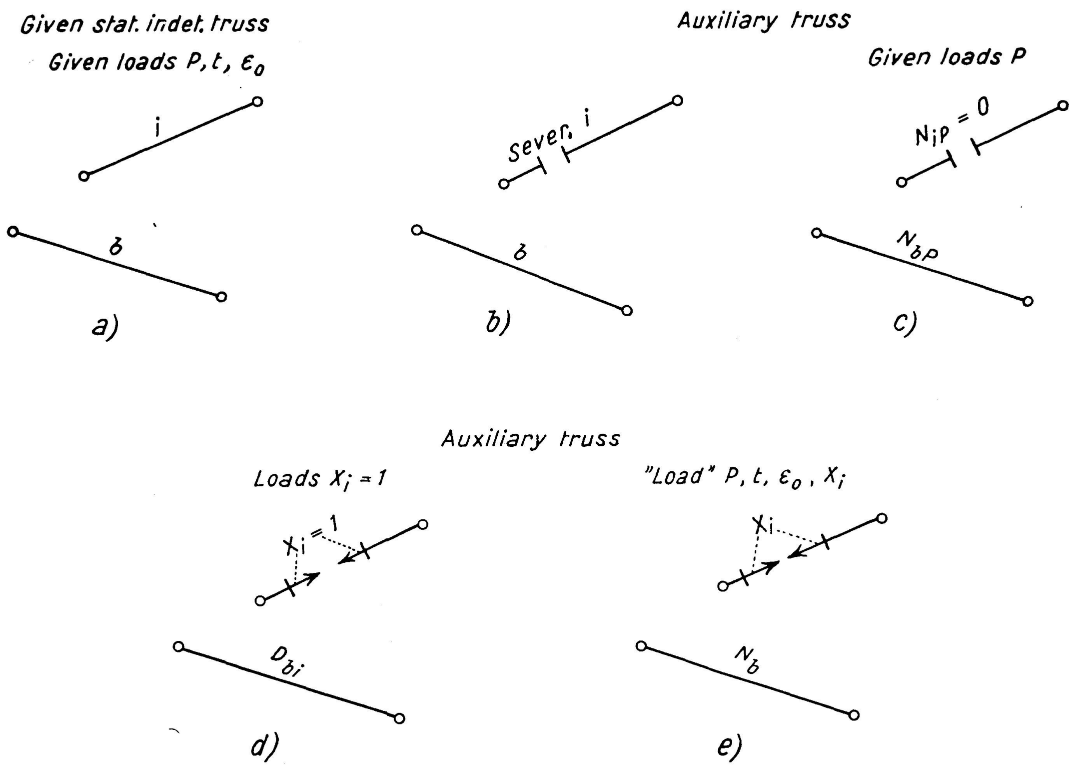

Fig. 11. Solution of Statically Indeterminate Truss.

3. This auxiliary truss is now loaded by the given loads P, fig. 11 c, and the bar forces N_{bP} evaluated by the methods given in articles 2a, 3a, 4a. In doing this, the bar-forces may be evaluated separately for each independent unit component of the load P and combined into a $B \times C$-matrix A, where C is the number of independent load components P. It is seen that the column N_P of bar-forces caused by the given loading P is equal to $A\,P$.

4. In the gap of a severed bar i, two axial loads $X_i = 1$ are inserted, fig. 11 d. With no other load on the auxiliary truss, the column D_{bi} of bar-forces are calculated. This is repeated for each severed bar i. All columns D_{bi} can be combined into a rectangular $B \times R$-matrix D. It is seen that each column of D is a linear aggregate of four columns of A, except that each element D_{ii} will be equal to one. (This D is not the same as D in article 4a but can be made to include it.)

5. Superposition upon the auxiliary truss of the given loading P and the severed bar loads X_i (instead of one), results in bar forces $N_b = N_{bP} + \sum D_{bi} X_i$,

or in matrices

$$N = N_P + D\,X = A\,P + D\,X. \tag{15}$$

6. The auxiliary truss is now loaded by the given "loads" P, t, ϵ_0 and by the external loads X that are adjusted to make the change u_i in gaps of these bars all zero. Maxwell-Mohr's work equation is applied to the auxiliary structure loaded as under point 4 and point 6. The work equation $\sum P\,u' = \sum N\,\Delta\,L'$ then becomes $1 \cdot 0 = \sum D_{bi} \Delta\,L'$, or

$$\sum D_{bi} \left(\frac{L_b}{E\,A_b} N_b + \omega\,t_b\,L_b + \epsilon_{0b}\,L_b \right) = 0. \tag{16}$$

7. Arraying $L_b/E\,A_b$, t_b, L_b, ϵ_{0b} into diagonal matrices F, t, L, ϵ_0 this last equation becomes

$$D^* (F\,A\,P + F\,D\,X + \omega\,t\,L + \epsilon_0\,L) = 0. \tag{17}$$

Premultiplication by $(D^*\,F\,D)^{-1} = C$ solves X:

$$X = -C\,D^* (F\,A\,P + \omega\,t\,L + \epsilon_0\,L). \tag{18}$$

This value is again inserted in the above expression for N, resulting in a final closed matrix expression for all bar forces in the structure, as a linear aggregate in the given loading P and the given temperature changes and initial strains. In this equation all reference to the redundants X is eliminated:

$$N = (I - D\,C\,D^*\,F)\,A\,P - D\,C\,D^* (\omega\,t + \epsilon_0)\,L. \tag{19}$$

For an example of the application to hand calculation of this method for calculating statically indeterminate trusses, the reader is referred to a previous publication by the writer[3]). The method is practically applicable to the computer calculation of indeterminate trusses, either in conjunction with the methods given here for solving statically determinate trusses or not.

Summary

A free simple truss is defined as a succession of tripods, each fixing a new joint by three bars from three previously fixed joints. The joints are numbered in order of definition. The "base" joints 1, 2, 3 (1, 2 in the plane) are connected by "base" bars. Six support bars, connecting "support joints" to foundation hinges, make this free simple truss a statically determinate "simple" truss. Any statically determinate ($B - 3J = 0$) truss that is not simple is called "complex". By adding bars to a complex truss any general statically indeterminate truss may be obtained.

In the simple truss a 6×6-matrix Q of support bar geometrical elements is established. It performs the counterpart in space to Ritter-Culmann's method of sections. Q^* also performs the counterpart to Mohr's method of

rigid displacement. For each tripod a geometric 3×3-matrix q expresses the counterparts of the main operations in Maxwell's force diagram and in Williot's displacement diagram.

In a loaded simple truss the support bar forces are first found by the matrix Q^{-1}. The bar forces are added to the support-joint loads. This makes a free simple truss in equilibrium under joint loads. By the use of its matrix q^{-1}, the load upon the highest numbered joint is distributed upon its tripod bars. The components of these bar-forces are added to the load components of the joint loads at either ends of the three tripod bars. The next lower-numbered joint is treated similarly, etc., down to the base. All these calculations of support bar and other bar forces are checked by that all joint load components are finally reduced to zero.

Base-joint displacements, compatible with the base-bar elongations but otherwise arbitrary, are assumed. Given elongations of the bars from the base-joints 1, 2, 3 to joint 4 permit the calculation of joint displacement 4 by the use of the matrix q^{-1*}. This analysis is continued in order of increasing joint numbers, 5, 6, etc., up to the highest numbered joint. The support displacement conditions are finally satisfied by superimposing Mohr's rigid displacement of the whole truss, found by application of the matrix Q^{-1*}.

A given statically indeterminate truss is analyzed by severing $B-3J$ well chosen bars to obtain a statically determinate truss that is as little complex as possible. By Henneberg bar exchanges it is further transformed into a simple truss. The given joint-load components P applied to the simple truss cause in its bars the forces AP, A being a (geometric) matrix obtained by the preceeding method. Pairs of unit loads on the ends of each severed Henneberg bar and on the ends of each severed redundant bar yield all rows in (geometric) matrices D of bar-forces. Application of Maxwell-Mohr's work equation combines A, D, P into a matrix expression for the column matrix N of the forces in all the bars in a given statically indeterminate truss. This expression gives N in terms of arbitrary joint-loads P, temperature increases t and initial strains ϵ_0.

Résumé

Le treillis libre du type le plus simple est défini comme une succession de trépieds, dans lesquels chaque nouveau nœud résulte de trois nœuds antérieurs. Les nœuds doivent être numérotés suivant l'ordre de détermination. Les nœuds de base 1, 2 et 3 (1 et 2 pour treillis plans) sont associés par des barres de base. Six barres qui assemblent les nœuds d'appui avec les articulations d'appui, font de ce treillis libre du type le plus simple un treillis isostatique du type le plus simple. Tout treillis isostatique ($B-3J=0$) qui n'est pas du type le plus

simple porte la désignation de «complexe». Par addition de barres dans un treillis complexe, il est possible de le transformer en tout treillis hyperstatique.

Pour ce treillis du type le plus simple, on considère une matrice Q 6×6 pour les élements géométriques des barres d'appui. Ceci correspond dans l'espace à la méthode des coupes virtuelles de Ritter-Culmann. Q^* est également la correspondante de la méthode de Mohr des déplacements rigides. Pour chaque trépied, une matrice géométrique 3×3 exprime la correspondante des opérations principales dans le diagramme d'efforts de Maxwell et dans le plan déplacement de Williot.

Dans un treillis chargé du type le plus simple, les efforts dans les barres d'appui sont tout d'abord définis par la matrice Q^{-1}. Ces efforts sont additionnées aux charges des nœuds d'appui. Ceci donne un treillis libre du type le plus simple en équilibre sous les charges nodales. Lorsque l'on emploie sa matrice q^{-1}, la charge sur le nœud d'ordre le plus élevé peut être répartie sur ses 3 barres formant trépied. Les composantes de ces efforts de barre sont superposées avec celles des charges des nœuds aux deux extrémités des 3 barres. Le nœud d'ordre immédiatement inférieur est ensuite traité d'une manière analogue et ainsi de suite jusqu'à la base. Tous ces calculs des efforts dans les barres d'appui et dans les autres barres sont vérifiés par la condition que toutes les charges nodales doivent en fin de compte être réduites à 0.

On considère en outre des déplacements des nœuds de base qui sont arbitraires, sous réserve de compatibilité avec les prolongements des barres de base. Les prolongements donnés des barres entre les points de base 1, 2 et 3 permettent, à l'aide de la matrice q^{-1*}, le calcul du deplacement du point 4. Cette investigation est poursuivie suivant la séquence d'ordre croissant sur 5, 6 et jusqu'à l'ordre le plus élevé. Les conditions de déplacement des appuis sont ensuite satisfaites par superposition à l'aide de la matrice Q^{-1*} du déplacement rigide de Mohr pour l'ensemble du treillis.

Un treillis hyperstatique donné est transformé en un treillis isostatique autant que possible moins complexe, par séparation de $B-3J$ barres opportunément choisies. La transformation en un treillis du type le plus simple est poursuivie par la méthode de Henneberg de permutation des barres. Les composantes données P des charges nodales appliquées à ce treillis du type le plus simple donnent les efforts AP dans les barres, A étant une matrice géométrique que l'on obtient d'après la méthode précédemment indiquée. Des efforts doubles aux extrémités de chaque barre de Henneberg coupée et aux extrémités de chaque barre surabondante donnent toutes les séries dans les matrices géométriques H et D des efforts dans les barres. Par application de l'équation de travail de Maxwell-Mohr, A, H et D sont combinés sous la forme d'une expression de matrice, pour la matrice en colonne N pour les efforts dans toutes les barres du treillis hyperstatique donné. Cette expression fournit N pour des données arbitraires concernant les charges nodales P, l'élévation de température ou l'allongement initial ϵ_0.

Zusammenfassung

Ein freies Fachwerk einfachster Art wird definiert als eine Folge von Dreifüßen, mit denen jeweils ein neuer Knotenpunkt, ausgehend von drei schon Gegebenen, bestimmt wird. Die Knotenpunkte sollen in der Reihenfolge des Aufbaues numeriert werden. Die Basisknotenpunkte 1, 2 und 3 (1 und 2 für ebene Fachwrerke) werden durch Basisstreben verbunden. Sechs Stäbe, die die Auflagerknotenpunkte mit den Auflagergelenken verbinden, machen aus diesem freien Fachwerk einfachster Art ein statisch bestimmtes Fachwerk einfachster Art. Jedes statisch bestimmte ($B-3J=0$) Fachwerk, das nicht einfachster Art ist, wird «komplex» genannt. Durch Addition von Stäben in einem komplexen Fachwerk kann jedes mögliche, statisch unbestimmte Fachwerk erhalten werden.

Für dasjenige einfachster Art wird eine 6×6-Matrix Q für die geometrischen Elemente der Stützstreben aufgestellt. Dies entspricht im Raume der Schnittmethode von Ritter-Culmann. Q^* ist ebenso das Entsprechende zur Mohrschen Methode der starren Verschiebungen. Für jeden Dreifuß drückt eine geometrische 3×3-Matrix das Entsprechende der Hauptoperationen im Maxwellschen Kräftediagramm und im Williotschen Verschiebungsplan aus.

In einem belasteten Fachwerk einfachster Art werden die Auflagerstabkräfte zuerst mit der Matrix Q^{-1} bestimmt. Diese Stabkräfte werden zu den Auflagerknotenlasten gezählt. Dies ergibt ein freies Fachwerk einfachster Art im Gleichgewicht unter Knotenlasten. Mit der Verwendung seiner Matrix q^{-1} kann die Last auf den Knoten mit der höchsten Ordnungszahl auf seine 3 Dreifußstreben verteilt werden. Die Komponenten dieser Stabkräfte werden mit denjenigen der Knotenlasten an beiden Enden der 3 Stäbe superponiert. Dann wird der Knoten mit der nächst kleineren Ordnungszahl analog behandelt usw., bis zur Basis. Alle diese Berechnungen der Auflagerstabkräfte und anderen Stabkräfte werden durch die Bedingung, daß sämtliche Knotenlasten endlich auf 0 reduziert werden müssen, kontrolliert.

Es werden weiterhin Basisknotenverschiebungen, nur verträglich mit den Verlängerungen der Basisstreben, sonst aber beliebig, angenommen. Gegebene Stabverlängerungen zwischen den Basispunkten 1, 2 und 3 gestatten mit Hilfe der Matrix q^{-1*} die Berechnung der Verschiebung des Punktes 4. Diese Untersuchung wird in der Reihenfolge der steigenden Ordnungszahlen 5, 6 bis zur höchsten durchgeführt. Die Auflagerverschiebungsbedingungen werden dann erfüllt durch Superposition mit Hilfe der Matrix Q^{-1*} von Mohrs starrer Verschiebung des ganzen Fachwerks.

Ein gegebenes, statisch unbestimmtes Fachwerk wird durch Auftrennung von $B-3J$ geschickt gewählten Stäben in ein möglichst wenig komplexes, statisch bestimmtes Fachwerk umgewandelt. Durch Hennebergs Stabvertauschungen wird die Transformation in ein Fachwerk einfachster Art weitergeführt. Die gegebenen Knotenlastkomponenten P an diesem Fachwerk ein-

fachster Art angesetzt, ergeben die Stabkräfte AP, wo A eine geometrische Matrix, die nach der vorangegangenen Methode erhalten wird, bedeutet. Doppelkräfte an den Enden jedes geschnittenen Henneberg-Stabes und an den Enden jedes überzähligen Stabes ergeben alle Reihen in den geometrischen Matrizen D der Stabkräfte. Durch Anwendung der Maxwell-Mohrschen Arbeitsgleichung werden A und D zu einem Matrizenausdruck für die Kolonnenmatrix N für die Kräfte in allen Stäben des gegebenen, statisch unbestimmten Fachwerkes kombiniert. Dieser Ausdruck ergibt N für beliebig gegebene Knotenlasten P, Temperatursteigerung t oder Initialdehnung ϵ_0.