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The Design of Frameworks to Give Specific Deflections

Etude d'un treillis à flèches imposées aux nœuds

Konstruktion eines Fachwerkes mit vorgegebenen Durchbiegungen

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The estimation of the deflected form of a loaded frame presents no major problems to the structural analyst. In the case of a statically-determinate (iso-static) frame the small extensions or contractions of the members are found from the internal loads in the members, and the resulting deflections of the frame are deduced. For a statically-indeterminate (hyperstatic) frame the internal loads are dependent on the relative stiffnesses of the members, and the estimation of the deflected form is more complex. In most problems of this type, the geometrical form of the frame, the stiffnesses of its component members, the external loading system, and the mode of support are usually defined completely.

Recently the authors encountered a deflection problem of a different type, and one which (so far as the authors are aware) has received little, or no, study in the past. In this problem, the system of external loads on the frame and the mode of support were defined completely; the general outline of the frame was also fixed, but within this outline the members were to be arranged, and given suitable stiffnesses, to ensure that the loaded joints of the frame deflected by specific amounts. The immediate problem is one of determining whether the specified deflections are physically attainable within the limitations imposed on the design of the frame. In general this problem is extremely complex, and especially so when both the geometrical form of the frame and the stiffnesses of the component members are adjustable.

The first problem studied is that of designing the component members of a *given* frame so that the loaded joints deflect by specific amounts. This problem is treated mathematically in Part I, and graphically in Part II.

When there is no restriction on the overall shape of the frame, then the

designer is free to dispose the members, as well as choose suitable stiffnesses, to give the specified deflections. The designer then becomes involved in fabricating, or "synthesising", the frame to give specific deflections. The problem of "synthesis" is discussed in Part III, in relation to a particular problem in which it was required to design a frame to give equal vertical deflections of the loaded joints.

Part I. Mathematical Analysis

1. *Introduction*

We discuss first the mathematical analysis of an iso-static frame which is to be designed to give specific deflections. We suppose that the relative positions of the members of the frame are fixed, as well as the external loading system and the conditions of support. The designer, however, is free to adjust the stiffnesses of the members to give the specific deflections.

2. *Frame Having two Members*

The simplest type of problem we can envisage is that of a pin-jointed frame consisting of two members, 1 and 2, which are attached to a rigid foundation at X and Y, fig. 1. When an external load P is applied at the joint A, the *tensile* loads induced in the members 1 and 2 are T_1 and T_2, respectively. If the resulting distortions of the frame are small, T_1 and T_2 are linearly related to P, and are determined by the geometry of the frame and the magnitude and direction of the force P; T_1 and T_2 are reckoned positive for members in tension, and negative for members in compression. Now a tensile member of a frame must extend; so that if we reckon an exten-

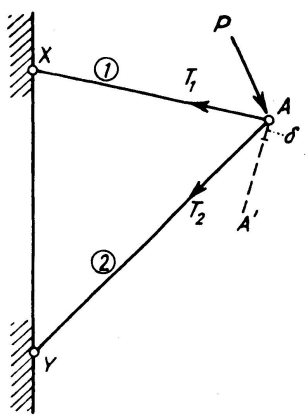


Fig. 1. Pin-Jointed Frame Having Two Members.

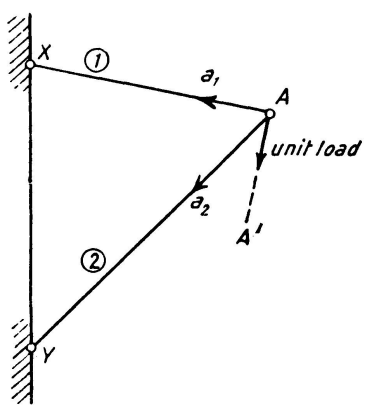


Fig. 2. Loads in the Members due to a Unit External Load at A in the Direction AA'.

sion of a member to correspond to a positive change of length, then we can always write the extension, e, of that member in the form

$$e = \lambda T, \tag{1}$$

where λ is positive. For a linear-elastic member λ is positive and constant; more generally, λ assumes a positive value which is a function of T.

Returning to the frame of fig. 1 suppose we wish to find the displacement, δ, of the joint A in the direction AA'. We evaluate first the loads in the members 1 and 2 due to a unit external load at A, in the direction AA', fig. 2; suppose the tensile loads induced in the members are a_1 and a_2, respectively. Positive values of a_1 and a_2 indicate tensile loads, and negative values compressive loads. On applying the principle of virtual work to the frames of figs. 1 and 2, we have

$$\delta = a_1 e_1 + a_2 e_2, \tag{2}$$

where e_1 and e_2 are the extensions of the members 1 and 2 due to the external load P. But if, from eq. (1), we write

$$e_1 = \lambda_1 T_1, \qquad e_2 = \lambda_2 T_2, \tag{3}$$

where λ_1 and λ_2 are *positive*, then eq. (2) becomes

$$\delta = (a_1 T_1)\lambda_1 + (a_2 T_2)\lambda_2. \tag{4}$$

On studying eq. (4) we can determine whether a specific value of δ can be attained in practice by a suitable choice of λ_1 and λ_2. Suppose first that the coefficients $(a_1 T_1)$, $(a_2 T_2)$ are both positive; a negative value of δ is then quite impracticable, but to give δ *any* positive value we have only to choose *any* value of λ_1 within the range

$$0 < \lambda_1 < \frac{\delta}{a_1 T_1}, \tag{5}$$

and deduce the corresponding value of λ_2 from eq. (4); alternatively, we can choose *any* value of λ_2 within the range

$$0 < \lambda_2 < \frac{\delta}{a_2 T_2}, \tag{6}$$

and deduce the corresponding value of λ_1 from eq. (4). If the coefficients $(a_1 T_1)$, $(a_2 T_2)$ are both negative, then δ must also be negative for a physically sensible solution of the problem; for this case λ_1 and λ_2 are again defined by the inequalities (5) and (6).

If one coefficient, $(a_1 T_1)$ say, in eq. (4) is positive, and the other, $(a_2 T_2)$, is negative, then for δ to be positive we must have

$$a_1 T_1 \lambda_1 > -a_2 T_2 \lambda_2. \tag{7}$$

If we now choose *any* positive value of λ_2 we can always derive a value of λ_1 satisfying the inequality (7); in this case, then, we have a wide choice of values of λ_1 and λ_2.

3. *Frame Having four Members*

We consider next a more complicated arrangement in which there are four members connecting two joints, A and B, to a rigid foundation at X and Y, fig. 3. We suppose the arrangement of members forms a just-stiff frame; then for loads P and Q at joints A and B, respectively, fig. 3 (I), the

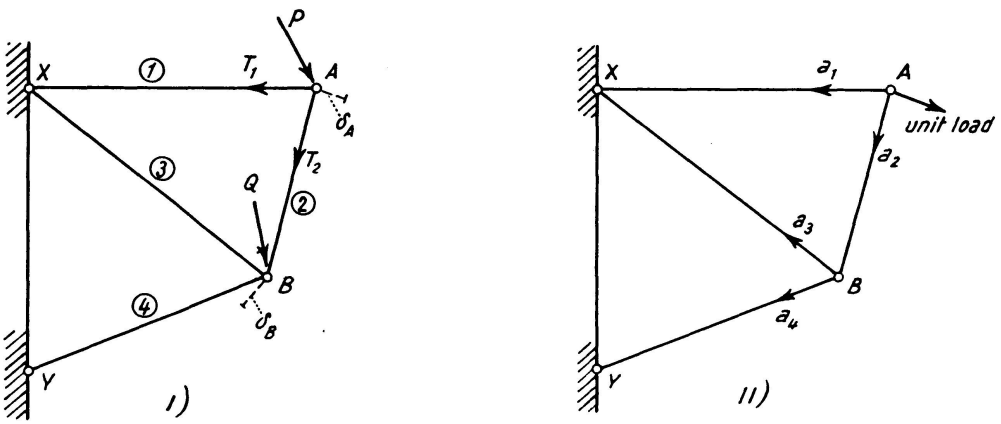


Fig. 3. Just-Stiff Frame Having Four Pin-Jointed Members.

frame is iso-static if distortions are small. Suppose the *tensile* loads in the members 1, 2, 3, 4, due to the external loads P and Q, are T_1, T_2, T_3, T_4, respectively; then these internal loads are linearly related to P and Q. We are interested in studying the deflection δ_A at A, and δ_B at B, having the directions shown in fig. 3 (I); to find δ_A, we apply a unit load at A in the direction of the displacement δ_A, as shown in fig. 3 (II). Suppose a unit load at A gives rise to *tensile* loads a_1, a_2, a_3, a_4, in the members of the frame. As before, we may write the *extensions* of the members of the frame of fig. 3 (I) in the forms

$$e_1 = \lambda_1 T_1, \qquad e_2 = \lambda_2 T_2, \qquad e_3 = \lambda_3 T_3, \qquad e_4 = \lambda_4 T_4, \tag{8}$$

in which λ_1, λ_2, λ_3, λ_4 are *positive* constants. Then for the deflection δ_A, we have, by the principle of virtual work,

$$\delta_A = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4, \tag{9}$$

or, on substituting for the extensions from eqs. (8),

$$\delta_A = (a_1 T_1)\lambda_1 + (a_2 T_2)\lambda_2 + (a_3 T_3)\lambda_3 + (a_4 T_4)\lambda_4. \tag{10}$$

Similarly, if b_1, b_2, b_3, b_4 are the *tensile* forces in the members due to a unit

external load at B, applied in the direction of the displacement δ_B in fig. 3 (I), then

$$\delta_B = (b_1 T_1)\lambda_1 + (b_2 T_2)\lambda_2 + (b_3 T_3)\lambda_3 + (b_4 T_4)\lambda_4 . \tag{11}$$

Consider now the design problem in which δ_A and δ_B are specified; is it then possible to find *positive* values of λ_1, λ_2, λ_3, λ_4 to give displacements δ_A and δ_B? We reduce eqs. (10) and (11) to the form

$$\begin{aligned}&(a_1 T_1 \delta_B - b_1 T_1 \delta_A)\lambda_1 + (a_2 T_2 \delta_B - b_2 T_2 \delta_A)\lambda_2 \\ &+ (a_3 T_3 \delta_B - b_3 T_3 \delta_A)\lambda_3 + (a_4 T_4 \delta_B - b_4 T_4 \delta_A)\lambda_4 = 0 .\end{aligned} \tag{12}$$

If, in eq. (12), at least one of the coefficients of the λ's is non-zero and positive, and at least one other is non-zero and negative, then *positive* values of λ_1, λ_2, λ_3, λ_4 can always be chosen to satisfy eq. (12). If all the coefficients of the λ's are of the same sign (positive or negative) at least one λ must be negative, which is physically inadmissible. When positive values of the λ's are possible, there is usually an infinite number of such solutions; this gives the range, or ranges, of values of the λ's within which the frame can be designed to give the specific deflections δ_A and δ_B. As an example of the application of the analysis developed so far, consider the simple framework of fig. 4; the whole system is symmetrical about a vertical centre-line. The frame is pinned to a rigid foundation at X, and supported on a roller at Y. From the conditions of symmetry there are only four relevant λ's, for the members 1, 2, 3, 4 indicated in fig. 4. (Members AB and BC are unloaded.) The specified design condition is that the joints A, B and C should deflect vertically by equal amounts δ.

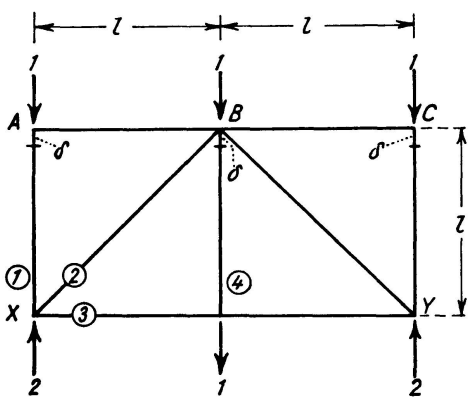


Fig. 4. Design of a Frame to Give Equal Vertical Deflections of Three Joints.

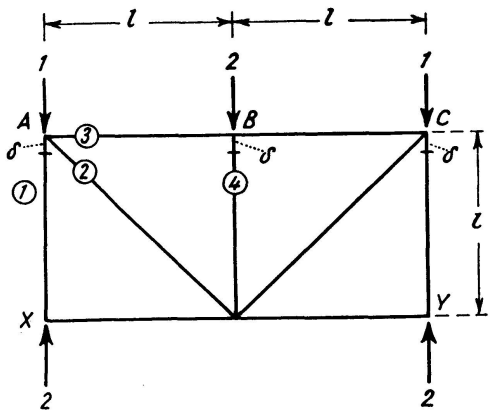


Fig. 5. Frame in Which Specific Displacements of Three Joints are Physically Unattainable.

We evaluate first the internal loads T_1, T_2, T_3, T_4, in the members of the frame due to the system of external loads; the extensions of the members are then given by

$$e_1 = \lambda_1 T_1 , \qquad e_2 = \lambda_2 T_2 , \quad \text{etc.,} \tag{13}$$

where λ_1, λ_2, etc., are positive constants. Unit vertical load is then applied at A, and the internal loads in the members evaluated; similarly, unit vertical loads are applied, in turn, at B and C. We have finally, by using the principle of virtual work, that the vertical deflections at joints A, B and C are

$$\delta_A = \lambda_1, \qquad \delta_B = 2\lambda_2 + \lambda_3, \qquad \delta_C = \lambda_1, \tag{14}$$

assuming X and Y to suffer no vertical displacements. If $\delta_A = \delta_B = \delta_C = \delta$, then

$$\lambda_1 = \delta, \; 2\lambda_2 + \lambda_3 = \delta. \tag{15}$$

If δ is defined, then only one value of λ_1 is acceptable, whereas λ_2 and λ_3 can have any suitable values within the ranges

$$0 < \lambda_2 < \tfrac{1}{2}\delta, \qquad 0 < \lambda_3 < \delta. \tag{16}$$

If δ is not defined, but we are interested only in the possibility that A, B and C suffer *equal* vertical deflections then we eliminate δ from eqs. (15), and have

$$\lambda_1 - 2\lambda_2 - \lambda_3 = 0. \tag{17}$$

The coefficients of the λ's in eq. (17) are not all of the same sign, and it is possible to design the members to give equal vertical deflections. From eq. (17), we note that if λ_2 and λ_3 are given *any* positive values, it is always possible to find a positive value of λ_1; however, we cannot always find a positive value of λ_2 if λ_1 and λ_3 assume *any* positive values.

As a further example, consider the simple frame shown in fig. 5; this is similar to the frame of fig. 4, except for the positions of the diagonal members. As before, the frame is supported at X and Y; there is complete symmetry of the system about a vertical centre-line. We have that the vertical displacements of the joints A, B and C are

$$\begin{aligned} \delta_A &= 2\lambda_1, \\ \delta_B &= 2\lambda_1 + 2\lambda_2 + \lambda_3 + 2\lambda_4, \\ \delta_C &= 2\lambda_1. \end{aligned} \tag{18}$$

For equal deflections of A, B and C, we must have

$$2\lambda_2 + \lambda_3 + 2\lambda_4 = 0. \tag{19}$$

Eq. (19) could only be satisfied if one λ at least is negative, which is physically unacceptable. If λ_2, λ_3 and λ_4 are always positive, we note from eqs. (18) that δ_B must always be greater than δ_A, a condition for which the frame can always be designed.

4. *More General Case*

When the number of joints, at which displacements are specified, is three or more the analysis becomes generally more complex. We assume as before that the frame is iso-static, and that for any given system of external loads

the forces in the members are determined; suppose T_m is the *tensile* load in a member m of the frame. Then the *extension* of the member can always be written in the form

$$e_m = \lambda_m T_m, \tag{20}$$

where λ_m is positive. Suppose $\delta_A, \delta_B, \delta_C, \ldots$ are the deflections specified for the joints $A, B, C, \ldots$ respectively. Then we can always write $\delta_A, \delta_B, \delta_C, \ldots$ in the forms

$$\delta_A = \sum (a_m T_m) \lambda_m, \qquad \delta_B = \sum (b_m T_m) \lambda_m, \qquad \delta_C = \sum (c_m T_m) \lambda_m, \ \ldots \tag{21}$$

where the summations are carried out for all members of the frame, and in which a_m is the *tensile* load in member m due to unit load at joint A applied in the direction of the deflection δ_A, b_m is the *tensile* load in member m due to unit load at joint B applied in the direction of the deflection δ_B, and so on for $c_m, \ldots$. Suppose we write

$$\alpha_m = a_m T_m, \qquad \beta_m = b_m T_m, \qquad \kappa_m = c_m T_m, \ldots \tag{22}$$

Then eqs. (21) become

$$\delta_A = \sum \alpha_m \lambda_m, \qquad \delta_B = \sum \beta_m \lambda_m, \qquad \delta_C = \sum \kappa_m \lambda_m, \ \ldots \tag{23}$$

Consider, for the sake of simplicity, the case in which deflections are specified at 3 joints and the frame has six members, suitably arranged to give a just-stiff structure. Then eqs. (23) become

$$\begin{bmatrix} \alpha_1 \alpha_2 \ldots \alpha_6 \\ \beta_1 \beta_2 \ldots \beta_6 \\ \kappa_1 \kappa_2 \ldots \kappa_6 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_6 \end{bmatrix} = \begin{bmatrix} \delta_A \\ \delta_B \\ \delta_C \end{bmatrix}. \tag{24}$$

We now write eqs. (24) in the forms

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \delta_A - \alpha_4 \lambda_4 - \alpha_5 \lambda_5 - \alpha_6 \lambda_6 \\ \delta_B - \beta_4 \lambda_4 - \beta_5 \lambda_5 - \beta_6 \lambda_6 \\ \delta_C - \kappa_4 \lambda_4 - \kappa_5 \lambda_5 - \kappa_6 \lambda_6 \end{bmatrix}. \tag{25}$$

So that

$$\lambda_1 = \begin{vmatrix} (\delta_A - \alpha_4 \lambda_4 - \alpha_5 \lambda_5 - \alpha_6 \lambda_6) & \alpha_2 & \alpha_3 \\ (\delta_B - \beta_4 \lambda_4 - \beta_5 \lambda_5 - \beta_6 \lambda_6) & \beta_2 & \beta_3 \\ (\delta_C - \kappa_4 \lambda_4 - \kappa_5 \lambda_5 - \kappa_6 \lambda_6) & \kappa_2 & \kappa_3 \end{vmatrix} \div \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{vmatrix}. \tag{26}$$

Then we may write

$$\lambda_1 = \left[\begin{vmatrix} \delta_A & \alpha_2 & \alpha_3 \\ \delta_B & \beta_2 & \beta_3 \\ \delta_C & \kappa_2 & \kappa_3 \end{vmatrix} - \lambda_4 \begin{vmatrix} \alpha_4 & \alpha_2 & \alpha_3 \\ \beta_4 & \beta_2 & \beta_3 \\ \kappa_4 & \kappa_2 & \kappa_3 \end{vmatrix} - \lambda_5 \begin{vmatrix} \alpha_5 & \alpha_2 & \alpha_3 \\ \beta_5 & \beta_2 & \beta_3 \\ \kappa_5 & \kappa_2 & \kappa_3 \end{vmatrix} - \lambda_6 \begin{vmatrix} \alpha_6 & \alpha_2 & \alpha_3 \\ \beta_6 & \beta_2 & \beta_3 \\ \kappa_6 & \kappa_2 & \kappa_3 \end{vmatrix} \right] \div \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{vmatrix}. \tag{27}$$

We have similarly that

$$\lambda_2 = \left[\begin{vmatrix} \alpha_1 & \delta_A & \alpha_3 \\ \beta_1 & \delta_B & \beta_3 \\ \kappa_1 & \delta_C & \kappa_3 \end{vmatrix} - \lambda_4 \begin{vmatrix} \alpha_1 & \alpha_4 & \alpha_3 \\ \beta_1 & \beta_4 & \beta_3 \\ \kappa_1 & \kappa_4 & \kappa_3 \end{vmatrix} - \lambda_5 \begin{vmatrix} \alpha_1 & \alpha_5 & \alpha_3 \\ \beta_1 & \beta_5 & \beta_3 \\ \kappa_1 & \kappa_5 & \kappa_3 \end{vmatrix} - \lambda_6 \begin{vmatrix} \alpha_1 & \alpha_6 & \alpha_3 \\ \beta_1 & \beta_6 & \beta_3 \\ \kappa_1 & \kappa_6 & \kappa_3 \end{vmatrix} \right] \div \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{vmatrix}, \quad (28)$$

and

$$\lambda_3 = \left[\begin{vmatrix} \alpha_1 & \alpha_2 & \delta_A \\ \beta_1 & \beta_2 & \delta_B \\ \kappa_1 & \kappa_2 & \delta_C \end{vmatrix} - \lambda_4 \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_4 \\ \kappa_1 & \kappa_2 & \kappa_4 \end{vmatrix} - \lambda_5 \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_5 \\ \beta_1 & \beta_2 & \beta_5 \\ \kappa_1 & \kappa_2 & \kappa_5 \end{vmatrix} - \lambda_6 \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_6 \\ \beta_1 & \beta_2 & \beta_6 \\ \kappa_1 & \kappa_2 & \kappa_6 \end{vmatrix} \right] \div \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{vmatrix}. \quad (29)$$

Suppose the determinants

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{vmatrix}, \quad \begin{vmatrix} \delta_A & \alpha_2 & \alpha_3 \\ \delta_B & \beta_2 & \beta_3 \\ \delta_C & \kappa_2 & \kappa_3 \end{vmatrix}, \quad \begin{vmatrix} \alpha_1 & \delta_A & \alpha_3 \\ \beta_1 & \delta_B & \beta_3 \\ \kappa_1 & \delta_C & \kappa_3 \end{vmatrix}, \quad \begin{vmatrix} \alpha_1 & \alpha_2 & \delta_A \\ \beta_1 & \beta_2 & \delta_B \\ \kappa_1 & \kappa_2 & \delta_C \end{vmatrix}, \quad (30)$$

are all of the same sign (positive or negative); then, in eqs. (27), (28) and (29), if we make λ_4, λ_5 and λ_6 positive, and sufficiently small, then we have positive values of λ_1, λ_2, λ_3. So that the specific deflections δ_A, δ_B, and δ_C are possible if we can find at least one set of determinants of the type (30), in which all the determinants are of the same sign (positive or negative). From eqs. (24), sets of determinants of the type (30) can be selected in 20 ways; positive values of the λ's are possible if in at least one set the determinants are all of the same sign. Any set of determinants is formed by taking three columns of coefficients α, β, κ from the left-hand sides of eqs. (24); the first determinant in any set is the determinant formed from these coefficients; the other determinants in the set are found by replacing each column in turn of the first determinant by the column of deflections forming the right-hand sides of eqs. (24).

More generally, if there are m values of λ, and displacements are specified at j joints, then there are

$$^{m}C_j = \frac{m(m-1)\ldots(m-j+1)}{1.2.3\ldots(j-1)j}, \quad (31)$$

sets of determinants to be studied. The specific deflections are possible if in one set the determinants have the same signs. However, to show the impracticability of a given set of specified deflections, it is essential to study *all* sets of determinants of the type (30).

As an example of a more complex problem, let us consider the deflection characteristics of the frame shown in fig. 6. The structure is supported at X and Y, and the whole system is symmetrical about a vertical centre-line. It is easily shown that the vertical deflections of the joints A, B and C, are

$$\begin{aligned} \delta_A &= 4\lambda_1 + 2\lambda_3 + \lambda_4 + 2\lambda_7, \\ \delta_B &= 3\lambda_2, \\ \delta_C &= 3\lambda_2 - \lambda_4 + 2\lambda_5 + 4\lambda_6. \end{aligned} \quad (32)$$

Suppose first that we are interested in designing for the case in which $\delta_A = \delta_B = \delta_C$. Then eqs. (32) may be written

$$\begin{aligned} 3\lambda_2 &= 4\lambda_1 + 2\lambda_3 + \lambda_4 + 2\lambda_7, \\ \lambda_4 &= 2\lambda_5 + 4\lambda_6. \end{aligned} \tag{33}$$

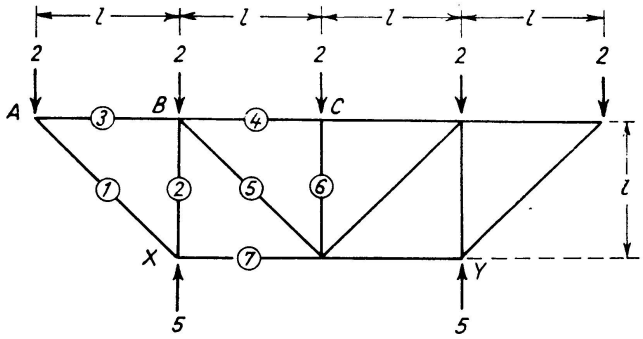


Fig. 6. Problem Involving Seven Adjustable Stiffnesses.

For any positive values of λ_1, λ_3, λ_4 and λ_7, positive values of λ_2, λ_5 and λ_6 can always be found; it is possible then to design the members to give equal vertical deflections of joints A, B and C.

Suppose now, however, that we wish to design for the condition

$$\delta_A = \tfrac{1}{2}\delta_B = \delta_C = 1, \quad \text{(say)}.$$

Then eqs. (32) become

$$1 = 4\lambda_1 + 2\lambda_3 + \lambda_4 + 2\lambda_7, \qquad 2 = 3\lambda_2, \qquad 1 = 3\lambda_2 - \lambda_4 + 2\lambda_5 + 4\lambda_6. \tag{34}$$

On eliminating λ_2 and λ_4, we have

$$4\lambda_1 + 2\lambda_3 + 2\lambda_5 + 4\lambda_6 + 2\lambda_7 = 0. \tag{35}$$

Since all the coefficients of eq. (35) are positive, at least one λ must be negative; the proposed design is impossible therefore.

Finally, suppose we are interested in the case when $\delta_A = 2\delta_B = \delta_C = 1$, (say). Then eqs. (32) become

$$\begin{bmatrix} 4 & 0 & 2 & 1 & 0 & 0 & 2 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1 & 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{36}$$

Consider the third order determinants which can be formed from the 1st, 2nd and 5th columns of the matrix of coefficients of the λ's, together with the single column of the right-hand side: four determinants can be formed from the columns

$$\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{37}$$

and the determinants (formed systematically) have the values

$$\begin{vmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 3 & 2 \end{vmatrix} = +48, \quad \begin{vmatrix} 1 & 0 & 0 \\ 1 & 6 & 0 \\ 1 & 3 & 2 \end{vmatrix} = +12, \quad \begin{vmatrix} 4 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = +8, \quad \begin{vmatrix} 4 & 0 & 1 \\ 0 & 6 & 1 \\ 0 & 3 & 1 \end{vmatrix} = +12, \tag{38}$$

which are all positive. Hence a solution is possible in which λ_3, λ_4, λ_6 and λ_7 assume small positive values, and λ_1, λ_2 and λ_5 assume larger positive values. The existence of at least one wholly positive set of λ's, satisfying eq. (36), is established therefore. This result could have been found more easily in the present simple problem; if, in eqs. (36), λ_3, λ_4, λ_6 and λ_7 are negligibly small, then

$$1 = 4\lambda_1, \qquad 1 = 6\lambda_2, \qquad 1 = 3\lambda_2 + 2\lambda_5, \ (\text{or } 1 = 4\lambda_5). \tag{39}$$

Then λ_1, λ_2 and λ_5 assume positive values.

3. *Physical Meaning of* λ

In eq. (1) we introduced λ in the form

$$e = \lambda T, \tag{1 bis}$$

in which T is the *tensile* load in a member, and e is the resulting *extension*. We are concerned only with structural members for which a tensile load results in an extension, and a compressive load in a contraction of the member. In this case λ in eq. (1) is always positive, and the general problem we have discussed becomes one of finding positive values of the λ's for all members to satisfy the deflection specifications. Eq. (1) is not limited to linear-elastic members; for the particular case of a linear-elastic members, λ is essentially a stiffness, and is constant for all values of e and T; more generally λ is a function of T.

Part II. Graphical Analysis

6. *Introduction*

In a practical structure, having a large number of members and in which the displacements are specified at a large number of joints, the mathematical analysis discussed in Part I, may become extremely laborious. The analysis can be simplified considerably by the use of simple graphical methods; one such method involves the construction of displacement diagrams similar in form to the standard Williot diagrams of deflection analysis.

7. *Frame Having two Members*

We consider first a simple frame consisting of two members which are pinned to a rigid foundation at X and Y, fig. 7. If the distortions of the frame, due to an external load P at joint A, are small, then the internal forces in the

members of the frame are statically-determinate, and are defined completely by the magnitude and direction of the external load P. We note that when the line of action of P lies between XA produced and YA produced, in fig. 7, then both members are in tension; if P lies between YA produced and XA, then member 1 is in compression and member 2 is in tension. The nature of the forces in the members (tension or compression) is determined entirely by the direction of the force P; in fig. 8, XAX' and YAY' are lines parallel to XA and YA, respectively, in fig. 7; the forces in the members 1 and 2 are tensile or compressive depending on which of the four sectors of fig. 8 contains the load vector P. For example, when the vector P lies between AX and AY', fig. 8, member 1 is in compression (−) and member 2 is in tension (+).

We consider now the forms of deflection which are possible for the joint A of fig. 7. Suppose that both members, 1 and 2, are in tension; then the load vector D lies in the sector between AX' and AY', fig. 8. We construct a "displacement" diagram in the following way:

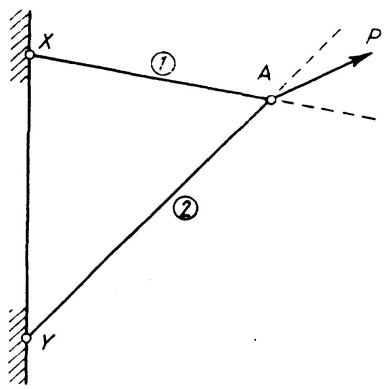


Fig. 7. Frame Having Two Pin-Jointed Members.

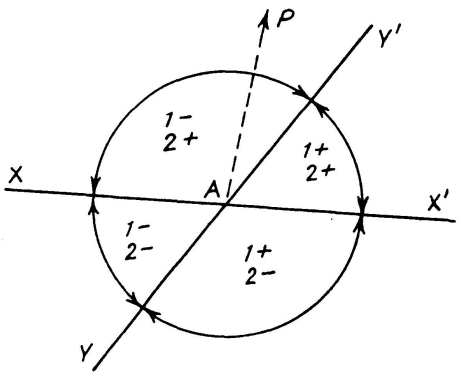


Fig. 8. Internal Load Systems Resulting from Different Directions of the External Load at Joint A.

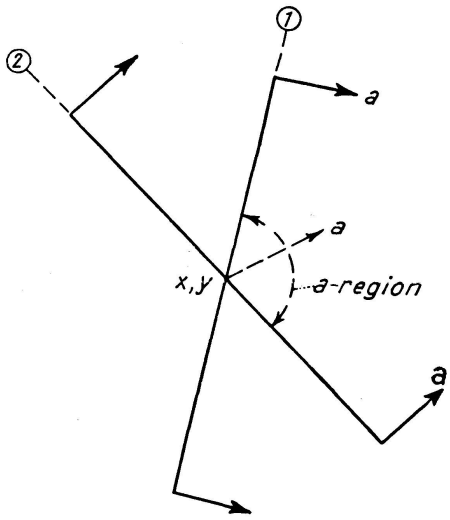


Fig. 9. Displacement Region for Joint A When Both Members are in Tension.

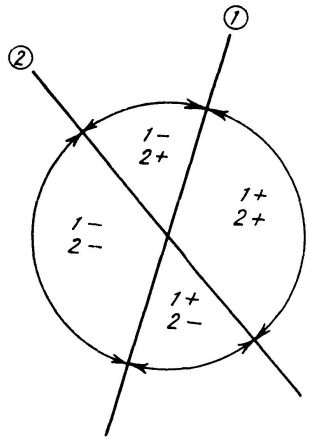


Fig. 10. Displacement Regions for Joint A.

Take a point x, y as origin, fig. 9; members 1 and 2 are in tension, and they must therefore extend under load. So that, relative to X, the joint A is always displaced to some point to the right of line 1, fig. 9, this line being perpendicular to XA in fig. 7. Similarly, joint A is always displaced to some point to the right of line 2, fig. 9, this line being perpendicular to YA in fig. 8. On the displacement diagram of fig. 9, the point a, defining the deflection of joint A relative to X and Y in fig. 7, must lie within the a-region indicated in fig. 9.

We note that for a given external load the members could be designed to give any *vertical* displacement of joint A; however, *horizontal* displacements must always be to the right. The displacement vector xa (or ya) in fig. 9 is limited to the a-region indicated when the load vector P lies between AX' and AY' in fig. 8; for other positions of the load vector P, the signs of the forces in the members 1 and 2 are defined, and these give the displacement regions of fig. 10, in which line 1 is perpendicular to XA in fig. 7, and line 2 is perpendicular to YA in fig. 7.

8. *Frame Having four Members*

When an iso-static frame carries a given system of external loads the forces in the members of the frame are determined independently of their stiffnesses. Suppose the frame of fig. 11 is loaded so that all the members are in tension; then the external load vectors P and Q at A and B, respectively, must lie within the limits shown. In the "displacement" diagram, fig. 12, we

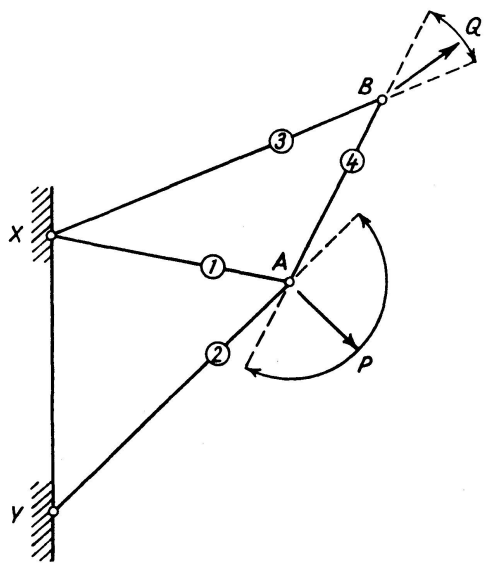


Fig. 11. Frame Having Four Pin-Jointed Members.

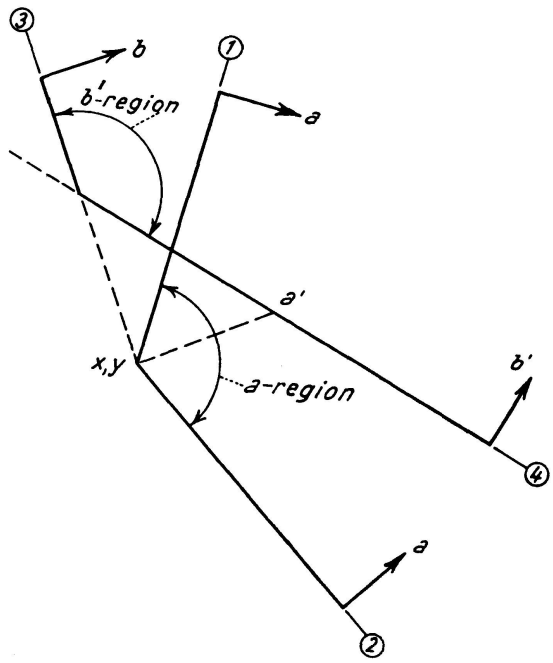


Fig. 12. "Displacement" Diagram for a Frame Having Four Members, all of Which are in Tension.

take an origin x, y; then a point a, defining the displacement of joint A, must lie to the right of line 1, which is perpendicular to XA in fig. 11. The point a also lies to the right of a line 2, which is perpendicular to YA in fig. 11. A vector $xa\,(ya)$, defining the displacement of joint A relative to X (or Y) must lie then in the a-region of fig. 12. Within this a-region consider a specific point a': relative to A, the joint B must always displace to the right of line 4, fig. 12, which is perpendicular to AB in fig. 11; relative to X, joint B must always displace to the right of line 3, fig. 12, which is perpendicular to XB in fig. 11. So that for a given point a', within the a-region of fig. 12, the point b' defining the displacement of joint B must lie within the b'-region shown.

As an example of the use of this type of displacement diagram, let us consider whether it is possible to design the members of the frame of fig. 11 so that the displacements of the joints A and B have the magnitudes and directions given by the vectors xa' and xb' of fig. 13. Taking the given position of the vector xa', we find that b' lies outside the region defining physically possible positions of b; the displacements xa' and xb' are impracticable, therefore.

As a further simple example, let us examine the possibility that the members can be designed to give equal horizontal deflections to the joints A and B, fig. 11. In the displacement diagram, fig. 14, we set off the required horizontal displacement xh. Then all points a must lie on a vertical line through h; the limiting positions for a are the points a' and a''. Again, b must lie on a vertical

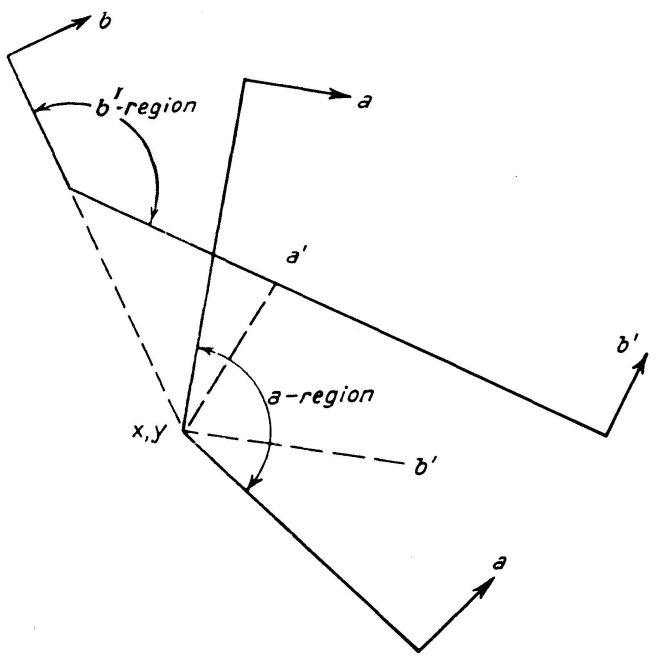


Fig. 13. Determination of the Practicability of Specific Displacements of the Joints of a Frame.

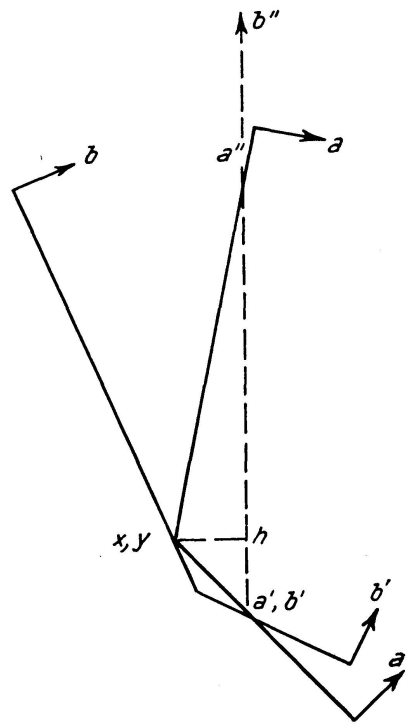


Fig. 14. Design of the Frame to Give Equal Horizontal Deflections to Joints A and B.

line through h; so that b' (which is coincident with a') is the "lower" of b, and b'' (at infinity) is the "upper" limit of b. Then there is a limited range of positions of the point a, and an infinitely large range of positions of the point b. Equal horizontal displacements of the two joints are physically possible.

9. *Solution of Some Simple Practical Problems*

In general the problem of designing a frame to give specific deflections of certain joints may take one of so many forms that it is difficult to establish a general pattern of displacement diagrams of the type already discussed. We restrict the discussion to a few particular cases which show the advantages of using the graphical method.

The pin-jointed frame and the loading system of fig. 15 (I) are symmetrical about a vertical centre-line; the frame is supported at A and A' and the external loads induce tensile (+), compressive (−), or zero forces (0) in the members as shown in fig. 15 (I). We are interested in the possibility that the frame can be designed to give equal vertical deflections of B and X. We consider the displacements of the frame relative to the unstressed member $X\,Y$, which we

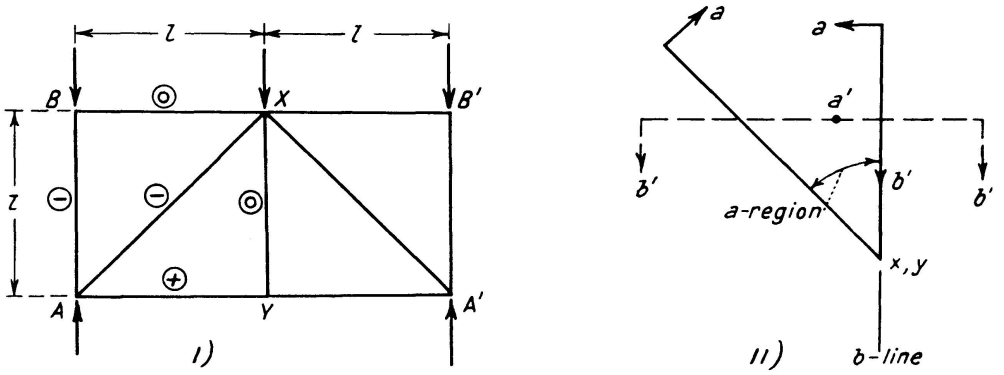


Fig. 15. Solution of a Frame to Give Equal Vertical Deflections of Certain Joints.

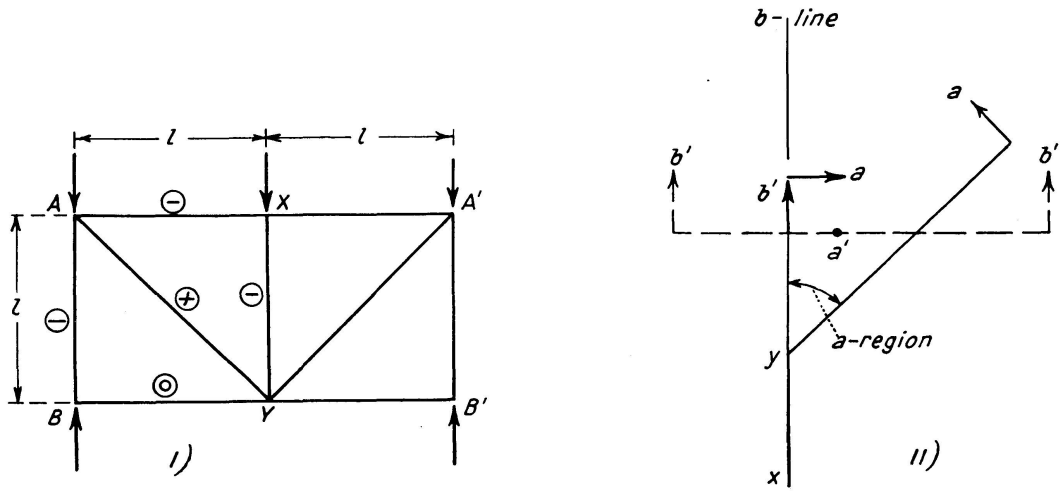


Fig. 16. Frame in Which Specific Displacements of Certain Joints are Impracticable.

assume remains vertical. In the displacement diagram, fig. 15 (II), the a-region forms a 45° octant; for any point a' in the a-region, b' lies on a vertical line through x, y, and below a horizontal line through a'. Obviously, for any position of a', within the a-region, b' could always be arranged to coincide with x, y, giving zero vertical deflection of B relative to X and Y.

We can study the frame of fig. 16 (I) in a similar way; the whole system is symmetrical about $X\,Y$; the frame is supported at B and B'. We are interested again in determining whether this frame can be designed to give equal vertical deflections of joints A, X and A'. From the displacement diagram, fig. 16 (II), we find that the a-region is such that equal vertical displacements of A and X are impossible.

As a final example we consider the frame of fig. 17 (I); the whole system is again symmetrical about $X\,Y$; the frame is supported at B and B'. For the external loads shown, the displacement diagram has the form shown in fig. 17 (II). We can deduce easily that the members can always be designed to give equal vertical deflections of X, A and C.

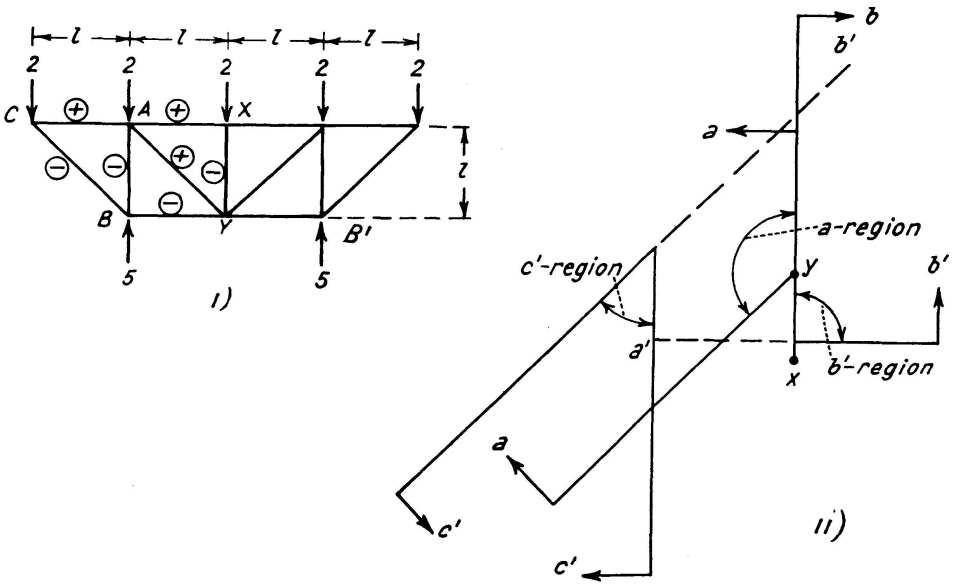


Fig. 17. Analysis of a Frame to Give Specific Deflections at a Number of Joints.

10. Graphical Determination of the "Extreme" Limits of Displacement of the Joints of a Frame

It is a relatively simple matter to find graphically the "extreme" limits of displacement of the joints of an iso-static frame. By an "extreme" limit we mean one beyond which displacements of a joint of a frame are certainly impracticable. Consider, for example, a plane frame, fig. 18 (I), which is pinned to a rigid foundation at X and Y; the external loads at joints A, B and C give rise to internal forces in the members which are tensile (+) or compressive (−), as indicated in fig. 18 (I). Member 1 is in compression, and must

shorten under load therefore; due to the shortening of member 1, the rest of the structure, $XABC$, rotates bodily about X. If the shortening of member 1 is small, we may assume that the displacements of the joints A, B and C take place in directions perpendicular to the lines AX, BX and CX, respectively; the magnitudes of the displacements of these joints are proportional to the lengths of the lines AX, BX and CX, respectively. Due to the shortening of member 1, the joint A is displaced a small amount given by the vector

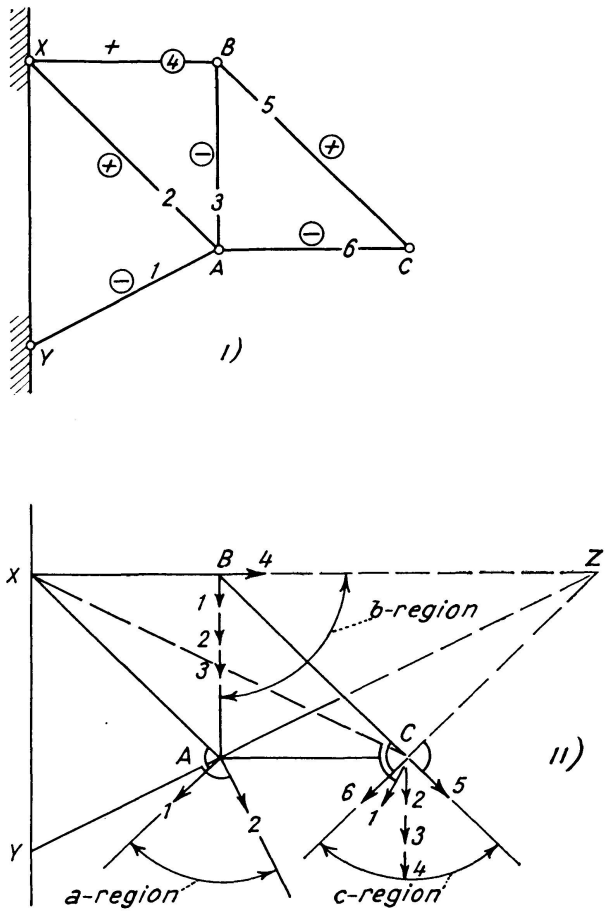


Fig. 18. Graphical Determination of the "Extreme" Limits of Displacement of the Joints of a Frame.

1 at A, fig. 18 (II); similarly, the joint B is displaced a small amount given by the vector 1 at B, and joint C a small amount given by the vector 1 at C. Consider now the extension of member 2; due to the extension of this member alone, A is displaced a small amount given by the vector 2 at A, which is perpendicular to YA; the rest of the structure, BAC, rotates about Z, which is the point of intersection of XB produced and YA produced; we then give joints B and C displacements defined by the vectors 2 at these joints, fig. 18 (II).

We then proceed to find the effect of the change in length of each member in turn on all joints of the frame. For example, joint A is unaffected by the

extension of member 4, whereas the effect of the extension of this member on joints B and C is given by the vectors 4 at these joints.

Now, the resultant displacement of any joint must be compounded of a set of vectors of the type indicated in fig. 18 (II). The actual lengths of the vectors depend on the extension properties of the members. Evidently, the displaced position of A must lie somewhere in the acute angle between vectors 1 and 2 at A, fig. 18 (II). At joint B, the displaced position of B must lie in the right angle between vectors 1, 2, 3 and vector 4. At joint C, the displaced position of C must lie between vectors 5 and 6, since the vectors 1, 2, 3, and 4 also lie within these limits.

We can fix "extreme" limits very simply by this graphical method; however, we cannot say that *any* displacements within these limits are attainable *simultaneously* by the joints of the frame. The limits are "extreme" therefore in a broad sense only; any design, specifying displacements outside these limits, is impracticable.

11. Redundant Frames

Our discussion so far has been limited to iso-static frames. When a structure contains redundant members, the analysis becomes considerably more difficult; the inclusion of a redundant member in a frame may not necessarily make attainable any specific displacements of the frame.

As a very simple example, consider a plane pin-jointed frame, fig. 19, consisting of three members, which are pinned to each other at A, and to a rigid foundation at X, Y and Z. In the initially unloaded condition, the members are unstressed. Suppose an external load is now applied at joint A; let us examine the possible forms of internal extensions and contractions of the members. If each member is free to be extended or contracted, then the members can be strained in 8 different ways; for example, member 1 may be in tension, and members 2 and 3 in compression, and so on. All possible types of behaviour are outlined in table 1; when the three members are in tension, as in (I) of that table, the displacement of A can always be represented by a vector somewhere within the right-angle sector defined by lines 1—1 and 3—3, in column (a); when the extensions of the bars have the forms given by (III)

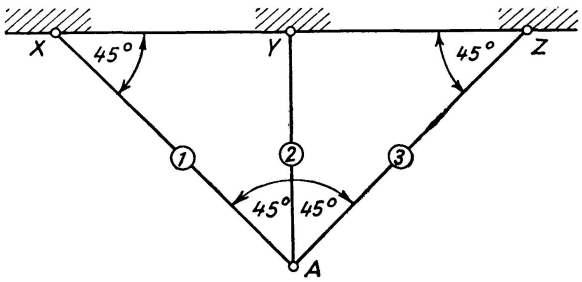
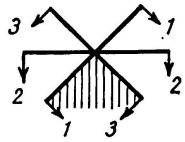
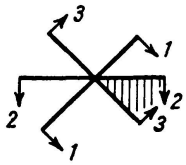
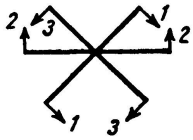
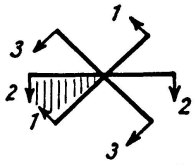
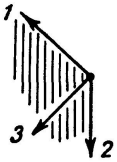
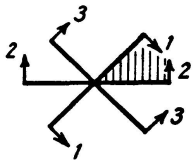
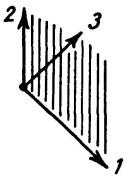
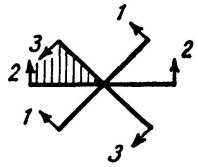
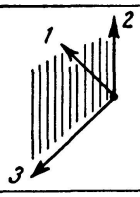
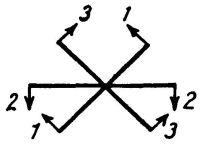
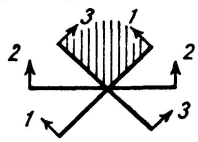
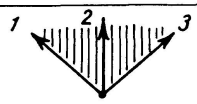


Fig. 19. Displacement Analysis of a Redundant Frame.

Table 1. Displacement and External Load Limitations for all Possible Systems of Internal Forces in the Members

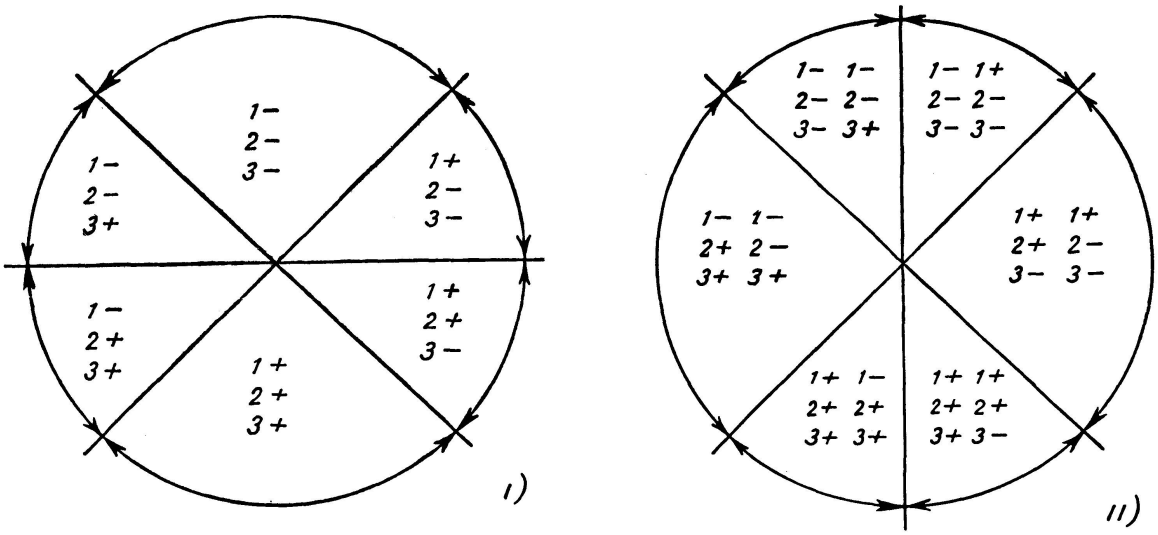
	Form of internal loads, or change of length			(a) Displacement region of joint A	(b) Limits of external load vector at joint A
	Member				
	1	2	3		
I)	+	+	+		
II)	+	+	−		
III)	+	−	+		no limits
IV)	−	+	+		
V)	+	−	−		
VI)	−	−	+		
VII)	−	+	−		no limits
VIII)	−	−	−		

extensions of members 1 and 3 are incompatible with a contraction of member 2; a similar incompatibility exists in (VII). We note first that the systems (III) and (VII) are physically impossible, and we need consider them no further.

Any given arrangement of internal changes of lengths of the members determines the signs of the internal forces in the members. If these internal forces are reversed in direction at the joint A, then we can find the limiting positions of the external load vector at A. For example, in arrangement (I), table 1, the three members are in tension, and the external load vector at A must lie within the right-angle between lines 1 and 3, as shown in column (b). Arrangements (III) and (VII), which we found to be physically incompatible from a strain standpoint, could be derived from an external load applied in any direction at A.

We can summarise the main features of table 1 in graphical form; in fig. 20 (I) are given the systems of internal strains (extension +, contraction −) for displacements of the joint A in various directions; in fig. 20 (II) are shown the internal force systems (tension +, compression −) for external loads in various directions. We note from fig. 20 (II) that there are two alternative internal load systems for any given external load; any given external load may give rise, therefore, to either of two systems of internal extensions.

The insertion of a redundant member into an iso-static system may increase the range of displacements for which the frame can be designed. For example, the simple frame of fig. 21 (I) is iso-static; for an external load P at joint A, in the direction indicated, the two members are in tension, and the joint A is displaced in the a-region of fig. 21 (I). When a redundant member YA is introduced into the frame, the joint A can be displaced anywhere in a larger a-region, fig. 21 (II), for the same external load P.



Summary of Displacement Limitations. Summary of External Load Limitations.

Fig. 20.

In general, the insertion of a redundant member cannot reduce the "extreme" limits of displacement of the joint A of the frame in fig. 21 (I). If the redundant member tends to reduce the limits of the a-region, then by making the redundant member very flexible compared with the other members of the frame, we approach the displacement limits of the original iso-static frame.

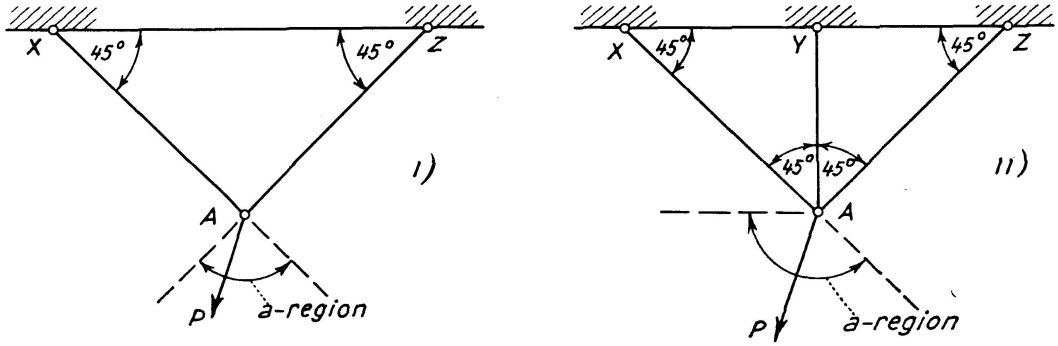


Fig. 21. Effect of the Inclusion of a Redundant Member on the Range of Displacements of a Loaded Joint of a Frame.

Part III. Design of a Frame to Give Equal Vertical Deflection of the Loaded Joints

12. Introduction

In Parts I and II we discussed certain examples in which it was physically impossible to design the members of a *given* frame to give deflections. In these cases the designer is led to experiment with alternative arrangements of the members of the frame.

In a recent structural problem, for example, it was required to fabricate a pin-jointed frame so that under the action of external loads, the loaded joints deflected vertically by equal amounts. In its simplest form the system of external loads consisted of a uniformly distributed vertical loading on the upper boom of a truss, fig. 22. The frame could be taken as symmetrical about a vertical centre-line. No horizontal thrusts could be induced at rigid supports on the underside of the frame. The problem reduced to designing a plane frame to give equal vertical deflections of the loaded joints of the upper boom.

Part III of the paper is limited to a discussion of iso-static plane frames having the property of equal deflections at the loaded joints. First, the poten-

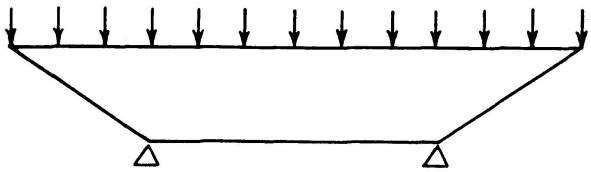


Fig. 22. Loading and Support Conditions for a Plane Frame.

tialities of Warren trusses are discussed; secondly, the properties of "spray" frames are studied; and, finally, a method of fabrication using basic structural units is introduced.

13. Use of Simple Warren Trusses

The authors were led first to examine the deflection characteristics of simple Warren trusses, as a possible means of meeting the design specification. If there are only three external loads applied to the upper boom, then the frame can be fabricated as shown in fig. 23. The distributed load can be treated

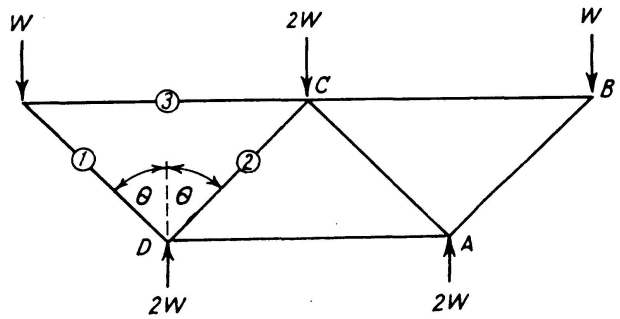


Fig. 23. Warren Truss Carrying Three Loads in the Upper Boom.

as concentrated vertical loads of W and $2\,W$ at B and C, respectively. Suppose the members are linear and elastic, and that the extension e of any member, carrying a tensile load T, is

$$e = \lambda\, T, \tag{1) bis}$$

where λ is constant for that member. If the frame is symmetrical about a vertical centre-line, and carries the external loads shown, then equal vertical displacements of the joints B and C are ensured if

$$\lambda_1 - \lambda_3 = \lambda_2 \sin^2\theta\,, \tag{40}$$

in which the subscripts refer to the members 1, 2, 3, respectively.

When there are four loaded joints in the upper boom, the frame can be fabricated as shown in fig. 24. Equal vertical deflections of the joints in the upper boom are ensured if

$$\lambda_1 - 2\,\lambda_2 = (2\,\lambda_3 - \lambda_4 + \lambda_5)\sin^2\theta\,. \tag{41}$$

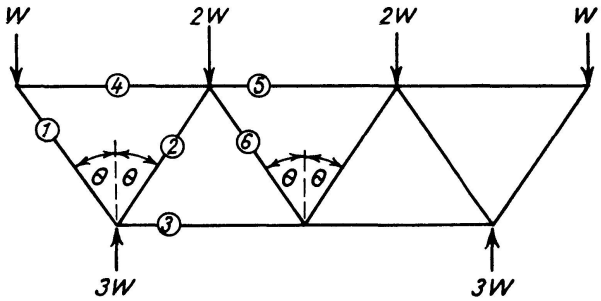


Fig. 24. Warren Truss With Four Loads in the Upper Boom.

Unfortunately, Warren trusses of this type cannot be extended to give equal vertical deflections of more than four loaded joints. For example, fig. 25 (I) shows a frame which is symmetrical about a vertical centre-line; it is required to design the members so that the loaded joints, D, B and X, suffer equal vertical deflections. A simple displacement diagram of the type shown in fig. 25 (II) (and discussed in Part II), indicates that there is always a vertical deflection of B relative to X, and the design is clearly impracticable.

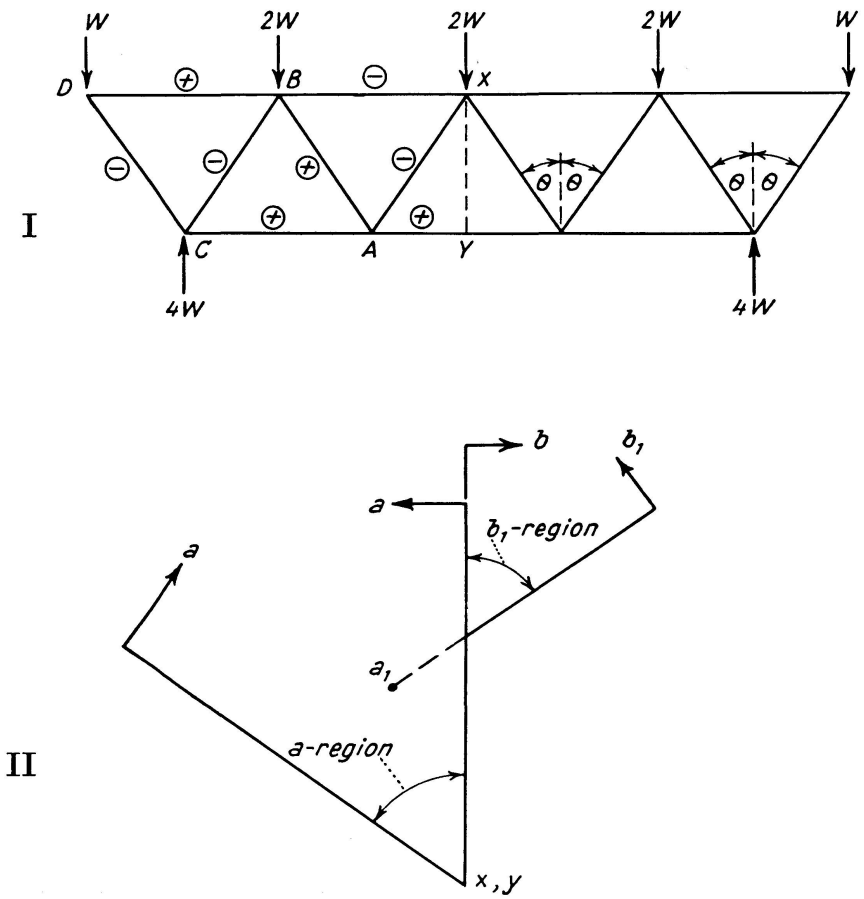


Fig. 25. Warren Truss With Five Loaded Joints.

14. *Spray Frames*

From § 13 it is clear that the specified deflections can be achieved in Warren trusses having only one or two bays between the supports. If in the Warren truss of fig. 23 two additional vertical members are introduced, the frame is capable of providing equal vertical deflections of five loaded joints; this is the case for the frame of fig. 26. The intermediate members between the upper and lower booms radiate from the joints at the supports; this arrangement we call a "spray" frame. If the whole system is symmetrical about a vertical centre-line, then for equal vertical deflections, δ, of the joints of the upper boom we have

$$\lambda_1 = \frac{\delta}{W}\cos^2\theta - (\lambda_4+\lambda_5)\sin^2\theta, \qquad \lambda_2 = \frac{\delta}{2W}, \qquad \lambda_3 = \frac{\delta}{W}\cos^2\theta, \tag{42}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are the extensional stiffnesses of the members $1, 2, \ldots 5$ of fig. 26. If h is the height of the frame, then the longitudinal strains in the members 1, 2 and 3 are, respectively,

$$\epsilon_1 = -\frac{\delta}{h}\cos^2\theta + \frac{W}{h}(\lambda_4+\lambda_5)\sin^2\theta, \qquad \epsilon_2 = -\frac{\delta}{h}, \qquad \epsilon_3 = -\frac{\delta}{h}\cos^2\theta. \tag{43}$$

If the members 1, 2 and 3 are made of the same material, then the compressive stresses are proportional to the strains; we note that the compressive stress varies as $\cos^2\theta$, so that for the long members 1 and 3, the stress is considerably less than in member 2. This implies that the outer members of the spray must be stressed rather inefficiently; since the outer members constitute the greater part of the weight of the frame, the system is generally inefficient from the weight standpoint. This property is true generally of spray frames; this leads us to consider alternative methods of fabricating frames to give equal vertical deflections of the loaded joints.

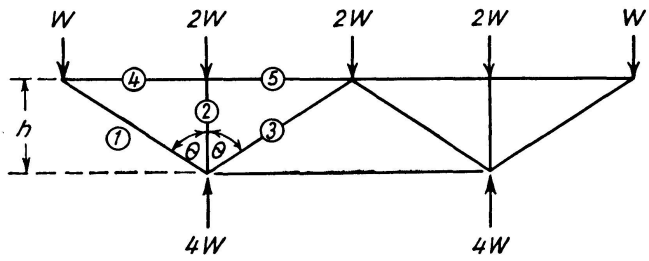


Fig. 26. Spray Frame Having Five Loaded Joints.

15. *Use of Basic Structural Units*

In dealing with the particular problem of a frame giving equal vertical deflections of the loaded joints, consider the simple triangular frame of fig. 27 (I). We suppose that the whole system — frame, external loads, and form of distortion — is symmetrical about a vertical centre-line through the support 0; then the member AB is displaced to the horizontal position $A'B'$. Suppose that the members are all equally strained, whether in tension or compression; then the displacement diagram has the form shown in fig. 27 (II); in this displacement diagram Oa, is the contraction of the member OA, and is drawn parallel to OA; ab is the extension of AB, and a_1a is perpendicular to Oa_1. For both tensile and compressive members to be equally strained, $b_1a_1 = ab$.

Suppose we now modify the frame of fig. 27 (I) by adding a central triangular frame CDE, as in fig. 28 (I), at the same time adding a vertical force $2W$ at the newly-formed pin-joint C, which is at the mid-point of AB. The newly-formed pin-joints D and E are at the mid-points of OB and OA, respectively. As before, we assume that all members are equally strained, whether in tension

or compression. Then in the displacement diagram, fig. 28 (II), we construct a small triangle cde within the triangle Oab. For symmetrical deflections, the frame gives equal vertical displacements of joints A, B and C.

In the frame of fig. 28 (I), the horizontal members AC and CB carry equal tensile loads; the tensile load in the member DE is twice as great as that in the members AC and CB. The compressive loads in the inclined members OD and OE are equal, and are twice as great as those in the inclined members AE, EC, CD and DB. The members of the triangular frame EOD are loaded twice as heavily as corresponding members of the triangular frames AEC and CDB.

We now consider further sub-divisions of the frame by introducing smaller triangular units FGH and IJK, as in fig. 29. Additional vertical loads $2W$ are applied at G and J, which are the mid-points of AC and CB, respectively. It is easily shown that no horizontal thrusts are transmitted across the pins

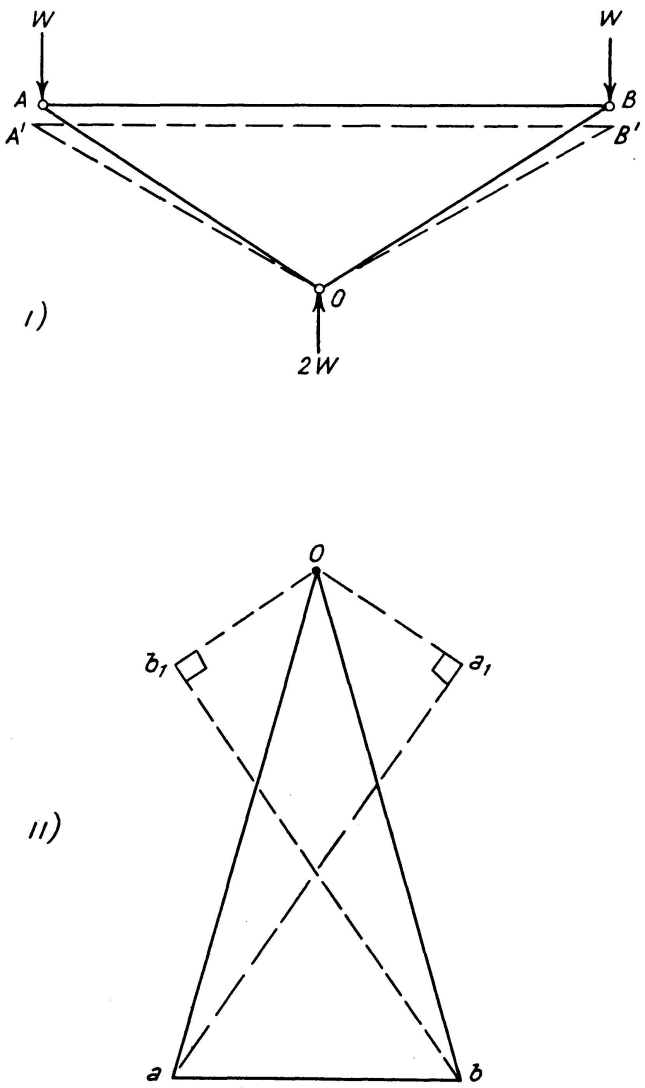


Fig. 27. Symmetrical Deflections of a Simple Structural Unit.

at G, C and J; so that all triangular units in the "first level" are loaded similarly. The loads at the "second level" are twice those in corresponding members in the "first level". At the "third level" the loads are twice the corresponding loads at the "second level", or four times those at the "first level". If all members are equally strained, as before, the loaded joints, A, B, C, G, J, displace vertically by equal amounts.

The sub-division introduced in fig. 29 involves the reduction of the upper levels of the frame of fig. 28. Continued sub-division gives the frame shown in fig. 30. The general pattern consists in introducing first a triangular frame a, secondly two triangular frames b, thirdly four frames c, and so on. At each stage the linear dimensions of the triangles are halved. The vertical loads at the upper panel points are all equal, except at the remote ends; this system of loads is, in the limit, equivalent to a uniformly-distributed vertical load on the upper boom.

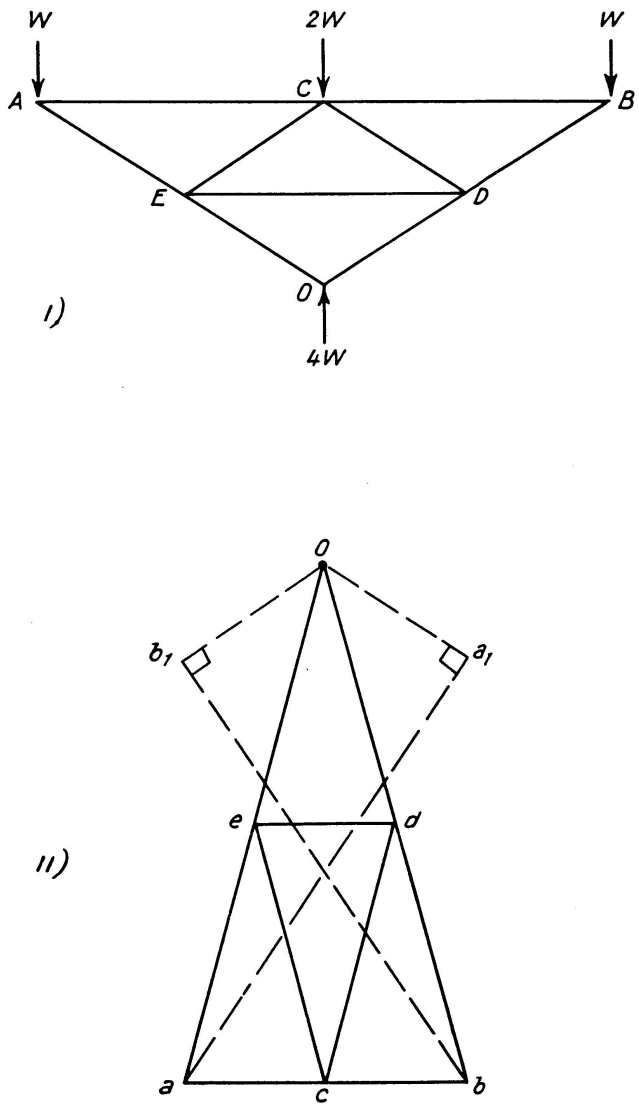


Fig. 28. Sub-Division of the Basic Unit to Provide for Another Loaded Joint.

No horizontal forces are transmitted across the pin-joints of the frame, and all units at any level are loaded in the same way.

In fig. 30 the sub-division gives rise to long members in the lower levels; these longer members can be braced to some extent by introducing further triangular units, as shown by the broken lines of fig. 31. The additional bracing members shown in fig. 31, are unloaded, and have no adverse effects on the deflection properties of the frame; moreover, all horizontal members are of the same length, as also are the inclined members.

Sub-divisions have been carried out on a triangular frame having only one support on the underside. The same process can be used to fabricate frames with two supports. For example, the frame of fig. 32 consists of two adjacent triangular units, ACD, BDE. Under the loading conditions shown there is no horizontal thrust between the two units at D. The vertical deflections of joints C, D and E are equal if the extensional stiffnesses of the members satisfy the relation

$$\lambda_2 - \lambda_1 = \lambda_3 \cos^2 \theta . \tag{44}$$

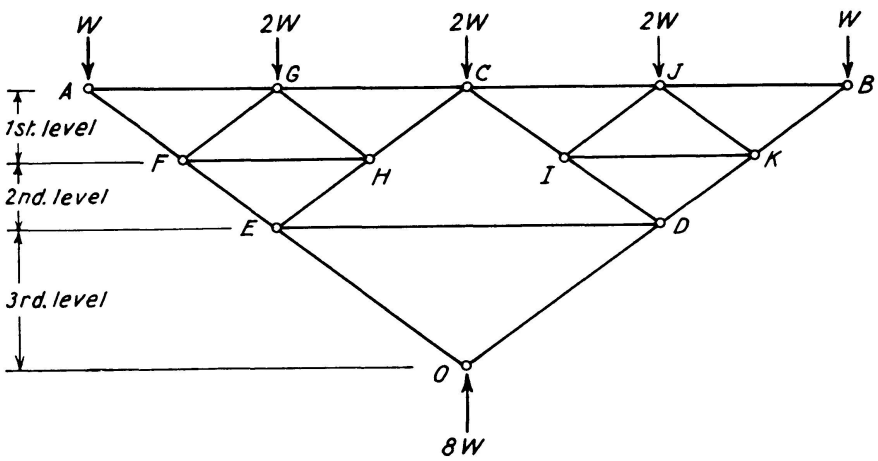


Fig. 29. Further Sub-Division to Provide 5 Loaded Joints.

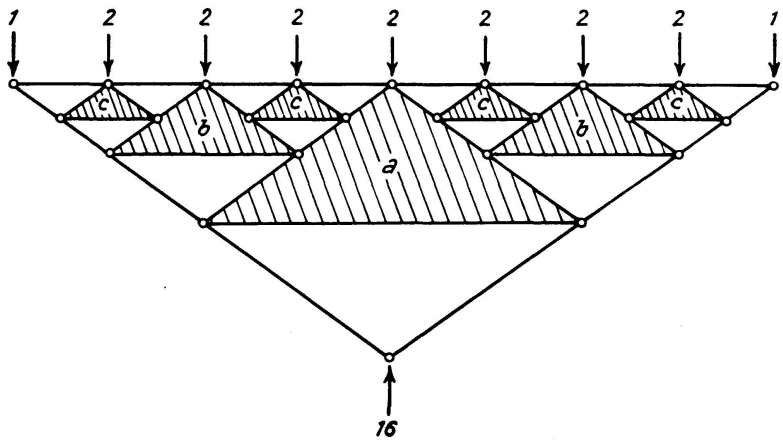


Fig. 30. Continued Sub-Division of the Basic Form.

The frame of fig. 32 is sub-divided by introducing small triangular units, as in fig. 33, and applying additional loads to the upper boom member. If the strains in the members of any unit are the same as the strains in the corresponding members of any other unit, then the structure retains its special deflection properties. The members in the lower levels are more heavily loaded than corresponding members in the upper levels of the frame.

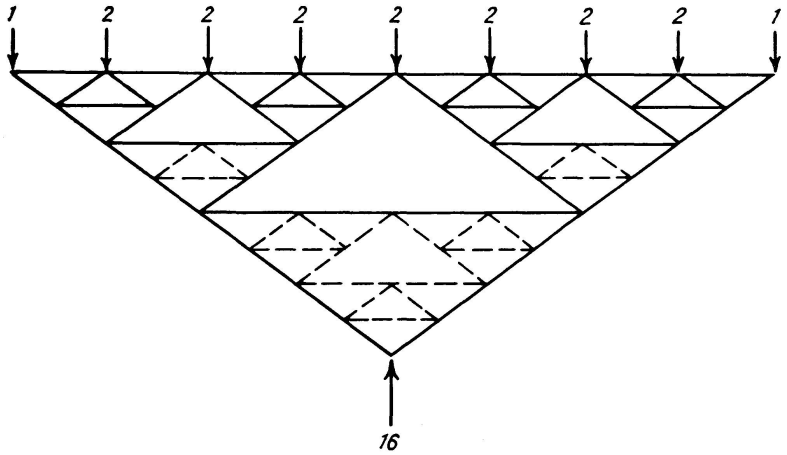


Fig. 31. Introduction of Bracing Members.

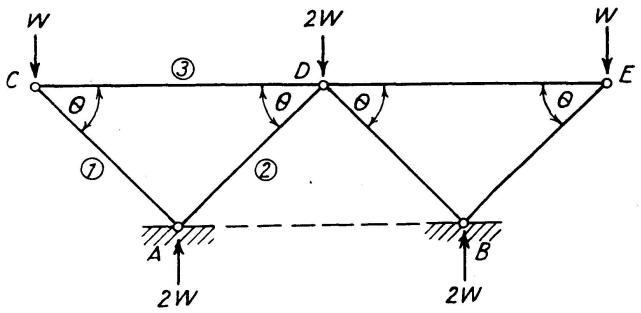


Fig. 32. Frame Having Two Supports.

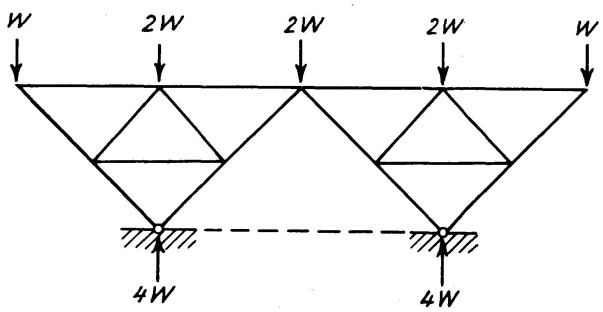


Fig. 33. Sub-Division of a Frame Having Two Supports.

16. Fabrication of a Space Frame Using Basic Units

The use of structural units in fabricating simple iso-static frames, with specific deflection properties, can be extended to space frames. The statically-determinate tetrahedron of fig. 34 (I) suffers equal vertical deflections of the

loaded joints if all the inclined members have the same stiffnesses, and if all the horizontal members are also of equal stiffness.

In the first subdivision of the structure hinges are introduced at the mid-points of all members; three tetrahedral units then rest, at the upper level,

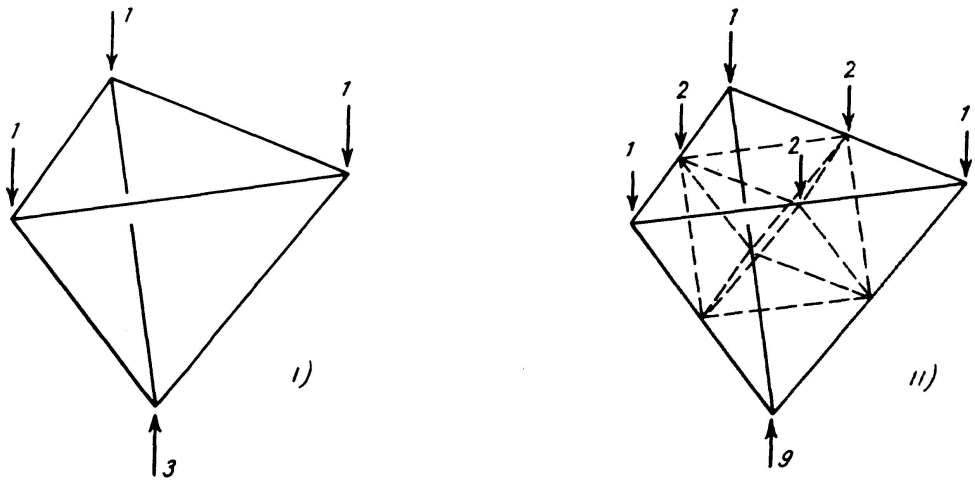


Fig. 34. Space Frame Having one Support.

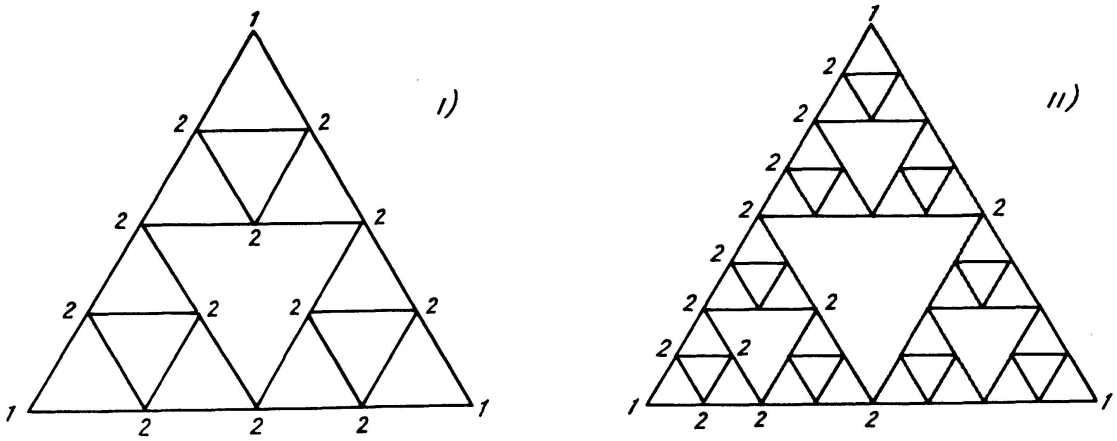


Fig. 35. Intensities of Loading on the Upper Surface of the Frame for Continued Sub-Division.

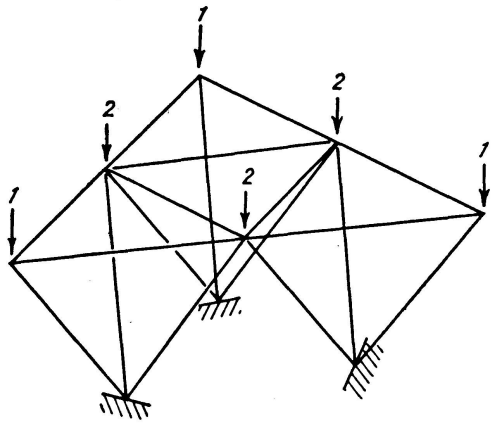


Fig. 36. Space Frame Having Three Supports.

on a single tetrahedral unit at the lower level, fig. 34 (II); the upper surface then carries the external loads shown in fig. 34 (II). When the sub-division is carried two stages further, the loads on the upper surface have the forms shown in fig. 35; there are two load units at each panel point, except at the extreme corners. The loading conditions do not correspond to uniform loading of the upper surface of the frame. As before, bracing of the space frame may be introduced at the lower levels, without impairing the deflection properties of the structure.

The space frame of fig. 36 has three supports; it consists of three basic tetrahedral units. The stiffnesses of the members can be arranged to give equal vertical deflections of the loaded joints.

Conclusions

The mathematical analysis of an iso-static frame, designed to give specific deflections, becomes in general a problem of finding positive values of a large number of variables (or stiffnesses), which satisfy a relatively small number of linear simultaneous equations.

In the general problem of a plane frame there are at least twice as many variables as linear equations, and this leads to certain difficulties in the mathematical analysis; a condition for the existence of a set of positive values of the variables (or stiffnesses) can be formulated in terms of determinants of the variables.

The analysis of the deflection properties of a frame can be simplified using graphical methods. But whereas the mathematical methods discussed in Part I could be extended to deal with problems of space frames, the usefulness of the graphical method is limited probably to plane frames. Graphical methods may be extended to study redundant plane frames; the analysis of redundant frames is complicated by the fact that alternative systems of internal loads are possible for any given system of external loads.

In an effort to simplify the design of a frame to give equal vertical deflections of the loaded joints, the structure has been built up from basic units which are loaded similarly. The composite structures developed are all iso-static; their analysis is relatively simple, compared with other types of structures giving the same specific deflection properties.

Acknowledgements

The authors are indebted to Atomic Power Projects for their interest in problems of uniformly deflecting structures, and to Dr. J. A. Todd, F.R.S., for his help in dealing with the mathematical analysis of Part I.

Summary

A study is made of the problem of designing a pin-jointed elastic structure so that the structure deflects under load in a given way. In the general problem of a plane iso-static frame there are at least twice as many variables as linear equations, and this leads to certain difficulties in the mathematical analysis. For this reason, graphical methods of solution are discussed; such methods can be extended to the study of hyper-static frames.

Finally, a method of "synthesis" is introduced to deal with the problem of designing a frame to give specific deflections. This is illustrated in relation to a plane pin-jointed truss on two supports.

Résumé

Les auteurs étudient le problème que pose un treillis élastique à nœuds articulés et devant présenter en charge des flèches déterminées.

Dans le cas général d'un treillis plan isostatique, il intervient au moins deux fois plus d'inconnues que d'équations linéaires, ce qui introduit certaines difficultés dans l'étude mathématique.

Pour cette raison, on a recours à des méthodes graphiques; ces méthodes peuvent être étendues à l'étude de systèmes hyperstatiques.

Enfin, les auteurs présentent une méthode de «synthèse», permettant de traiter le problème d'un treillis à flèches imposées. A titre d'exemple, ils étudient le cas d'un treillis plan à nœuds articulés, portant sur deux appuis.

Zusammenfassung

Hier wird das Problem, ein mit Gelenkbolzen verbundenes elastisches Tragwerk zu entwerfen, das unter Belastung Durchbiegungen in vorgegebener Art erleidet, untersucht.

In der allgemeinen Aufgabe eines ebenen statisch bestimmten Fachwerks sind mindestens zweimal soviel Unbekannte vorhanden als lineare Gleichungen, was zu gewissen Schwierigkeiten in der mathematischen Untersuchung führt.

Aus diesem Grunde werden graphische Lösungsmethoden vorgeschlagen; diese Methoden können für das Studium von statisch unbestimmten Systemen erweitert werden.

Zuletzt wird eine Methode der «Synthese» eingeführt, um das Problem des Entwurfs eines Fachwerks mit vorgegebenen Durchbiegungen zu behandeln. — Als Beispiel wird ein ebenes Gelenkbolzenfachwerk auf 2 Auflager untersucht.