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The Influence of Residual Stress on the Carrying Capacity of Eccentrically Loaded Columns

*Influence des contraintes propres sur la capacité portante des barres
chargées excentriquement*

*Der Einfluß der Eigenspannung auf die Tragfähigkeit von
außermittig belasteten Stäben*

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Abstract

The general method of solution for the maximum carrying capacity of columns containing residual stresses is first presented. Application is then made to a specific example of an *H*-column. The influence of an asymmetric residual stress distribution is considered and the results are presented in the usual forms as column curves for axially and eccentrically applied loads. Next, approximate solutions are presented for beam-columns having an idealized elastic-plastic stress-strain relationship and which contain residual stresses whose patterns have at least one axis of symmetry. The results of these studies are also shown in the form of column curves. Finally, the theory is compared with test results which have been reported in various publications.

I. Introduction

It is known that residual stresses are set up in beams and columns due to non-uniform cooling, cold straightening and other manufacturing processes. The influence of these initial stresses may be considerable for axially loaded steel columns depending on the geometry of the section and the residual stress distribution [1, 2].

KETTER, KAMINSKY and BEEDLE [3] and GALAMBOS and KETTER [7] are the first, to the authors' knowledge, who have studied the influence of residual stress on the behavior of beam-columns. They have shown how the thrust-moment-curvature relationships of wide-flange members are influenced by a symmetrical, "cooling" type residual stress pattern. Having this information, it is possible to determine the ultimate carrying capacity of eccentrically loaded members, which contain such residuals, along the lines of the classical inelastic column theory [4].

This paper is concerned with the determination of the influence of residual stress on the strength of columns of equal end eccentricities. These end eccentricities are assumed to lie on the same side of the member and result in a single curvature type of deformation. In addition, it will be presupposed that the member contains a residual stress distribution having at least one axis of symmetry.

In the presentation of the analytical solution to the problem, there is first discussed a general method suitable for any material. This rigorous treatment is applicable to axially loaded columns as well as the eccentrically loaded ones. The approximate solutions which are considered later, however, are only applicable when the material has a stress-strain curve which closely approximates the idealized elastic-fully plastic case, as is typical of mild structural steel.

II. General Method of Solution

If loads are applied eccentrically to a member or if, on the other hand, residual stresses do not have axial symmetry and a thrust is applied axially; a simple solution to the problem of determining the ultimate carrying capacity of the given member is not possible. It is first necessary to assume deflections and by successive adjustment to arrive at the correct deformation for a given loading. In general such a procedure is very similar to the classical method of solution as outlined by KARMAN [4]. Approximate solutions, on the other hand may be obtained by working with assumed deflection curves, as did JEZEK [4] and others. Such procedures will be presented and materially reduces the amount of numerical work that is required to obtain an answer to a given problem.

At first no stipulation will be made as to the shape of the cross-section nor to the stress-strain relationship of the material. It will be assumed, however, that the deflections are symmetric with respect to the center of the member and that the ends are pinned. Furthermore, it will be assumed that the residual stresses have at least one axis of symmetry and are uniform throughout the length. The possibilities of lateral-torsional or local instability are specifically excluded. Instability (maximum load) will be considered as a condition of excessive bending in the plane of the applied moments.

Consider the eccentrically loaded column shown in fig. 1a. For any section along the length of the member, strains can be described in terms of the compressive strain, ϵ_c , and the bending strain (i.e. curvature), ϕ_x , at that section (see fig. 1c). Knowing the stress-strain relationship of the material in question (e.g. fig. 1b), it is possible to compute the internal force, P , and moment, M , for various assumed values of ϵ_c and ϕ_x . The results of such calculations can be plotted as u versus ϕ and P versus ϕ curves for constant values of the parameter ϵ_c . u is the deflection plus the eccentricity at the section in question. Typical curves showing the relationships to be expected are given in figs. 1d and 1e. Eliminating ϵ_c from these two series of curves, the three variables u , P and ϕ can be combined into one graph as shown in fig. 1f.

To determine the critical length for a column subjected to a given thrust, P , applied at an eccentricity, e ; a deflected shape, $u(x)$, must be assumed (see fig. 1g). Entering fig. 1f with the value u , a value of ϕ_2 is obtained for

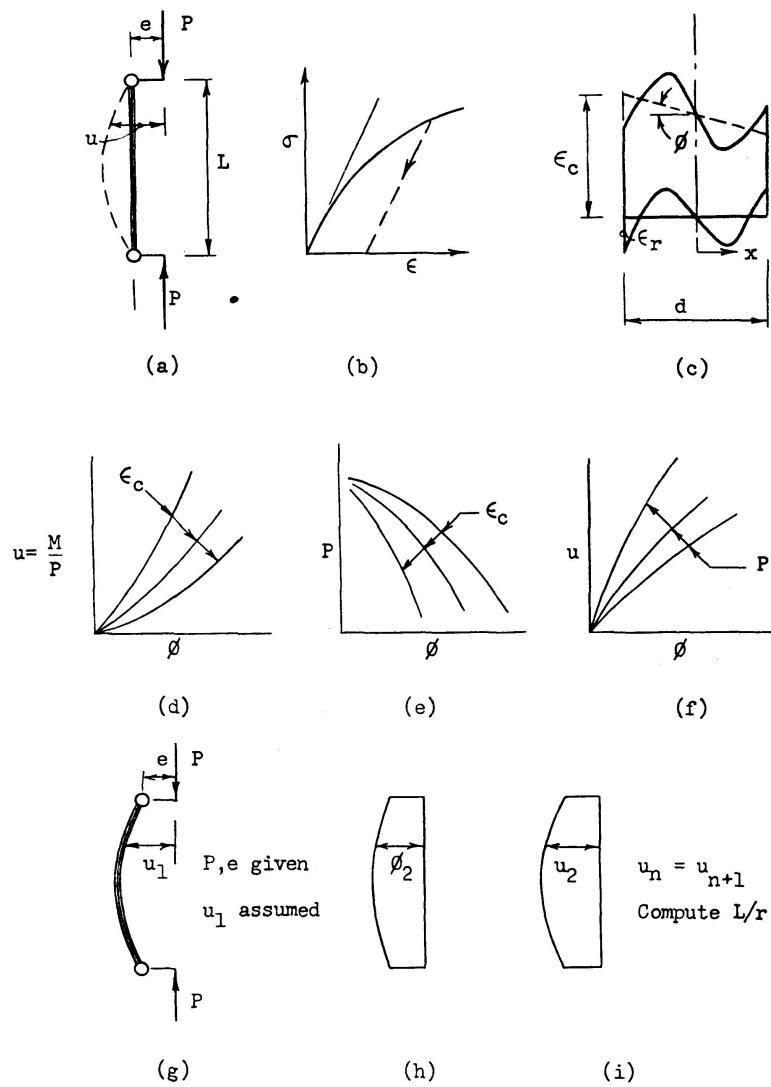


Fig. 1. Procedure for the Solution of Eccentrically Loaded Columns.

the given load P (fig. 1h). Double integration of ϕ_2 yields another deflection, u_2 , that contains the length, L , as a parameter. This process can be repeated until $u_n = u_{n+1}$ from which L is obtained.

In determining the critical length, L_{cr} , it is necessary to vary the magnitude of the center deflection, u_0 , and go through the same process as described above until a maximum value of the length has been realized [4].

Residual stresses influence the u - P - ϕ relationship and thereby influence the carrying capacity of a member containing such stresses. The procedure for determining the critical length when the magnitude of the load and residual stresses are given may be summarized as follows:

1. Determine P and M for various values of ϵ_c and ϕ . Calculate $u = \frac{M}{P}$.
2. Plot $P - \phi$ and $M - \phi$ graphs, where ϵ_c is a parameter for both relationships.
3. Combine the two graphs into one u - P - ϕ graph by the elimination of ϵ_c .
4. Assume a deflected shape of the column (u_1).
5. From the u - P - ϕ graph of step 3, determine ϕ for the given P value.
6. Numerically integrate to obtain a new deflection (u_2) and continue until $u_n = u_{n+1}$.
7. Determine the critical length by varying the magnitude of the center deflection. The maximum length obtained by this process is the critical length.

III. Application of the General Method of Solution to Axially and Eccentrically Loaded H-COLUMNS

The general method outlined in the previous section will now be applied to the specific solution of an *H*-column which contains a coldbending type of residual stress pattern in the flanges. Since sections of this type usually fail by combined bending and twist, when bending is imposed about the major axis of the cross-section, only weak axis loading and instability will be considered. It should be noted that the residual stress pattern for cold bending being asymmetric would tend to increase this possibility of lateral-torsional instability.

The column to be investigated has a cross-section as shown in fig. 2a. The flanges are assumed to contain the residual stress distribution shown in fig. 2b. The initial stresses are further presupposed to be constant along the length of the column. The material properties and stress-strain curve are as given in fig. 2c.

Firstly, it is necessary to calculate the axial thrust and the bending moment corresponding to various assumed values of the compressive strain, ϵ_c , and

curvature, ϕ (see fig. 2d). The limiting curves of applied strain, shown by the light solid lines in fig. 2d, correspond to that condition of yield point stress minus the initial residual stress. For a large number of combinations of these two parameters the following can be written after integrating and simplifying:

$$\frac{P}{P_y} = \frac{3}{4} \left[0.656 - \frac{2x_1}{b} + 1.375 \frac{x_1^2}{b^2} + \left(\frac{\epsilon_c}{\epsilon_y} - \frac{\phi b}{2\epsilon_y} \right) \left(\frac{x_1}{b} + 0.5 \right) \right. \\ \left. - \frac{\phi b}{2\epsilon_y} \left(0.25 - \frac{x_1^2}{b^2} \right) + \frac{1}{3} \left(\frac{\epsilon_c}{\epsilon_y} - \frac{\phi b}{2\epsilon_y} \right) \right] \quad (1)$$

and

$$\frac{u}{b} = \frac{M}{Pb} = \frac{3}{4} \frac{P_y}{P} \left[0.1335 - \frac{x_1^2}{b^2} + 0.9170 \frac{x_1^3}{b^3} - \left(\frac{\epsilon_c}{\epsilon_y} - \frac{\phi b}{2\epsilon_y} \right) \left(0.1250 - \frac{x_1^2}{2b^2} \right) \right. \\ \left. + \frac{\phi b}{3\epsilon_y} \left(\frac{x_1^3}{b^3} + 0.1250 \right) \right], \quad (2)$$

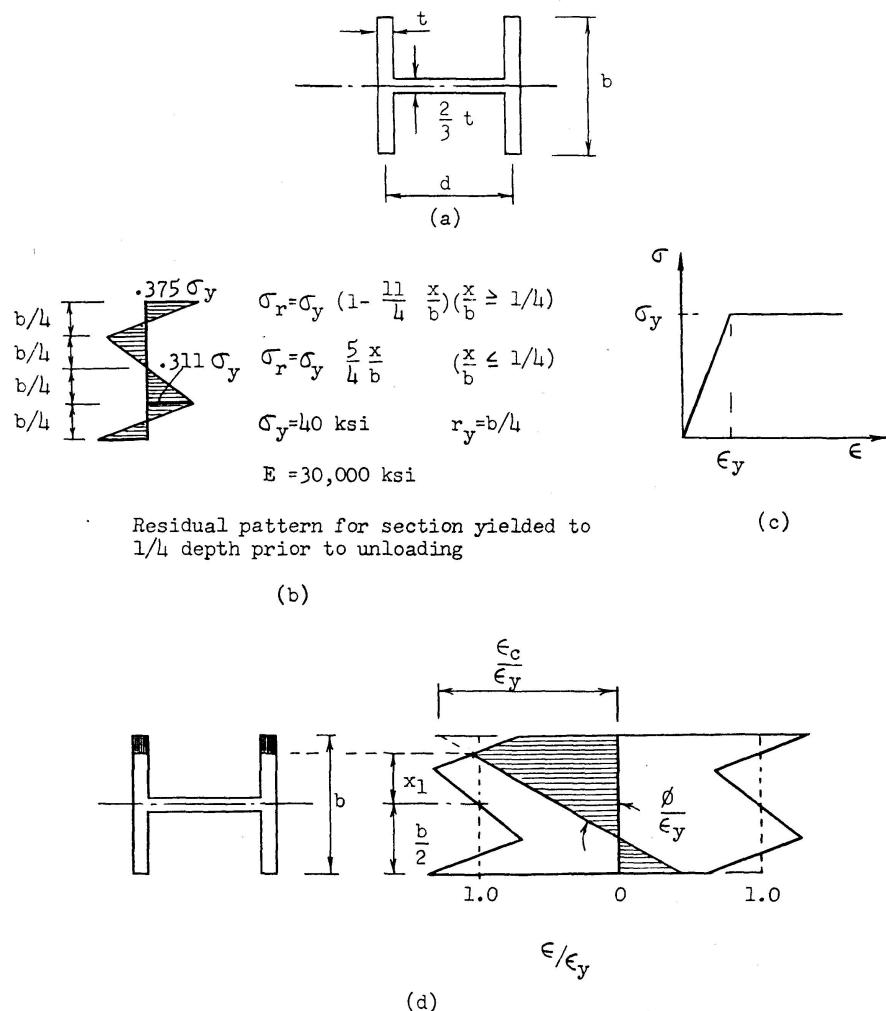


Fig. 2. Residual Stress and Applied Stress in Steel Column.

where

$$\frac{x_1}{b} = \left[\frac{2 - \left(\frac{\epsilon_c}{\epsilon_y} - \frac{\phi b}{2 \epsilon_y} \right)}{2.75 + \frac{\phi b}{\epsilon_y}} \right] \quad (3)$$

is a measure of the distance from the flange center to the yield zone created due to the applied loading. Eqs. (1), (2) and (3) are valid in the range.

$$\frac{x_1}{b} \geq 0.250 \quad \text{and} \quad \frac{b\phi}{\epsilon_y} < 2.581.$$

The results of the calculations using these equations are given in fig. 3 where ratios of the three parameters u , P and ϕ are plotted for constant values of the variable ϵ_c/ϵ_y . Eliminating ϵ_c/ϵ_y as indicated by the dashed lines in fig. 3, the direct relationship between u and ϕ with P as the variable is obtained as shown in fig. 4.

With this information the numerical procedure outlined in the preceding section can be applied. A sample calculation is shown in fig. 5.

The final results of these calculations can be presented in the conventional form as column curves with the eccentricity ratio, ec/r^2 , as the variable. These have been shown in fig. 6. It should be noted that they are valid only for the

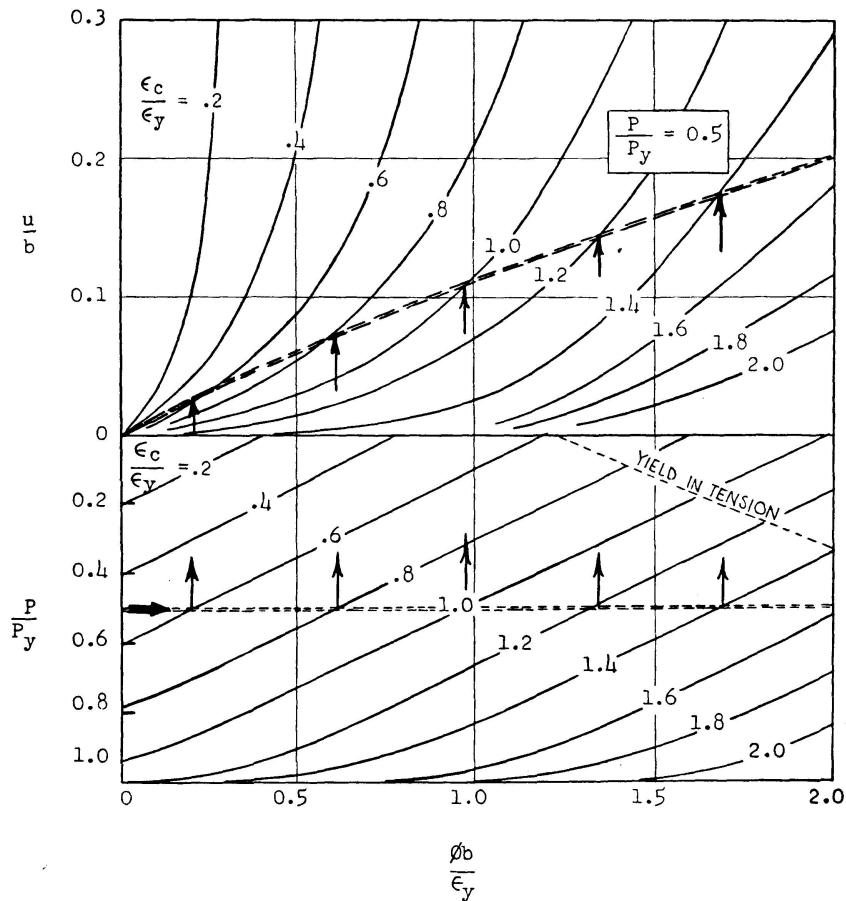


Fig. 3. Relations Between $\frac{P}{P_y} - \frac{u}{b} - \frac{\phi b}{\epsilon_y} - \frac{\epsilon_c}{\epsilon_y}$.

assumed residual stress distribution. Also shown on this figure as dashed curves, are the corresponding elastic solutions of the second order stress problem (secant solution) with the outer fiber at the yield point stress (no residual stress). In addition, certain approximate solutions and test points are shown that will be discussed later in this paper.

Even with the $u\phi P$ relationship of fig. 4 given, the numerical work required to obtain a sufficient number of values of $(L/r)_{cr}$ is formidable. An approximate solution which requires much less work can be realized by assuming a deflected shape of the member. It is then only necessary to vary the magnitude of the center-line deflection to obtain the critical slenderness ratio.

Assuming a sine curve for the approximate deflected shape of the member, the curvature at the centerline section is given by the expression

$$\phi_0 = (u_0 - e) \frac{\pi^2}{L^2}, \quad (4)$$

from which

$$L = \pi \sqrt{\frac{u_0 - e}{\phi_0}}, \quad (5)$$

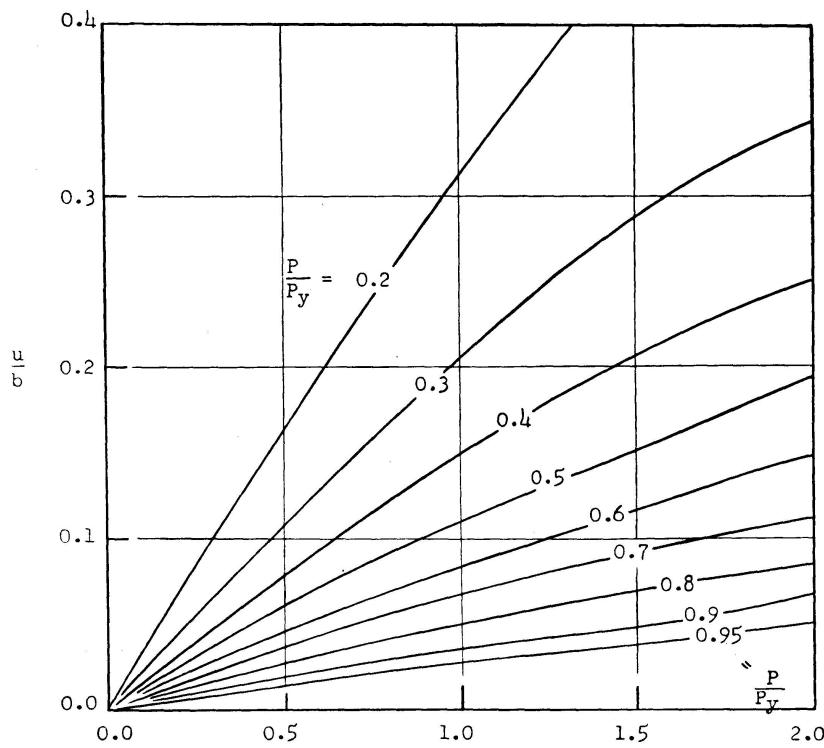


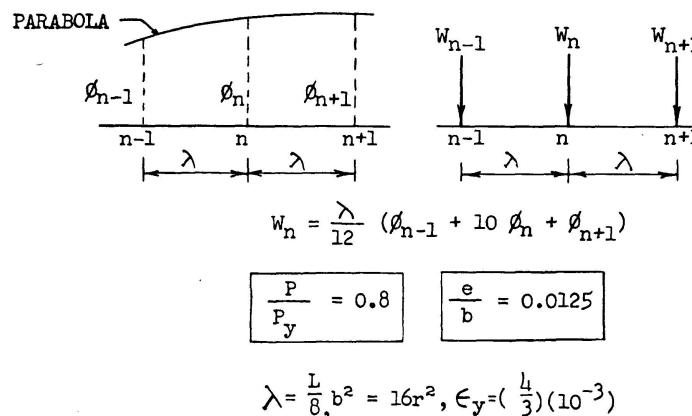
Fig. 4. Relations Between $\frac{P}{P_y} - \frac{u}{b} - \frac{b\phi}{\epsilon_y}$.

or in terms of non-dimensional variables

$$L = \frac{\pi b}{\sqrt{\epsilon_y}} \sqrt{\frac{\left(\frac{u_0 - e}{b}\right)}{\frac{\phi b}{\epsilon_y}}}. \quad (6)$$

Substituting in eq. (6) corresponding values of u_0 and ϕ_0 (from fig. 4) for constant values of the axial thrust, P , L_{max} may be obtained within a few trials. The results of this approximation are shown in fig. 6 for particular values of P/P_y by the open circles.

In spite of the simplification obtained by assuming a particular type of deflected shape, the work required is still considerable. This is especially true if the influence of several different residual stress patterns is desired. This follows from the fact that for each pattern a new set of curves similar to those shown in fig. 4 (for $u-\phi-P$) would be required. In the following, a relatively simple solution to the problem will be obtained which is applicable to beam-columns of both rectangular and simplified H -shape cross-sections.



POINT	0	1	2	3	$\frac{L}{r}$	FACTOR
$u_1 - e/b$	0	0.0273	0.0498	0.0647	0.0700	1.0
u_1/b	0.0125	0.0398	0.0623	0.0772	0.0825	1.0
ϕ_2	0.240	0.805	1.310	1.770	1.930	ϵ_y/b
w_n		9.600	15.675	20.940	11.420	$\epsilon_y/12b$
SHEAR	57.635	48.035	32.360	11.420		$\epsilon_y/12b$
$u_2 - e/b$	0	57.635	105.670	138.030	149.450	$\epsilon_y^2/12b^2$
CHECK	0	0.0270	0.0495	0.0646	0.0700	1.0

$$0.07 = 149.45 \frac{\epsilon_y \lambda^2}{12b^2} \quad \left(\frac{L}{r}\right)^2 = \frac{0.07}{149.45} 9.22 \cdot 10^6 = (66.4)^2$$

Fig. 5. Example of Numerical Deflection Calculation.

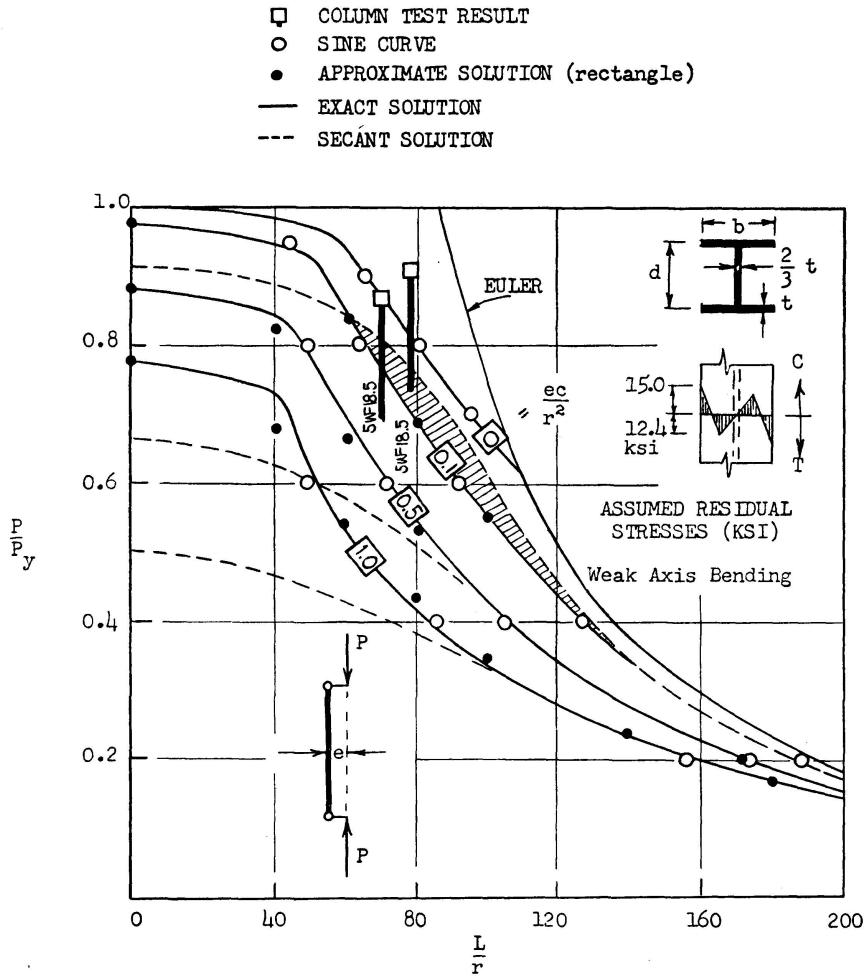


Fig. 6. Axial and Eccentric Column Curves — Cold Bending Residual Stresses.

IV. Approximate Solutions

Bending About the Strong Axis

An idealized section is assumed which is composed of two rectangular flanges and a shear resisting web of negligible area. For a particular wide-flange shape the properties of the idealized section (marked with the subscript i) can be computed as follows:

$$A = A_i, \quad I = I_i = A r^2, \quad r = r_i = \frac{d_i}{2}, \quad \frac{ec}{r^2} = 2 \frac{ed}{d_i^2}.$$

(See fig. 7.)

The residual stress distribution is assumed to be piecewise linear and constant along the length. The deflection curve is assumed to be a sine curve with the maximum deflection, u_0 , at the center of the member. Furthermore, the moment diagram is symmetric with respect to the center, where the

moment has the value $M_0 = P u_0 + m_0$. The other suppositions are the same as those listed earlier.

The equilibrium equations at the center of the member can be written in the following form¹⁾:

$$P = P_1(\epsilon_1, \sigma_r) + P_2(\epsilon_1, \phi) \quad (7)$$

and

$$M_0 = P u_0 + m_0 = \frac{d_i}{2} [P_1(\epsilon_1, \sigma_r) - P_2(\epsilon_1, \phi)], \quad (8)$$

where P_1 and P_2 are the forces taken by the flanges. The curvature at the centerline section is

$$\phi_0 = \frac{\pi^2}{L^2} u_0. \quad (9)$$

Combining eqs. (7), (8) and (9) an expression of the following form is obtained:

$$F = F(P, u_0, L, \sigma_r, m_0). \quad (10)$$

To determine the critical load corresponding to a given situation, the maximum value of

$$P = P(u_0, L, \sigma_r, m_0) \quad (11)$$

must be ascertained. For a given length, L , residual stresses, σ_r , and moment, m_0 , the maximum value is determined from

$$\frac{dP}{du_0} = 0. \quad (12)$$

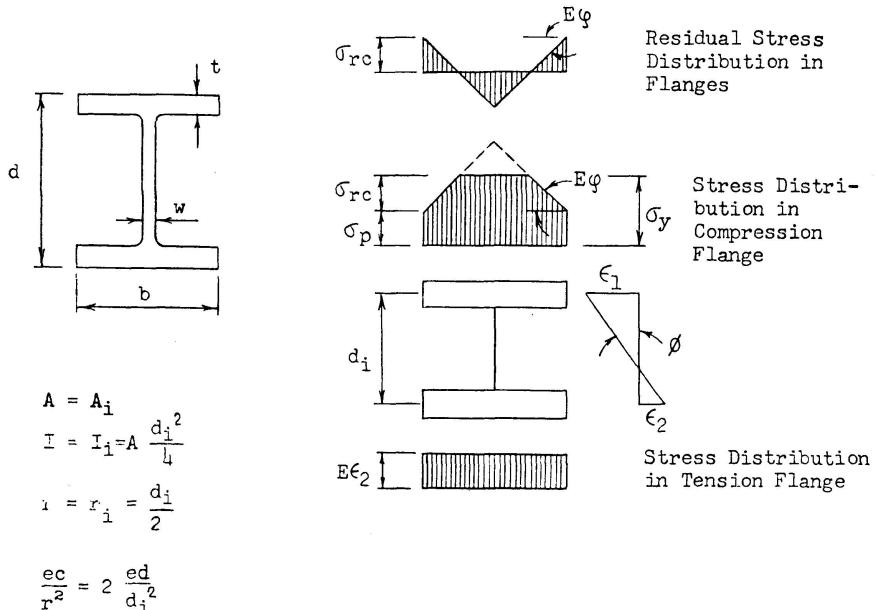


Fig. 7. Actual and Idealized Section with Stress Distribution.

¹⁾ A detailed derivation of these expressions is given in the appendix.

Using the implicit form of the function F

$$\frac{dF}{du_0} = \frac{\partial F}{\partial P} \frac{dP}{du_0} + \frac{\partial F}{\partial u_0} = 0$$

or

$$\frac{\partial F}{\partial u_0} \equiv 0. \quad (13)$$

Eq. (13) then defines the critical value of u_0 , which when substituted into eqs. (10) and (11) gives

$$F = F(P, L, \sigma_r, m_0) \quad (14)$$

or

$$P = P(L, \sigma_r, m_0).$$

To obtain the critical length, L_{cr} , of a column subjected to a given loading, P , residual stress, σ_r , and moment, m_0 , the maximum value of

$$L = L(u_0, P, \sigma_r, m_0) \quad (15)$$

must be determined. The condition for a maximum is

$$\frac{dL}{du_0} = 0 \quad (16)$$

and

$$\frac{dF}{du_0} = \frac{\partial F}{\partial L} \frac{dL}{du_0} + \frac{\partial F}{\partial u_0} = 0 \quad (17)$$

or as in the previous case

$$\frac{dF}{du_0} \equiv 0. \quad (18)$$

Thereby the proof is given that $L_{max} = L_{cr}$.

For a linear variation of residual stresses between the flange edges and the flange center the analytical evaluation corresponding to eq. (14) becomes

$$\sigma_{cr} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2} \left[1 - \frac{E \varphi b \left\{ \frac{1 - \frac{\sigma_{cr}}{\sigma_E}}{2 - \frac{\sigma_y}{\sigma_E}} \right\} + \frac{4m_0}{A d_i}}{2(\sigma_p - \sigma_{cr}) + E \varphi b \left\{ \frac{\left(2 - \frac{\sigma_{cr}}{\sigma_E}\right)^2 - 1}{\left(2 - \frac{\sigma_{cr}}{\sigma_E}\right)^2} \right\}} \right], \quad (19)$$

where σ_p is the pseudo-proportional limit; that is, $\sigma_y - \sigma_{rc}$ (see fig. 7), and σ_E is the stress corresponding to the Euler load.

For the case where $\sigma_r = 0$, $\varphi = 0$ and $\sigma_p = \sigma_y$, eq. (19) takes a form that is expressable as the quadratic equation (see also eq. (m), Appendix)

$$\left(\frac{\sigma}{\sigma_y}\right)^2 - \left(\frac{\sigma}{\sigma_y}\right) \left(1 + \frac{\sigma_E}{\sigma_y}\right) - \left(\frac{2m_0}{d_i P_y}\right) \left(\frac{\sigma_E}{\sigma_y}\right) + \frac{\sigma_E}{\sigma_y} = 0. \quad (20)$$

When $\left(\frac{L}{r}\right) = 0$ the stability problem reduces to a stress equilibrium problem and the influence of residual stress, which is in equilibrium within itself, is wiped out for full yielding of the cross-section. For an *H*-section the simple plastic theory gives for bending about the strong axis:

$$\left(\frac{\sigma}{\sigma_y}\right)^2 + \frac{4w}{A} \frac{m_0}{P_y} - \frac{4w}{A^2} \left[b t (d - t) + w \left(\frac{d}{2} - t\right)^2 \right] = 0 \quad (21)$$

$$\text{for } \frac{\sigma}{\sigma_y} \leq \frac{w(d - 2t)}{A}$$

and

$$\left(\frac{\sigma}{\sigma_y}\right)^2 + 2 \left(\frac{\sigma}{\sigma_y}\right) \left(\frac{bd}{A} - 1\right) + \frac{4b}{A} \frac{m_0}{P_y} - \left[\frac{2bd}{A} - 1\right] = 0 \quad (22)$$

$$\text{for } \frac{\sigma}{\sigma_y} \geq \frac{w(d - 2t)}{A}.$$

Bending About the Weak Axis

The idealized section may in this case be assumed to consist of a single rectangle made up of the two flanges. The derivation follows closely the previous one.

Two basic type stress distributions are possible. Due to axial thrust and bending moment the general pattern will be either of the cases shown in fig. 8. In case I (fig. 8b) it is assumed that yielding is only in compression; whereas, in case II (fig. 8c) yielding occurs in both tension and compression. The rate of change of the residual stress at the outer fiber is given as before by the angle φ .

It is possible from the internal stress distribution to express the corresponding forces and moments as follows²⁾:

Case I

$$P = P(\phi_0, x_1, \sigma_r) \quad (23)$$

$$M_0 = M_0(\phi_0, x_1, \sigma_r) \quad (24)$$

$$\phi_0 = \frac{\pi^2}{L^2} u_0 \quad (25)$$

Case II

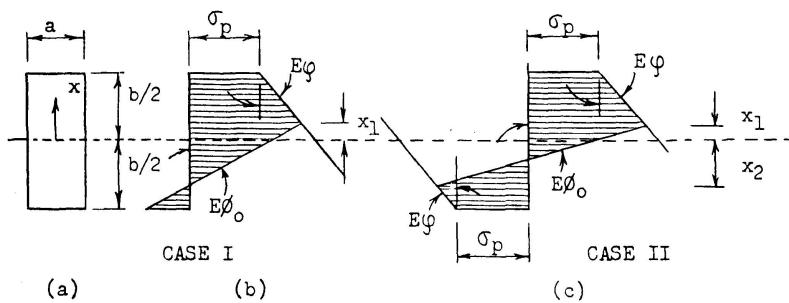


Fig. 8. Approximate Analysis — Rectangular Column.

²⁾ See Appendix (b).

where x_1 and x_2 are measures of the distance from the center of the section to the beginning of the plastic zones (see fig. 8). The zero subscript refers to the section at the middle of the member.

For both cases I and II the expressions can be reduced by combining eqs. (23), (24), and (25). There results

$$F = F(P, u_0, L, \sigma_r, m_0). \quad (26)$$

It should be noted that this equation has the same form as eq. (10). Therefore, the critical centerline deflection and the solution of the function, F , are obtained in the same manner as that given earlier. The resulting analytical expressions for critical stress are

$$\sigma_{cr} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2} \left[1 - \frac{\frac{E\varphi b}{\sigma_{cr}} - \frac{\varphi b}{\pi^2} \left(\frac{L}{r}\right)^2 + \frac{12m_0}{Pb}}{\left\{ \frac{3E\varphi b + 6\sigma_p}{\sigma_{cr}} \right\} - 6} \right]^3 \quad (\text{Case I}) \quad (27)$$

$$\text{and } \sigma_{cr} = \left[\frac{(\sigma_p + \frac{1}{2}E\varphi b) \left\{ \frac{\left(\frac{L}{r}\right)^2}{E\pi^2} \right\}^2}{\left\{ \left(\frac{3\sigma_p + \frac{1}{2}E\varphi b}{3\sigma_{cr}} \right) - \frac{4m_0}{Pb} - \left(\frac{\sigma_{cr}}{\sigma_p + \frac{1}{2}E\varphi b} \right) + \frac{\varphi b}{3\pi^2} \left(\frac{L}{r} \right)^2 \right\}^3} \right] \quad (\text{Case II}), \quad (28)$$

where as before σ_p is the pseudo-proportional limit, $\sigma_y - \sigma_{rc}$. Eqs. (27) and (28) are the solutions to the problem. They must be solved by trial and error.

If no residual stresses are present $\varphi = 0$, $\sigma_p = \sigma_y$ and the equations become

$$\sigma_{cr} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2} \left[1 - \frac{\frac{2m_0}{Pb}}{\left(\frac{\sigma_y}{\sigma_{cr}} - 1 \right)} \right]^3 \quad (\text{Case I}) \quad (29)$$

$$\text{for } \left(\frac{2m_0}{Pb} \leq 1 \right)$$

$$\text{and } \sigma_{cr} = \frac{\sigma_y^3 \left[\frac{\left(\frac{L}{r}\right)^2}{E\pi^2} \right]^2}{\left[\frac{\sigma_y}{\sigma_{cr}} - \frac{4m_0}{Pb} - \frac{\sigma_{cr}}{\sigma_y} \right]^3} \quad (\text{Case II}) \quad (30)$$

$$\text{for } \left(\frac{2m_0}{Pb} \geq 1 \right).$$

After rewriting in a slightly different form, eqs. (29) and (30) are identical to the solutions given by JEZEK [4].

A simple limit for the range of applicability of eqs. (27) and (28) cannot be given because of their complexity. The following, however, will aid in the estimation of that limit. If case I applies, but eq. (28) for case II was used, the actual critical slenderness ratio and critical stress will be larger than the computed values. If on the other hand case II applies, but case I was used, the

actual critical slenderness ratio and critical stress will be smaller than the computed values.

For short columns (where L/r is less than about 50) eqs. (27) and (28) do not hold since the developed stress distribution at the centerline section will be different from that assumed. However, for the limiting case of $L/r=0$ the section becomes fully yielded, the stability problem reduces to one of a stress consideration and correspondingly the influence of residual stress vanishes. For the *H*-profile the simple plastic theory for weak axis bending becomes

$$\left(\frac{\sigma}{\sigma_y}\right)^2 - \left(\frac{\sigma}{\sigma_y}\right) \left(\frac{2w(d-2t)}{A}\right) + \left(\frac{8t}{A} \frac{m_0}{P_y}\right) - \left(\frac{4bt}{A} - 1\right) = 0. \quad (31)$$

Interpolation between the results of eqs. (27) or (28) and eq. (31) for critical stresses in short columns ($0 \leq \frac{L}{r} \leq 50$) can be done without appreciable error.

Of particular interest is the application of this approximate solution to the problem solved previously by the more exact methods. The results of these calculations are shown in fig. 6. The approximate solution (filled circles) is in very good agreement with the "exact" solution.

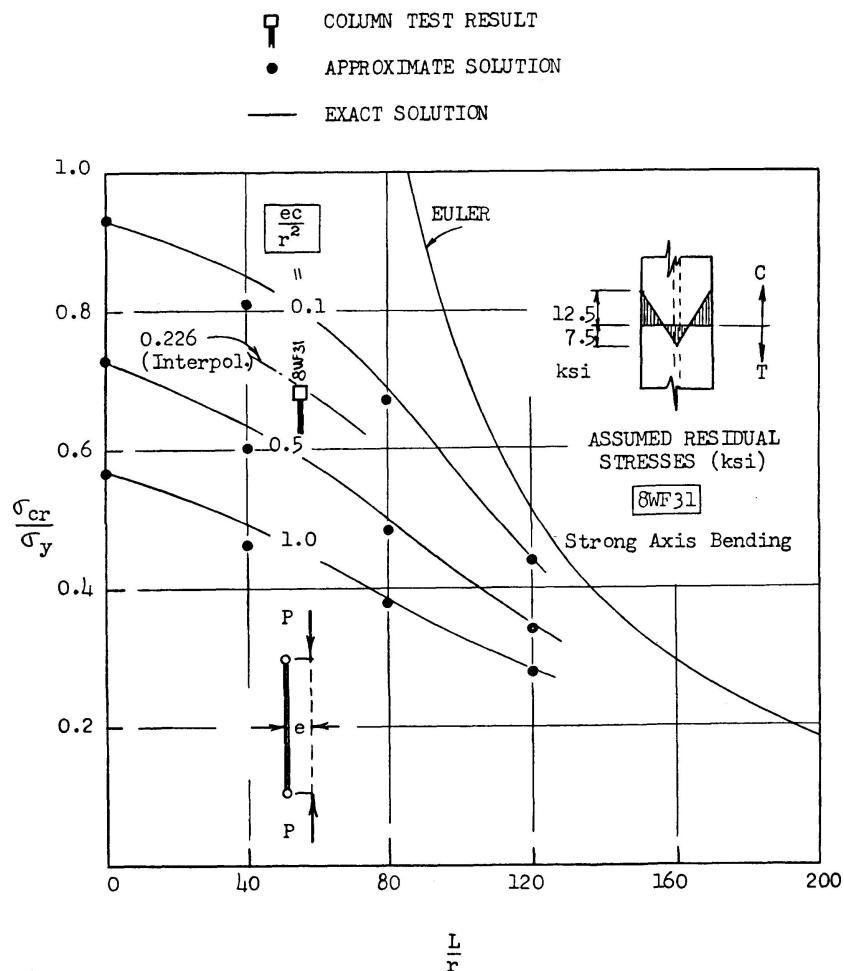


Fig. 9. Eccentric Column Curves [3] — Comparison with Approximate Solution and Tests.

V. Discussion

The rigorous development for the carrying capacity of axially and eccentrically loaded columns forced to bend about their weak axis is given by the solid lines in fig. 6. The solution to the second order stress problem (secant solution) is shown by the dashed line for corresponding values of $\frac{e^c}{r^2}$ (0.1, 0.5 and 1.0). For an eccentricity ratio $\frac{e^c}{r^2} = 0.1$ and slenderness values $60 \leq \frac{L}{r} \leq 130$ the secant solution (neglecting any initial imperfections) would be unsafe as shown by the cross-hatched area; whereas, for slenderness ratios below 60 it is too conservative³). Fig. 6 also shows that the assumption of a sine curve for the deflected shape of the column gives good results (open circles).

Because of the large amount of time required to obtain the rigorous solution, the approximate methods were developed to afford a rapid study of the influence of various residual stress distributions. It is necessary, however, to first determine how closely these approximate solutions are to the rigorous solution. Good agreement is demonstrated for several residual stress distributions

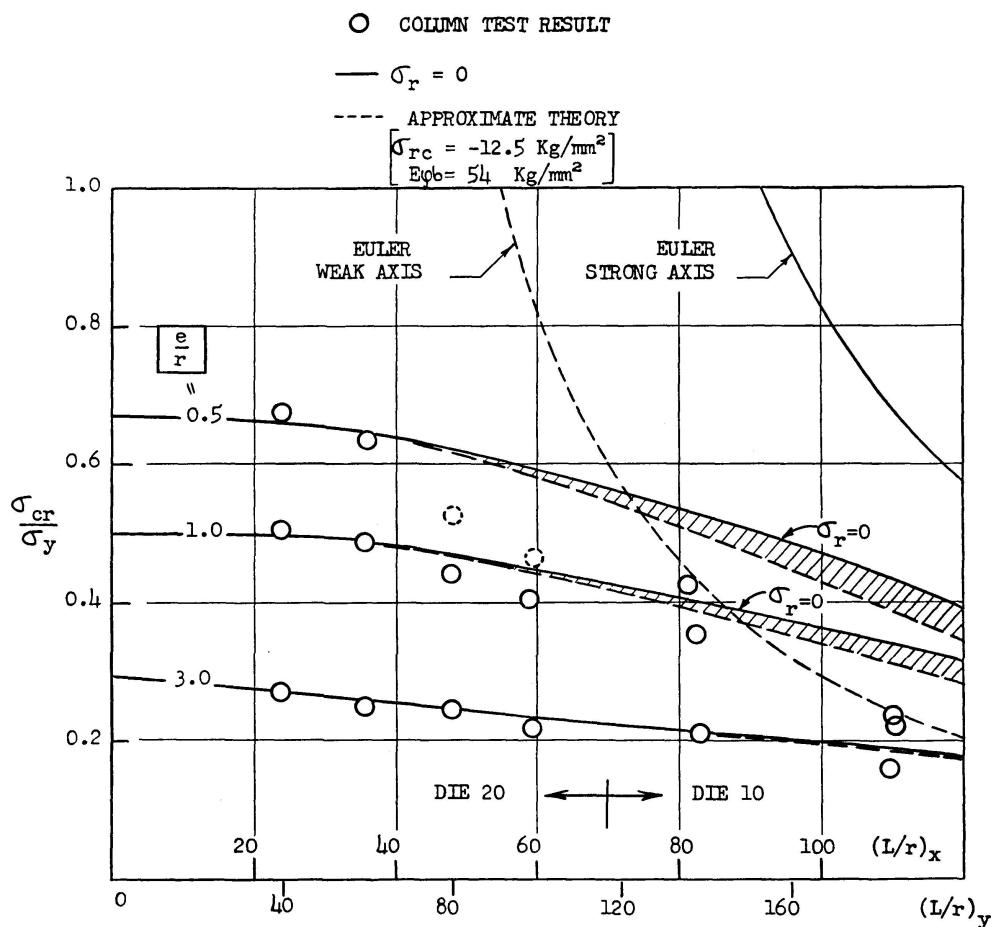


Fig. 10. Eccentric Column Curves and Tests [5].

³) It should be emphasized that these statements hold for the one particular pattern of residual stress assumed. Other distributions, however, would show similar results.

and axes of bending (figs. 6, 9 and 12). In each of these figures the approximate values have been shown as filled circles. For $L/r = 0$ the values were calculated for the actual cross-sectional shape by the simple plastic theory. For all cases shown, the approximate solutions were in close agreement with the more exact ones. (The rigorous solutions of figs. 9 and 12 were taken from Reference 3 where a slightly different method was used.)

The effects of varying the initial residual stress distribution in the flange of an idealized *H*-shape are shown in figs. 11 and 13 for bending about the strong and weak axes respectively. A constant value of the eccentricity ratios $e/r = 0.5$ and $e^2/r^2 = 0.5$ was assumed. The value of $E\varphi b$ is varied while the compressive residual stress at the flange tips is assumed to be 20 ksi and the yield point stress is 40 ksi. A value of $E\varphi b = 60$ would result in a residual stress distribution that would not be unlike those observed in typical wide-flange shapes [2]. It should be noted by comparison of the column curves corresponding to $E\varphi b = 60$ (shown by the heavy solid line) and $\sigma_r = 0$, that case which neglects residual stress, the influence of residual stress is still pronounced (see figs. 11 and 13). Fig. 10 shows the diminishing effect of residual stress on eccentric column strength as the eccentricity is increased.

In addition to the theoretical conclusions discussed above it is possible to

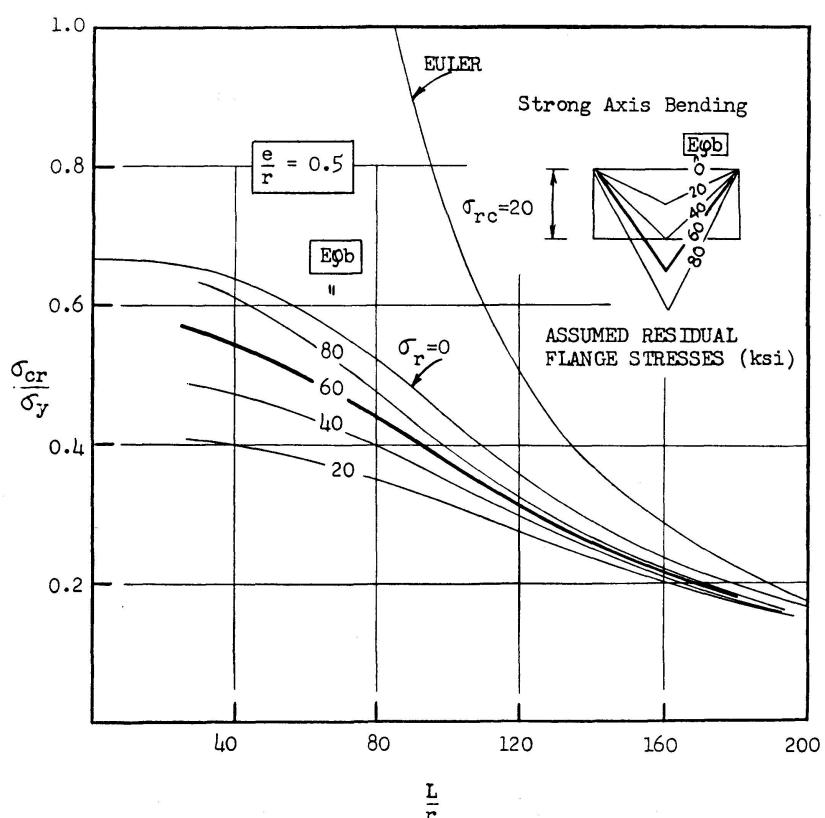


Fig. 11. Influence of Residual Stresses on Eccentric Column Curve.
(Strong Axis Bending.)

compare predicted strengths as presented in this report to column test results that have been reported in the literature and for which residual stress measurements were made.

Although the residual stress patterns correspond only roughly to the assumed pattern of fig. 6, the result of two cold straightened axial column tests ($L/r = 70$ and 79.5) [6] have been shown on fig. 6 by the open squares. The latter column ($L/r = 79.5$) was straightened in the laboratory under a loading which subjected the members to uniform moment. The measured values of maximum residual stress were 8 and 6 ksi which is about one half of the value assumed in the rigorous solution. As would be expected, the column test result lies above the theoretical solution. The other column ($L/r = 70$) was also cold-straightened in the laboratory but by a single concentrated load at the center of the member. For such a condition of loading, the residual stress pattern along the member will not be uniform. Furthermore, a higher residual stress will result at the center-line section of the column. The column test result is shown in fig. 6.

The result of an additional column test is shown in fig. 9. This test was part of a separate investigation on "Welded Continuous Frames and Their

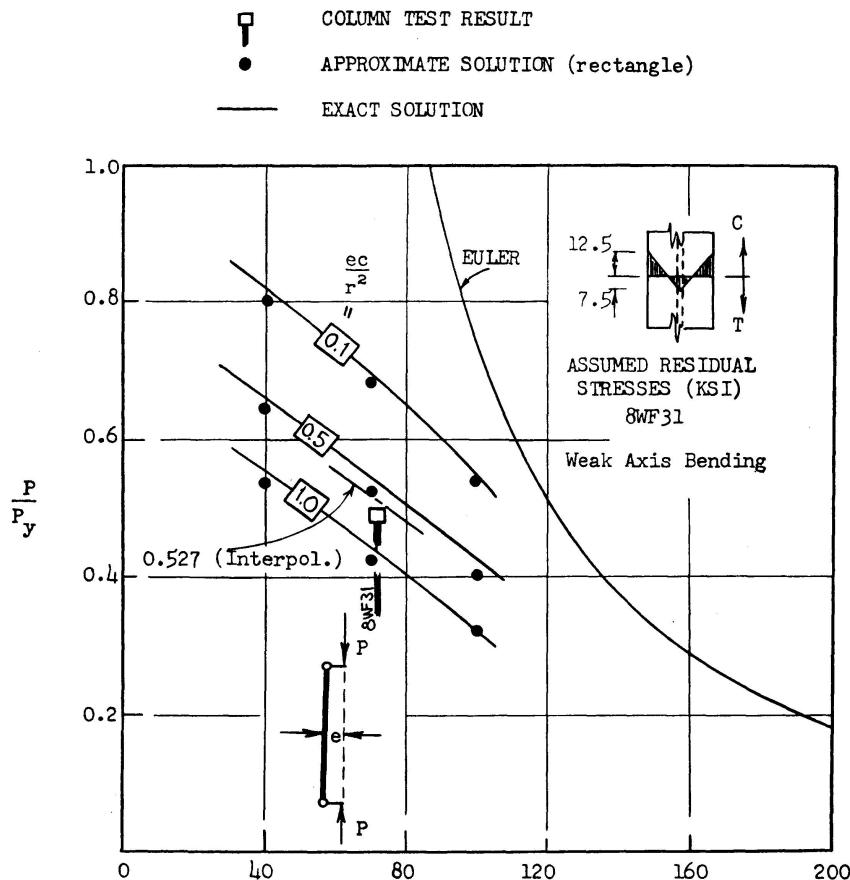


Fig. 12. Eccentric Column Curves — Comparison with Approximate Solution and Tests.

Components" currently being carried out at Fritz Engineering Laboratory. Interpolation shows good agreement with the theory.

The results of the column tests reported in Reference [5] are shown in fig. 10. These tests cover relatively low slenderness ratios due to the fact that the end supports were also pin-ended in the weak direction. Although these tests fall in slenderness and eccentricity ratios which do not demonstrate the full significance of residual stress, and even though the members failed by lateral-torsional buckling, there is reasonable agreement with the theory. The possibility of differences between the residual stresses measured on one piece of the material as opposed to those present in the various columns is also a condition that would tend to alter the test correlation [1].

For an eccentricity normal to the plane of the web one test result is available for comparison. This test is also due to the previously cited study being carried out at Fritz Laboratory on "Welded Continuous Frames". Here again, interpolation shows good agreement with the theory (fig. 12).

One variable that has not as yet been stressed in this report, but which is also very important, is the yield point stress. The eccentric column curves for $\varphi = 0$ in figs. 11 and 13 could also be interpreted as column curves for a yield point of 20 ksi, where residual stresses are zero and the reference stress, σ_y , is taken as 40 ksi.

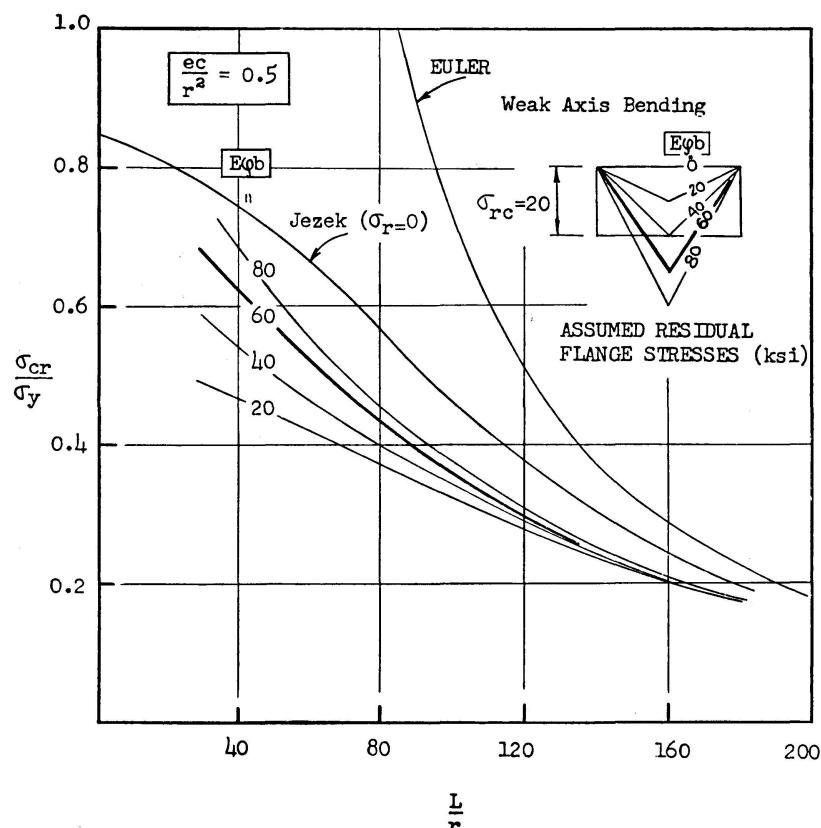


Fig. 13. Influence of Residual Stresses on Eccentric Column Curve.
(Weak Axis Bending.)

VI. Acknowledgements

This report presents a part of the theoretical and experimental studies made during the course of a three year research investigation of the influence of residual stresses on column strength, which was carried out under the general direction of Lynn S. Beedle. The work was done at Fritz Engineering Laboratory, Lehigh University, of which William J. Eney is director.

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VIII. Nomenclature

A Cross-sectional area.
b Flange width or width of rectangle.
d Depth of *H*-section or rectangle.
e Eccentricity of loads.
E Young's modulus of elasticity.
I Moment of inertia.
L Total length of a pin-ended column.
L/r Slenderness ratio.
M Moment.
m Moment due to lateral loads on column.

P	Load on a column.
P_{cr}	Critical or maximum load on a column.
P_E	Euler buckling load for a pin-ended column.
P_y	Axial load corresponding to yield point stress across entire section.
r	Radius of gyration in the plane of bending.
t	Flange thickness.
u	Deflection in x -direction.
x_1, x_2	Distances from the center of the cross-section to the beginning of the yielded area (see figs. 2d and 8).
ϵ	Unit strain.
ϵ_c	Maximum compressive strain.
ϵ_y	Strain corresponding to the yield point.
σ	Normal stress.
σ_p	Proportional limit stress.
σ_{cr}	Critical applied average stress on a column.
σ_E	Euler buckling stress.
σ_r	Residual stress.
σ_{rc}	Residual stress at flange edges.
σ_{r_0}	Residual stress at flange centers.
σ_y	Yield stress level; average stress in the plastic range.
φ	Rate of residual stress variation (see fig. 7 or 8).
ϕ	Curvature.

IX. Appendix: Derivation of Approximate Solutions

In this Appendix the derivations are given for the approximate solutions for the idealized H and rectangular section discussed in Section IV. The assumptions and limitation are listed in the earlier section and will not be repeated here.

a) Bending About the Strong Axis of an H -Shape

Referring to fig. 7, the higher stressed flange of the idealized section is assumed to carry a load P_1 which is a function of the strain ϵ_1 and the linear residual stress pattern.

$$P_1 = \frac{A}{2E\varphi b} [E\epsilon_1(E\varphi b) - (E\epsilon_1)^2 + 2(E\epsilon_1)\sigma_p - \sigma_p^2]. \quad (a)$$

The load carried by the other flange is given by

$$P_2 = \frac{A}{2} [E\epsilon_1 - E\phi d_i]. \quad (b)$$

The stress in this less highly stressed flange will, for all intents and purposes, remain elastic.

The total axial thrust on the column is then given by the sum of eqs. (a) and (b). That is,

$$P = P_1 + P_2$$

or
$$P = \frac{A}{2E\varphi b} [2E\epsilon_1(\sigma_p + E\varphi b) - (E\epsilon_1)^2 - \sigma_p^2 - (E\varphi b)(E\phi_0 d_i)]. \quad (c)$$

The corresponding internal moment at the centerline section is

$$M_0 = P u_0 + m_0 = \frac{d_i}{2} (P_1 - P_2) \quad (d)$$

or
$$M_0 = \frac{A d_i}{4 E \varphi b} [(E\varphi b)(E\phi_0 d_i) + 2E\epsilon_1\sigma_p - (E\epsilon_1)^2 - (\sigma_p)]. \quad (e)$$

From eq. (c), $E\epsilon_1$ can be expressed as follows:

$$E\epsilon_1 = (E\varphi b + \sigma_p) - \sqrt{(E\varphi b)^2 + (E\varphi b) \left(2\sigma_p - E\phi_0 d_i - \frac{2P}{A} \right)}, \quad (f)$$

where
$$\phi_0 = \frac{\pi^2}{L^2} u_0. \quad (g)$$

Writing eq. (e) in implicit form F and performing a partial differentiation with respect to u_0 , the following is obtained:

$$0 = P - \frac{A d_i}{4 E \varphi b} \left[(E\varphi b) \frac{E\pi^2}{L^2} + 2(\sigma_p - E\epsilon_1) \frac{\partial(E\epsilon_1)}{\partial u_0} \right]. \quad (h)$$

From eq. (f)

$$\frac{\partial(E\epsilon_1)}{\partial u_0} = \frac{(E\varphi b) \frac{E\pi^2}{L^2} d_i}{2 \sqrt{(E\varphi b)^2 + (E\varphi b) \left(2\sigma_p - E\phi_0 d_i - \frac{2P}{A} \right)}}, \quad (i)$$

which when substituted into eq. (h) gives

$$0 = P - P_E \left[2 - \frac{E\varphi b}{(E\varphi b)^2 + E\varphi b \left(2\sigma_p - \frac{E\pi^2 d_i}{L^2} u_0 - \frac{2P}{A} \right)} \right]. \quad (j)$$

The Euler load, P_E , for the idealized section is given by:

$$P_E = \frac{\pi^2 E A d_i^2}{4 L^2}. \quad (k)$$

From eq. (j) u_0 is obtained.

$$u_0 = \frac{L^2}{Ed_i\pi^2} \left[E\varphi b + 2\sigma_p - \frac{2P}{A} - \frac{E\varphi b}{\left(2 - \frac{P}{P_E}\right)^2} \right]. \quad (l)$$

Substituting u_0 into eq. (d)

$$\begin{aligned} \left(\frac{P}{P_y}\right)^2 - \left(\frac{P}{P_y}\right) \left[\frac{\sigma_p}{\sigma_y} + \frac{E\varphi b}{2\sigma_y} + \frac{P_E}{P_y} - \frac{E\varphi b}{2\sigma_y \left(2 - \frac{P}{P_E}\right)^2} \right] - \frac{2m_0}{d_i P_y} \frac{P_E}{P_y} \\ + \frac{P_E}{P_y} \left[\frac{\sigma_p}{\sigma_y} + \frac{E\varphi b}{\sigma_y \left(2 - \frac{P}{P_E}\right)} - \frac{E\varphi b}{\sigma_y \left(2 - \frac{P}{P_E}\right)^2} \right] = 0. \end{aligned} \quad (m)$$

Solving this pseudo-quadratic equation and rearranging terms gives finally eq. (19). Setting $\sigma_p = \sigma_y$ and $\varphi = 0$ eq. (20), which is the approximate solution when no residual stresses are present in the column, is obtained.

b) Bending About the Weak Axis of an H-Shape

Case I. Yielding only in Compression

Axial equilibrium requires that at the center of the column the following equation must be satisfied:

$$\frac{P}{a} = \int_{x_1}^{b/2} \left[\sigma_p + E\varphi \left(\frac{b}{2} - x \right) \right] dx + \int_{-b/2}^{x_1} \left[\sigma_p + \frac{E\varphi b}{2} - E x_1 (\varphi + \phi_0) + E\phi_0 x \right] dx. \quad (n)$$

When integrated and corresponding terms combined, this becomes

$$0 = x_1^2 + x_1 b - \left[\frac{2\sigma_p b + E\varphi b^2}{E(\varphi + \phi_0)} \right] + \frac{b^2}{4} + \frac{2P}{Ea} \left[\frac{1}{\varphi + \phi_0} \right]. \quad (o)$$

Solving for x_1 ,

$$x_1 = -\frac{b}{2} + \sqrt{\frac{2\sigma_p b + E\varphi b^2 - \frac{2P}{A}}{E(\varphi + \phi_0)}}. \quad (p)$$

Moment equilibrium at the center section requires that

$$\begin{aligned} \frac{M_0}{a} = \frac{P}{a} u_0 + \frac{m_0}{a} = \int_{x_1}^{b/2} \left[\left(\sigma_p + \frac{E\varphi b}{2} \right) x - E\varphi x^2 \right] dx \\ + \int_{-b/2}^{x_1} \left[\left(\sigma_p + \frac{E\varphi b}{2} - E x_1 (\varphi + \phi_0) \right) x + E\phi_0 x^2 \right] dx. \end{aligned} \quad (q)$$

After integrating and combining terms, the following expression can be written:

$$F = x_1^3(\varphi + \phi_0) - \frac{3}{4}x_1 b^2(\varphi + \phi_0) + \frac{b^3}{4}(\varphi - \phi_0) + \frac{6P u_0}{Ea} + \frac{6m_0}{Ea} \equiv 0, \quad (r)$$

where

$$\phi_0 = \frac{\pi^2}{L^2} u_0$$

follows from the assumed sine deflection curve. The partial derivation of F with respect to u_0 gives

$$\left(x_1 + \frac{b}{2}\right)^3 = \left[\frac{2\sigma_p b + E\varphi b^2 - \frac{2P}{a}}{E\left(\varphi + \frac{\pi^2}{L^2} u_0\right)} \right]^{1/3} = \frac{12PL^2}{Ea\pi^2}. \quad (s)$$

Expressing u_0 from eq. (s) and substituting its value into eq. (r) the following is obtained:

$$0 = \left[\frac{12PL^2}{Ea\pi^2} \right]^{1/3} - (b) + \left[\frac{\frac{1}{3}E\varphi b^3 - 4\frac{P}{a} \frac{L^2}{\pi^2} + 4\frac{m_0}{a}}{2\sigma_p b + E\varphi b^2 - \frac{2P}{a}} \right], \quad (t)$$

from which eq. (27) was directly obtained.

Case II. Yielding in Tension and Compression

As in the preceeding case the equations for axial thrust and moment equilibrium at the column center are as follows:

$$\begin{aligned} \frac{P}{a} &= \int_{x_1}^{b/2} \left[\sigma_p + E\varphi \left(\frac{b}{2} - x \right) \right] dx + \int_{-x_2}^{x_1} \left[\sigma_p + \frac{1}{2}E\varphi b - E x_1 (\varphi + \phi_0) + E\phi_0 x \right] dx \\ &\quad + \int_{-b/2}^{-x_2} \left[\sigma_p + E\varphi \left(\frac{b}{2} - x \right) \right] dx, \end{aligned} \quad (u)$$

from which

$$0 = -\frac{2P}{Ea} - (\varphi + \phi_0) x_1^2 + (\varphi + \phi_0) x_2^2. \quad (v)$$

However, x_1 and x_2 are related by geometry (see fig. 8c).

$$\frac{1}{E} (2\sigma_p + E\varphi b) = x_1(\varphi + \phi_0) + x_2(\varphi + \phi_0). \quad (w)$$

Solving eqs. (v) and (w) simultaneously:

$$\begin{aligned} x_1 &= \left[\frac{\sigma_p + \frac{1}{2} E \varphi b}{E(\varphi + \phi_0)} - \frac{P}{2 \sigma_p a + E \varphi a b} \right], \\ x_2 &= \left[\frac{\sigma_p + \frac{1}{2} E \varphi b}{E(\varphi + \phi_0)} - \frac{P}{2 \sigma_p a + E \varphi a b} \right]. \end{aligned} \quad (\text{x})$$

The moment at the center of the column equals

$$\begin{aligned} \frac{M_0}{a} &= \frac{P}{a} u_0 + \frac{m_0}{a} = \int_{x_1}^{b/2} \left[\left(\sigma_p + \frac{1}{2} E \varphi b \right) x - E \varphi x^2 \right] dx \\ &+ \int_{-x_2}^{x_1} \left[\left(\sigma_p + \frac{1}{2} E \varphi b - E x_1 (\varphi + \phi_0) \right) x + E \phi_0 x^2 \right] dx \\ &+ \int_{-b/2}^{-x_2} \left[\left(\sigma_p + \frac{1}{2} E \varphi b \right) x - E \varphi x^2 \right] dx, \end{aligned} \quad (\text{y})$$

from which

$$\begin{aligned} F &= \frac{3}{2} \frac{b^3}{E} \left(\sigma_p + \frac{1}{2} E \varphi b \right) - \frac{1}{2} \varphi b^3 - x_1^3 (\varphi + \phi_0) - x_2^3 (\varphi + \phi_0) \\ &- \frac{6 P}{E a} u_0 - \frac{6 m_0}{E a} \equiv 0. \end{aligned} \quad (\text{z})$$

As before F is a linear function of u_0 . The partial derivative of F gives:

$$\left(\varphi + \frac{\pi^2}{L^2} u_0 \right)^3 = \frac{2 \left(\frac{\sigma_p}{E} + \frac{1}{2} b \varphi \right)^3}{\frac{3 P}{E a} \frac{L^2}{\pi^2}}. \quad (\text{aa})$$

Expressing u_0 from eq. (aa) and substituting into eq. (z).

$$\begin{aligned} 0 &= \left[\frac{2 \left(\frac{\sigma_p}{E} + \frac{1}{2} \varphi b \right)^3}{\frac{3 P}{E a} \frac{L^2}{\pi^2}} \right]^{2/3} \left[\frac{3}{2} b^2 \left(\frac{\sigma_p}{E} + \frac{1}{2} b \varphi \right) - \frac{1}{2} \varphi b^3 - \frac{6 m_0}{E a} \right. \\ &\quad \left. - \left\{ \frac{3 P^2}{2 E a^2} \right\} + \frac{6 P}{E a} \frac{L^2}{\pi^2} \varphi \right] - 6 \left(\frac{\sigma_p}{E} + \frac{1}{2} b \varphi \right)^3, \end{aligned} \quad (\text{bb})$$

from which eq. (28) can be directly obtained.

Summary

A general method, applicable to both eccentrically and axially loaded columns was developed along the lines of the classical inelastic column theory. In the development it was assumed that the residual stress distribution had at least one axis of symmetry which was further assumed to be normal to the plane of bending.

Approximate solutions, based on the same concepts as Jezek's solution for beam-columns, were then presented for the rectangular and idealized *H*-section of elasto-plastic material.

Results compare well with the rigorous solutions and also with test results. It was shown that the influence of residual stress decreased with increasing eccentricities. Up to a value of the eccentricity ratio of about $\frac{ec}{r^2} = 1.0$, the influence of residual stress in determining the carrying capacity of the member may be considerable.

Résumé

Un procédé de validité générale, applicable aux barres comprimées chargées aussi bien excentriquement que centriquement, a été mis au point sur les bases de la théorie classique des barres dans le domaine inélastique. Il a été supposé que la répartition des contraintes propres présente tout au moins un axe de symétrie, que l'on a en outre considéré comme perpendiculaire au plan du moment fléchissant.

Des solutions approchées, reposant sur les mêmes hypothèses que la solution de Jezek pour les barres fléchies sous pression axiale, ont été indiquées pour des piliers constitués par des matériaux plasto-élastiques, avec section rectangulaire et section idealisée en *H*.

La comparaison des résultats obtenus avec des solutions rigoureuses et des résultats expérimentaux est très satisfaisante. Il a été indiqué que l'influence des contraintes propres diminue lorsque l'excentricité augmente. Jusqu'à une excentricité $ec/r^2 = 1,0$, l'influence des contraintes propres sur la capacité portante de la barre peut être considérable.

Zusammenfassung

Ein allgemeingültiges Verfahren, anwendbar sowohl auf außermittig als auch auf zentrisch belastete Druckstäbe, wurde nach den Grundlinien der klassischen Theorie für Stäbe im unelastischen Bereich entwickelt. In der Ableitung wurde vorausgesetzt, daß die Eigenspannungsverteilung mindestens eine Symmetriearchse aufweist, die ferner als senkrecht zur Ebene des Biegemomentes angenommen wurde.

Näherungslösungen, fußend auf denselben Voraussetzungen wie Jezeks Lösung für Biegestäbe mit Axialdruck, wurden dann für Stützen aus elastisch-plastischem Material mit Rechteck- und idealisiertem *H*-Querschnitt gegeben.

Der Vergleich der Ergebnisse mit strengen Lösungen und Versuchsresultaten ist sehr zufriedenstellend. Es wurde gezeigt, daß der Einfluß der Eigenspannungen mit zunehmender Außermittigkeit abnimmt. Bis zu einer Exzentrizität von $\frac{e c}{r^2} = 1.0$ kann der Einfluß der Eigenspannungen auf die Tragfähigkeit des Stabes beträchtlich sein.